Research Article

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Solutions to problems about potentially $K_{s,t}$-bigraphic pair

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Abstract: Let $S = (a_1,\ldots,a_m;\ b_1,\ldots,b_n)$, where $a_1,\ldots,\ a_m$ and $b_1,\ldots,\ b_n$ are two nonincreasing sequences of nonnegative integers. The pair $S = (a_1,\ldots,a_m;\ b_1,\ldots,b_n)$ is said to be a *bigraphic pair* if there is a simple bipartite graph $G = (X \cup Y, E)$ such that $a_1,\ldots,\ a_m$ and $b_1,\ldots,\ b_n$ are the degrees of the vertices in X and Y, respectively. In this case, G is referred to as a *realization* of S. Given a bigraphic pair S, and a complete bipartite graph $K_{s,t}$, we say that S is a *potentially $K_{s,t}$-bigraphic pair* if some realization of S contains $K_{s,t}$ as a subgraph (with s vertices in the part of size m and t in the part of size n). Ferrara et al. (*Potentially H-bigraphic sequences*, Discuss. Math. Graph Theory **29** (2009), 583–596) defined $\sigma(K_{s,t}, m, n)$ to be the minimum integer k such that every bigraphic pair $S = (a_1,\ldots,a_m;b_1,\ldots,b_n)$ with $\sigma(S) = a_1 + \cdots + a_m \geq k$ is a potentially $K_{s,t}$-bigraphic pair. This problem can be viewed as a "potential" degree sequence relaxation of the (forcible) Turán problem. Ferrara et al. determined $\sigma(K_{s,t}, m, n)$ for $n \geq m \geq 9s^4t^4$. In this paper, we further determine $\sigma(K_{s,t}, m, n)$ for $n \geq m \geq s$ and $n + m \geq 2t^2 + t + s$. As two corollaries, if $n \geq m \geq t^2 + \frac{t+s}{2}$ or if $n \geq m \geq s$ and $n \geq 2t^2 + t$, the values $\sigma(K_{s,t}, m, n)$ are determined completely. These results give a solution to a problem due to Ferrara et al. and a solution to a problem due to Yin and Wang.

Keywords: bigraphic pair, realization, potentially $K_{s,t}$-bigraphic pair

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1 Introduction

The study of vertex degrees in graphs has a long history, often asking when an n-tuple of nonnegative integers is realizable as the vertex degrees of a simple n-vertex graph with specified properties. Analogous problems are also studied for bipartite graphs. Let $S = (a_1,\ldots,a_m;\ b_1,\ldots,b_n)$, where $a_1,\ldots,\ a_m$ and $b_1,\ldots,\ b_n$ are two sequences of nonnegative integers with $a_1 \geq \cdots \geq a_m$ and $b_1 \geq \cdots \geq b_n$. We say that S is a *bigraphic pair* if there is a simple bipartite graph G with partite sets $\{x_1,\ldots,x_m\}$ and $\{y_1,\ldots,y_n\}$ such that the degree of x_i is a_i and the degree of y_j is b_j. In this case, we say that G is a *realization* of S. Two methods to determine if S is a bigraphic pair are the Gale-Ryser criteria [1,2] and the Havel-Hakimi-type algorithm [3].

Theorem 1.1. [1,2] *S is a bigraphic pair if and only if $\sum_{i=1}^m a_i = \sum_{i=1}^n b_i$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^n \min\{k, b_i\}$ for $k = 1,\ldots, m$ (or $\sum_{i=1}^k b_i \leq \sum_{i=1}^m \min\{k, a_i\}$ for $k = 1,\ldots, n$).*

For $1 \leq p \leq m$ and $1 \leq q \leq n$, let $S(a_p) = (a_1,\ldots,a_{p-1}, a_{p+1},\ldots,a_m;\ b_1',\ldots,b_n')$ and $S(b_q) = (a_1',\ldots,a_m';\ b_1,\ldots,b_{q-1}, b_{q+1},\ldots,b_n)$, where $b_1' \geq \cdots \geq b_n'$ is a rearrangement in nonincreasing order of $b_1 - 1,\ldots,$

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$b_{a_p} - 1, b_{a_p+1}, \dots, b_n$ and $a'_1 \geq \dots \geq a'_m$ is a rearrangement in nonincreasing order of $a_1 - 1, \dots, a_{b_q} - 1, a_{b_q+1}, \dots, a_m$. We say that $S(a_p)$ (resp. $S(b_q)$) is the residual pair obtained from S by laying off a_p (resp. b_q).

Theorem 1.2. [3] *S is a bigraphic pair if and only if $S(a_p)$ (or $S(b_q)$) is a bigraphic pair.*

We can also ask whether there is a realization satisfying a particular property. Let $S = (a_1, \dots, a_m;\ b_1, \dots, b_n)$ be a bigraphic pair, and let $K_{s,t}$ be the complete bipartite graph with partite sets of size s and t. We say that S is a *potentially $K_{s,t}$-bigraphic pair* if some realization of S contains $K_{s,t}$ (with s vertices in the part of size m and t in the part of size n). If some realization of S contains $K_{s,t}$ on those vertices having degree $a_1, \dots, a_s, b_1, \dots, b_t$, we say that S is a *potentially $A_{s,t}$-bigraphic pair*. Ferrara et al. [4] proved that S is a potentially $A_{s,t}$-bigraphic pair if and only if it is a potentially $K_{s,t}$-bigraphic pair. Yin and Wang [5] developed a Havel-Hakimi-type algorithm to determine if S is a potentially $K_{s,t}$-bigraphic pair. This algorithm can also be used to construct a graph with degree sequence pair S and containing $K_{s,t}$ on those vertices having degree $a_1, \dots, a_s, b_1, \dots, b_t$.

Let $S = (a_1, \dots, a_m;\ b_1, \dots, b_n)$, where $a_1, \dots, a_m$ and $b_1, \dots, b_n$ are two nonincreasing sequences of nonnegative integers. Let $1 \leq s \leq m$, $1 \leq t \leq n$, $a_s \geq t$ and $b_t \geq s$. We first define pairs $S_0, \dots, S_s$ as follows. Let $S_0 = S$. Let

$$S_1 = (a_2, \dots, a_m;\ b_1 - 1, \dots, b_t - 1, b_{t+1}^{(1)}, \dots, b_n^{(1)}),$$

where $b_{t+1}^{(1)} \geq \dots \geq b_n^{(1)}$ is a rearrangement in nonincreasing order of $b_{t+1} - 1, \dots, b_{a_1} - 1, b_{a_1+1}, \dots, b_n$. For $2 \leq i \leq s$, given $S_{i-1} = (a_i, \dots, a_m;\ b_1 - i + 1, \dots, b_t - i + 1, b_{t+1}^{(i-1)}, \dots, b_n^{(i-1)})$, let

$$S_i = (a_{i+1}, \dots, a_m;\ b_1 - i, \dots, b_t - i, b_{t+1}^{(i)}, \dots, b_n^{(i)}),$$

where $b_{t+1}^{(i)} \geq \dots \geq b_n^{(i)}$ is a rearrangement in nonincreasing order of $b_{t+1}^{(i-1)} - 1, \dots, b_{a_i}^{(i-1)} - 1, b_{a_i+1}^{(i-1)}, \dots, b_n^{(i-1)}$.

We now define pairs $S'_0, \dots, S'_t$ as follows. Let $S'_0 = S$. Let

$$S'_1 = (a_1 - 1, \dots, a_s - 1, a_{s+1}^{(1)}, \dots, a_m^{(1)};\ b_2, \dots, b_n),$$

where $a_{s+1}^{(1)} \geq \dots \geq a_m^{(1)}$ is a rearrangement in nonincreasing order of $a_{s+1} - 1, \dots, a_{b_1} - 1, a_{b_1+1}, \dots, a_m$. For $2 \leq i \leq t$, given $S'_{i-1} = (a_1 - i + 1, \dots, a_s - i + 1, a_{s+1}^{(i-1)}, \dots, a_m^{(i-1)};\ b_i, \dots, b_n)$, let

$$S'_i = (a_1 - i, \dots, a_s - i, a_{s+1}^{(i)}, \dots, a_m^{(i)};\ b_{i+1}, \dots, b_n),$$

where $a_{s+1}^{(i)} \geq \dots \geq a_m^{(i)}$ is a rearrangement in nonincreasing order of $a_{s+1}^{(i-1)} - 1, \dots, a_{b_i}^{(i-1)} - 1, a_{b_i+1}^{(i-1)}, \dots, a_m^{(i-1)}$.

Theorem 1.3. [5] *S is a potentially $A_{s,t}$-bigraphic pair if and only if S_s (or S'_t) is a bigraphic pair.*

Motivated by the problem due to Erdős et al. [6] of finding the minimum integer k such that every realizable n-tuple with a sum of at least k is potentially K_r-graphic, Ferrara et al. [4] investigated analogous problem for bipartite graphs. They defined $\sigma(K_{s,t}, m, n)$ to be the minimum integer k such that every bigraphic pair $S = (a_1, \dots, a_m;\ b_1, \dots, b_n)$ with $\sigma(S) = a_1 + \dots + a_m \geq k$ is a potentially $K_{s,t}$-bigraphic pair. They determined $\sigma(K_{s,t}, m, n)$ when m and n are sufficiently large in terms of s and t. This problem can be viewed as a "potential" degree sequence relaxation of the (forcible) Turán problem.

Theorem 1.4. [4] *If $t \geq s \geq 1$ and $n \geq m \geq 9s^4t^4$, then $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$.*

Ferrara et al. proposed a problem as follows.

Problem 1.1. [4] This would be useful if one were interested in finding smaller bounds on the n and m necessary to assure Theorem 1.4.

Yin and Wang proved a new result as follows.

Theorem 1.5. [5] *If $t \geq s \geq 1$, $n \geq m \geq s$ and $n \geq (s+1)t^2 - (2s-1)t + s - 1$, then $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$.*

Yin and Wang also proposed a problem as follows.

Problem 1.2. [5] It would be meaningful to investigate a lower bound on $n + m$ necessary to assure Theorem 1.5.

The purpose of this paper is to improve Theorem 1.5 and determine $\sigma(K_{s,t}, m, n)$ for $n \geq m \geq s$ and $n + m \geq 2t^2 + t + s$, that is, a solution to Problems 1.2. As two corollaries, if $n \geq m \geq t^2 + \frac{t+s}{2}$ or if $n \geq m \geq s$ and $n \geq 2t^2 + t$, the values $\sigma(K_{s,t}, m, n)$ are determined completely, that is, a solution to Problem 1.1.

Theorem 1.6. *If $t \geq s \geq 1$, $n \geq m \geq s$ and $n + m \geq 2t^2 + t + s$, then $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$.*

Corollary 1.1. *If $t \geq s \geq 1$ and $n \geq m \geq t^2 + \frac{t+s}{2}$, then $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$.*

Corollary 1.2. *If $t \geq s \geq 1$, $n \geq m \geq s$ and $n \geq 2t^2 + t$, then $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$.*

2 Proof of Theorem 1.6

In order to prove Theorem 1.6, we need some lemmas.

Lemma 2.1. [7] Theorem 1.1 *remains valid if $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{n} \min\{k, b_i\}$ is assumed only for those k for which $a_k > a_{k+1}$ or $k = m$ (or $\sum_{i=1}^{k} b_i \leq \sum_{i=1}^{m} \min\{k, a_i\}$ is assumed only for those k for which $b_k > b_{k+1}$ or $k = n$).*

Lemma 2.2. [5] *Let $S = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ be a bigraphic pair with $a_s \geq t$, $b_t \geq s$, $m - 1 \geq b_1 \geq \cdots \geq b_t = \cdots = b_{a_1+1} \geq b_{a_1+2} \geq \cdots \geq b_n$ and $n - 1 \geq a_1 \geq \cdots \geq a_s = \cdots = a_{b_1+1} \geq a_{b_1+2} \geq \cdots \geq a_m$. For each $S_i = (a_{i+1}, \ldots, a_m; b_1 - i, \ldots, b_t - i, b_{t+1}^{(i)}, \ldots, b_n^{(i)})$ with $0 \leq i \leq s$,* let *$t_i = \max\{j | b_{t+1}^{(i)} - b_{t+j}^{(i)} \leq 1\}$. Then*

(1) *$t_s \geq t_{s-1} \geq \cdots \geq t_0 \geq a_1 + 1 - t$.*

(2) *For each i with $1 \leq i \leq s$, we have $b_{t+k}^{(i)} = b_{t+k}^{(i-1)}$ for $k > t_i$. Consequently, $b_{t+k}^{(s)} = b_{t+k}$ for $k > t_s$.*

Lemma 2.3. *Let $S = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ be a bigraphic pair with $a_s \geq t$, $b_t \geq s$, $m - 1 \geq b_1 \geq \cdots \geq b_t = \cdots = b_{a_1+1} \geq b_{a_1+2} \geq \cdots \geq b_n$ and $n - 1 \geq a_1 \geq \cdots \geq a_s = \cdots = a_{b_1+1} \geq a_{b_1+2} \geq \cdots \geq a_m$. If $\sum_{i=1}^{t}(b_i - b_t) + \sum_{i=a_{s+1}+1}^{n} b_i \geq ts$, then S is a potentially $A_{s,t}$-bigraphic pair.*

Proof. It is trivial for $s = 1$. Assume $s \geq 2$. By Theorem 1.3, we only need to check that $S_s = (a_{s+1}, \ldots, a_m; b_1 - s, \ldots, b_t - s, b_{t+1}^{(s)}, \ldots, b_n^{(s)})$ is a bigraphic pair. Clearly, $a_{s+1} + \cdots + a_m = (b_1 - s) + \cdots + (b_t - s) + b_{t+1}^{(s)} + \cdots + b_n^{(s)}$. Denote $\ell = b_t$ and $p = \max\{i | a_{s+i} = a_s\}$. Then $s + p \geq b_1 + 1$, i.e., $p \geq b_1 + 1 - s$. By Lemma 2.1, it is enough to check that $\sum_{i=1}^{k} a_{s+i} \leq \sum_{i=1}^{t} \min\{k, b_i - s\} + \sum_{i=t+1}^{n} \min\{k, b_i^{(s)}\}$ for $p \leq k \leq m - s$. Denote $x = b_{t+1}^{(s)}$. By $b_{t+1}^{(s)} \leq b_{t+1} = \ell$, we have $x \leq \ell$. If $k \geq x$, by $k \geq p \geq b_1 + 1 - s > b_i - s$ for $1 \leq i \leq t$, then $\sum_{i=1}^{t} \min\{k, b_i - s\} + \sum_{i=t+1}^{n} \min\{k, b_i^{(s)}\} = \sum_{i=1}^{t}(b_i - s) + \sum_{i=t+1}^{n} b_i^{(s)} = a_{s+1} + \cdots + a_m \geq \sum_{i=1}^{k} a_{s+i}$. Assume $p \leq k \leq x - 1$. If $t_s \geq a_{s+1}$, by $b_{a_{s+1}+t} \geq b_{t+t_s} \geq x - 1 \geq k$, then $\sum_{i=1}^{t} \min\{k, b_i - s\} + \sum_{i=t+1}^{n} \min\{k, b_i^{(s)}\} \geq \sum_{i=t+1}^{a_{s+1}+t} \min\{k, b_i^{(s)}\} = k a_{s+1} \geq \sum_{i=1}^{k} a_{s+i}$. Assume $t_s < a_{s+1}$. Then by Lemma 2.2, $b_{t+j}^{(s)} = b_{t+j}$ for $j \geq a_{s+1}$. If $k \leq b_{a_{s+1}+t}$, then

$\sum_{i=1}^{t}\min\{k, b_i - s\} + \sum_{i=t+1}^{n}\min\{k, b_i^{(s)}\} \geq \sum_{i=t+1}^{a_{s+1}+t}\min\{k, b_i^{(s)}\} = ka_{s+1} \geq \sum_{i=1}^{k}a_{s+i}$. Assume $k > b_{a_{s+1}+t}$. For each i with $a_{s+1} + 1 \leq i \leq t + t_s$, we have $\min\{k, b_i^{(s)}\} = k = \ell - (\ell - k) \geq b_i - (\ell - k)$. Also, for each i with $t + t_s + 1 \leq i \leq a_{s+1} + t$, by Lemma 2.2, we have $\min\{k, b_i^{(s)}\} = \min\{k, b_i\} = \min\{\ell - (\ell - k), b_i\} \geq \min\{b_i - (\ell - k), b_i\} = b_i - (\ell - k)$. Therefore, $\sum_{i=1}^{t}\min\{k, b_i - s\} + \sum_{i=t+1}^{n}\min\{k, b_i^{(s)}\} = \sum_{i=1}^{t}(b_i - s) + \sum_{i=t+1}^{a_{s+1}}\min\{k, b_i^{(s)}\} + \sum_{i=a_{s+1}+1}^{a_{s+1}+t}\min\{k, b_i^{(s)}\} + \sum_{i=a_{s+1}+t+1}^{n}\min\{k, b_i^{(s)}\} \geq \sum_{i=1}^{t}((b_i - \ell) + (\ell - s)) + k(a_{s+1} - t) + \sum_{i=a_{s+1}+1}^{a_{s+1}+t}(b_i - (\ell - k)) + \sum_{i=a_{s+1}+t+1}^{n} b_i = \left(\sum_{i=1}^{t}(b_i - \ell) + \sum_{i=a_{s+1}+1}^{n}b_i\right) + (\ell - s)t + k(a_{s+1} - t) - (\ell - k)t \geq ts + (\ell - s)t + k(a_{s+1} - t) - (\ell - k)t = ka_{s+1} \geq \sum_{i=1}^{k}a_{s+i}$. □

Lemma 2.4. *Let $S = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ be a bigraphic pair with $a_s \geq t$, $b_t \geq s$, $m - 1 \geq b_1 \geq \cdots \geq b_t = \cdots = b_{a_1+1} \geq b_{a_1+2} \geq \cdots \geq b_n$ and $n - 1 \geq a_1 \geq \cdots \geq a_s = \cdots = a_{b_1+1} \geq a_{b_1+2} \geq \cdots \geq a_m$. If $\sum_{i=1}^{s}(a_i - a_s) + \sum_{i=b_{t+1}+1}^{m}a_i \geq ts$, then S is a potentially $A_{s,t}$-bigraphic pair.*

Proof. By the symmetry, the proof of Lemma 2.4 is similar to that of Lemma 2.3. □

Lemma 2.5. [4] *Suppose that $S = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ is not a potentially $A_{s,t}$-bigraphic pair. Let G be a realization of S with partite sets X and Y, with $|X| = m$ and $|Y| = n$. Let X_s be the set of s highest degree vertices of X, and Y_t be the set of t highest degree vertices of Y. Assume that G is a realization of S that maximizes the number of edges between X_s and Y_t. Let x and y be nonadjacent members of X_s and Y_t, and let $A = N_G(x)\backslash Y_t$ and $B = N_G(y)\backslash X_s$. Then both A and B contain at most $(s - 1)(t - 1)$ vertices.*

Lemma 2.6. [7] *If $S = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ is a bigraphic pair with $a_s \geq 2t - 1$ and $b_t \geq 2s - 1$, then S is a potentially $A_{s,t}$-bigraphic pair.*

Lemma 2.7. *Let $S = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ be a bigraphic pair with $m - 1 \geq b_1 \geq \cdots \geq b_t = \cdots = b_{a_1+1} \geq b_{a_1+2} \geq \cdots \geq b_n$ and $n - 1 \geq a_1 \geq \cdots \geq a_s = \cdots = a_{b_1+1} \geq a_{b_1+2} \geq \cdots \geq a_m$. If $n(s - 1) + m(t - 1) \geq \max\{2st^2 - 2t^2 + t - s, 2ts^2 - 2s^2 + s - t\}$ and $\sigma(S) \geq n(s - 1) + m(t - 1) - (t - 1)(s - 1) + 1$, then S is a potentially $A_{s,t}$-bigraphic pair.*

Proof. By $\sigma(S) \geq n(s - 1) + m(t - 1) - (t - 1)(s - 1) + 1$, it is straightforward to show that $a_s \geq t$ and $b_t \geq s$. On the contrary, we assume that S is not a potentially $A_{s,t}$-bigraphic pair. Let G be a realization of S with partite sets X and Y, with $|X| = m$ and $|Y| = n$. Let X_s be the set of s highest degree vertices of X, and Y_t be the set of t highest degree vertices of Y. Assume that G is a realization of S that maximizes the number of edges between X_s and Y_t. Let x and y be nonadjacent members of X_s and Y_t, and let $A = N_G(x)\backslash Y_t$ and $B = N_G(y)\backslash X_s$. By Lemma 2.5, both A and B contain at most $(s - 1)(t - 1)$ vertices. This implies $a_s \leq d_G(x) \leq |A| + |Y_t| - 1 \leq (s - 1)(t - 1) + t - 1 = st - s$ and $b_t \leq d_G(y) \leq |B| + |X_s| - 1 \leq (s - 1)(t - 1) + s - 1 = st - t$. By Lemma 2.6, we have $a_s \leq 2t - 2$ or $b_t \leq 2s - 2$, and so we may consider the following two cases.

Case 1. $a_s \leq 2t - 2$.

It follows from Lemma 2.3 that $\sigma(S) = \sum_{i=1}^{t}b_i + \sum_{i=t+1}^{a_{s+1}}b_i + \sum_{i=a_{s+1}+1}^{n}b_i = \sum_{i=1}^{t}(b_i - b_t + b_t) + \sum_{i=t+1}^{a_{s+1}}b_i + \sum_{i=a_{s+1}+1}^{n} b_i \leq \left(\sum_{i=1}^{t}(b_i - b_t) + \sum_{i=a_{s+1}+1}^{n} b_i\right) + tb_t + (a_{s+1} - t)b_t \leq ts - 1 + a_{s+1} b_t \leq ts - 1 + (2t - 2)(st - t) < (2st^2 - 2t^2 + t - s) - (t - 1)(s - 1) + 1 \leq n(s - 1) + m(t - 1) - (t - 1)(s - 1) + 1$, a contradiction.

Case 2. $b_t \leq 2s - 2$.

It follows from Lemma 2.4 that $\sigma(S) = \sum_{i=1}^{s}a_i + \sum_{i=s+1}^{b_{t+1}}a_i + \sum_{i=b_{t+1}+1}^{m}a_i = \sum_{i=1}^{s}(a_i - a_s + a_s) + \sum_{i=s+1}^{b_{t+1}}a_i + \sum_{i=b_{t+1}+1}^{m}a_i \leq \left(\sum_{i=1}^{s}(a_i - a_s) + \sum_{i=b_{t+1}+1}^{m}a_i\right) + sa_s + (b_{t+1} - s)a_s \leq ts - 1 + b_{t+1}a_s \leq ts - 1 + (2s - 2)(st - s) < (2ts^2 - 2s^2 + s - t) - (t - 1)(s - 1) + 1 \leq n(s - 1) + m(t - 1) - (t - 1)(s - 1) + 1$, a contradiction. □

Lemma 2.8. *Let $S = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ be a bigraphic pair. If $n + m \geq 2\max\{t^2, s^2\} + t + s$ and $\sigma(S) \geq n(s - 1) + m(t - 1) - (t - 1)(s - 1) + 1$, then S is a potentially $A_{s,t}$-bigraphic pair.*

Proof. It is straightforward to show that $a_s \geq t$ and $b_t \geq s$. We use induction on $s+t$. It is trivial for $s=1$ or $t=1$. Assume $s \geq 2$ and $t \geq 2$. If $a_1 = n$ or there exists an integer k with $t \leq k \leq a_1$ such that $b_k > b_{k+1}$, then the residual pair $S(a_1) = (a_2, \ldots, a_m; b_1', \ldots, b_n')$ obtained from S by laying off a_1 satisfies $n + (m-1) \geq 2\max\{t^2, s^2\} + t + (s-1) \geq 2\max\{t^2, (s-1)^2\} + t + (s-1)$, $\sigma(S(a_1)) = \sigma(S) - a_1 \geq n(s-1) + m(t-1) - (t-1)(s-1) + 1 - n = n(s-2) + (m-1)(t-1) - (t-1)(s-2) + 1$ and $b_1' = b_1 - 1, \ldots, b_t' = b_t - 1$. By Theorem 1.2 and the induction hypothesis, $S(a_1)$ is a potentially $A_{s-1,t}$-bigraphic pair, and hence S is a potentially $A_{s,t}$-bigraphic pair. So we may assume $a_1 \leq n-1$ and $b_1 \geq \cdots \geq b_t = \cdots = b_{a_1+1} \geq b_{a_1+2} \geq \cdots \geq b_n$. If $b_1 = m$ or there exists an integer k with $s \leq k \leq b_1$ such that $a_k > a_{k+1}$, then the residual pair $S(b_1) = (a_1', \ldots, a_m'; b_2, \ldots, b_n)$ obtained from S by laying off b_1 satisfies $(n-1) + m \geq 2\max\{t^2, s^2\} + (t-1) + s \geq 2\max\{(t-1)^2, s^2\} + (t-1) + s$, $\sigma(S(b_1)) = \sigma(S) - b_1 \geq n(s-1) + m(t-1) - (t-1)(s-1) + 1 - m = (n-1)(s-1) + m(t-2) - (t-2)(s-1) + 1$ and $a_1' = a_1 - 1, \ldots, a_s' = a_s - 1$. By Theorem 1.2 and the induction hypothesis, $S(b_1)$ is a potentially $A_{s,t-1}$-bigraphic pair, and hence S is a potentially $A_{s,t}$-bigraphic pair. So we may further assume $b_1 \leq m-1$ and $a_1 \geq \cdots \geq a_s = \cdots = a_{b_1+1} \geq a_{b_1+2} \geq \cdots \geq a_m$. If $s \leq t$, then $2st^2 - 2t^2 + t - s \geq 2ts^2 - 2s^2 + s - t$ and $n(s-1) + m(t-1) \geq (n+m)(s-1) \geq (2t^2 + t + s)(s-1) = 2st^2 - 2t^2 + (t+s)(s-1) \geq 2st^2 - 2t^2 + t + s$, implying that $n(s-1) + m(t-1) \geq \max\{2st^2 - 2t^2 + t - s, 2ts^2 - 2s^2 + s - t\}$. Similarly, if $t \leq s$, then $2ts^2 - 2s^2 + s - t \geq 2st^2 - 2t^2 + t - s$ and $n(s-1) + m(t-1) \geq (n+m)(t-1) \geq (2s^2 + t + s)(t-1) = 2ts^2 - 2s^2 + (t+s)(t-1) \geq 2ts^2 - 2s^2 + t + s$, implying that $n(s-1) + m(t-1) \geq \max\{2st^2 - 2t^2 + t - s, 2ts^2 - 2s^2 + s - t\}$. Thus by Lemma 2.7, S is a potentially $A_{s,t}$-bigraphic pair. □

Proof of Theorem 1.6. To show the lower bound, Ferrara et al. [4] considered the bigraphic pair $S = (n^{s-1}, (t-1)^{m-s+1}; m^{s-1}, (t-1)^{m-s+1}, (s-1)^{n-m})$, where the symbol x^y stands for y consecutive terms, each equal to x. Clearly, neither partite set in any realization of S has s vertices of degree t. Hence, S is not a potentially $K_{s,t}$-bigraphic pair. Thus, $\sigma(K_{s,t}, m, n) \geq \sigma(S) + 1 = n(s-1) + m(t-1) - (t-1)(s-1) + 1$. The upper bound directly follows from Lemma 2.8. □

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