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Research Article

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Isoperimetric and Brunn-Minkowski inequalities for the (p, q)-mixed geominimal surface areas

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Abstract: Motivated by the celebrated work of Lutwak, Yang and Zhang [1] on (p, q)-mixed volumes and that of Feng and He [2] on (p, q)-mixed geominimal surface areas, we in the present paper establish and confirm the affine isoperimetric and Brunn-Minkowski-type inequalities with respect to (p, q)-mixed geominimal surface areas.

Keywords: (p, q)-mixed volume, (p, q)-mixed geominimal surface area, monotonic inequality, isoperimetric inequality, Brunn-Minkowski-type inequality

MSC 2020: 52A20, 52A39, 52A40

1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in the Euclidean n-space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of convex bodies whose centroid lie at the origin, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. For the set of star bodies (about the origin) in $\mathbb{R}^n$, we write $\mathcal{S}_o^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, its volume is written by $\omega_n = V(B)$.

If $K \in \mathcal{K}^n$, then the support function of K, $h_K = h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$, is defined by [3]

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$.

Let K be a compact star shaped (about the origin) in $\mathbb{R}^n$, the radial function of K, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to [0, +\infty)$, is defined by [4]

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (with respect to the origin). Two star bodies K and L are dilated (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

For $p \geq 1$, the L_p mixed volume, $V_p(K, L)$, of $K, L \in \mathcal{K}_o^n$ is defined by [5]

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v) \mathrm{d}S_p(K, v), \tag{1.1}$$

where $S_p(K, \cdot)$ is the L_p surface area measure.

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Based on the L_p mixed volume (1.1), Lutwak [6] in 1996 introduced the notion of L_p geominimal surface areas: For $p \geq 1$ and $K \in \mathcal{K}_o^n$, the L_p geominimal surface area, $G_p(K)$, is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K, Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\},$$

where Q^* denotes the polar of Q. When $p = 1$, $G_1(K)$ is just Petty's classical geominimal surface area [7]. In particular, Lutwak [6] proved the following inequalities.

Theorem 1.A. *If $K \in \mathcal{K}_o^n$ and $1 \leq p < q$, then*

$$\left(\frac{G_p(K)^n}{n^n V(K)^{n-p}}\right)^{\frac{1}{p}} \leq \left(\frac{G_q(K)^n}{n^n V(K)^{n-q}}\right)^{\frac{1}{q}},$$

with equality if and only if K is p-self-minimal.

Theorem 1.B. *If $K \in \mathcal{K}_o^n$ and $p \geq 1$, then*

$$\omega_n \left(\frac{G_p(K)^n}{n^n V(K)^{n-p}}\right)^{\frac{1}{p}} \leq V(K)V(K^*),$$

with equality if and only if K is p-self-minimal.

Very recently, Lutwak et al. [1] introduced the L_p dual curvature measures as follows: Suppose $p, q \in \mathbb{R}$. If $K \in \mathcal{K}_o^n$ while $L \in \mathcal{S}_o^n$, then the L_p dual curvature measures, $\widetilde{C}_{p,q}(K, L, \cdot)$, on S^{n-1} are defined by

$$\int_{S^{n-1}} g(v)\mathrm{d}\widetilde{C}_{p,q}(K, L, v) = \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))h_K(\alpha_K(u))^{-p}\rho_K(u)^q\rho_L(u)^{n-q}\mathrm{d}u,$$

for each continuous $g : S^{n-1} \to \mathbb{R}$, where α_K is the radial Gauss map (see [1], Section 3).

Using the L_p dual curvature measures, Lutwak et al. [1] defined the (p, q)-mixed volumes as follows: For $p, q \in \mathbb{R}$, $K, Q \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$, the (p, q)-mixed volume, $\widetilde{V}_{p,q}(K, Q, L)$, is defined by

$$\widetilde{V}_{p,q}(K, Q, L) = \int_{S^{n-1}} h_Q^p(v)\mathrm{d}\widetilde{C}_{p,q}(K, L, v).$$

Meanwhile, Lutwak et al. [1] gave the following formula of (p, q)-mixed volume:

$$\widetilde{V}_{p,q}(K, Q, L) = \frac{1}{n}\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)(\alpha_K(u))^p\rho_K(u)^q\rho_L(u)^{n-q}\mathrm{d}u. \tag{1.2}$$

In [1], Lutwak et al. introduced the L_p mixed volume $V_p(K, Q)$ of $K, Q \in \mathcal{K}_o^n$ for all $p \in \mathbb{R}$ by

$$V_p(K, Q) = \frac{1}{n}\int_{S^{n-1}} h_Q^p(v)\mathrm{d}S_p(K, v).$$

The case $p \geq 1$ is just Lutwak's L_p mixed volume (1.1). Moreover, for $q \in \mathbb{R}$ and $K, Q \in \mathcal{S}_o^n$, they in [1] also defined the qth dual mixed volume, $\widetilde{V}_q(K, Q)$, by

$$\widetilde{V}_q(K, Q) = \frac{1}{n}\int_{S^{n-1}} \rho_K^q(v)\rho_Q^{n-q}(v)\mathrm{d}v.$$

By (1.2), the following special cases are showed:

$$\widetilde{V}_{p,q}(K, Q, K) = V_p(K, Q), \tag{1.3}$$

$$\widetilde{V}_{p,q}(K, K, L) = \widetilde{V}_q(K, L), \tag{1.4}$$

$$\widetilde{V}_{p,n}(K, Q, L) = V_p(K, Q). \tag{1.5}$$

The (p,q)-mixed volumes unify the mixed volumes of convex bodies in the L_p Brunn-Minkowski theory and the dual mixed volumes of star bodies in the L_p dual Brunn-Minkowski theory. In the last 20 years, the L_p Brunn-Minkowski theory and its dual theory have been developed very rapidly, see e.g., [3–6,8–23].

Based on the (p,q)-mixed volumes, Feng and He in [2] introduced the concept of (p,q)-mixed geominimal surface areas as follows:

Definition 1.1. For $p, q \in \mathbb{R}$, $K \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$, the (p,q)-mixed geominimal surface areas, $\widetilde{G}_{p,q}(K,L)$, of K and L are defined by

$$\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K,L) = \inf\{n\widetilde{V}_{p,q}(K,Q,L)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}. \tag{1.6}$$

If $L = K$ or $q = n$ in (1.6), then from (1.3) or (1.5) we see that for $p \geq 1$ the definition is just Lutwak's L_p geominimal surface area in [6]. For the studies of L_p geominimal surface areas, some results have been obtained in these articles (see, e.g., [24–35]).

According to the definition of (p,q)-mixed geominimal surface areas, Feng and He [2] extended Theorem 1.A to the following result:

Theorem 1.C. *Suppose $q \in \mathbb{R}$. If $K \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$, then for $0 < r < s$,*

$$\left(\frac{\widetilde{G}_{r,q}(K,L)^n}{n^n\widetilde{V}_q(K,L)^{n-r}}\right)^{\frac{1}{r}} \leq \left(\frac{\widetilde{G}_{s,q}(K,L)^n}{n^n\widetilde{V}_q(K,L)^{n-s}}\right)^{\frac{1}{s}},$$

with equality if and only if K and L are dilates.

Obviously, Theorem 1.C for $L = K$ and $1 \leq r < s$ implies Theorem 1.A.

In addition, they [2] gave a Brunn-Minkowski-type inequality for the (p,q)-mixed geominimal surface areas as follows:

Theorem 1.D. *Suppose $p, q \in \mathbb{R}$ such that $0 < \frac{n-q}{q} < 1$ and $q \neq n$, and let $\lambda, \mu \in \mathbb{R}$. If $K \in \mathcal{K}_o^n$, and $L_1, L_2 \in \mathcal{S}_o^n$, then*

$$\widetilde{G}_{p,q}(K, \lambda \cdot L_1 \tilde{+}_q \mu \cdot L_2)^{\frac{q}{n-q}} \geq \lambda\widetilde{G}_{p,q}(K,L_1)^{\frac{q}{n-q}} + \mu\widetilde{G}_{p,q}(K,L_2)^{\frac{q}{n-q}},$$

with equality if and only if L_1 and L_2 are dilates.

2 Main results

Here, $\lambda \cdot L_1 \tilde{+}_q \mu \cdot L_2$ is the L_q-radial combination of L_1 and L_2 (see (2.1)).

In this article, associated with the (p,q)-mixed geominimal surface areas, we first establish two monotonic inequalities as follows:

Theorem 1.1. *Suppose $p, q \in \mathbb{R}$. If $K \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$, then for $1 \leq p < q \leq n-1$,*

$$\left(\frac{\widetilde{G}_{p,n-p}(K,L)^n}{n^nV(K)^{n-p}}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{G}_{q,n-q}(K,L)^n}{n^nV(K)^{n-q}}\right)^{\frac{1}{q}}, \tag{1.7}$$

with equality if and only if K and L are dilates.

Theorem 1.2. *Suppose $p, q \in \mathbb{R}$ such that $1 \leq p < q$. If $K \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$, then*

$$\left(\frac{\widetilde{G}_{p,p}(K,L)^n}{n^nV(L)^{n-p}}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{G}_{q,q}(K,L)^n}{n^nV(L)^{n-q}}\right)^{\frac{1}{q}}, \tag{1.8}$$

with equality if and only if K and L are dilates.

Remark 1.1. When $L = K$ and K is p-self-minimal, Theorems 1.1 and 1.2 both become Theorem 1.A.

Next, we obtain an affine isoperimetric inequality for the (p, q)-mixed geominimal surface areas.

Theorem 1.3. *Suppose $p, q \in \mathbb{R}$ such that $p > 0$ and $0 < q \leq n$. If $K \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$, then*

$$\omega_n \left(\frac{\widetilde{G}_{p,q}(K, L)^n}{n^n V(K)^{q-p} V(L)^{n-q}} \right)^{\frac{1}{p}} \leq V(K)V(K^*), \tag{1.9}$$

for $0 < q < n$ equality holds when K and L are dilates.

Remark 1.2. If $q = n$ and $p \geq 1$, Theorem 1.3 only contains Lutwak's inequality.

Finally, together with the L_q harmonic Blaschke combination, we give the following Brunn-Minkowski-type inequality for the (p, q)-mixed geominimal surface areas.

Theorem 1.4. *Suppose $p, q \in \mathbb{R}$ such that $0 < q < n$, and let $\lambda, \mu \geq 0$ (not both zero). If $K \in \mathcal{K}_o^n$ and $L_1, L_2 \in \mathcal{S}_o^n$, then*

$$\frac{\widetilde{G}_{p,q}(K, \lambda * L_1 \hat{+}_q \mu * L_2)^{\frac{n+q}{n-q}}}{V(\lambda * L_1 \hat{+}_q \mu * L_2)} \geq \lambda \frac{\widetilde{G}_{p,q}(K, L_1)^{\frac{n+q}{n-q}}}{V(L_1)} + \mu \frac{\widetilde{G}_{p,q}(K, L_2)^{\frac{n+q}{n-q}}}{V(L_2)}, \tag{1.10}$$

with equality if and only if L_1 and L_2 are dilates.

Here, $\lambda * L_1 \hat{+}_q \mu * L_2$ is the L_q harmonic Blaschke combination of $L_1, L_2 \in \mathcal{S}_o^n$ (see (2.2)).

3 Background

In order to complete the proofs of Theorems 1.1–1.4, we collect some basic facts about convex bodies and star bodies.

If E is a nonempty subset in $\mathbb{R}^n$, then the polar set, E^*, of E is defined by [1,2]

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \quad y \in E\}.$$

Meanwhile, it is easy to get that $(K^*)^* = K$ for all $K \in \mathcal{K}_o^n$.

For $K, L \in \mathcal{S}_o^n$, $q \neq 0$ and $\lambda, \mu \geq 0$ (not both zero), the L_q radial combination, $\lambda \cdot K \tilde{+}_q \mu \cdot L \in \mathcal{S}_o^n$, of K and L is defined by [1]

$$\rho(\lambda \cdot K \tilde{+}_q \mu \cdot L, \cdot)^q = \lambda \rho(K, \cdot)^q + \mu \rho(L, \cdot)^q, \tag{2.1}$$

where the operation "$\tilde{+}_q$" is called L_q radial addition, $\lambda \cdot K$ denotes L_q radial scalar multiplication and $\lambda \cdot K = \lambda^{\frac{1}{q}} K$.

For $K, L \in \mathcal{S}_o^n$, $q > 0$ and $\lambda, \mu \geq 0$ (not both zero), the L_q harmonic Blaschke combination, $\lambda * K \hat{+}_q \mu * L \in \mathcal{S}_o^n$, of K and L is defined by [36]

$$\frac{\rho(\lambda * K \hat{+}_q \mu * L, \cdot)^{n+q}}{V(\lambda * K \hat{+}_q \mu * L)} = \lambda \frac{\rho(K, \cdot)^{n+q}}{V(K)} + \mu \frac{\rho(L, \cdot)^{n+q}}{V(L)}, \tag{2.2}$$

where the operation "$\hat{+}_q$" is called L_q harmonic Blaschke addition, $\lambda * K$ denotes L_q harmonic Blaschke scalar multiplication and $\lambda * K = \lambda^{\frac{1}{q}} K$. When $\lambda = \mu = 1$, $K \hat{+}_q L$ is called L_q harmonic Blaschke sum.

4 Proofs of main theorems

In this section, we will prove Theorems 1.1–1.4. To complete the proof of Theorem 1.1, we require the following lemma.

Lemma 3.1. [37] *Suppose $p, q \in \mathbb{R}$. If $K, Q \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$, then for $0 < p < q \leq n-1$,*

$$\left[\frac{\widetilde{V}_{p,n-p}(K,Q,L)}{V(K)}\right]^{\frac{1}{p}} \leq \left[\frac{\widetilde{V}_{q,n-q}(K,Q,L)}{V(K)}\right]^{\frac{1}{q}}, \tag{3.1}$$

with equality if and only if K and L are dilates.

Proof of Theorem 1.1. Together (1.6) with inequality (3.1), we obtain that for $1 \leq p < q \leq n-1$,

$$\begin{aligned}\omega_n\left(\frac{\widetilde{G}_{p,n-p}(K,L)^n}{n^nV(K)^{n-p}}\right)^{\frac{1}{p}} &= \inf\left\{\left(\frac{\widetilde{V}_{p,n-p}(K,Q,L)}{V(K)}\right)^{\frac{n}{p}}V(K)V(Q^*) : Q \in \mathcal{K}_o^n\right\}\\ &\leq \inf\left\{\left(\frac{\widetilde{V}_{q,n-q}(K,Q,L)}{V(K)}\right)^{\frac{n}{q}}V(K)V(Q^*) : Q \in \mathcal{K}_o^n\right\}\\ &= \omega_n\left(\frac{\widetilde{G}_{q,n-q}(K,L)^n}{n^nV(K)^{n-q}}\right)^{\frac{1}{q}}.\end{aligned}$$

This is

$$\left(\frac{\widetilde{G}_{p,n-p}(K,L)^n}{n^nV(K)^{n-p}}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{G}_{q,n-q}(K,L)^n}{n^nV(K)^{n-q}}\right)^{\frac{1}{q}}.$$

This yields inequality (1.7). According to the equality condition of inequality (3.1), we see that the equality of the above inequality holds if and only if K and L are dilates. □

Lemma 3.2. [37] *Suppose $p, q \in \mathbb{R}$ satisfy $1 \leq p < q$. If $K, Q \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$, then*

$$\left[\frac{\widetilde{V}_{p,p}(K,Q,L)}{V(L)}\right]^{\frac{1}{p}} \leq \left[\frac{\widetilde{V}_{q,q}(K,Q,L)}{V(L)}\right]^{\frac{1}{q}}, \tag{3.2}$$

with equality if and only if K and L are dilates.

Proof of Theorem 1.2. From Definition 1.1 and (3.2) we see that for $1 \leq p < q$,

$$\begin{aligned}\omega_n\left(\frac{\widetilde{G}_{p,p}(K,L)^n}{n^nV(L)^{n-p}}\right)^{\frac{1}{p}} &= \inf\left\{\left(\frac{\widetilde{V}_{p,p}(K,Q,L)}{V(L)}\right)^{\frac{n}{p}}V(L)V(Q^*) : Q \in \mathcal{K}_o^n\right\}\\ &\leq \inf\left\{\left(\frac{\widetilde{V}_{q,q}(K,Q,L)}{V(L)}\right)^{\frac{n}{q}}V(L)V(Q^*) : Q \in \mathcal{K}_o^n\right\}\\ &= \omega_n\left(\frac{\widetilde{G}_{q,q}(K,L)^n}{n^nV(L)^{n-q}}\right)^{\frac{1}{q}}.\end{aligned}$$

This is, for $1 \leq p < q$,

$$\left(\frac{\widetilde{G}_{p,p}(K,L)^n}{n^nV(L)^{n-p}}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{G}_{q,q}(K,L)^n}{n^nV(L)^{n-q}}\right)^{\frac{1}{q}}.$$

This gives inequality (1.8). From the equality condition of inequality (3.2), we see that the equality holds in (1.8) if and only if K, L are dilates. □

Lemma 3.3. [2] *If $K, L \in \mathcal{S}_o^n$, $0 < q \le n$, then*

$$\widetilde{V}_q(K, L) \le V(K)^{\frac{q}{n}} V(L)^{\frac{n-q}{n}}. \tag{3.3}$$

For $0 < q < n$, equality holds in (3.3) if and only if K and L are dilates; for $q = n$, (3.3) becomes an equality.

Proof of Theorem 1.3. Since $p > 0$, it follows from (1.6) that

$$\omega_n \widetilde{G}_{p,q}(K, L)^{\frac{n}{p}} = \inf\{n^{\frac{n}{p}} \widetilde{V}_{p,q}(K, Q, L)^{\frac{n}{p}} V(Q^*) : Q \in \mathcal{K}_o^n\}. \tag{3.4}$$

Taking $Q = K$ in (3.4), it follows from (1.4) and (3.3) that for $0 < q \le n$,

$$\omega_n \widetilde{G}_{p,q}(K, L)^{\frac{n}{p}} \le n^{\frac{n}{p}} \widetilde{V}_{p,q}(K, K, L)^{\frac{n}{p}} V(K^*) = n^{\frac{n}{p}} \widetilde{V}_q(K, L)^{\frac{n}{p}} V(K^*) \le n^{\frac{n}{p}} V(K)^{\frac{q}{p}} V(L)^{\frac{n-q}{p}} V(K^*),$$

i.e.,

$$\omega_n \left(\frac{\widetilde{G}_{p,q}(K, L)^n}{n^n V(K)^{q-p} V(L)^{n-q}} \right)^{\frac{1}{p}} \le V(K) V(K^*).$$

By the equality condition of inequality (3.3) and Definition 1.1, for $0 < q < n$ equality holds in (1.9) when K and L are dilates. □

Lemma 3.4. *Suppose $p, q \in \mathbb{R}$ such that $0 < q < n$, and let $\lambda, \mu > 0$. If $K, Q \in \mathcal{K}_o^n$ and $L_1, L_2 \in \mathcal{S}_o^n$, then*

$$\frac{\widetilde{V}_{p,q}(K, Q, \lambda * L_1 \widetilde{+}_q \mu * L_2)^{\frac{n+q}{n-q}}}{V(\lambda * L_1 \widetilde{+}_q \mu * L_2)} \ge \lambda \frac{\widetilde{V}_{p,q}(K, Q, L_1)^{\frac{n+q}{n-q}}}{V(L_1)} + \mu \frac{\widetilde{V}_{p,q}(K, Q, L_2)^{\frac{n+q}{n-q}}}{V(L_2)}, \tag{3.5}$$

with equality if and only if L_1 and L_2 are dilates.

Proof. Since $0 < q < n$, thus $0 < \frac{n-q}{n+q} < 1$. From (1.2), (2.2) and Minkowski's integral inequality [38], we get that for any $Q \in \mathcal{K}_o^n$,

$$\begin{aligned}
\frac{\widetilde{V}_{p,q}(K, Q, \lambda * L_1 \widetilde{+}_q \mu * L_2)^{\frac{n+q}{n-q}}}{V(\lambda * L_1 \widetilde{+}_q \mu * L_2)} &= \frac{\left[\frac{1}{n}\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u)) \rho_K^q(u) \rho_{\lambda * L_1 \widetilde{+}_q \mu * L_2}^{n-q}(u) \mathrm{d}u\right]^{\frac{n+q}{n-q}}}{V(\lambda * L_1 \widetilde{+}_q \mu * L_2)} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\left(\frac{h_Q}{h_K}\right)^{\frac{p(n+q)}{n-q}} (\alpha_K(u)) \rho_K(u)^{\frac{q(n+q)}{n-q}} \frac{\rho_{\lambda * L_1 \widetilde{+}_q \mu * L_2}^{n+q}(u)}{V(\lambda * L_1 \widetilde{+}_q \mu * L_2)} \right)^{\frac{n-q}{n+q}} \mathrm{d}u\right]^{\frac{n+q}{n-q}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\left(\frac{h_Q}{h_K}\right)^{\frac{p(n+q)}{n-q}} (\alpha_K(u)) \rho_K(u)^{\frac{q(n+q)}{n-q}} \left(\lambda \frac{\rho_{L_1}^{n+q}(u)}{V(L_1)} + \mu \frac{\rho_{L_2}^{n+q}(u)}{V(L_2)}\right) \right)^{\frac{n-q}{n+q}} \mathrm{d}u\right]^{\frac{n+q}{n-q}} \\
&\ge \lambda \frac{\left[\frac{1}{n}\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u)) \rho_K^q(u) \rho_{L_1}^{n-q}(u) \mathrm{d}u\right]^{\frac{n+q}{n-q}}}{V(L_1)} \\
&\quad + \mu \frac{\left[\frac{1}{n}\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p (\alpha_K(u)) \rho_K^q(u) \rho_{L_2}^{n-q}(u) \mathrm{d}u\right]^{\frac{n+q}{n-q}}}{V(L_2)} \\
&= \lambda \frac{\widetilde{V}_{p,q}(K, Q, L_1)^{\frac{n+q}{n-q}}}{V(L_1)} + \mu \frac{\widetilde{V}_{p,q}(K, Q, L_2)^{\frac{n+q}{n-q}}}{V(L_2)}.
\end{aligned}$$

This yields inequality (3.5).

By the equality condition of Minkowski's integral inequality, we see that equality holds in (3.5) if and only if L_1 and L_2 are dilates. □

Proof of Theorem 1.4. Because of $0 < q < n$, thus $\frac{n+q}{n-q} > 0$. Hence, by (1.6) and (3.5) we have

$$\begin{aligned}
\frac{\left[\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K, \lambda * L_1 \hat{+}_q \mu * L_2)\right]^{\frac{n+q}{n-q}}}{V(\lambda * L_1 \hat{+}_q \mu * L_2)} &= \inf\left\{\frac{[n\widetilde{V}_{p,q}(K, Q, \lambda * L_1 \hat{+}_q \mu * L_2)]^{\frac{n+q}{n-q}}}{V(\lambda * L_1 \hat{+}_q \mu * L_2)}\left[V(Q^*)^{\frac{p}{n}}\right]^{\frac{n+q}{n-q}} : Q \in \mathcal{K}_o^n\right\} \\
&\geq \lambda \inf\left\{\frac{[n\widetilde{V}_{p,q}(K, Q, L_1)]^{\frac{n+q}{n-q}}}{V(L_1)}\left[V(Q^*)^{\frac{p}{n}}\right]^{\frac{n+q}{n-q}} : Q \in \mathcal{K}_o^n\right\} \\
&\quad + \mu \inf\left\{\frac{[n\widetilde{V}_{p,q}(K, Q, L_2)]^{\frac{n+q}{n-q}}}{V(L_2)}\left[V(Q^*)^{\frac{p}{n}}\right]^{\frac{n+q}{n-q}} : Q \in \mathcal{K}_o^n\right\} \\
&= \lambda\frac{\left[\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K, L_1)\right]^{\frac{n+q}{n-q}}}{V(L_1)} + \mu\frac{\left[\omega_n^{\frac{p}{n}}\widetilde{G}_{p,q}(K, L_2)\right]^{\frac{n+q}{n-q}}}{V(L_2)},
\end{aligned}$$

i.e.,

$$\frac{\widetilde{G}_{p,q}(K, \lambda * L_1 \hat{+}_q \mu * L_2)^{\frac{n+q}{n-q}}}{V(\lambda * L_1 \hat{+}_q \mu * L_2)} \geq \lambda\frac{\widetilde{G}_{p,q}(K, L_1)^{\frac{n+q}{n-q}}}{V(L_1)} + \mu\frac{\widetilde{G}_{p,q}(K, L_2)^{\frac{n+q}{n-q}}}{V(L_2)}.$$

This gives inequality (1.10).

According to the equality condition of inequality (3.5), we see that the equality holds in (1.10) if and only if L_1 and L_2 are dilates. □

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