Research Article

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Some results on the total proper *k*-connection number

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Abstract: In this paper, we first investigate the total proper connection number of a graph G according to some constraints of $\overline{G}$. Next, we investigate the total proper connection numbers of graph G with large clique number $\omega(G) = n - s$ for $1 \le s \le 3$. Finally, we determine the total proper k-connection numbers of circular ladders, Möbius ladders and all small cubic graphs of order 8 or less.

Keywords: total coloring, total proper path, total proper k-connected, total proper k-connection number, complement graph, clique number

MSC 2020: 05C15, 05C35, 05C40

1 Introduction

In this paper, all graphs under our consideration are simple, finite and undirected. We follow the notation and terminology of [1]. For a graph G, we denote by $V(G)$, $E(G)$ and $\text{diam}(G)$ the vertex set, edge set and diameter of G, respectively. The *distance* between two vertices u and v in a connected graph G, denoted by $\text{dist}(u, v)$, is the length of a shortest path between them in G. The *eccentricity* of a vertex v in G is defined as $\text{ecc}_G(v) = \max_{x \in V(G)} \text{dist}(v, x)$. For convenience, a set of internally pairwise vertex disjoint paths will be called *disjoint*.

In recent years, colored notions of connectivity in graphs becomes a new and active subject in graph theory. Stating from rainbow connection [2], rainbow vertex connection [3] and total rainbow connection [4,5] appeared later. Many researchers are working in this field, and a lot of papers have been published in journals, see [6–16] for details. The reader can also see [17] for a survey, [18] for a dynamic survey and [19] for a new monograph on this topic.

In 2012, Borozan et al. [20] introduced the concept of proper k-connection number. A path in an edge-colored graph is a *proper path* if any two adjacent edges on the path differ in color. An edge-colored graph is *proper k-connected* if any two distinct vertices of the graph are connected by k disjoint proper paths. *The proper k-connection number* of a k-connected graph G, denoted by $pc_k(G)$, is defined as the smallest number of colors that are needed in order to make G proper k-connected. For more results, the reader can see [21–24] for details.

As a natural generalization, Jiang et al. [25] presented the concept of proper vertex k-connection number. A path in a vertex-colored graph is a *vertex proper path* if any two internal adjacent vertices of the path differ in color. A vertex-colored graph is *proper vertex k-connected* if any two distinct vertices of the graph are connected by k disjoint vertex proper paths. For a k-connected graph G, *the proper vertex*

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k-connection number of G, denoted by $pvc_k(G)$, is defined as the smallest number of colors required to make G proper vertex k-connected.

Motivated by the concept of total chromatic number of graph, now for proper connection and proper vertex connection, the concept of total proper connection was introduced by Jiang et al. [26]. A *total coloring* of a graph G is a mapping from the set $V(G) \cup E(G)$ to some finite set of colors. A path in a total-colored graph is a *total proper path* if the coloring of the edges and internal vertices is proper, that is, any two adjacent or incident elements of edges and internal vertices on the path differ in color. A total-colored graph is *total proper k-connected* if any two distinct vertices of the graph are connected by k disjoint total proper paths. For a connected graph G, the *total proper k-connection number* of a k-connected graph G, denoted by $\mathrm{tpc}_k(G)$, is defined as the smallest number of colors that are needed in order to make G total proper k-connected. For convenience, we write $\mathrm{tpc}(G)$ for $\mathrm{tpc}_1(G)$. Obviously, $\mathrm{tpc}(G) \le \mathrm{tpc}_2(G) \le \mathrm{tpc}_3(G)$. By [26], if G is complete, then $\mathrm{tpc}(G) = 1$; if G has a Hamiltonian path that is not complete, then $\mathrm{tpc}(G) = 3$. Note that if G is a nontrivial connected graph and H is a connected spanning subgraph of G, then $\mathrm{tpc}(G) \le \mathrm{tpc}(H)$.

In this paper, we investigate the total proper connection number of a graph G under some constraints on its complement graph $\overline{G}$.

Theorem 1.1. *Let G be a connected graph of order $n \ge 3$, if* $\mathrm{diam}(\overline{G})$ *does not belong to the following two cases*: (*i*) $\mathrm{diam}(\overline{G}) = 2, 3$, (*ii*) $\overline{G}$ *contain exactly two components and one of them is trivial, then* $\mathrm{tpc}(G) \le 4$.

For the remaining cases, $\mathrm{tpc}(G)$ can be very large as discussed in Section 2. Then we add a constraint, i.e., we let $\overline{G}$ be triangle-free. Hence, G is claw-free, and we can derive our next main result:

Theorem 1.2. *For a connected graph G, if $\overline{G}$ is triangle-free, then* $\mathrm{tpc}(G) = 3$.

Recall that a clique of a graph is a set of mutually adjacent vertices, and that a maximum clique is a clique of the largest possible size in a given graph. The clique number $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G. Let G be a connected graph, and let X be a maximum clique of G. We say that $N_X(u)$ is the set of neighbors of u in $G[X]$ and $d_X(u) = |N_X(u)|$. Let $F = G[V(G)\backslash X]$. Kemnitz and Schiermeyer [9] considered graphs with $rc(G) = 2$ and large clique number. In this paper, we characterize graphs with small total proper connection number with respect to their large clique number. If $\omega(G) = n$, then G is a complete graph, which implies $\mathrm{tpc}(G) = 1$. If G is connected and $\omega(G) = n - 1$, then G has a Hamiltonian path, and so $\mathrm{tpc}(G) = 3$. For the cases $\omega(G) = n - 2, n - 3$, we obtain the following three main results.

Theorem 1.3. *Let G be a connected graph of order n. If $\omega(G) = n - 2$ and X is a maximum clique of G with $V(G)\backslash X = \{u_1, u_2\}$, then* $\mathrm{tpc}(G) = 3$.

Theorem 1.4. *Let G be a connected graph of order n,* $\mathrm{diam}(G) = 2$. *If $\omega(G) = n - 3$ and X is a maximum clique of G with $V(G)\backslash X = \{u_1, u_2, u_3\}$, then* $\mathrm{tpc}(G) = 3$ *or* $\mathrm{tpc}(G) = 4$ *for the following case $F \cong 3K_3$, $|N_X(u_1) \cap N_X(u_2) \cap N_X(u_3)| = 1$ and $d_X(u_1) = d_X(u_2) = d_X(u_3) = 1$.*

Theorem 1.5. *Let G be a connected graph of order n,* $\mathrm{diam}(G) \ge 3$. *If $\omega(G) = n - 3$ and X is a maximum clique of G with $V(G)\backslash X = \{u_1, u_2, u_3\}$, then* $\mathrm{tpc}(G) = 3$, *or* $\mathrm{tpc}(G) = 4$ *and one of the following holds.*

(*i*) *$F \cong 3K_3$, $N_X(u) \cap N_X(v) \ne \emptyset$ and $d_X(u) = d_X(v) = 1$, where u and v are any two distinct vertices in $V(G)\backslash X$.*
(*ii*) *$F \cong 3K_3$, $d_X(u_1) = d_X(u_2) = d_X(u_3) = 1$ and for any two vertices in $V(G)\backslash X$, there is no common neighbor in $G[X]$.*

For an integer $n \ge 3$, the circular ladder CL_{2n} of order $2n$ is a cubic graph constructed by taking two copies of the cycle C_n on disjoint vertex sets $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$, then joining the corresponding vertices u_iv_i for $1 \le i \le n$. The Möbius ladder M_{2n} of order $2n$ is obtained from the ladder by deleting the edges u_1u_n and v_1v_n, and then inserting two edges u_1v_n and u_nv_1. Subscripts are considered modulo n, and we can derive our next main result:

Theorem 1.6. *Let n be an integer with $n \geq 3$. Then*
(*i*) $\text{tpc}(\text{CL}_{2n}) = \text{tpc}_2(\text{CL}_{2n}) = 3$, $\text{tpc}_3(\text{CL}_{2n}) = 4$.
(*ii*) $\text{tpc}(M_{2n}) = \text{tpc}_2(M_{2n}) = 3$, $\text{tpc}_3(M_{2n}) = 4$.

In [7], Fujie-Okamoto et al. investigated the rainbow k-connection numbers of all small cubic graphs of order 8 or less. In this paper, we determine the total proper k-connection numbers of all small cubic graphs of order 8 or less. We can easily verify that all such cubic graphs have orders 4, 6, or 8, and those with orders 4 or 6 are K_4, $K_{3,3}$, and $K_3 \square K_2$ (where $\square$ denotes Cartesian product). In [27], it was shown that all connected cubic graphs of order 8 are Q_3, M_8, F_1, F_2, and F_3, and these graphs are depicted in Figure 1. Our last main result is stated as follows:

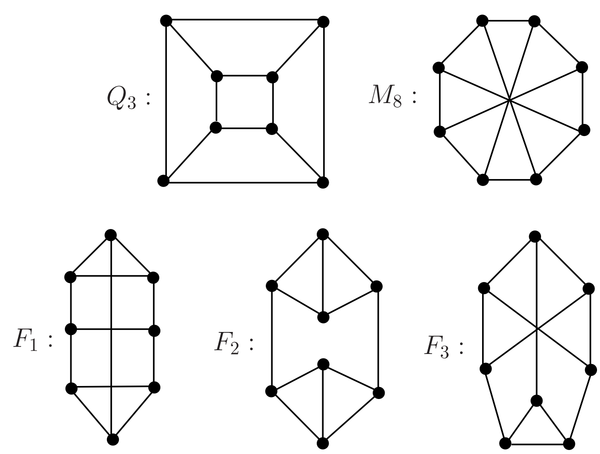

Figure 1: All connected cubic graphs of order 8.

Theorem 1.7.
(*i*) $\text{tpc}(K_4) = 1$, $\text{tpc}_2(K_4) = 3$, $\text{tpc}_3(K_4) = 4$.
(*ii*) $\text{tpc}(K_{3,3}) = \text{tpc}_2(K_{3,3}) = 3$, $\text{tpc}_3(K_{3,3}) = 4$.
(*iii*) $\text{tpc}(K_3 \square K_2) = \text{tpc}_2(K_3 \square K_2) = 3$, $\text{tpc}_3(K_3 \square K_2) = 4$.
(*iv*) $\text{tpc}(Q_3) = \text{tpc}_2(Q_3) = 3$, $\text{tpc}_3(Q_3) = 4$.
(*v*) $\text{tpc}(M_8) = \text{tpc}_2(M_8) = 3$, $\text{tpc}_3(M_8) = 4$.
(*vi*) $\text{tpc}(F_1) = \text{tpc}_2(F_1) = 3$, $\text{tpc}_3(F_1) = 4$.
(*vii*) $\text{tpc}(F_2) = \text{tpc}_2(F_2) = 3$.
(*viii*) $\text{tpc}(F_3) = \text{tpc}_2(F_3) = 3$, $\text{tpc}_3(F_3) = 4$.

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1. [26] *For $2 \leq m \leq n$, we have* $\text{tpc}(K_{m,n}) = 3$.

Lemma 2.2. [26] *If G is a complete multipartite graph that is neither a complete graph nor a tree, then* $\text{tpc}(G) = 3$.

Let $N^i_{\overline{G}}(x) = \{v : \text{dist}_{\overline{G}}(x, v) = i\}$, where $0 \leq i \leq 3$, and $N^4_{\overline{G}}(x) = \{v : \text{dist}(x, v) \geq 4\}$. In this paper, we use $N^i_{\overline{G}}$ instead of $N^i_{\overline{G}}(x)$ for convenience. Then $N^0_{\overline{G}} = \{x\}$ and $N^1_{\overline{G}} = N_{\overline{G}}(x)$.

Lemma 2.3. *For a connected graph G, if $\overline{G}$ is connected and* $\mathrm{diam}(\overline{G}) \geq 4$, *then* $\mathrm{tpc}(G) \leq 4$.

Proof. Choose a vertex x with $\mathrm{ecc}_{\overline{G}}(x) = \mathrm{diam}(\overline{G})$. By the definition of $N^i_{\overline{G}}$, we know $uv \in E(G)$ for any $u \in N^i_{\overline{G}}$, $v \in N^j_{\overline{G}}$ with $|i-j| \geq 2$. Now we define a total coloring of G as follows: assign color 1 to the edges xu for $u \in N^2_{\overline{G}}$, all edges between $N^1_{\overline{G}}$ and $N^3_{\overline{G}}$, and all vertices and edges in $N^4_{\overline{G}}$; assign color 2 to the edges xu for $u \in N^3_{\overline{G}}$, all edges between $N^2_{\overline{G}}$ and $N^4_{\overline{G}}$, and all vertices and edges in $N^1_{\overline{G}}$; assign color 3 to the edges xu for $u \in N^4_{\overline{G}}$, all edges between $N^1_{\overline{G}}$ and $N^4_{\overline{G}}$, and all vertices and edges in $N^2_{\overline{G}}$, $N^3_{\overline{G}}$; assign color 4 to the vertex x.

We prove that there is a total proper path between any two vertices u and v of G. It is trivial when $uv \in E(G)$. Thus, we only need to consider the pairs $u, v \in N^i_{\overline{G}}$ or $u \in N^i_{\overline{G}}$, $v \in N^{i+1}_{\overline{G}}$. Note that $P = xx_3x_1x_4x_2$ is a total proper path, where $x_i \in N^i_{\overline{G}}$. By means of the path P, we can find that u and v are connected by some total proper path for any $u \in N^i_{\overline{G}}$, $v \in N^{i+1}_{\overline{G}}$. If $i = 1$, then $P = ux_3xx_2x_4v$ is a total proper path, where $x_i \in N^i_{\overline{G}}$. If $i = 2$, then $P = uxx_4v$ is a total proper path, where $x_4 \in N^4_{\overline{G}}$. If $i = 3$, then $P = ux_1x_4x_2xv$ is a total proper path, where $x_i \in N^i_{\overline{G}}$. If $i = 4$, then $P = uxx_2v$ is a total proper path, where $x_2 \in N^2_{\overline{G}}$. Hence, $\mathrm{tpc}(G) \leq 4$. □

Proof of Theorem 1.1. Assume that $\overline{G}$ is connected. Since $\mathrm{diam}(\overline{G}) \geq 4$, we have $\mathrm{tpc}(G) \leq 4$ by Lemma 2.3. Assume that $\overline{G}$ is disconnected. By the assumption, we know that there exist either at least three connected components or exactly two nontrivial components. Let $\overline{G_i}$ be the components of $\overline{G}$ with $t_i = |V(\overline{G_i})|$, where $1 \leq i \leq h$. Then G contains a connected spanning subgraph $K_{t_1,t_2,\dots,t_h}$, and we have $\mathrm{tpc}(G) \leq \mathrm{tpc}(K_{t_1,t_2,\dots,t_h}) = 3$ by Lemma 2.2. Note that G is not complete. Thus, $\mathrm{tpc}(G) \geq 3$, and so $\mathrm{tpc}(G) = 3$. □

Next, we will give three examples to show that $\mathrm{tpc}(G)$ can be arbitrarily large if one of the following three conditions holds: $\mathrm{diam}(\overline{G}) = 2$, $\mathrm{diam}(\overline{G}) = 3$, $\overline{G}$ contains exactly two connected components and one of them is trivial.

Example 2.4. For the graph $\overline{G}$ shown in Figure 2, we choose a vertex x with $\mathrm{ecc}_{\overline{G}}(x) = \mathrm{diam}(\overline{G})$. Let $N^1_{\overline{G}}(x) = \{u_i | 1 \leq i \leq k\}$, $N^2_{\overline{G}}(x) = \{v_j | 1 \leq j \leq k\}$, and let $E(\overline{G}) = \{xu_i | 1 \leq i \leq k\} \cup \{u_{i_1}u_{i_2} | 1 \leq i_1, i_2 \leq k\} \cup \{v_{j_1}v_{j_2} | 1 \leq j_1, j_2 \leq k\} \cup \{u_iv_j | 1 \leq i, j \leq k\} \backslash \{u_iv_i | 1 \leq i \leq k\}$, where $k \geq 3$. Obviously, $\mathrm{diam}(\overline{G}) = 2$ and G is a tree. Then $\mathrm{tpc}(G) = \Delta(G) + 1 = k + 1$ by [26, Theorem 1]. Observe that $\mathrm{tpc}(G)$ will be arbitrarily large based on the increase of k.

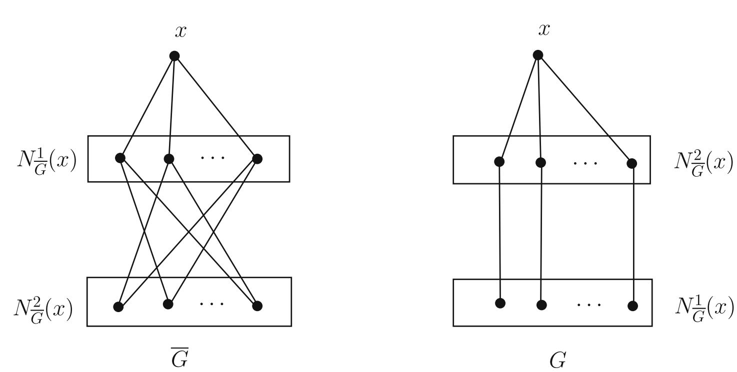

Figure 2: The graph of Example 2.4.

Example 2.5. For the graph $\overline{G}$ shown in Figure 3, we choose a vertex x with $\mathrm{ecc}_{\overline{G}}(x) = \mathrm{diam}(\overline{G})$. Let $N^1_{\overline{G}}(x) = \{u_i | 1 \leq i \leq k\}$, $N^2_{\overline{G}}(x) = \{v_j | 1 \leq j \leq k\}$, and $N^3_{\overline{G}}(x) = \{w_s | 1 \leq s \leq k\}$, where $k \geq 3$. Furthermore, let $E(\overline{G}) = \{xu_i | 1 \leq i \leq k\} \cup \{u_iv_j | 1 \leq i, j \leq k\} \cup \{v_jw_s | 1 \leq j, s \leq k\} \cup \{v_{j_1}v_{j_2} | 1 \leq j_1, j_2 \leq k\}$. Obviously, $\mathrm{diam}(\overline{G}) = 3$ and G is a connected graph. Note that $N^2_{\overline{G}}(x)$ is a stable set in $\overline{G}$, and each edge between x and $N^2_{\overline{G}}(x)$ is a cut edge in G. Therefore, $\mathrm{tpc}(G) \geq k + 1$ by [26, Proposition 2], and so $\mathrm{tpc}(G)$ will be arbitrarily large based on the increase of k.

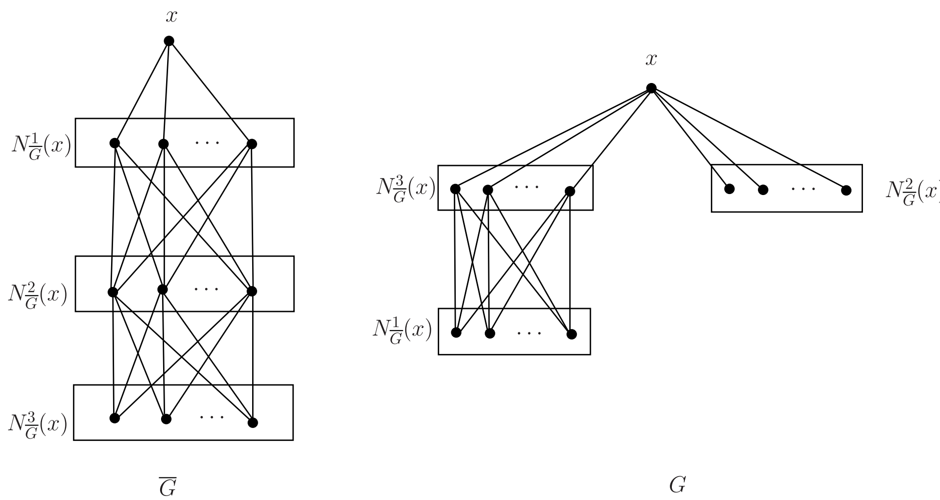

Figure 3: The graph of Example 2.5.

Example 2.6. Let $\overline{G}$ contains exactly two components $\overline{G_1}$ and $\overline{G_2}$, where $\overline{G_1}$ is trivial and $\overline{G_2}$ is a clique of $\overline{G}$. Clearly, G is a star, and $\mathrm{tpc}(G) = |V(\overline{G_2})| + 1$. Thus, $\mathrm{tpc}(G)$ can be made arbitrarily large by increasing $|V(\overline{G_2})|$.

3 Proof of Theorem 1.2

Lemma 3.1. *For a connected graph G, if $\overline{G}$ is connected with* $\mathrm{diam}(\overline{G}) \geq 4$, *and $\overline{G}$ is triangle-free, then* $\mathrm{tpc}(G) = 3$.

Proof. Choose a vertex x with $\mathrm{ecc}_{\overline{G}}(x) = \mathrm{diam}(\overline{G})$. Since $\overline{G}$ is triangle-free, we know that $N^1_{\overline{G}}$ is a clique in G. Now we define a total coloring of G as follows: assign color 1 to the edges xu for $u \in N^2_{\overline{G}}$, all edges between $N^1_{\overline{G}}$ and $N^3_{\overline{G}}$, and all vertices and edges in $N^4_{\overline{G}}$; assign color 2 to the edges between $N^2_{\overline{G}}$ and $N^4_{\overline{G}}$, all vertices and edges in $N^1_{\overline{G}}$, and the vertex x; assign color 3 to the edges xu for $u \in N^3_{\overline{G}}, N^4_{\overline{G}}$, all edges between $N^1_{\overline{G}}$ and $N^4_{\overline{G}}$, and all vertices and edges in $N^2_{\overline{G}}, N^3_{\overline{G}}$.

We prove that there is a total proper path between any two distinct vertices u and v in G. Note that $P = xx_2x_4x_1x_3$ is a total proper path, where $x_i \in N^i_{\overline{G}}$. By means of the path P, we can find that u and v are connected by some total proper path for any $u \in N^i_{\overline{G}}, v \in N^{i+1}_{\overline{G}}$. Thus, we only need to consider the pairs $u, v \in N^i_{\overline{G}}$. For $i = 2$, $P = uxx_4v$ is a total proper path, where $x_4 \in N^4_{\overline{G}}$. For $i = 4$, $P = uxx_2v$ is a total proper path, where $x_2 \in N^2_{\overline{G}}$. For $i = 3$, $P = ux_1x_4x_2xv$ is a total proper path, where $x_i \in N^i_{\overline{G}}$. Thus, G is total proper connected with the above coloring, and so $\mathrm{tpc}(G) = 3$. □

Lemma 3.2. *Let G be a connected graph. If* $\mathrm{diam}(\overline{G}) = 3$ *and $\overline{G}$ is triangle-free, then* $\mathrm{tpc}(G) = 3$.

Proof. For a vertex x of $\overline{G}$ satisfying $\mathrm{ecc}_{\overline{G}}(x) = \mathrm{diam}(\overline{G}) = 3$, let n_i represent the number of vertices with distance i from x. If $n_1 = n_2 = n_3 = 1$, then $G \cong P_4$, and so $\mathrm{tpc}(G) = 3$.

Case 1. Two of n_1, n_2, n_3 are equal to 1. Without loss of generality, we may assume $n_1 = n_2 = 1$. Since $\overline{G}$ is triangle-free, we have that $N^3_{\overline{G}}$ is a stable set in $\overline{G}$, and so a clique in G. We can find that G has a Hamiltonian path. Thus, $\mathrm{tpc}(G) = 3$.

Case 2. One of n_1, n_2, n_3 is equal to 1. Suppose $n_2 = 1$. Since $\overline{G}$ is triangle-free, we know that $N^1_{\overline{G}}$ and $N^3_{\overline{G}}$ is a stable set in $\overline{G}$, and so a clique in G. Note that G has a Hamiltonian path, and so $\mathrm{tpc}(G) = 3$.

Subcase 2.1. $n_1 = 1$. Since $\overline{G}$ is triangle-free, we obtain that $N^2_{\overline{G}}$ is a clique in G. Define a total coloring of G as follows: assign color 3 to the vertex x, all edges between $N^2_{\overline{G}}$ and $N^3_{\overline{G}}$, and all edges between $N^1_{\overline{G}}$ and $N^3_{\overline{G}}$; assign color 2 to the edges xu for $u \in N^2_{\overline{G}}$, and all vertices and edges in $N^3_{\overline{G}}$; assign color 1 to the edges xu for $u \in N^3_{\overline{G}}$, and all vertices and edges in $N^1_{\overline{G}}, N^2_{\overline{G}}$. We prove that there is a total proper path between any two distinct vertices u and v in G. Note that $P = x_1x_3xx_2$ is a total proper path, where $x_i \in N^i_{\overline{G}}$. By means of the path P, we know that u and v are connected by some total proper path for any $u \in N^i_{\overline{G}}, v \in N^{i+1}_{\overline{G}}$. For any two vertices $u, v \in N^3_{\overline{G}}$, it is trivial if $uv \in E(G)$. If $uv \notin E(G)$, since $u, v \in N^3_{\overline{G}}$, there exist two vertices $u', v' \in N^3_{\overline{G}}$ such that $uu', vv' \in E(\overline{G})$. Since $\overline{G}$ is triangle-free, we can see that $u' \neq v'$ and $vu', uv' \in E(G)$. Then $P = uxu'v$ ia a total proper path. Hence, G is total proper connected with the above coloring, and so $\text{tpc}(G) = 3$.

Subcase 2.2. $n_3 = 1$. Since $\overline{G}$ is triangle-free, we know that $N^1_{\overline{G}}$ is a stable set in $\overline{G}$, and so a clique in G. Define a total coloring of G as follows: assign color 3 to the vertex x, and all edges between $N^1_{\overline{G}}$ and $N^3_{\overline{G}}$; assign color 2 to the edges xu for $u \in N^2_{\overline{G}}$, and all vertices and edges in $N^3_{\overline{G}}$; assign color 1 to the edges xu for $u \in N^3_{\overline{G}}$, and all vertices and edges in $N^1_{\overline{G}}, N^2_{\overline{G}}$. We prove that there is a total proper path between any two distinct vertices u and v in G. Note that $P = x_1x_3xx_2$ is a total proper path, where $x_i \in N^i_{\overline{G}}$. By means of the path P, we obtain that u and v are connected by some total proper path for any $u \in N^i_{\overline{G}}, v \in N^{i+1}_{\overline{G}}$. Let u, v be any two distinct vertices of $N^2_{\overline{G}}$, and $N^3_{\overline{G}} = \{y\}$. If y is adjacent to any vertex of $N^2_{\overline{G}}$ in $\overline{G}$, then $N^2_{\overline{G}}$ is a clique in G, and so G has a Hamiltonian path. Otherwise, let V_y denote the set of neighbors of y in $N^2_{\overline{G}}$ in G. We can check that $P = uyxv$ is a total proper path, where $u, v \in V_y$. If $|N^2_{\overline{G}} \backslash V_y| = 1$, then $P = uyxv$ is a total proper path, where $u \in V_y, v \in N^2_{\overline{G}} \backslash V_y$. If $|N^2_{\overline{G}} \backslash V_y| \geq 2$, then G is claw-free since $\overline{G}$ is triangle-free, and $G[x \cup N^2_{\overline{G}} \backslash V_y]$ is a complete graph. Note that $P = uyxv$ is a total proper path, where $u \in V_y, v \in N^2_{\overline{G}}$. Thus, G is total proper connected with the above coloring, and so $\text{tpc}(G) = 3$.

Case 3. $n_1, n_2, n_3 \geq 2$. Since $\overline{G}$ is triangle-free, we have that $N^1_{\overline{G}}$ is a stable set in $\overline{G}$, and so a clique in G. If any vertex in $N^3_{\overline{G}}$ is adjacent to all vertices of $N^2_{\overline{G}}$ in $\overline{G}$, then both $N^2_{\overline{G}}$ and $N^3_{\overline{G}}$ are stable sets in $\overline{G}$, and so cliques in G. Thus, G has a Hamiltonian path, and so $\text{tpc}(G) = 3$.

Otherwise, we choose a vertex $u \in N^3_{\overline{G}}$, let V_u denote the set of neighbors of u in $N^2_{\overline{G}}$ in G, we have $V_u \neq \emptyset, N^2_{\overline{G}}$. Define a total coloring of G: assign color 2 to the vertex x, all vertices and edges in $N^1_{\overline{G}}$, and all edges between $N^2_{\overline{G}}, N^3_{\overline{G}}$; assign color 3 to the vertex u, all edges between $N^1_{\overline{G}}$ and $N^3_{\overline{G}} \backslash \{u\}$, and all edges between x and V_u; assign color 2 to the remaining vertices and edges. Note that $P = xvux_1$ is a total proper path, where $v \in V_u, x_1 \in N^1_{\overline{G}}$. For any two vertices $w, z \in N^3_{\overline{G}}$, $P = wx_1uvxz$ is a total proper path, where $x_1 \in N^1_{\overline{G}}, v \in V_u$. For any two vertices $w, z \in N^2_{\overline{G}}$, $P = wuxv$ is a total proper path, where $u, v \in V_y$. If $|N^2_{\overline{G}} \backslash V_y| = 1$, then $P = uxv$ is a total proper path, where $u \in V_y, v \in N^2_{\overline{G}} \backslash V_y$. If $|N^2_{\overline{G}} \backslash V_y| \geq 2$, since $\overline{G}$ is triangle-free, we know that G is claw-free, and the subgraph $G[x \cup N^2_{\overline{G}} \backslash V_y]$ is a complete graph. Note that $P = uxv$ is a total proper path, where $u \in V_y, v \in N^2_{\overline{G}} \backslash V_y$. For any $w \in N^2_{\overline{G}}, z \in N^3_{\overline{G}}$, $P = wxvux_1z$ is a total proper path, where $v \in V_y, x_1 \in N^1_{\overline{G}}$. Similarly, there is a total proper path connecting any two vertices $w \in N^2_{\overline{G}}, z \in N^1_{\overline{G}}$. Hence, G is total proper connected, and so $\text{tpc}(G) = 3$. □

Lemma 3.3. *For a connected graph G, if $\overline{G}$ is triangle-free and* $\text{diam}(\overline{G}) = 2$, *then* $\text{tpc}(G) = 3$.

Proof. Choose a vertex x with $\text{ecc}_{\overline{G}}(x) = \text{diam}(\overline{G}) = 2$. Since G is connected, we have $n_1 \geq 2$, $n_2 = 1$ or $n_1, n_2 \geq 2$, and there exist two vertices $u \in N^1_{\overline{G}}, v \in N^2_{\overline{G}}$ such that $uv \in E(G)$. Assume $n_1 \geq 2$ and $n_2 = 1$. Since $\overline{G}$ is triangle-free, we know that $N^1_{\overline{G}}$ is a stable set in $\overline{G}$, and so a clique in G. Note that G has a Hamiltonian path, and so $\text{tpc}(G) = 3$.

Assume $n_1, n_2 \geq 2$. Observe that $N^1_{\overline{G}}$ is a stable set in $\overline{G}$ since $\overline{G}$ is triangle-free, and so a clique in G. We show a total coloring of G as follows: assign color 1 to the vertex x, the edge uv and all vertices in $N^1_{\overline{G}} \backslash u$; assign color 3 to the vertex v and all edges in $N^1_{\overline{G}}$; assign color 2 to the remaining vertices and edges. If there exist some vertices $w \in N^2_{\overline{G}}$ with $d_{\overline{G}}(w) = n - 2$, then w is adjacent to the remaining vertices except x in $\overline{G}$.

Since $\mathrm{diam}(\overline{G}) = 2$, there exists an edge $w_1w_2 \in E(\overline{G})$ with $w_1 \in N^1_{\overline{G}}$, $w_2 \in N^2_{\overline{G}}$. Thus, w, w_1, w_2 is a triangle in $\overline{G}$, a contradiction. Hence, $d_{\overline{G}}(w) < n-2$ for all $w \in N^2_{\overline{G}}$, and so $d_G(w) \geq 2$. For any $z \in N^1_{\overline{G}}$, we know that $P = xvuz$ is a total proper path. For any $y \in N^2_{\overline{G}} \backslash \{v\}$ and $z \in N^1_{\overline{G}}$, if $N_G(y) \cap N^1_{\overline{G}} \neq \varnothing$, let $w \in N_G(y) \cap N^1_{\overline{G}}$. Then ywz is a total proper path. Otherwise, let $N_G(y) \cap N^1_{\overline{G}} = \varnothing$. We claim that y is adjacent to all the other vertices of $N^2_{\overline{G}}$ in G. In fact, for any vertex $w \in N^2_{\overline{G}} \backslash \{y\}$, there exists a vertex $w' \in N^1_{\overline{G}}$ such that $ww' \in E(\overline{G})$. Since $yw' \in E(\overline{G})$, we know that $yw \in E(G)$. Then $yvuz$ is a total proper path. Next we consider $w, z \in N^2_{\overline{G}}$ such that $wz \notin E(G)$. Since $\overline{G}$ is triangle-free, we have that G is claw-free, and at least one of w and z is adjacent to the v, without loss of generality, assume that $wv \in E(G)$. Since $w, z \in N^2_{\overline{G}}$, there exist two vertices $w', z' \in N^2_{\overline{G}}$ such that $ww', zz' \in E(\overline{G})$, and $w' \neq z'$. Then $zw', wz' \in E(G)$ and $P = wvuw'z$ is a total proper path. Thus, G is total proper connected with the above coloring. Hence, $\mathrm{tpc}(G) = 3$. □

Lemma 3.4. *Let G be a connected graph of order $n \geq 3$. If $\overline{G}$ is disconnected and triangle-free, then* $\mathrm{tpc}(G) = 3$.

Proof. Suppose $\overline{G}$ is triangle-free and contains two connected components one of which is trivial. Let $\overline{G_1}$ and $\overline{G_2}$ be the two components of $\overline{G}$, where $V(\overline{G_1}) = \{u\}$. Then u is adjacent to any other vertex in G. We will consider two cases according to the value of δ, where δ is the minimum degree of G. If $\delta = 1$, let $d(v) = \delta$. Since $\overline{G}$ is triangle-free, we know that G is claw-free, and the subgraph $G[V(G)\backslash\{v\}]$ is a complete graph. Thus, G has a Hamiltonian path, and so $\mathrm{tpc}(G) = 3$. If $\delta \geq 2$, let $d(v) = \delta$, $D = V(G)\backslash\{u, v\}$, and V_v be the set of neighbors of v in G. Now we define a total coloring of G as follows: assign color 1 to the vertex v and all the edges between u and V_v; assign color 3 to the vertex u and all the edges between v and V_v; assign color 2 to the remaining vertices and edges. Since G is claw-free, we can find that the subgraph $G[V(G)\backslash\{v\} \cup V_v]$ is a complete graph, and $P = v_1uvv_2$ is a total proper path, where $v_1, v_2 \in V_v$. For any $w \in V_v$, $z \in D\backslash V_v$, we obtain that $P = wuz$ is a total proper path. Thus, G is total proper connected with the above coloring, and so $\mathrm{tpc}(G) = 3$. Suppose $\overline{G}$ contains at least three connected components or exactly two nontrivial components. Then we have $\mathrm{tpc}(G) = 3$ by the similar proof of Theorem 1.1. □

Proof of Theorem 1.2. If $\overline{G}$ is connected, the result holds for the case $\mathrm{diam}(\overline{G}) \geq 4$ by Lemma 3.1, the case $\mathrm{diam}(\overline{G}) = 3$ by Lemma 3.2, and the case $\mathrm{diam}(\overline{G}) = 2$ by Lemma 3.3. If $\overline{G}$ is disconnected, the result holds by Lemma 3.4. □

4 Proof of Theorem 1.3

Suppose $F \cong K_2$. Note that G has a Hamiltonian path, and thus $\mathrm{tpc}(G) = 3$. Next, we compute the total proper connection number of G by proving the following claim.

Claim 1. Let G be a graph obtained by adding two pendant vertices $\{u_1, u_2\}$ to a vertex v_1 of a complete graph K_t. Then $\mathrm{tpc}(G) = 3$.

Proof. Since G is not a complete graph, we have $\mathrm{tpc}(G) \geq 3$. Now we only need to prove $\mathrm{tpc}(G) \leq 3$ by the following cases.

Case 1. $t \equiv 0 \pmod 3$. Assign a total coloring c to G as follows: Let $c(u_1v_1) = 1$, $c(u_2v_1) = 3$; $c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = 2$, $c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = 1$, and $c(v_{3i+3}) = c(v_{3i+1}u_{3i+2}) = 3$, where $0 \leq i \leq \frac{t}{3} - 1$. Observe that $P_1 = u_1v_1v_2 \cdots v_{t-1}v_t$ and $P_2 = u_2v_1v_tv_{t-1} \cdots v_3v_2$ are two total proper paths.

Case 2. $t \equiv 1 \pmod 3$. Assign a total coloring c to G as follows: Let $c(u_1v_1) = c(v_{t-1}v_1) = 1$, $c(v_t) = 2$, $c(u_2v_1) = c(v_tv_1) = 3$. Let i be an integer with $0 \leq i \leq \left\lfloor \frac{t}{3} \right\rfloor - 1$, $c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = 2$, $c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = 1$, and $c(v_{3i+3}) = c(v_{3i+1}u_{3i+2}) = 3$. We can find that $P_1 = u_1v_1v_2 \cdots v_{t-1}v_t$ and $P_2 = u_2v_1v_{t-1} \cdots v_3v_2v_t$ are two total proper paths.

Case 3. $t \equiv 2 \pmod 3$. Assign a total coloring c to G as follows: Let $c(u_1v_1) = c(v_tv_1) = c(v_t) = c(v_{t-2}v_1) = 1$, $c(v_{t-1}) = 2$, $c(u_2v_1) = c(v_tv_{t-1}) = c(v_{t-1}v_2) = 3$. Let i be an integer with $0 \le i \le \lfloor \frac{t}{3} \rfloor - 1$, $c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = 2$, $c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = 1$, and $c(v_{3i+3}) = c(v_{3i+1}u_{3i+2}) = 3$. We can easily verify that $P_1 = u_1v_1v_2 \cdots v_{t-1}v_t$, $P_2 = u_2v_1v_{t-2}v_{t-3} \cdots v_3v_2v_{t-1}$, and $P_3 = v_tv_1u_2$ are three total proper paths.

Thus, G is total proper connected with the above coloring, and so $\mathrm{tpc}(G) = 3$. This completes the proof of Claim 1. □

Suppose $F \cong 2K_1$. Assume that $N_X(u_1) \cap N_X(u_2) = \varnothing$. Then G has a Hamiltonian path, and so $\mathrm{tpc}(G) = 3$. Otherwise, if $|N_X(u_1) \cap N_X(u_2)| = d_X(u_1) = d_X(u_2) = 1$, then we know that $\mathrm{tpc}(G) = 3$ from Claim 1. If $|N_X(u_1) \cap N_X(u_2)| \ge 2$, or $|N_X(u_1) \cap N_X(u_2)| = 1$, and $\max\{d_X(u_1), d_X(u_2)\} \ge 2$, then we can find that G has a Hamiltonian path. Thus, $\mathrm{tpc}(G) = 3$.

5 Proof of Theorem 1.4

Suppose $F \cong K_3$ or P_3. Note that G has a Hamiltonian path, and so $\mathrm{tpc}(G) = 3$. The following three claims will be used later.

Claim 2. Let G be a graph obtained by adding a pendant vertex u_3 adjacent to vertex u_1 or u_2 of graph in Claim 1. Then $\mathrm{tpc}(G) = 3$.

Proof. Without loss of generality, assume that u_3 is adjacent to u_1. Let $c(u_1) = \{1, 2, 3\} \setminus \{c(u_1v_1), c(v_1)\}$, $c(u_1u_3) = c(v_1)$, and the remaining vertices and edges are assigned the same color as Claim 1. We can verify that G is total proper connected with the above coloring. Then $\mathrm{tpc}(G) = 3$. This completes the proof of Claim 2. □

Claim 3. Let G be a graph obtained by adding three vertices $\{u_1, u_2, u_3\}$ to a complete graph K_t such that $d(u_1) = d(u_3) = 1$, $d(u_2) = 2$, $N(u_1) \cap N(u_3) = \varnothing$, and $|N(u_2) \cap N(u_i)| = 1$, where $1 \le i \le 2$. Then $\mathrm{tpc}(G) = 3$.

Proof. Without loss of generality, assume that $N(u_2) \cap N(u_1) = v_1$, $N(u_2) \cap N(u_3) = v_i$. Let $c(u_3v_i) = c(v_iv_{i+1})$, $c(v_1u_2) = c(v_iv_{i-1})$, and the remaining vertices and edges are assigned the same color as Claim 1. Suppose $t \equiv 0 \pmod 3$. Note that $P_1 = u_1v_1v_2 \cdots v_{t-1}v_t$, $P_2 = u_2v_1v_tv_{t-1} \cdots v_3v_2$, $P_3 = u_3v_iu_2$, $P_4 = u_1v_1v_2 \cdots v_iu_3$, and $P_5 = u_3v_iv_{i-1} \cdots v_1v_tv_{t-1} \cdots v_{i+1}$ are five total proper paths. Suppose $t \equiv 1 \pmod 3$. Note that $P_1 = u_1v_1v_2 \cdots v_{t-1}v_t$, $P_2 = u_2v_1v_{t-1} \cdots v_3v_2v_t$, $P_3 = u_3v_iu_2$, $P_4 = u_1v_1v_2 \cdots v_iu_3$ and $P_5 = u_3v_iv_{i-1} \cdots v_2v_tv_{t-1} \cdots v_{i+1}$ are five total proper paths. Suppose $t \equiv 2 \pmod 3$. We can find that $P_1 = u_1v_1v_2 \cdots v_{t-1}v_t$, $P_2 = u_2v_1v_{t-2} \cdots v_3v_2v_{t-1}$, $P_3 = u_3v_iu_2$, $P_4 = u_1v_1v_2 \cdots v_iu_3$, $P_5 = u_3v_iv_{i-1} \cdots v_2v_{t-1}v_{t-2} \cdots v_{i+1}$, and $P_6 = u_3v_iv_{i-1}v_t$ are six total proper paths. Therefore, G is total proper connected with the above coloring, and so $\mathrm{tpc}(G) = 3$. This completes the proof of Claim 3. □

Claim 4. Let G be a graph obtained by adding a vertex u to the graph in Claim 1 such that $d(u) = 2$ and v_1 is adjacent to u. Then $\mathrm{tpc}(G) = 3$.

Proof. Since $d(u) = 2$, without loss of generality, we assume that v_i is adjacent to u. Let $c(uv_1) = 1$, $c(uv_i) = \{1, 2, 3\} \setminus \{c(v_i), c(v_iv_{i-1})\}$, and the remaining vertices and edges are assigned the same color as Claim 1. Suppose $t \equiv 0 \pmod 3$. Note that $P_1 = uv_iv_{i-1} \cdots v_1v_tv_{t-1} \cdots v_{i+1}$, $P_2 = uv_iv_{i-1} \cdots v_1u_1$ and $P_3 = uv_1u_2$ are three total proper paths. Suppose $t \equiv 1 \pmod 3$. Note that $P_1 = uv_iv_{i-1} \cdots v_1u_1$, $P_2 = uv_iv_{i-1} \cdots v_2v_tv_{t-1} \cdots v_{i+1}$, and $P_3 = uv_1u_2$ are three total proper paths. Suppose $t \equiv 2 \pmod 3$. We can find that $P_1 = uv_iv_{i-1} \cdots v_1u_1$, $P_2 = uv_iv_{i-1} \cdots v_2v_{t-1}v_{t-2} \cdots v_{i+1}$, $P_3 = uv_1u_2$, and $P_4 = uv_iv_{i-1} \cdots v_1v_t$ are four total proper paths. Hence, G is total proper connected with the above coloring, and so $\mathrm{tpc}(G) = 3$. This completes the proof of Claim 4. □

Suppose $F \cong K_2 + K_1$. Let $V(K_2) = \{u_1, u_2\}$ and $V(K_1) = \{u_3\}$. Since $\text{diam}(G) = 2$, we have $N_X(u_1) \cap N_X(u_2) \cap N_X(u_3) = \{v\}$, and so $\text{tpc}(G) = 3$ by Claim 2. Suppose $F \cong 3K_1$. Assume $N_X(u_1) \cap N_X(u_2) \cap N_X(u_3) = \emptyset$. Then $\text{tpc}(G) = 3$ by Claim 3. Assume $N_X(u_1) \cap N_X(u_2) \cap N_X(u_3) \neq \emptyset$. If $d_X(u_1) = d_X(u_2) = d_X(u_3) = 1$, then $\text{tpc}(G) \geq 4$ by [26, Proposition 2]. Define a total coloring of G as follows: $c(u_3v) = 4$ with $v \in N_X(u_3)$, and the remaining vertices and edges are assigned the same color as Claim 1. We check that any two vertices have a total proper path, and so $\text{tpc}(G) = 4$. Otherwise, we have $d_X(u_1) + d_X(u_2) + d_X(u_3) \geq 4$. Without loss of generality, let $d_X(u_1) \geq 2$, and $u \in X\backslash\{v\}$ where $v \in N_X(u_1) \cap N_X(u_2) \cap N_X(u_3)$. Thus, $\text{tpc}(G) = 3$ by Claim 4.

6 Proof of Theorem 1.5

Case 1. $\text{diam}(G) = 3$. We prove Case 1 by analyzing the structure of F.

Subcase 1.1. $F \cong K_3$ or P_3. Note that G has a Hamiltonian path, and so $\text{tpc}(G) = 3$.

Subcase 1.2. $F \cong K_2 + K_1$. Denote $V(K_2) = \{u_1, u_2\}$ and $V(K_1) = \{u_3\}$. Suppose $N_X(u_1) \cap N_X(u_3) \neq \emptyset$ or $N_X(u_2) \cap N_X(u_3) \neq \emptyset$. Without of loss generality, we may assume that $N_X(u_1) \cap N_X(u_3) \neq \emptyset$. Then $\text{tpc}(G) = 3$ by Claim 2. Suppose $N_X(u_1) \cap N_X(u_3) = \emptyset$ and $N_X(u_2) \cap N_X(u_3) = \emptyset$. Since $\text{diam}(G) = 3$, we have $N_X(u_1) \neq \emptyset$ and $N_X(u_3) \neq \emptyset$. Then G has a Hamiltonian path, and so $\text{tpc}(G) = 3$.

Subcase 1.3. $F \cong 3K_1$. Let $V(F) = \{u_1, u_2, u_3\}$. Since $\text{diam}(G) = 3$, we have $N_X(u_1) \cap N_X(u_2) \cap N_X(u_3) = \emptyset$. Suppose there exists two vertices $u_i, u_j \in V(F)$ satisfy $N_X(u_i) \cap N_X(u_j) \neq \emptyset$. Without loss of generality, let u_1 and u_2 satisfy $N_X(u_1) \cap N_X(u_2) \neq \emptyset$ and $v_1 \in N_X(u_1) \cap N_X(u_2)$. Assume $d_X(u_1) = d_X(u_2) = 1$ and $v_i \in N(u_3)$. Since G is not complete, we have $\text{tpc}(G) \geq 3$. To the contrary, suppose there exists a total coloring c of G using three colors such that G is total proper connected. Since any two vertices of G are connected by a total proper path, we have $c(u_1v_1) \neq c(v_1) \neq c(u_2v_1)$. Without loss of generality, let $c(u_1v_1) = 1$, $c(v_1) = 2$ and $c(u_2v_1) = 3$. Consider the total proper path P between u_1 and u, then the color of vertices and edges in P follows the sequence $1, 2, 3, \ldots, 1, 2, 3, \ldots$. Thus, the value of $(c(v_i), c(v_iu))$ is $(1, 2)$, $(2, 3)$, or $(3, 1)$. Consider the total proper path Q between u_2 and u, then the color of vertices and edges in Q follows the sequence $3, 2, 1, \ldots, 3, 2, 1, \ldots$. But the value of $(c(v_i), c(v_iu))$ is $(3, 2)$, $(2, 1)$, or $(1, 3)$, a contradiction. Assign a total coloring c to G as follows: $c(u_1v_1) = 1$, $c(v_1) = c(v_iu) = 2$, $c(u_2v_1) = 3$, assign 4 to the remaining edges, and assign 1 to the remaining vertices. We can check that G is total proper connected with the above coloring, and so $\text{tpc}(G) = 4$. Assume $d_X(u_1) + d_X(u_2) \geq 3$, without loss of generality, let $d_X(u_1) \geq 2$. If $d_X(u_3) = 1$ and $N_X(u_1) \cap N_X(u_3) \neq \emptyset$, then we have $\text{tpc}(G) = 3$ by Claim 3; if $N_X(u_1) \cap N_X(u_3) = \emptyset$, or $N_X(u_1) \cap N_X(u_3) \neq \emptyset$ and $d_X(u_3) \geq 2$, then G has a Hamiltonian path, and so $\text{tpc}(G) = 3$.

Now, we may suppose $N_X(u_1) \cap N_X(u_2) = \emptyset$, $N_X(u_1) \cap N_X(u_3) = \emptyset$, and $N_X(u_2) \cap N_X(u_3) = \emptyset$. Since G is not complete, we have $\text{tpc}(G) \geq 3$. To the contrary, assume that $v_i \in N(u_i)$, and there exists a total coloring c of G using three colors such that G is total proper connected. Since any two vertices of G are connected by a total proper path, we have $c(u_1v_1) \neq c(v_1)$. Without loss of generality, let $c(u_1v_1) = 1$ and $c(v_1) = 2$. Consider the total proper path P between u_1 and u_2, then the color of vertices and edges in P follows the sequence $1, 2, 3, \ldots, 1, 2, 3, \ldots$. Thus, the value of $(c(v_2), c(v_2u_2))$ is $(1, 2)$, $(2, 3)$, or $(3, 1)$. Consider the total proper path Q between u_1 and u_3, then the color of vertices and edges in Q follows the sequence $1, 2, 3, \ldots, 1, 2, 3, \ldots$. Hence, the value of $(c(v_3), c(v_3u_3))$ is $(1, 2)$, $(2, 3)$, or $(3, 1)$. Consider the total proper path W between u_2 and u_3. If $c(v_2) = 1$ and $c(v_2u_2) = 2$, then the color of vertices and edges in W follows the sequence $2, 1, 3, \ldots, 2, 1, 3, \ldots$. Note that the value of $(c(v_3), c(v_3u_3))$ is $(2, 1)$, $(1, 3)$, or $(3, 2)$, a contradiction. If $c(v_2) = 2$ and $c(v_2u_2) = 3$, then the color of vertices and edges in W follows the sequence $3, 2, 1, \ldots, 3, 2, 1, \ldots$. Note that the value of $(c(v_3), c(v_3u_3))$ is $(2, 1)$, $(1, 3)$, or $(3, 2)$, a contradiction. If $c(v_2) = 3$ and $c(v_2u_2) = 1$, then the color of vertices and edges in W follows the sequence $1, 3, 2, \ldots, 1, 3, 2, \ldots$. Note that the value of $(c(v_3), c(v_3u_3))$ is $(2, 1), (1, 3)$, or $(3, 2)$, a contradiction. Assign a total coloring c to G as follows: $c(u_1v_1) = c(v_2) = 1$, $c(v_1) = c(u_2v_2) = c(u_3v_3) = 2$, assign 4 to the remaining vertices, and assign 3 to the remaining edges. We can verify that G is total proper connected with the above coloring, and so $\text{tpc}(G) = 4$.

Case 2. $\text{diam}(G) \geq 4$. Thus, $F \cong P_3$ or $F \cong K_2 + K_1$. Assume $F \cong P_3$. Obviously, G has a Hamiltonian path, and so $\text{tpc}(G) = 3$. Assume $F \cong K_2 + K_1$. Denote $V(K_2) = \{u_1, u_2\}$ and $V(K_1) = \{u_3\}$, without loss of generality, we have $d_X(u_2) = 0$, $d_X(u_1) \geq 1$, $d_X(u_3) \geq 1$ satisfying $N_X(u_1) \cap N_X(u_2) = \emptyset$. Hence, we can find that G has a Hamiltonian path, and so $\text{tpc}(G) = 3$.

7 Proof of Theorem 1.6

The proof of Theorem 1.6 follows from the next two lemmas. First, we shall determine the total proper k-connection numbers of the circular ladders.

Lemma 7.1. *Let n be an integer with $n \geq 3$. Then* $\mathrm{tpc}(\mathrm{CL}_{2n}) = \mathrm{tpc}_2(\mathrm{CL}_{2n}) = 3$, $\mathrm{tpc}_3(\mathrm{CL}_{2n}) = 4$.

Proof. Let n be an integer with $n \geq 3$. Since CL_{2n} contains a Hamiltonian path that is not complete, we have $\mathrm{tpc}(\mathrm{CL}_{2n}) = 3$. Since $\mathrm{tpc}_2(\mathrm{CL}_{2n}) \geq \mathrm{tpc}(\mathrm{CL}_{2n}) = 3$, we only need to prove $\mathrm{tpc}_2(\mathrm{CL}_{2n}) \leq 3$.

Case 1. $n \equiv 0 \pmod 3$. Let $n = 3t$. Assign a total coloring c to CL_{2n} as follows: Let i be an integer with $0 \leq i \leq t-1$, $c(u_{3i+1}) = c(u_{3i+2}u_{3i+3}) = c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = 1$, $c(u_{3i+2}) = c(u_{3i+3}u_{3i+4}) = c(v_{3i+3}) = c(v_{3i+1}v_{3i+2}) = 2$, and $c(u_{3i+3}) = c(u_{3i+1}u_{3i+2}) = c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = 3$; $c(u_iv_i), c(u_i), c(v_i) \in \{1, 2, 3\}$ with $c(u_iv_i) \neq c(u_i) \neq c(v_i)$ for $1 \leq i \leq n$. Let x and y be any two distinct vertices of CL_{2n}. By symmetry, we may assume that $x = u_1$. If $y = u_i$ for $2 \leq i \leq n$, then $xu_2u_3 \cdots u_{i-1}y$ and $xu_nu_{n-1} \cdots u_{i+1}y$ are two total proper paths connecting x and y. If $y = v_1$, then xy and $xu_nu_{n-1} \cdots u_2y$ are two total proper paths connecting x and y. If $y = v_i$, then $xv_1v_nv_{n-1} \cdots v_{i+1}y$ and $xu_nu_{n-1} \cdots u_iy$ are two total proper paths connecting x and y. Thus, CL_{2n} is total proper 2-connected with the above coloring.

Case 2. $n \equiv 1 \pmod 3$. Let $n = 3t+1$. Define a total coloring c of CL_{2n} as follows: Let i be an integer with $0 \leq i \leq t-1$, $c(u_{3i+1}) = c(u_{3i+2}u_{3i+3}) = c(v_{3i+3}) = c(v_{3i+1}v_{3i+2}) = 1$, $c(u_{3i+2}) = c(u_{3i+3}u_{3i+4}) = c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = 3$, and $c(u_{3i+3}) = c(u_{3i+1}u_{3i+2}) = c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = 2$; $c(u_n) = c(v_1v_n) = 1$, $c(u_nv_n) = 2$, $c(v_n) = c(u_1u_n) = 3$, and $c(u_jv_j) = 3$ for $1 \leq j \leq n-1$. Let x and y be any two distinct vertices of CL_{2n}. We may assume that $x = u_i$ for $1 \leq i \leq n-1$. If $y = u_j$ for $i \leq j \leq n-1$, then $xu_{i+1}u_{i+2} \cdots u_{j-1}y$ and $xu_{i-1}u_{i-2} \cdots u_1v_1v_2 \cdots v_{n-1}u_{n-1}u_{n-2} \cdots u_{j+1}y$ are two total proper paths connecting x and y. If $y = v_j$ for $1 \leq j \leq n-1$, then $xu_{i-1}u_{i-2} \cdots u_1v_1v_2 \cdots v_{j-1}y$ and $xu_{i+1} \cdots u_{n-1}v_{n-1}v_{n-2} \cdots v_{j+1}y$ are two total proper paths connecting x and y. If $y = u_n$, then $xu_{i+1} \cdots u_{n-1}y$ and $xu_{i-1}u_{i-2} \cdots u_1y$ are two total proper paths connecting x and y. If $y = v_n$, then $xu_{i+1} \cdots u_{n-1}u_ny$ and $xu_{i-1}u_{i-2} \cdots u_1v_1v_2 \cdots v_{n-1}y$ are two total proper paths connecting x and y. Assume that $x = u_n$. If $y = v_j$ for $1 \leq j \leq n-1$, then $xu_1u_2 \cdots u_{n-1}v_{n-1}v_{n-2} \cdots v_{j+1}y$ and $xu_nv_nv_1v_2 \cdots v_{j-1}y$ are two total proper paths connecting x and y. If $y = v_n$, then xy and $xu_{n-1}u_{n-2} \cdots u_1v_1y$ are two total proper paths connecting x and y. Thus, CL_{2n} is total proper 2-connected with the above coloring.

Case 3. $n \equiv 2 \pmod 3$. Let $n = 3t+2$. Define a total coloring c of CL_{2n} as follows: Let i be an integer with $0 \leq i \leq t-2$, $c(u_{3i+1}) = c(u_{3i+2}u_{3i+3}) = c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = 1$, $c(u_{3i+2}) = c(u_{3i+3}u_{3i+4}) = c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = 3$, and $c(u_{3i+3}) = c(u_{3i+1}u_{3i+2}) = c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = 2$; $c(u_n) = c(u_{n-4}) = c(v_{n-2}) = c(v_n) = 1$, $c(u_{n-1}) = c(v_{n-1}) = c(u_{n-3}) = c(v_{n-3}) = 3$, $c(u_{n-2}) = c(v_{n-4}) = 2$, $c(u_{n-4}u_{n-3}) = c(u_{n-1}u_n) = c(v_{n-1}v_n) = c(v_{n-3}v_{n-2}) = c(v_{n-2}v_{n-1}) = 2$, $c(u_{n-3}u_{n-2}) = c(u_{n-2}u_{n-1}) = c(v_{n-4}v_{n-3}) = c(u_2v_2) = 1$, $c(u_1u_1) = c(u_1u_n) = c(v_1v_n) = 3$; $c(u_{4j}v_{4j}) = 1$, $c(u_{4j-1}v_{4j-1}) = 2$, $c(u_{4j+1}v_{4j+1}) = 3$, where $1 \leq j \leq t-1$. Note that $u_1u_2 \cdots u_{n-2}v_{n-2}v_{n-3} \cdots v_1u_1$ is a total proper cycle, any two distinct vertices of the cycle have two disjoint total proper paths. Now, we may assume that $x = u_i$ for $1 \leq i \leq n-2$. If $y = u_{n-1}$, then $xu_{i+1} \cdots u_{n-2}v_{n-2}v_{n-1}y$ and $xu_{i-1} \cdots u_1u_ny$ are two total proper paths connecting x and y. If $y = u_n$, then $xu_{i+1} \cdots u_{n-2}v_{n-2}v_{n-1} \cdots v_1v_ny$ and $xu_{i-1} \cdots u_1v_1v_ny$ are two total proper paths connecting x and y. If $y = v_{n-1}$, then $xu_{i-1} \cdots u_3v_3v_2v_1v_ny$ and $xu_{i+1} \cdots u_{n-2}v_{n-2}y$ are two total proper paths connecting x and y. If $y = v_n$, then $xu_{i-1} \cdots u_1u_ny$ and $xu_{i+1} \cdots u_{n-2}v_{n-2}v_{n-1}y$ are two total proper paths connecting x and y. Assume that $x = v_i$ for $1 \leq i \leq n-2$. If $y = v_{n-1}$, then $xv_{i+1} \cdots v_{n-2}u_{n-2}u_{n-1} \cdots u_1u_nu_{n-1}y$ and $xv_{i-1} \cdots v_1v_ny$ are two total proper paths connecting x and y. If $y = v_n$, then $xv_{i+1} \cdots v_{n-2}u_{n-2}u_{n-1}u_ny$ and $xv_{i-1} \cdots v_1y$ are two total proper paths connecting x and y. Thus, CL_{2n} is total proper 2-connected with the above coloring.

To the contrary, suppose there exists a total proper 3-connected coloring c of CL_{2n} using three colors. Considering u_1 and v_2, $u_1u_2v_2$, $u_1v_1v_2$, and $u_1u_nv_nv_{n-1} \cdots v_3v_2$ must be three total proper paths connecting u_1 and v_2. Then $c(u_1u_2) \neq c(u_2v_2) \neq c(u_2)$. Considering u_1 and u_3, $u_1u_2u_3$, $u_1u_nu_{n-1} \cdots u_4u_3$, and $u_1v_1v_2v_3u_3$ must be three total proper paths connecting u_1 and u_3. Hence, $c(u_1u_2) \neq c(u_2u_3) \neq c(u_2)$, and so $c(u_2v_2) = c(u_2u_3)$. But then, there is no set of three disjoint total proper paths connecting u_3 and v_2, a contradiction. Hence, $\mathrm{tpc}_3(\mathrm{CL}_{2n}) \geq 4$. Now we only need to prove $\mathrm{tpc}_3(\mathrm{CL}_{2n}) \leq 4$.

Case 1. $n \equiv 0 \pmod 3$. Let $n = 3t$. Assign a total coloring c to CL_{2n} as follows: Let i be an integer with $0 \leq i \leq t-1$, $c(u_{3i+1}) = c(u_{3i+2}u_{3i+3}) = c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = 1$, $c(u_{3i+2}) = c(u_{3i+3}u_{3i+4}) = c(v_{3i+3}) = c(v_{3i+1}v_{3i+2}) = 2$, and $c(u_{3i+3}) = c(u_{3i+1}u_{3i+2}) = c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = 3$; $c(u_jv_j) = 4$ for $1 \le j \le n$. Let x and y be any two distinct vertices of CL_{2n}. By symmetry, we may assume that $x = u_1$. If $y = u_i$ for $2 \le i \le n$, then $xu_2u_3 \cdots u_{i-1}y, xu_nu_{n-1} \cdots u_{i+1}y$, and $xv_1v_2 \cdots v_iy$ are three total proper paths connecting x and y. If $y = v_1$, then xy, xu_2v_2y and xu_nv_ny are three total proper paths connecting x and y. If $y = v_i$ for $2 \le i \le n$, then $xu_1u_2 \cdots u_iy, xv_1v_2 \cdots v_{i-1}y$, and $xu_nv_nv_{n-1} \cdots u_iy$ are three total proper paths connecting x and y. Hence, CL_{2n} is total proper 3-connected with the above coloring.

Case 2. $n \equiv 1 \pmod 3$. Let $n = 3t + 1$. Define a total coloring c of CL_{2n} as follows: $c(u_{3i+1}) = c(u_{3i+2}u_{3i+3}) = c(v_{3i+3}) = c(v_{3i+1}v_{3i+2}) = 1$, $c(u_{3i+2}) = c(u_{3i+3}u_{3i+4}) = c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = 3$, and $c(u_{3i+3}) = c(u_{3i+1}u_{3i+2}) = c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = 2$, where $0 \le i \le t-1$; $c(u_n) = c(v_1v_n) = 1$, $c(u_nv_n) = 2$, $c(v_n) = c(u_1u_n) = 4$, and $c(u_jv_j) = 4$ for $2 \le j \le n-1$. Let x and y be any two distinct vertices of CL_{2n}. By symmetry, we may assume that $x = u_1$. If $y = u_i$ for $2 \le i \le n$, then $xu_2u_3 \cdots u_{i-1}y$, $xu_nu_{n-1} \cdots u_{i+1}y$, and $xv_1v_2 \cdots v_iy$ are three total proper paths connecting x and y. If $y = v_1$, then xy, xu_2v_2y and xu_nv_ny are three total proper paths connecting x and y. If $y = v_i$ for $2 \le i \le n$, then $xu_1u_2 \cdots u_iy$, $xv_1v_2 \cdots v_{i-1}y$, and $xu_nv_nv_{n-1} \cdots u_iy$ are three total proper paths connecting x and y. Hence, CL_{2n} is total proper 3-connected with the above coloring.

Case 3. $n \equiv 2 \pmod 3$. Let $n = 3t + 2$. Define a total coloring c of CL_{2n} as follows: $c(u_{3i+1}) = c(u_{3i+2}u_{3i+3}) = c(v_{3i+4}) = c(v_{3i+2}v_{3i+3}) = 1$, $c(u_{3i+2}) = c(u_{3i+3}u_{3i+4}) = c(v_{3i+3}) = c(v_{3i+1}v_{3i+2}) = 3$, and $c(u_{3i+3}) = c(u_{3i+1}u_{3i+2}) = c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = 2$, where $0 \le i \le t-1$; $c(u_n) = c(v_{n-1}v_n) = 3$, $c(v_n) = c(u_{n-1}u_n) = c(u_1v_1) = 2$, $c(u_{n-1}) = c(v_1v_n) = c(u_nu_1) = 1$, $c(v_1) = 4$, and $c(u_jv_j) = 4$ for $2 \le j \le n$. Let x and y be any two distinct vertices of CL_{2n}. By symmetry, we may assume that $x = u_1$. If $y = u_i$ for $2 \le i \le n$, then $xu_2u_3 \cdots u_{i-1}y$, $xu_nu_{n-1} \cdots u_{i+1}y$, and $xv_1v_2 \cdots v_iy$ are three total proper paths connecting x and y. If $y = v_1$, then xy, xu_2v_2y, and xu_nv_ny are three total proper paths connecting x and y. If $y = v_i$ for $2 \le i \le n$, then $xu_1u_2 \cdots u_iy$, $xv_1v_2 \cdots v_{i-1}y$ and $xu_nv_nv_{n-1} \cdots u_iy$ are three total proper paths connecting x and y. Hence, CL_{2n} is total proper 3-connected with the above coloring. ☐

Next, we shall determine the total proper k-connection numbers of the Möbius ladders.

Lemma 7.2. *Let n be an integer with $n \ge 3$. Then* $\mathrm{tpc}(M_{2n}) = \mathrm{tpc}_2(M_{2n}) = 3$, $\mathrm{tpc}_3(M_{2n}) = 4$.

Proof. Since M_{2n} contains a Hamiltonian path and is not complete, we have $\mathrm{tpc}(M_{2n}) = 3$. Since $\mathrm{tpc}_2(M_{2n}) \ge \mathrm{tpc}(M_{2n}) = 3$, we only need to prove $\mathrm{tpc}_2(M_{2n}) \le 3$.

Case 1. $n \equiv 0 \pmod 3$. Define a total coloring c of M_{2n} as follows: Let i be an integer with $1 \le i \le n-2$, $c(u_i) = c(v_{n-i+1}) = 1$, $c(u_{i+1}) = c(v_{n-i}) = 3$, and $c(u_{i+2}) = c(v_{n-i-1}) = 2$; $c(u_iu_{i+1}), c(u_i), c(u_{i+1}) \in \{1, 2, 3\}$ with $c(u_iu_{i+1}) \ne c(u_i) \ne c(u_{i+1})$ for $1 \le i \le n-1$; $c(v_iv_{i+1}), c(v_i), c(v_{i+1}) \in \{1, 2, 3\}$ with $c(v_iv_{i+1}) \ne c(v_i) \ne c(v_{i+1})$ for $1 \le i \le n-1$; $c(u_1v_1) = 3$, $c(u_nv_n) = 2$, and $c(u_jv_{n-j+1}) = 3$ for $1 \le j \le n$. Let x and y be any two distinct vertices of M_{2n}. By symmetry, we may assume that $x = u_1$. If $y = u_i$ for $2 \le i \le n$, then $xu_2u_3 \cdots u_{i-1}y$ and $xv_1v_2 \cdots v_nu_nu_{n-1} \cdots u_{i+1}y$ are two total proper paths connecting x and y. If $y = v_1$, then xy and xu_2u_3y are two total proper paths connecting x and y. If $y = v_i$ for $2 \le i \le n$, then $xu_1u_2 \cdots u_nv_nv_{n-1} \cdots v_{i+1}y$ and $xv_1v_2 \cdots v_{i-1}y$ are two total proper paths connecting x and y. Thus, M_{2n} is total proper 2-connected with the above coloring.

Case 2. $n \equiv 1 \pmod 3$. Let $n = 3t + 1$. Define a total coloring c of M_{2n} as follows: Let i be an integer with $0 \le i \le t-1$, $c(u_{3i+1}) = c(u_{3i+2}u_{3i+3}) = c(v_{3i+3}) = c(v_{3i+1}v_{3i+2}) = 1$, $c(u_{3i+2}) = c(u_{3i+3}u_{3i+4}) = c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = 3$, and $c(u_{3i+3}) = c(u_{3i+1}u_{3i+2}) = c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = 2$; if $j = 3i$ for $1 \le i \le t$, then $c(u_jv_{n-j+1}) = 3$; if $j \ne 3i$ for $1 \le i \le t$, then $c(u_jv_{n-j+1}), c(u_j), c(v_{n-j+1}) \in \{1, 2, 3\}$ with $c(u_jv_{n-j+1}) \ne c(u_j) \ne c(v_{n-j+1})$; $c(u_n) = 1$, $c(u_nv_n) = c(u_1v_1) = 2$, $c(v_n) = 3$. Let x and y be any two distinct vertices of M_{2n}. We may assume that $x = u_i$ for $1 \le i \le n$. If $y = u_j$ for $i \le j \le n$, then $xu_{i+1}u_{i+2} \cdots u_{j-1}y$ and $xu_{i-1}u_{i-2} \cdots u_1v_nu_nu_{n-1} \cdots u_{j+1}y$ are two total proper paths connecting x and y. If $y = v_j$ for $1 \le j \le n$, then $xu_{i-1}u_{i-2} \cdots u_1v_nv_{n-1} \cdots v_{j+1}y$ and $xu_{i+1} \cdots u_{n-j+1}y$ are two total proper paths connecting x and y. Thus, M_{2n} is total proper 2-connected with the above coloring.

Case 3. $n \equiv 2 \pmod 3$. Let $n = 3t + 2$. Define a total coloring c of M_{2n} as follows: Let i be an integer with $0 \le i \le t-1$, $c(u_{3i+1}) = c(u_{3i+2}u_{3i+3}) = c(v_{3i+3}) = c(v_{3i+1}v_{3i+2}) = c(u_{3i+2}v_{n-3i-1}) = 1$, $c(u_{3i+2}) = c(u_{3i+3}u_{3i+4}) = c(v_{3i+1}) = c(v_{3i+2}v_{3i+3}) = c(u_{3i+3}v_{n-3i-2}) = 3$, and $c(u_{3i+3}) = c(u_{3i+1}u_{3i+2}) = c(v_{3i+2}) = c(v_{3i+3}v_{3i+4}) = c(u_{3i+4}v_{n-3i-3}) = 2$; $c(u_{n-1}) = c(u_nv_1) = c(v_{n-1}v_n) = 1$, $c(u_n) = c(v_{n-1}) = c(u_1v_n) = 3$, $c(v_n) = c(u_{n-1}u_n) = 2$. Let x and y be any two distinct vertices of M_{2n}. We may assume that $x = u_i$ for $1 \le i \le n$. If $y = u_j$ for $i \le j \le n$, then

$xu_{i+1}u_{i+2}\cdots u_{j-1}y$ and $xu_{i-1}u_{i-2}\cdots u_1v_nu_nu_{n-1}\cdots u_{j+1}y$ are two total proper paths connecting x and y. If $y = v_j$ for $1 \le j \le n$, then $xu_{i-1}u_{i-2}\cdots u_1v_nv_{n-1}\cdots v_{j+1}y$ and $xu_{i+1}\cdots u_{n-j+1}y$ are two total proper paths connecting x and y. Thus, M_{2n} is total proper 2-connected with the above coloring.

To the contrary, suppose there exists a total proper 3-connected coloring c of M_{2n} using three colors. By considering the pair $\{u_2, v_n\}$, $u_2u_3\cdots u_nv_n$, $u_2u_1v_n$, and $u_2v_{n-1}v_n$ must be three total proper paths connecting u_2 and v_n. Then $c(u_2v_{n-1}) \ne c(v_{n-1}v_n) \ne c(v_{n-1})$. By considering the pair $\{u_2, v_{n-2}\}$, $u_2u_1v_nv_{n-1}v_{n-2}$, $u_2u_3v_{n-2}$, and $u_2v_{n-1}v_n$ must be three total proper paths connecting u_2 and v_{n-2}. Thus, $c(u_2v_{n-1}) \ne c(v_{n-1}v_{n-2}) \ne c(v_{n-1})$, and hence $c(v_{n-1}v_{n-2}) = c(v_{n-1}v_n)$. But then, there is no set of three disjoint total proper paths connecting v_{n-2} and v_n, a contradiction. Thus, $\mathrm{tpc}_3(M_{2n}) \ge 4$. Now we only need to prove $\mathrm{tpc}_3(M_{2n}) \le 4$.

Case 1. $n \equiv 0 \pmod 2$. Let $n = 2t$. Assign a total coloring c of M_{2n} as follows: $c(u_{2i+1}) = c(v_{n-2i}) = c(u_{2i+2}v_{n-2i-1}) = 1$, $c(u_{2i+2}) = c(u_{n-2i-1}) = c(u_{2i+1}v_{n-2i}) = 3$, $c(v_{n-2i}v_{n-2i-1}) = c(u_{2i+1}u_{2i+2}) = 2$, where $0 \le i \le t-1$; $c(u_{2i+2}u_{2i+3}) = c(v_{n-2i-1}v_{n-2i-2}) = 4$ for $0 \le i \le t-2$; $c(u_nv_n) = c(u_1v_1) = 4$. Let x and y be any two distinct vertices of M_{2n}. By symmetry, we may assume that $x = u_1$. If $y = u_i$ for $2 \le i \le n$, then $xu_2\cdots u_{i-1}y$, $xv_nu_nu_{n-1}\cdots u_{i+1}y$, and $xv_1v_2\cdots v_{n-i+1}u_i$ are three total proper paths connecting x and y. If $y = v_1$, then xy, $xu_2u_3\cdots u_ny$, and $xv_nv_{n-1}\cdots v_2y$ are three total proper paths connecting x and y. If $y = v_i$ for $2 \le i \le n$, then $xv_1v_2\cdots v_{i-1}y$, $xv_nv_{n-1}\cdots v_{i+1}y$ and $xu_2u_3\cdots u_{n-i+1}y$ are three total proper paths connecting x and y. Hence, M_{2n} is total proper 3-connected with the above coloring.

Case 2. $n \equiv 1 \pmod 2$. For $n = 3$, we assign a total coloring c to M_6 as follows: $c(u_1) = c(v_3) = c(u_2u_3) = c(v_1v_2) = 1$, $c(u_2) = c(v_2) = c(u_3v_3) = c(u_1v_1) = 2$, $c(u_3) = c(v_1) = c(u_1u_2) = c(v_2v_3) = 3$, $c(u_iv_{n-i+1}) = 4$ for $1 \le i \le 3$. We can verify that M_6 is total proper 3-connected with the above total coloring, and so $\mathrm{tpc}_3(M_6) = 4$.

For $n = 5$, define a total coloring c of M_{10} as follows: $c(u_1) = c(v_5) = 1$, $c(u_2) = c(v_4) = 2$, $c(u_3) = c(v_3) = c(u_5) = c(v_1) = 3$, $c(u_4) = c(v_2) = 4$, $c(u_2u_3) = c(u_4u_5) = c(v_1v_2) = c(v_3v_4) = 1$, $c(u_3u_4) = c(u_5v_5) = c(u_1v_1) = c(v_2v_3) = 2$, $c(u_4v_2) = c(u_1u_2) = c(v_4v_5) = 3$, $c(u_iv_{n-i+1}) = 4$ for $i = 1, 2, 3, 5$. We can check that M_{10} is total proper 3-connected with the above total coloring, and so $\mathrm{tpc}_3(M_{10}) = 4$.

Subcase 2.1. Let $n \equiv 1 \pmod 4$ for $n \ge 7$. Let $n = 2t+1$. Assign a total coloring c to M_{2n} as follows: $c(u_{4i+1}) = c(v_{n-4i}) = 1$, $c(u_{4i+2}) = c(v_{n-4i-1}) = 2$, $c(u_{4i+3}) = c(v_{n-4i-2}) = 3$, and $c(u_{4i+4}) = c(v_{n-4i-3}) = 4$, where $0 \le i \le \frac{t-2}{2}$; $c(u_{4i}u_{4i+1}) = c(v_{n-4i+1}v_{n-4i}) = c(u_{4i+2}v_{n-4i-1}) = 3$, $c(u_{4i+1}u_{4i+2}) = c(v_{n-4i}v_{n-4i-1}) = c(u_{4i+3}v_{n-4i-2}) = 4$, $c(u_{4i+2}u_{4i+3}) = c(v_{n-4i-1}v_{n-4i-2}) = c(u_{4i}v_{n-4i+1}) = 1$, and $c(v_{n-4i-2}v_{n-4i-3}) = c(u_{4i+3}u_{4i+4}) = c(u_{4i+1}v_{n-4i}) = 2$, where $1 \le i \le \frac{t-2}{2}$; $c(u_n) = c(v_1) = c(u_1u_2) = c(v_nv_{n-1}) = 3$, $c(u_2u_3) = c(v_{n-1}v_{n-2}) = c(v_1v_2) = c(u_{n-1}u_n) = c(u_{n-1}v_2) = 1$, $c(u_3u_4) = c(v_{n-2}v_{n-3}) = c(u_1v_1) = c(u_nv_n) = 2$, and $c(u_1v_n) = c(u_2v_{n-1}) = c(u_3v_{n-2}) = 4$. Let x and y be any two distinct vertices of M_{2n}. By symmetry, we may assume that $x = u_1$. If $y = u_i$ for $2 \le i \le n$, then $xu_2u_3\cdots u_{i-1}y$, $xv_nu_nu_{n-1}\cdots u_{i+1}y$, and $xv_1v_2\cdots v_{n-i+1}u_i$ are three total proper paths connecting x and y. If $y = v_1$, then xy, $xu_2u_3\cdots u_ny$, and $xv_nv_{n-1}\cdots v_2y$ are three total proper paths connecting x and y. If $y = v_i$ for $2 \le i \le n$, then $xv_1v_2\cdots v_{i-1}y$, $xv_nv_{n-1}\cdots v_{i+1}y$ and $xu_2u_3\cdots u_{n-i+1}y$ are three total proper paths connecting x and y. Hence, M_{2n} is total proper 3-connected with the above coloring.

Subcase 2.2. Let $n \equiv 3 \pmod 4$ for $n \ge 7$. Let $n = 2t+1$. Assign a total coloring c to M_{2n} as follows: Let i be an integer with $0 \le i \le \frac{t-3}{2}$, $c(u_{4i+1}) = c(v_{n-4i}) = 1$, $c(u_{4i+2}) = c(v_{n-4i-1}) = 2$, $c(u_{4i+3}) = c(v_{n-4i-2}) = 3$, and $c(u_{4i+4}) = c(v_{n-4i-3}) = 4$. Let i be an integer with $1 \le i \le \frac{t-1}{2}$, $c(u_{4i}u_{4i+1}) = c(v_{n-4i-1}v_{n-4i}) = c(u_{4i+2}v_{n-4i-1}) = 3$, $c(u_{4i+1}u_{4i+2}) = c(v_{n-4i}v_{n-4i-1}) = c(u_{4i+3}v_{n-4i-2}) = 4$, $c(u_{4i+2}u_{4i+3}) = c(u_{4i}v_{n-4i+1}) = c(v_{n-4i-1}v_{n-4i-2}) = 1$, and $c(u_{4i+1}v_{n-4i}) = c(u_{4i-1}u_{4i}) = c(v_{n-4i+2}v_{n-4i+1}) = 2$; $c(u_n) = c(v_1) = c(u_1u_2) = c(v_nv_{n-1}) = 3$, $c(u_{n-1}) = c(v_2) = c(u_1v_1) = c(u_nv_n) = 2$, and $c(u_{n-2}) = c(v_3) = c(u_2u_3) = c(v_{n-1}v_{n-2}) = 1$. By the similar proof of the above subcase, we can verify M_{2n} is total proper 3-connected with the above coloring. □

8 Proof of Theorem 1.7

Note that $K_3 \square K_2 = \mathrm{CL}_6$, $K_{3,3} = M_6$, $Q_3 = \mathrm{CL}_8$ and M_8. By means of Theorem 1.6, we can obtain their total proper k-connection numbers. Now, we only need to consider K_4, F_1, F_2, F_3.

Lemma 8.1. $\text{tpc}(K_4) = 1$, $\text{tpc}_2(K_4) = 3$, $\text{tpc}_3(K_4) = 4$.

Proof. By [26], we know that $\text{tpc}(K_4) = 1$. Suppose $\text{tpc}_2(K_4) = 2$, then there is no set of two disjoint total proper paths connecting u_1 and u_2, a contradiction. Thus, $\text{tpc}_2(K_4) \geq 3$. Let $V(K_4) = \{u_1, u_2, u_3, u_4\}$, we assign a total coloring c to K_4 as follows: $c(u_1) = c(u_2u_3) = c(u_3u_4) = c(u_2u_4) = 1$, $c(u_2) = c(u_4) = c(u_1u_3) = 2$, $c(u_3) = c(u_1u_2) = c(u_1u_4) = 3$. We can verify that the K_4 is total proper 2-connected, so $\text{tpc}_2(K_4) = 3$.

Now, we suppose there exists a total proper 3-connected coloring c of K_4 using three colors. Considering u_1 and u_2, u_1u_2, $u_1u_4u_2$, and $u_1u_3u_2$ must be the three total proper paths connecting u_1 and u_2. Then $c(u_1u_4) \neq c(u_2u_4) \neq c(u_4)$. Considering u_1 and u_3, u_1u_3, $u_1u_2u_3$, and $u_1u_4u_3$ must be the three total proper paths connecting u_1 and u_3. Thus, $c(u_1u_4) \neq c(u_3u_4) \neq c(u_4)$, and so $c(u_2u_4) = c(u_4u_3)$. But then, there is no set of three disjoint total proper paths connecting u_2 and u_3, a contradiction. Hence, $\text{tpc}_3(K_4) \geq 4$. Define a total coloring c of K_4 as follows: $c(u_1) = c(u_3) = 1$, $c(u_2) = c(u_4) = 2$, $c(u_1u_2) = c(u_3u_4) = 4$, $c(u_1u_4) = c(u_2u_3) = 3$, $c(u_1u_3) = 2$, $c(u_2u_4) = 1$. We can easily check that K_4 is total proper 3-connected, and so $\text{tpc}_3(K_4) = 4$. □

Lemma 8.2. $\text{tpc}(F_1) = \text{tpc}_2(F_1) = 3$, $\text{tpc}_3(F_1) = 4$.

Proof. Since F_1 has a Hamiltonian path that is not complete, we know that $\text{tpc}(F_1) = 3$. It is easy to verify that F_1 is total proper 2-connected depicted in Figure 4(a), and so $\text{tpc}_2(F_1) = 3$. Now, we suppose there exists a total proper 3-connected coloring c of F_1 using three colors. By considering the pair $\{u_2, u_8\}$, u_2u_8, $u_2u_1u_8$, and $u_2u_3 \cdots u_8$ must be the three total proper paths connecting u_2 and u_8. Then $c(u_1u_2) \neq c(u_1u_8) \neq c(u_1)$. By considering the pair $\{u_5, u_8\}$, $u_5u_6u_7u_8$, $u_5u_1u_8$, and $u_5u_4u_3u_2u_8$ must be the three total proper paths connecting u_5 and u_8. Then $c(u_1u_5) \neq c(u_1u_8) \neq c(u_1)$, and hence $c(u_1u_2) = c(u_1u_5)$. But then, there is no set of three disjoint total proper paths connecting u_2 and u_5, a contradiction. Hence, $\text{tpc}_3(F_1) \geq 4$. By Figure 4(b), we know that F_1 is total proper 3-connected, and so $\text{tpc}_3(F_1) = 4$. □

Lemma 8.3. $\text{tpc}(F_2) = \text{tpc}_2(F_2) = 3$.

Proof. Since F_1 has a Hamiltonian path that is not complete, we know that $\text{tpc}(F_2) = 3$. We can check that the coloring shown in Figure 4(c) is total proper 2-connected. Thus, $\text{tpc}_2(F_2) = 3$. □

Lemma 8.4. $\text{tpc}(F_3) = \text{tpc}_2(F_3) = 3$, $\text{tpc}_3(F_3) = 4$.

Proof. Since F_3 has a Hamiltonian path that is not complete, we know that $\text{tpc}(F_3) = 3$. It is easy to check that the coloring shown in Figure 4(d) is total proper 2-connected using three colors. Thus, $\text{tpc}_2(F_3) = 3$. Now, we suppose there exists a total proper 3-connected coloring c of F_3 using three colors. Considering u_2 and u_8, $u_2u_8, u_2u_1u_8$ and $u_2u_3 \cdots u_8$ must be the three total proper paths connecting u_2 and u_8. Then $c(u_1u_2) \neq c(u_1u_8) \neq c(u_1)$. Considering u_5 and u_8, $u_5u_6u_7u_8$, $u_5u_1u_8$, and $u_5u_4u_3u_2u_8$ must be the three

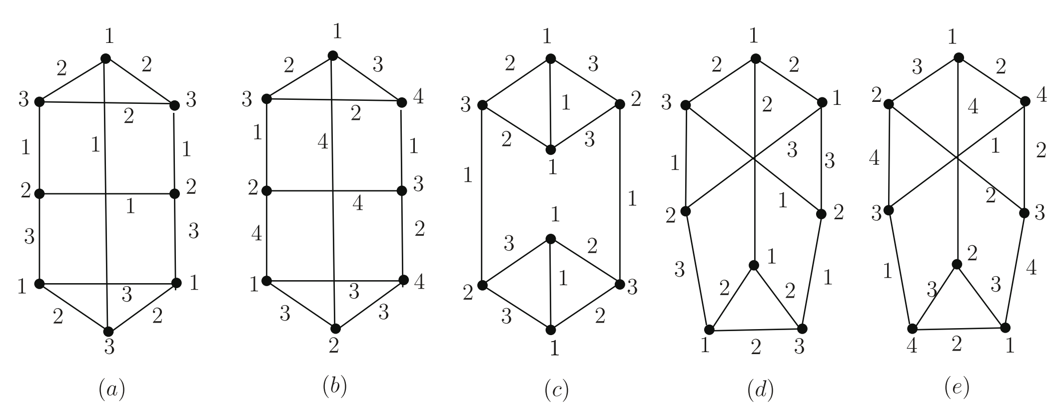

Figure 4: The total proper k-connected coloring of F_1, F_2 and F_3.

total proper paths connecting u_5 and u_8. Thus, $c(u_1u_5) \neq c(u_1u_8) \neq c(u_1)$, and so $c(u_1u_2) = c(u_1u_5)$. But then, there is no set of three disjoint total proper paths connecting u_2 and u_5, a contradiction. Hence, $\mathrm{tpc}_3(F_3) \geq 4$. By Figure 4(e), we know that F_3 is total proper 3-connected, and so $\mathrm{tpc}_3(F_3) = 4$. □

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References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, Berlin, 2008.
[2] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133** (2008), no. 1, 85–98.
[3] M. Krivelevich, and R. Yuster, *The rainbow connection of a graph is (at most) reciprocal to its minimum degree*, J. Graph Theory, **63** (2010), no. 3, 185–191, DOI: http://dx.doi.org/10.1002/jgt.20418.
[4] H. Liu, Â. Mestre, and T. Sousa, *Total rainbow k-connection in graphs*, Discrete Appl. Math. **174** (2014), 92–101, DOI: http://dx.doi.org/10.1016/j.dam.2014.04.012.
[5] K. Uchizawa, T. Aoki, T. Ito, A. Suzuki, and X. Zhou, *On the rainbow connectivity of graphs: complexity and FPT algorithms*, Algorithmica **67** (2013), no. 2, 161–179, DOI: http://dx.doi.org/10.1007/s00453-012-9689-4.
[6] L. S. Chandran, A. Das, D. Rajendraprasad, and N. M. Varma, *Rainbow connection number and connected dominating sets*, J. Graph Theory **71** (2012), no. 2, 206–218, DOI: http://dx.doi.org/10.1016/j.endm.2011.09.040.
[7] F. Fujie-Okamoto, G. L. Johns, and P. Zhang, *The rainbow connectivities of small cubic graphs*, Ars Combin. **105** (2012), 129–147.
[8] X. Huang, X. Li, Y. Shi, J. Yue, and Y. Zhao, *Rainbow connections for outerplanar graphs with diameter 2 and 3*, Appl. Math. Comput. **242** (2014), 277–280, DOI: http://dx.doi.org/10.1016/j.amc.2014.05.066.
[9] A. Kemnitz and I. Schiermeyer, *Graphs with rainbow connection number two*, Discuss. Math. Graph Theory **31** (2011), no. 2, 313–320, DOI: https://doi.org/10.7151/dmgt.1547.
[10] H. Lei, S. Li, H. Liu, and Y. Shi, *Rainbow vertex connection of digraphs*, J. Comb. Optim. **35** (2018), no. 1, 86–107, DOI: https://doi.org/10.1007/s10878-017-0156-7.
[11] H. Li and Y. Ma, *Rainbow connection number and graph operations*, Discrete Appl. Math. **230** (2017), 91–99, DOI: http://dx.doi.org/10.1016/j.dam.2017.06.004.
[12] Y. Ma, *Total rainbow connection number and complementary graph*, Results Math. **70** (2016), no. 1–2, 173–182, DOI: http://dx.doi.org/10.1007/s00025-015-0469-8.
[13] Y. Ma, L. Chen, and X. Li, *Graphs with small total rainbow connection number*, Front. Math. China **12** (2017), no. 4, 921–936, DOI: http://dx.doi.org/10.1007/s11464-017-0651-2.
[14] Y. Ma and Z. Lu, *Rainbow connection numbers of Cayley graphs*, J. Combin. Optim. **34** (2017), no. 1, 182–193, DOI: http://dx.doi.org/10.1007/s10878-016-0052-6.
[15] Y. Ma and Z. Lu, *Rainbow connection numbers of Cayley digraphs on abelian groups*, Appl. Math. Comput. **311** (2017), 178–183, DOI: http://dx.doi.org/10.1016/j.amc.2017.05.024.
[16] I. Schiermeyer, *On minimally rainbow k-connected graphs*, Discrete Appl. Math. **161** (2013), no. 4–5, 702–705, DOI: http://dx.doi.org/10.1016/j.dam.2011.05.001.
[17] X. Li, Y. Shi, and Y. Sun, *Rainbow connections of graphs: A survey*, Graphs Combin. **29** (2013), no. 1, 1–38, DOI: http://dx.doi.org/10.1007/s00373-012-1243-2.
[18] X. Li and Y. Sun, *An updated survey on rainbow connections of graphs – A dynamic survey*, Theory Appl. Graphs **2017** (2017), no. 1, 3, DOI: http://dx.doi.org/10.20429/tag.2017.000103.
[19] X. Li and Y. Sun, *Rainbow Connections of Graphs*, Springer, New York, 2012.

[20] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero, et al., *Proper connection of graphs*, Discrete Math. **312** (2012), no. 17, 2550–2560, DOI: http://dx.doi.org/10.1016/j.disc.2011.09.003.
[21] Y. Mao, F. Yanling, Z. Wang, and C. Ye, *Proper connection number of graph products*, Bull. Malays. Math. Sci. Soc. **41** (2018), no. 4, 2041–2051, DOI: https://doi.org/10.1007/s40840-016-0442-z.
[22] F. Huang, X. Li, and S. Wang, *Proper connection numbers of complementary graphs*, Bull. Malays. Math. Sci. Soc. **41** (2018), no. 3, 1199–1209, DOI: https://doi.org/10.1007/s40840-016-0381-8.
[23] X. Li, M. Wei, and J. Yue, *Proper connection number and connected dominating sets*, Theoret. Comput. Sci. **607** (2015), no. 3, 480–487, DOI: http://dx.doi.org/10.1016/j.tcs.2015.06.006.
[24] S. A. Aardt, C. Brause, A. P. Burger, M. Frick, A. Kemnitz, and I. Schiermeyer, *Proper connection and size of graphs*, Discrete Math. **340** (2017), no. 11, 2673–2677, DOI: http://dx.doi.org/10.1016/j.disc.2016.09.021.
[25] H. Jiang, X. Li, Y. Zhang, and Y. Zhao, *On* (*strong*) *proper vertex-connection of graphs*, Bull. Malays. Math. Sci. Soc. **41** (2018), no. 1, 415–425, DOI: https://doi.org/10.1007/s40840-015-0271-5.
[26] H. Jiang, X. Li, Y. Zhang, and Y. Zhao, *On the total proper connection of graphs*, Bull. Malays. Math. Sci. Soc. **42** (2019), no. 5, 2105–2117, DOI: https://doi.org/10.1007/s40840-019-00767-z.
[27] R. Read and R. Wilson, *An Atlas of Graphs*, Oxford University Press, New York, 1998.