Research Article

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Generic uniqueness of saddle point for two-person zero-sum differential games

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Abstract: The generic uniqueness of saddle point for two-person zero-sum differential games, within the class of open-loop, against the perturbation of the right-hand side function of the control system is investigated. By employing set-valued mapping theory, it is proved that the majority of the two-person zero-sum differential games have unique saddle point in the sense of Baire's category.

Keywords: generic uniqueness, saddle point, zero-sum game, USC mapping with compact

MSC 2020: 91A23, 49N70, 91A05, 91A10

1 Introduction

In the 1950s, Isaacs [1] initiated the study of two-person zero-sum differential games. Later in the 1960s and 1970s, Berkovitz [2], Elliott-Kalton [3], Fleming [4], and Friedman [5] also made contributions. Two-person zero-sum differential games were investigated extensively in the literature as they are widely used in many fields, such as biology, finance, and engineering, and also play a key role in the research of general differential games. Ramaswamy and Shaiju [6] proved convergence theorems for the approximate value functions by Yosida type approximations and constructed approximate saddle-point strategies within the sense of feedback in Hilbert Space. Berkovitz [7] defined differential games of fixed duration and showed that games of fixed duration that satisfy Isaacs condition have saddle point. Ghosh and Shaiju [8] proved the existence of saddle point equilibrium for two-player zero-sum differential games in Hilbert space. Ammar et al. [9] derived sufficient and necessary conditions for an open-loop saddle point of rough continuous differential games for two-person zero-sum rough interval continuous differential games. In particular, Sun [10] derived a sufficient condition of the existence of an open-loop saddle point for two-person zero-sum stochastic linear quadratic differential games in 2021. We refer the reader to [11,12] and references therein.

It is worth noting that uniqueness is important in both practice and theory, especially in mathematical problems including two-person zero-sum differential games. However, how many problems have a unique solution? In fact, most mathematical problems cannot guarantee the uniqueness of the solution. So, we have to settle for the second thing: generic uniqueness (see Remark 3.1).

Regarding the generic uniqueness, many results have been investigated. Kenderov [13] studied the solutions of optimization problems and obtained an important result: most optimization problems have a unique solution. Ribarska and Kenderov [14] in their work proved that most two-person zero-sum continuous games have a unique solution in the sense of Baire's category. Tan et al. [15] studied the saddle point for general functions and derived the generic uniqueness of saddle points by the set-valued analysis method. Yu et al. [16] considered the generic uniqueness of equilibrium points for general equilibrium problems.

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On the other hand, Yu et al. [17] presented the existence and stability of optimal control problems using set-valued analysis theory in 2014 and showed that most of the optimal control problems are generic stable. After that, Deng and Wei [18,19] proved that generic stability result of optimal control problems governed by semi-linear evolution equation and nonlinear optimal control problems with 1-mean equilibrium controls, respectively. In 2020, the generic stability of Nash equilibria is investigated by Yu and Peng in their work [20] on noncooperative differential games in the sense of Baire's category.

To the best of our knowledge, there is no published result for the generic uniqueness of saddle point for two-person sum-person differential games. The purpose of this paper is to study such problems. We point out that the main idea of the present paper comes from the works of Kenderov [13], Ribarska and Kenderov [14], and Yu et al. [15,20].

The remainder of this paper is organized as follows. The next section is devoted to formulating the game model, collecting some basic preliminary, and stating some properties of a saddle point. In Section 3, we formulate a space of problem and introduce a set-valued mapping. We then state some continuous dependence of state trajectory and cost functional and present some main results in this paper. Finally, some conclusions are given in Section 4.

2 Model and preliminaries

We begin with classical differential games governed by ordinary equations. Let R^p and R^q be Euclidean space, $U \subset R^p$ and $V \subset R^q$ be bounded closed and convex set. Let $T > 0$, for initial state $x_0 \in R^n$, consider the following control systems:

$$\begin{cases} \dot{X}(t) = f(t, X(t), u(t), v(t)), \quad t \in [0, T], \\ X(0) = x_0, \end{cases} \tag{1}$$

where $f : [0, T] \times R^n \times U \times V \to R^n$ is a given map. $X(\cdot)$ is called the state trajectory, $u(\cdot)$ and $v(\cdot)$ are control functions valued in U and V, respectively. We denote

$$\begin{aligned} \mathcal{U}[t, s] &= \{u : [t, s] \to U \mid u(\cdot) \text{ is continuous}\}, \\ \mathcal{V}[t, s] &= \{v : [t, s] \to V \mid v(\cdot) \text{ is continuous}\}. \end{aligned} \tag{2}$$

Under some mild conditions, for initial pair $(0, x_0)$ and any $(u(\cdot), v(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T]$, control system (1) admits a unique solution.

Remark 2.1. It is obvious that $X(\cdot)$, which is the solution of control system (1), depends on f, u, and v. Thus, let $X(\cdot) \equiv X^f_{u,v}(\cdot)$. See the below section for more description with respect to continuous dependence.

We now introduce the following cost functionals which measures the performance of the control $u(\cdot)$ and $v(\cdot)$.

$$J_i(u(\cdot), v(\cdot)) = \int_0^T \varphi_i(t, X(t), u(t), v(t))\mathrm{d}t + \psi_i(X(T)), \quad i = 1, 2, \tag{3}$$

for some given maps $\varphi_i : [0, T] \times R^n \times U \times V \to R$ and $\psi_i : R^n \to R$ $(i = 1, 2)$. The following two-person differential games is posed.

Problem (**DG**). For a given initial pair $(0, x_0)$, Player 1 finds a control $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ and Player 2 finds a control $\bar{v}(\cdot) \in \mathcal{V}[0, T]$ such that

$$\begin{aligned} J_1(\bar{u}(\cdot), \bar{v}(\cdot)) &= \inf_{u(\cdot) \in \mathcal{U}[0,T]} J_1(u(\cdot), \bar{v}(\cdot)), \\ J_2(\bar{u}(\cdot), \bar{v}(\cdot)) &= \inf_{v(\cdot) \in \mathcal{V}[0,T]} J_2(\bar{u}(\cdot), v(\cdot)). \end{aligned} \tag{4}$$

Any $(\bar{u}(\cdot), \bar{v}(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T]$ satisfying (4) is called an open-loop Nash equilibrium control.

Now, we let cost functionals (3) satisfies

$$\begin{cases} \varphi_1(t, X(t), u, v) + \varphi_2(t, X(t), u, v) = 0, \\ \psi_1(X(T)) + \psi_2(X(T)) = 0, \end{cases}$$

where $\varphi_i(t, X(t), u(t), v(t)) = h_i(t, X(t)) + Wu(t) + Zv(t)$ $(i = 1, 2)$, and W, Z are constant positive definite matrix. $h_i : [0, T] \times R^n \to R$ is the given mapping. Then, one has

$$J_1(u(\cdot), v(\cdot)) + J_2(u(\cdot), v(\cdot)) = 0.$$

In this case, Problem(DG) is a two-person zero-sum differential game. For convenience, we call it **Problem(ZDG)**. Define

$$\begin{cases} \varphi(t, X(t), u, v) = \varphi_1(t, X(t), u, v) = -\varphi_2(t, X(t), u, v), \\ \psi(X(T)) = \psi_1(X(T)) = -\psi_2(X(T)), \end{cases}$$

and

$$J(u(\cdot), v(\cdot)) = J_1(u(\cdot), v(\cdot)) = -J_2(u(\cdot), v(\cdot)).$$

This yields that

$$J(\bar{u}(\cdot), \bar{v}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} J(u(\cdot), \bar{v}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} J_1(u(\cdot), \bar{v}(\cdot)) = J_1(\bar{u}(\cdot), \bar{v}(\cdot)).$$

and

$$J(\bar{u}(\cdot), \bar{v}(\cdot)) = \inf_{v(\cdot)\in\mathcal{V}[0,T]} J(\bar{u}(\cdot), v(\cdot)) = -\inf_{v(\cdot)\in\mathcal{V}[0,T]} J_2(\bar{u}(\cdot), v(\cdot)) = -J_2(\bar{u}(\cdot), \bar{v}(\cdot)).$$

Remark 2.2. In this paper, our objective is to investigate generic uniqueness of Problem(ZDG) against the perturbation of the right-hand side function of control system. To this end, we assume that cost functional is linear with regard to $u(\cdot)$ and $v(\cdot)$, which does not impact our main idea.

Definition 2.1. Let initial pair $(0, x_0)$ be fixed. A control pair $(\bar{u}(\cdot), \bar{v}(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T]$ is called an open-loop saddle point of Problem(ZDG), if for any $(u(\cdot), v(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T]$, it satisfies

$$J(\bar{u}(\cdot), v(\cdot)) \le J(\bar{u}(\cdot), \bar{v}(\cdot)) \le J(u(\cdot), \bar{v}(\cdot)).$$

In this paper, $\|\cdot\|$ represents a Euclidean norm.

We make the following assumptions.

[F] The map $f : [0, T] \times R^n \times U \times V \to R^n$ is measured in t and continuous with respect to u and v. There exist constant $L > 0$ and $\phi(\cdot) \in L^p([0, T]; R)$ $(p \ge 1)$ such that

$$\begin{cases} \|f(t, x, u, v) - f(t, y, u, v)\| \le L\|x - y\|, \\ \|f(t, 0, u, v)\| \le \phi(t), \end{cases} \quad \forall (t, x, u, v) \in [0, T] \times R^n \times U \times V.$$

[H1] The maps $\psi : R^n \to R$ and $\varphi : [0, T] \times R^n \times U \times V \to R$ are continuous in $(t, x, u, v) \in [0, T] \times R^n \times U \times V$. There exists constant $K > 0$ such that

$$\varphi(t, x, u, v), \psi(x) \ge -K, \quad \forall (t, x, u, v) \in [0, T] \times R^n \times U \times V.$$

[H2] For $0 \le t \le T$, the map $\varepsilon(t, \cdot) : R^n \to 2^{R \times R^n}$ has Cesari properties, i.e.,

$$\bigcap_{\delta>0} \bar{co}\varepsilon(t, O_\delta(x)) = \varepsilon(t, x), \tag{5}$$

for all $x \in R^n$, where $O_\delta(x)$ is a δ-neighborhood of $x \in R^n$, and for any $(t, x) \in [0, T] \times R^n$.

$$\varepsilon(t,x)=\left\{(z^0,z)\in R\times R^n \;\middle|\; \begin{array}{l} z^0\geq \varphi(t,x,u,v),\\ z=f(t,x,u,v),\\ (u,v)\in U\times V, \end{array}\right\}. \tag{6}$$

[I] The following condition holds for any $(t,x)\in[0,T]\times R^n$,

$$\inf_{u\in U}\sup_{v\in V}(\langle p, f(t,x,u,v)\rangle+\varphi(t,x,u,v))=\sup_{v\in V}\inf_{u\in U}(\langle p, f(t,x,u,v)\rangle+\varphi(t,x,u,v)),\quad \forall p\in R^n.$$

Remark 2.3. Under the assumptions [F], [I], and [H1]–[H2], Problem(ZDG) admits open-loop saddle point (see [6–8] and references therein).

Next, we state some property on saddle point.

Property 2.1. Let $(\bar{u}(\cdot),\bar{v}(\cdot))\in\mathcal{U}[0,T]\times\mathcal{V}[0,T]$. Then $(\bar{u}(\cdot),\bar{v}(\cdot))$ is a saddle point of Problem(ZDG) if and only if (for short, iff)

$$\inf_{u(\cdot)\in\mathcal{U}[0,T]}\sup_{v(\cdot)\in\mathcal{V}[0,T]}J(u(\cdot),v(\cdot))=\sup_{v(\cdot)\in\mathcal{V}[0,T]}\inf_{u(\cdot)\in\mathcal{U}[0,T]}J(u(\cdot),v(\cdot)). \tag{7}$$

Proof. Let $(\bar{u}(\cdot),\bar{v}(\cdot))$ be a saddle point, then for any $u(\cdot)\in\mathcal{U}[0,T]$, we have $J(\bar{u}(\cdot),\bar{v}(\cdot))\leq J(u(\cdot),\bar{v}(\cdot))$. This implies that $J(\bar{u}(\cdot),\bar{v}(\cdot))\leq \sup_{v(\cdot)\in\mathcal{V}[0,T]}J(u(\cdot),v(\cdot))$, which results in

$$J(\bar{u}(\cdot),\bar{v}(\cdot))\leq \inf_{u(\cdot)\in\mathcal{U}[0,T]}\sup_{v(\cdot)\in\mathcal{V}[0,T]}J(u(\cdot),v(\cdot)).$$

Similarly, we can prove that

$$J(\bar{u}(\cdot),\bar{v}(\cdot))\geq \sup_{v(\cdot)\in\mathcal{V}[0,T]}\inf_{u(\cdot)\in\mathcal{U}[0,T]}J(u(\cdot),v(\cdot)).$$

From the above, (7) holds.

Conversely, let $\omega=J(\bar{u}(\cdot),\bar{v}(\cdot))$, that is $\omega=\inf_{u(\cdot)\in\mathcal{U}[0,T]}\sup_{v(\cdot)\in\mathcal{V}[0,T]}J(u(\cdot),v(\cdot))=\sup_{v(\cdot)\in\mathcal{V}[0,T]}\inf_{u(\cdot)\in\mathcal{U}[0,T]}J(u(\cdot),v(\cdot))$. Then for any $u(\cdot)\in\mathcal{U}[0,T]$ and $v(\cdot)\in\mathcal{V}[0,T]$, we have

$$\omega=\inf_{u(\cdot)\in\mathcal{U}[0,T]}J(u(\cdot),\bar{v}(\cdot))=\sup_{v(\cdot)\in\mathcal{V}[0,T]}J(\bar{u}(\cdot),v(\cdot)).$$

So,

$$J(\bar{u}(\cdot),v(\cdot))\leq\omega\leq J(u(\cdot),\bar{v}(\cdot)),$$

i.e.,

$$J(\bar{u}(\cdot),v(\cdot))\leq J(\bar{u}(\cdot),\bar{v}(\cdot))\leq J(u(\cdot),\bar{v}(\cdot)).$$

This completes the proof. □

Property 2.2. Let $(\bar{u}_1(\cdot),\bar{v}_1(\cdot))$, $(\bar{u}_2(\cdot),\bar{v}_2(\cdot))\in\mathcal{U}[0,T]\times\mathcal{V}[0,T]$ be saddle point of Problem(ZDG). Then $(\bar{u}_1(\cdot),\bar{v}_2(\cdot))$, $(\bar{u}_2(\cdot),\bar{v}_1(\cdot))\in\mathcal{U}[0,T]\times\mathcal{V}[0,T]$ are also saddle point and

$$J(\bar{u}_1(\cdot),\bar{v}_1(\cdot))=J(\bar{u}_2(\cdot),\bar{v}_1(\cdot))=J(\bar{u}_1(\cdot),\bar{v}_2(\cdot))=J(\bar{u}_2(\cdot),\bar{v}_2(\cdot)). \tag{8}$$

Proof. Since $(\bar{u}_1(\cdot),\bar{v}_1(\cdot))$, $(\bar{u}_2(\cdot),\bar{v}_2(\cdot))$ are saddle points, then for any $(u(\cdot),v(\cdot))\in\mathcal{U}[0,T]\times\mathcal{V}[0,T]$, we have

$$J(u(\cdot),\bar{v}_1(\cdot))\geq J(\bar{u}_1(\cdot),\bar{v}_1(\cdot))\geq J(\bar{u}_1(\cdot),v(\cdot)). \tag{9}$$

$$J(u(\cdot),\bar{v}_2(\cdot))\geq J(\bar{u}_2(\cdot),\bar{v}_2(\cdot))\geq J(\bar{u}_2(\cdot),v(\cdot)). \tag{10}$$

We denote $u(\cdot)=\bar{u}_2(\cdot)$, $v(\cdot)=\bar{v}_2(\cdot)$, and $u(\cdot)=\bar{u}_1(\cdot)$, $v(\cdot)=\bar{v}_1(\cdot)$ in (9) and (10), respectively.

$$\begin{aligned} J(\bar u_2(\cdot), \bar v_1(\cdot)) &\geq J(\bar u_1(\cdot), \bar v_1(\cdot)) \geq J(\bar u_1(\cdot), \bar v_2(\cdot)), \\ J(\bar u_1(\cdot), \bar v_2(\cdot)) &\geq J(\bar u_2(\cdot), \bar v_2(\cdot)) \geq J(\bar u_2(\cdot), \bar v_1(\cdot)). \end{aligned} \tag{11}$$

It follows from (11)) that

$$J(\bar u_1(\cdot), \bar v_1(\cdot)) = J(\bar u_2(\cdot), \bar v_1(\cdot)) = J(\bar u_1(\cdot), \bar v_2(\cdot)) = J(\bar u_2(\cdot), \bar v_2(\cdot)).$$

Namely, (8) holds. From (8) and (11), we obtain that

$$\begin{aligned} J(u(\cdot), \bar v_2(\cdot)) &\geq J(\bar u_1(\cdot), \bar v_2(\cdot)) \geq J(\bar u_1(\cdot), v(\cdot)), \\ J(u(\cdot), \bar v_1(\cdot)) &\geq J(\bar u_2(\cdot), \bar v_1(\cdot)) \geq J(\bar u_2(\cdot), v(\cdot)), \end{aligned}$$

for any $(u(\cdot), v(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T]$. This completes the proof. □

3 Generic uniqueness

To investigate the generic uniqueness of open-loop saddle point for Problem(ZDG), we construct the following model. Let

$$\Omega = \{f \mid f \text{ satisfy } [F]\}. \tag{12}$$

We denote the following set of open-loop saddle points of Problem(ZDG).

$$\begin{aligned} E(f) = \{(\bar u(\cdot), \bar v(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T] | (\bar u(\cdot), \bar v(\cdot)) \text{ is open-loop} \\ \text{saddle point of Problem(ZDG), for any } f \in \Omega\}. \end{aligned} \tag{13}$$

Then, the correspondence $f \to E(f)$ yields a set-valued mapping $E : \Omega \to 2^{U\times V}$. We shall study the generic uniqueness of $E(f)$. The associated metric $d : \Omega \times \Omega \to R$ is defined by

$$d(f, g) = \sup_{(t,x,u,v)\in[0,T]\times R^n\times U\times V} \|f(t, x, u, v) - g(t, x, u, v)\|, \quad \forall f, g \in \Omega.$$

Then, one can easily prove that (Ω, d) is a complete metric space.

Next, we recall a series of definitions on set-valued mapping from [21] to study the generic uniqueness of Problem(ZDG).

Let $U \times V$ be a metric space. A set-valued mapping $E : \Gamma \to 2^{U\times V}$ is called (1) upper (respectively, lower) semi-continuous at $f \in \Omega$ iff for each open set O in $U \times V$ with $E(f) \subset O$ (respectively, $O \cap E(f) \neq \varnothing$), there exists $\delta > 0$ such that $E(g) \subset O$ (respectively, $O \cap E(g) \neq \varnothing$) for any $g \in \Omega$ with $\rho(f, g) < \delta$; (2) continuous at $f \in \Omega$ iff E is both upper and lower semi-continuous at f; (3) an usc mapping with compact values iff E is upper semi-continuous and $E(f)$ is nonempty compact for each $f \in \Omega$; and (4) closed iff Graph(E) is closed, where Graph$(E) \equiv \{(f, u, v) \in \Omega \times U \times V : (u, v) \in E(f)\}$ is the graph of Ω. Also recall that a subset $Q \subset \Omega$ is called a residual set iff it contains countably many intersections of open and dense subsets of Ω. If Ω is a complete space, any residual subset of Ω must be dense in Ω and it is a second category set.

Lemma 3.1. [22] *Let set-valued mapping $E : \Omega \to 2^{U\times V}$ be closed and $U \times V$ be compact, then E is upper semi-continuous at each $f \in \Omega$.*

Lemma 3.2. [23] *Let Ω be a complete metric space, $U \times V$ be a metric space, and $E : \Omega \to 2^{\mathbf{U}}$ be an usc mapping with compact. Then there exists a dense residual subset Q of Ω such that E is lower semi-continuous at every point in Q.*

Remark 3.1. Let $Q \subset \Omega$ be a dense residual set, if for any $\beta \in Q$, a certain property P depending on β holds. Then P is called generic property on Ω. Since Q is a second category, we may say that the property P holds for most of the points of Ω in the sense of Baire's category.

In what follows, inspired by the literature [18] and [20], we give some basic property about continuous dependence for state trajectory.

Property 3.1. Let $\{f_k\} \subset \Omega$ with $f_k \to f \in \Omega$. For any $(u_k(\cdot), v_k(\cdot)) \in \mathcal{U}[0,T] \times \mathcal{V}[0,T]$ with $(u_k(\cdot), v_k(\cdot)) \to (\bar{u}(\cdot), \bar{v}(\cdot)) \in \mathcal{U}[0,T] \times \mathcal{V}[0,T]$, one has $X^{f_k}_{u_k,v_k}(\cdot) \to X^{f}_{\bar{u},\bar{v}}(\cdot)$ as $k \to \infty$.

Proof. For any $t \in [0,T]$, according to control system (1), we have

$$\begin{cases} X_k(t) = x_0 + \int\limits_0^T f_k(t, X_k(t), u_k(t), v_k(t))\mathrm{d}t, \\ X(t) = x_0 + \int\limits_0^T f(t, X(t), u(t), v(t))\mathrm{d}t. \end{cases}$$

Since $f_k \to f$, for any $\varepsilon > 0$, there exists $N_1 > 0$ such that for any $k > N_1$, $d(f_k, f) < \frac{\varepsilon}{3T}$. $X(t)$ is continuous at $[0,T]$, then there exists constant $a_1 > 0$ such that $\max_{t\in[0,T]}\|X(t)\| \le a_1$. $U \subset R^p$ and $V \subset R^q$ are bounded closed and convex set. That is, U and V are also compact. Because $u(\cdot)$ and $v(\cdot)$ are continuous in $[0,T]$, there exist constants $a_2 > 0$ and $a_3 > 0$ such that $\max_{t\in[0,T]}\|u(t)\| \le a_2$ and $\max_{t\in[0,T]}\|v(t)\| \le a_3$. Thus, f is uniformly continuous on the set

$$\Sigma = [0,T] \times \{X \in R^n \mid \|X(t)\| \le a_1\} \times \{u \in U \mid \|u(t)\| \le a_2\} \times \{v \in V \mid \|v(t)\| \le a_3\}.$$

Owing to $(u_k(\cdot), v_k(\cdot)) \to (\bar{u}(\cdot), \bar{v}(\cdot))$, there exists constant $N_2 > 0$ such that for any $t \in [0,T]$, when $k \ge N_2$, one has

$$\|f(t, X(t), u_k(t), v_k(t)) - f(t, X(t), u(t), v_k(t))\| < \frac{\varepsilon}{3T}.$$

There exists constant $N_3 > 0$ such that for any $t \in [0,T]$, when $k \ge N_3$, one has

$$\|f(t, X(t), u(t), v_k(t)) - f(t, X(t), u(t), v(t))\| < \frac{\varepsilon}{3T}.$$

Therefore, choose $N = \max\{N_1, N_2, N_3\}$ such that for any $t \in [0,T]$, when $k \ge N$, one has

$$\begin{aligned} \|X_{f_k}(\cdot) - X_f(\cdot)\| &\le \int\limits_0^T \|f_k(t, X_k(t), u_k(t), v_k(t)) - f(t, X(t), u(t), v(t))\|\mathrm{d}t \\ &\le \|f_k(t, X_k(t), u_k(t), v_k(t)) - f(t, X_k(t), u_k(t), v_k(t))\|\mathrm{d}t \\ &\quad + \int\limits_0^T \|f(t, X_k(t), u_k(t), v_k(t)) - f(t, X(t), u_k(t), v_k(t))\|\mathrm{d}t \\ &\quad + \int\limits_0^T \|f(t, X(t), u_k(t), v_k(t)) - f(t, X(t), u(t), v_k(t))\|\mathrm{d}t \\ &\quad + \int\limits_0^T \|f(t, X(t), u(t), v_k(t)) - f(t, X(t), u(t), v(t))\|\mathrm{d}t \\ &\le \int\limits_0^T \frac{\varepsilon}{3T}\mathrm{d}t + \int\limits_0^T L\|X_k(t) - X(t)\|\mathrm{d}t + \int\limits_0^T \frac{\varepsilon}{3T}\mathrm{d}t + \int\limits_0^T \frac{\varepsilon}{3T}\mathrm{d}t \\ &\le \varepsilon + L\int\limits_0^T \|X_k(t) - X(t)\|\mathrm{d}t. \end{aligned}$$

Thanks to Gronwall's inequality, we have

$$\|X_{f_k} - X_f\| \le \varepsilon e^{LT}.$$

From the arbitrary of $\varepsilon > 0$, it yields $X^{f_k}_{u_k,v_k}(\cdot) \to X^{f}_{\bar{u},\bar{v}}(\cdot)$. □

From Property 3.1, the following result is easily obtained.

Corollary 3.1. *Let* $\{f_k\} \subset \Omega$ *with* $f_k \to f \in \Omega$.

(1) *For any* $u_k(\cdot) \in \mathcal{U}[0,T]$ *with* $u_k(\cdot) \to \bar{u}(\cdot) \in \mathcal{U}[0,T]$. *Then for any* $v(\cdot) \in \mathcal{V}[0,T]$, $X^{f_k}_{u_k,v}(\cdot) \to X^{f}_{\bar{u},v}(\cdot)$ *as* $k \to \infty$.
(2) *For any* $v_k(\cdot) \in \mathcal{V}[0,T]$ *with* $v_k(\cdot) \to \bar{v}(\cdot) \in \mathcal{V}[0,T]$. *Then for any* $u(\cdot) \in \mathcal{U}[0,T]$, $X^{f_k}_{u,v_k}(\cdot) \to X^{f}_{u,\bar{v}}(\cdot)$ *as* $k \to \infty$.
(3) *For any* $(u(\cdot), v(\cdot)) \in \mathcal{U}[0,T] \times \mathcal{V}[0,T]$. *Then* $X^{f_k}_{u,v}(\cdot) \to X^{f}_{\bar{u},\bar{v}}(\cdot)$ *as* $k \to \infty$.

Corollary 3.2. *Let* $f \in \Omega$.

(1) *For any* $(u_k(\cdot), v_k(\cdot)) \in \mathcal{U}[0,T] \times \mathcal{V}[0,T]$ *with* $(u_k(\cdot), v_k(\cdot)) \to (\bar{u}(\cdot), \bar{v}(\cdot)) \in \mathcal{U}[0,T] \times \mathcal{V}[0,T]$. *Then* $J_f(u_k(\cdot), v_k(\cdot)) \to J_f(\bar{u}(\cdot), \bar{v}(\cdot))$ *as* $k \to \infty$.
(2) *For any* $u_k(\cdot) \in \mathcal{U}[0,T]$ *with* $u_k(\cdot) \to \bar{u}(\cdot) \in \mathcal{U}[0,T]$. *Then for any* $v(\cdot) \in \mathcal{V}[0,T]$, $J_f(u_k(\cdot), v(\cdot)) \to J_f(\bar{u}(\cdot), v(\cdot))$ *as* $k \to \infty$.
(3) *For any* $v_k(\cdot) \in \mathcal{V}[0,T]$ *with* $v_k(\cdot) \to \bar{v}(\cdot) \in \mathcal{V}[0,T]$. *Then for any* $u(\cdot) \in \mathcal{U}[0,T]$, $J_f(u(\cdot), v_k(\cdot)) \to J_f(u(\cdot), \bar{v}(\cdot))$ *as* $k \to \infty$.

Now, we present the main results in this paper.

Theorem 3.1. *Set-valued mapping* $E : \Omega \to 2^{U\times V}$ *is an usc mapping with compact.*

Proof. Since $U \subset R^p$ and $V \subset R^q$ are bounded closed and convex set, then $U \times V \subset R^{p+q}$ is also bounded closed and convex set, i.e., $U \times V$ is compact and convex set. By Lemma 3.2, it suffices to show that the graph of E is closed, where $\text{Graph}(E) = \{(f, u, v) \in \Omega \times U \times V \mid (u, v) \in E(f)\}$. Suppose that $\{f_k\} \subset \Omega$ with $f_k \to f \in \Omega$, for any $(u_k(\cdot), v_k(\cdot)) \in E(f_k)$ with $(u_k(\cdot), v_k(\cdot)) \to (\bar{u}(\cdot), \bar{v}(\cdot))$. Let us show that $(\bar{u}(\cdot), \bar{v}(\cdot)) \in E(f)$.

By $(u_k(\cdot), v_k(\cdot)) \in E(f_k)$, then for any $(u, v) \in \mathcal{U}[0,T] \times \mathcal{V}[0,T]$, we have

$$J_{f_k}(u(\cdot), v_k(\cdot)) \geq J_{f_k}(u_k(\cdot), v_k(\cdot)) \geq J_{f_k}(u_k(\cdot), v(\cdot)).$$

Since $(u_k(\cdot), v_k(\cdot)) \to (\bar{u}(\cdot), \bar{v}(\cdot))$, and $f_k \to f$, by Property 3.1 and its Corollaries, we obtain that

$$\begin{aligned} J_{f_k}(u(\cdot), v_k(\cdot)) &\to J_f(u(\cdot), \bar{v}(\cdot)), \\ J_{f_k}(u_k(\cdot), v_k(\cdot)) &\to J_f(\bar{u}(\cdot), \bar{v}(\cdot)), \quad \text{as} k \to \infty. \\ J_{f_k}(u_k(\cdot), v(\cdot)) &\to J_f(\bar{u}(\cdot), v(\cdot)), \end{aligned}$$

Therefore, for any $(u(\cdot), v(\cdot)) \in \mathcal{U}[0,T] \times \mathcal{V}[0,T]$, it results in

$$J_f(u(\cdot), \bar{v}(\cdot)) \geq J_f(\bar{u}(\cdot), \bar{v}(\cdot)) \geq J_f(\bar{u}(\cdot), v(\cdot)),$$

which yields $(\bar{u}(\cdot), \bar{v}(\cdot)) \in E(f)$. This completes the proof. □

Theorem 3.2. *There exists a dense residual subset* Q *of* Ω *such that for any* $\varpi \in Q$, $E(\varpi)$ *is a singleton set.*

Proof. Since $U \times V$ is compact and (Ω, d) is a complete metric space, according to Theorem 3.1, set-valued mapping E is an usc mapping with compact. By using Lemma 3.2, there exists a dense residual subset Q such that for any $\varpi \in Q$, E is lower semi-continuous at ϖ, which implies E is continuous at ϖ.

Assume that $E(\varpi)$ is not a singleton set for some $\varpi \in Q$. Then there exists $(u_1, v_1), (u_2, v_2) \in E(\varpi)$, and $(u_1, v_1) \neq (u_2, v_2)$. Without loss of generality, let $u_1 \neq u_2$. By separation theorem of convex set, there exists continuous linear functional η in E such that $\eta(u_1) \neq \eta(u_2)$, let $g : U \to R$ be defined by

$$g(u) = \frac{\eta(u) - \eta(u_2)}{\eta(u_1) - \eta(u_2)}, \quad \text{for any } u \in U.$$

Then $g(u_1) = 1$, $g(u_2) = 0$, and g is continuous and bounded in U. Take $(u, v) \in U \times V$, for any $\varepsilon > 0$, define a function $\varpi_\varepsilon(u, v) = \varpi(u, v) - \varepsilon g(u)$. It is easy to prove that $\varpi_\varepsilon \in \Omega$ and $\varpi_\varepsilon \to \varpi$ as $\varepsilon \to 0$.

Let $G = \left\{u \in U \mid g(u) > \frac{1}{2}\right\} \times V$, then $G \subset U \times V$ is an open set. Since $g(u_1) = 1, (u_1, v_1) \in G, G \cap E(\varpi) \neq \varnothing$. Since set-valued mapping E is lower semi-continuous, thus, when $\varepsilon > 0$ is very small, we have $G \cap E(f_\varepsilon) \neq \varnothing$. Take $(\bar{u}, \bar{v}) \in G \cap E(\varpi_\varepsilon)$, that is, $(\bar{u}, \bar{v}) \in E(\varpi_\varepsilon)$ and $g(\bar{u}) > \frac{1}{2}$,

$$\begin{aligned} V_\varepsilon &= \inf_{u\in U} \sup_{v\in V} \varpi_\varepsilon(u, v) \geq \inf_{u\in U} \varpi_\varepsilon(u, \bar{v}) = \varpi_\varepsilon(\bar{u}, \bar{v}) = \sup_{v\in V} \varpi_\varepsilon(\bar{u}, v) \\ &= \sup_{v\in V}[\varpi(\bar{u}, v) - \varepsilon g(\bar{u})] = \sup_{v\in V} \varpi(\bar{u}, v) - \varepsilon g(\bar{u}) \\ &> \inf_{u\in U} \sup_{v\in V} \varpi(u, v) - \frac{\varepsilon}{2} = \omega - \frac{\varepsilon}{2}, \end{aligned}$$

where $\omega = \inf_{u\in U} \sup_{v\in V} \varpi(u, v)$.

On the other hand, since $g(u_2) = 0$ and $(u_1, v_1), (u_2, v_2) \in E(\varpi)$, by Property 2.2, $(u_2, v_1) \in E(\varpi)$.

$$\begin{aligned} \omega &= \inf_{u\in U} \sup_{v\in V} \varpi(u, v) \geq \inf_{u\in U} \varpi(u, \bar{v}) = \varpi(\bar{u}, \bar{v}) = \sup_{v\in V} \varpi(\bar{u}, v) \\ &= \sup_{v\in V}[\varpi(\bar{u}, v) - \varepsilon g(\bar{u})] = \sup_{v\in V} \varpi_\varepsilon(\bar{u}, v) \\ &\geq \inf_{u\in U} \sup_{v\in V} \varpi_\varepsilon(u, v) = V_\varepsilon, \end{aligned}$$

which is a contradiction with $V_\varepsilon > \omega - \frac{\varepsilon}{2}$. Thus, the proof is complete. □

4 Conclusion

By constructing a complete metric space, based on the theory of set-valued mappings, this paper investigates the generic uniqueness of saddle point with respect to the right-hand side functions of the control system for two-person zero-sum differential games within the class of open-loop. That is, most of the two-person zero-sum differential games have unique saddle point in the sense of Baire's category. However, it is great that our cost functional is linear with respect to control functions $u(\cdot)$ and $v(\cdot)$. We will investigate the corresponding stability for a general cost functional in the future.

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