Research Article

Zeng Lingli*

# The transfer ideal under the action of orthogonal group in modular case

https://doi.org/10.1515/math-2022-0021
received July 19, 2021; accepted February 9, 2022

**Abstract:** In this paper, we study the structures of the invariant subspaces under the action of orthogonal group $O_{2\nu}(F_q, S)$. In particular, we give a detailed description of 2-codimensional invariant subspaces. Moreover, we show that the height of transfer ideal $\mathrm{Im}(\mathrm{Tr}^{O_{2\nu}(F_q,S)})$ is 2 and give a primary decomposition for the radical ideal of this transfer ideal.

**Keywords:** transfer ideal, transfer variety, invariant subspace, orthogonal group

**MSC 2020:** 47A15, 20M12

## 1 Introduction

Let $V$ be a vector space of dimension $n$ over a field $F$ of characteristic $p$ and let $F[V]$ be the symmetric algebra of $V^*$ (the dual of $V$). If $\{x_1, \ldots, x_n\}$ is a basis for $V$, then $F[V]$ can be identified with the polynomial ring $F[x_1, \ldots, x_n]$. Let $G \subseteq GL(V)$ be a finite group. Then the elements of $G$ act on $F[V]$ as algebra automorphisms and we form the subring

$$F[V]^G \triangleq \{f \in F[V] | gf = f, \quad \forall g \in G\}$$

of $G$-invariant polynomials. The image of transfer map

$$\mathrm{Tr}^G : F[V] \to F[V]^G; \quad f \mapsto \sum_{g \in G} g \cdot f$$

is an ideal of $F[V]^G$. We call it the *transfer ideal* under the action of $G$ and denoted by $\mathrm{Im}(\mathrm{Tr}^G)$. If the order of $G$ is invertible in $F$, then the transfer map $\mathrm{Tr}^G$ is a surjection onto $F[V]^G$. When the characteristic of $F$ divides the order of $G$, the transfer ideal is a proper, nonzero ideal in $F[V]^G$. The transfer ideal is of considerable interest in modular invariant theory.

In 1999 Shank and Wehlau [1] proved that $\mathrm{Im}(\mathrm{Tr}^G)$ is a principal ideal if $G$ is a $p$-group defined over $F_p$ and $F[V]^G$ is a polynomial ring. They also showed that $\mathrm{Im}(\mathrm{Tr}^G)$ are principal for $G = \mathrm{SL}_n(F_q)$ and $\mathrm{GL}_n(F_q)$ with natural actions. Later, Neusel [2,3] studied the transfer ideal $\mathrm{Im}(\mathrm{Tr}^G)$ for permutation group. In addition, she proved that the ideal $\mathrm{Im}(\mathrm{Tr}^G)$ is a prime ideal for cyclic $p$-groups and determined an upper bound of its height. Moreover, Kuhnigt and Smith studied the transfer ideal for the symplectic group $\mathrm{Sp}_{2\nu}(F_q)$ and showed that the radical ideal of transfer is a principal ideal. These detailed proofs can be found on page 276 of [4].

Along this research route, we focus on the transfer ideal for the orthogonal groups. Let $q = p^t$ be a positive odd prime power, $F_q$ be the Galois field with $q$ elements. Let $S$ be an $n \times n$ nonsingular symmetric matrix over $F_q$. Then the set of all matrices $A$ such that $ASA' = S$ forms a group with respect to matrix

*** Corresponding author: Zeng Lingli,** School of Mathematics, Northwest University, Xi'an, Shaanxi, 710127, P. R. China, e-mail: zengll929@nwu.edu.cn

multiplication, where $A'$ denotes the transpose of $A$. We call it the *orthogonal group* of degree $n$ with respect to $S$ and denote it by $O_n(F_q, S)$, i.e.,

$$O_n(F_q, S) = \{A \in \mathrm{GL}_n(F_q) | ASA' = S\}.$$

By [5, Theorem 6.4] we know that the nonsingular symmetric matrix $S$ is one of the following forms:

$$S_{2\nu} = \begin{pmatrix} & I^{(\nu)} \\ I^{(\nu)} & \end{pmatrix}; \quad S_{2\nu+1,1} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & 1 \end{pmatrix}; \quad S_{2\nu+1,z} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & z \end{pmatrix}; \quad S_{2\nu+2} = \begin{pmatrix} 0 & I^{(\nu)} & & \\ I^{(\nu)} & 0 & & \\ & & 1 & 0 \\ & & 0 & -z \end{pmatrix},$$

where $I^{(\nu)}$ is a $\nu \times \nu$ identity matrix and $z$ is a non square element in $F_q$. Then, up to isomorphism, the orthogonal groups are four types. In this paper, we shall focus attention on the orthogonal group $O_{2\nu}(F_q, S)$ with respect to the nonsingular symmetric matrix $S = S_{2\nu}$. The other cases are similar and are omitted.

The paper is organized as follows. After this introductory section, in Sections 2 and 3 we discuss the structures of invariant subspaces under the action of the orthogonal group $O_{2\nu}(F_q, S)$. In Section 4, we determine the structures of the transfer variety $\Omega_{O_{2\nu}(F_q,S)}$ and give a primary decomposition for the radical ideal of transfer ideal, and show that the height of this transfer ideal is 2. In addition, we give a detailed example for $q = 3$ and $\nu = 2$ in Section 5.

## 2 Types of 2-codimensional invariant subspaces

Let $e_1, e_2, \dots, e_{2\nu}$ be the standard basis of the vector space $V = F_q^{2\nu}$. For each $v = k_1e_1 + k_2e_2 + \cdots + k_{2\nu}e_{2\nu} \in V$, $k_i \in F_q$, there is an action of $O_{2\nu}(F_q, S)$ on $V$ defined as

$$\begin{gathered} V \times O_{2\nu}(F_q, S) \longrightarrow V \\ ((k_1, k_2, \dots, k_{2\nu}), A) \longmapsto (k_1, k_2, \dots, k_{2\nu})A. \end{gathered}$$

Then the vector space $V$ together with this action is called the $2\nu$-dimensional *orthogonal space* over $F_q$ with respect to $S$.

Let $P$ be an $m$-dimensional vector subspace of $V$. We use the same symbol $P$ to denote the matrix representation of the vector subspace $P$, i.e., $P$ is an $m \times 2\nu$ matrix whose rows form a basis of the vector subspace $P$. Two $n \times n$ matrices $A$ and $B$ are said to be cogredient, if there is a $n \times n$ nonsingular matrix $Q$ such that $QAQ' = B$. It is well known that $PSP'$ is cogredient to one of the following normal forms [5]:

$$M(m, 2s, s) = \begin{pmatrix} 0 & I^{(s)} & \\ I^{(s)} & 0 & \\ & & 0^{(m-2s)} \end{pmatrix}, \qquad M(m, 2s+1, s, 1) = \begin{pmatrix} 0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & 1 & \\ & & & 0^{(m-2s-1)} \end{pmatrix},$$

$$M(m, 2s+1, s, z) = \begin{pmatrix} 0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & z & \\ & & & 0^{(m-2s-1)} \end{pmatrix}, \quad M(m, 2s+2, s) = \begin{pmatrix} 0 & I^{(s)} & & & \\ I^{(s)} & 0 & & & \\ & & 1 & & \\ & & & -z & \\ & & & & 0^{(m-2s-2)} \end{pmatrix}.$$

We use the symbol $M(m, 2s + \gamma, s, \Gamma)$ to represent any one of these four normal forms, where $s$ is its *index*, $\gamma = 0, 1$, or $2$, and $\Gamma$ represents the definite part in these normal forms. If $PSP'$ is cogredient to $M(m, 2s + \gamma, s, \Gamma)$, then $P$ is called a *subspace of type* $(m, 2s + \gamma, s, \Gamma)$ with respect to $S$ in $V$. Subspaces of type $(m, 2s, s)$, $(m, 2s + 1, s, 1)$, $(m, 2s + 1, s, z)$, and $(m, 2s + 2, s)$ are also called *subspace of the hyperbolic type*, *the square type*, *the nonsquare type*, and *the elliptic type*, respectively.

By the proof of Theorem 6.3 in [5], we have the following lemma

**Lemma 2.1.** [5] *Subspaces of types* $(m, 2s+\gamma, s, \Gamma)$ *exist in the* $2\nu$*-dimensional orthogonal space* $V = F_q^{2\nu}$ *with respect to the nonsingular symmetric matrix* $S$ *if and only if*

$$2s+\gamma \leq m \leq \nu + s.$$

Then, we can determine the types of 2-codimensional subspaces.

**Lemma 2.2.** *There are five types of* 2*-codimensional subspaces of the* $2\nu$*-dimensional orthogonal space* $V = F_q^{2\nu}$.

**Proof.** Let $m = 2\nu - 2$. By Lemma 2.1, $2s+\gamma \leq m \leq \nu + s$, it follows that if $r = 0$ then $s = \nu - 2$ or $\nu - 1$; if $r = 1$ then $s = \nu - 2$; if $r = 2$ then $s = \nu - 2$. Hence, we obtain the following five types of 2-codimensional subspaces: $(2\nu-2, 2(\nu-2), \nu-2)$, $(2\nu-2, 2(\nu-1), \nu-1)$, $(2\nu-2, 2(\nu-2)+1, \nu-2, 1)$, $(2\nu-2, 2(\nu-2)+1, \nu-2, z)$, and $(2\nu-2, 2(\nu-2)+2, \nu-2)$. □

**Remark 2.3.** For convenience, let **type I of 2-codim**, **type II of 2-codim**, **type III of 2-codim**, **type IV of 2-codim**, and **type V of 2-codim** denote the subspaces of type $(2\nu-2, 2(\nu-2), \nu-2)$, $(2\nu-2, 2(\nu-1), \nu-1)$, $(2\nu-2, 2(\nu-2)+1, \nu-2, 1)$, $(2\nu-2, 2(\nu-2)+1, \nu-2, z)$, and $(2\nu-2, 2(\nu-2)+2, \nu-2)$, respectively.

The following two lemmas celebrated Witt's transitivity theorem will be often used.

**Lemma 2.4.** ([5], Theorem 6.4) *Let* $P_1$ *and* $P_2$ *be two* $m$*-dimensional subspaces of* $V$. *Then there is an* $A \in O_{2\nu}(F_q, S)$ *such that* $P_1 = BP_2A$, *where* $B$ *is an* $m \times m$ *nonsingular matrix, if and only if* $P_1$ *and* $P_2$ *are of the same type with respect to* $S$. *In other words,* $O_{2\nu}(F_q, S)$ *acts transitively on each set of subspaces of the same type.*

**Lemma 2.5.** [5, Lemma 6.8] *Let* $P_1$ *and* $P_2$ *be two* $m \times m$ *matrices of rank* $m$. *Then there exists an element* $A \in O_{2\nu}(F_q, S)$ *such that* $P_1 = P_2A$ *if and only if* $P_1SP_1' = P_2SP_2'$.

Now, let us study the structures of the type I of 2-codim subspaces.

**Definition 2.6.** ([6], Section 9.2 Definition) An element $T \in O_{2\nu}(F_q, S)$ is called a 2-transvection if $T = I + N$, where $I$ is the identity matrix, the rank of $N$ is 2 and $NSN' = 0$.

**Lemma 2.7.** ([6], Section 9.2 Theorem 1) *In the orthogonal group* $O_{2\nu}(F_q, S)$, *each* 2*-transvection is similar to*

$$\left(\begin{array}{c|c} I^{(\nu)} & K \\ \hline & I^{(\nu)} \end{array}\right), \text{ where } K = \left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & \\ \hline & & 0^{(\nu-2)} \end{array}\right).$$

Let $V^A = \{v \in V | vA = v\}$ where $A \in GL_{2\nu}(F_q)$. Then it is easy to check that $V^A$ is a subspace of vector space $V$ and $V^{A^{-1}BA} = V^BA$ for each $B \in GL_{2\nu}(F_q)$.

**Lemma 2.8.** *Let* $T \in O_{2\nu}(F_q, S)$. *Then* $T$ *is a* 2*-transvection if and only if the invariant subspace* $V^T$ *is a type I of 2-codim subspace.*

**Proof.** Suppose that $T$ is a 2-transvection. Let $T_0 = \left(\begin{array}{c|c} I^{(\nu)} & K \\ \hline & I^{(\nu)} \end{array}\right)$ in Lemma 2.7. Then $T_0$ is also a 2-transvection and $ATA^{-1} = T_0$ for some $A \in O_{2\nu}(F_q, S)$. For each $v = k_1e_1 + \cdots + k_{2\nu}e_{2\nu} \in V$, we have

$$vT_0 = k_1e_1 + \cdots + k_\nu e_\nu + (k_{\nu+1} - k_2)e_{\nu+1} + (k_{\nu+2} + k_1)e_{\nu+2} + k_{\nu+3}e_{\nu+3} + \cdots + k_{2\nu}e_{2\nu}.$$

If $v \in V^{T_0}$, then $vT_0 = v$, whence $k_1 = k_2 = 0$. Therefore,

$$V^{T_0} = \{k_3e_3 + k_4e_4 + \cdots + k_{2v}e_{2v} \mid k_i \in F_q\}$$

and $\dim V^{T_0} = 2v - 2$. We denote the vector invariant subspace $V^{T_0}$ as the $(2v-2) \times 2v$ matrix

$$\hat{T}_0 = (e_3, \ e_4, \ldots, \ e_{2v})'.$$

By computing, it follows that

$$\hat{T}_0 S \hat{T}_0' = \left(\begin{array}{c|c} & 0^{2\times(v-2)} \\ & I^{(v-2)} \\ \hline 0^{(v-2)\times 2} \quad I^{(v-2)} & \end{array}\right) \text{ is cogredient to } \left(\begin{array}{cc|cc} 0 & I^{(v-2)} & & \\ I^{(v-2)} & 0 & & \\ \hline & & 0 & 0 \\ & & 0 & 0 \end{array}\right).$$

Then the type of $V^{T_0}$ is $(2v-2, 2(v-2), v-2)$. This implies that $V^{T_0}$ is a type I of 2-codim subspace. Since $V^T = V^{A^{-1}T_0A} = V^{T_0}A$, the invariant subspace $V^T$ has the same type with $V^{T_0}$ by Lemma 2.4. Consequently, $V^T$ is also a type I of 2-codim subspace.

Conversely, suppose that $V^T$ is a type I of 2-codim subspace. With the preceding discussion, the invariant subspace $V^{T_0}$ is a type I of 2-codim subspace and $\{e_3, e_4, \ldots, e_{2v}\} \subseteq V^{T_0}$. By Lemma 2.4, there exists an element $A \in O_{2v}(F_q, S)$ such that $V^{T_0} = V^TA = V^{A^{-1}TA}$. Let $T_1 = A^{-1}TA$. Then $T_1 \in O_{2v}(F_q, S)$ and the elements $e_3, e_4, \ldots, e_{2v}$ are invariants under the action of $T_1$. Therefore, we may assume that

$$T_1 = \left(\begin{array}{ccc|ccc} a_{11} & a_{12} & H_{11} & a_{1v+1} & a_{1v+2} & H_{12} \\ a_{21} & a_{22} & H_{21} & a_{2v+1} & a_{2v+2} & H_{22} \\ 0 & 0 & I^{(v-2)} & 0 & 0 & 0^{(v-2)} \\ \hline & & & & I^{(v)} & \end{array}\right),$$

where $H_{ij}$, $i, j = 1, 2$, are $1 \times (v-2)$ matrices over $F_q$. Since $T_1 \in O_{2v}(F_q, S)$, it must satisfy $T_1ST_1' = S$, then we get the following equations:

$$\begin{cases} a_{11} = a_{22} = 1 \\ a_{1v+2} + a_{2v+1} = 0 \\ a_{12} = a_{21} = a_{1v+1} = a_{2v+2} = 0 \\ H_{ij} = 0, \ \ i, j = 1, 2. \end{cases} \tag{2.1}$$

Hence,

$$T_1 = \left(\begin{array}{c|c} I^{(v)} & a_{1v+2}K \\ \hline & I^{(v)} \end{array}\right),$$

where $a_{1v+2} \in F_q^*$.

It is easy to check that $(T_1 - I)S(T_1 - I)' = (0)$ and $\operatorname{rank}(T_1 - I) = 2$, thus $T_1$ is a 2-transvection. Consequently, $T = AT_1A^{-1}$ is also a 2-transvection. ☐

Next, we are going to study the structures of the type II of 2-codim subspaces.

**Definition 2.9.** ([6], Section 9.3 Definition) Let $v \geq 1$. A subspace $P$ of $V$ is called a *hyperbolic place* if $\dim(P) = 2$ and $P$ has a basis $\{u, v\}$ such that $uSu' = vSv' = 0$, $uSv' = 1$. An element $R \in O_{2v}(F_q, S)$ is called *hyperbolic motion* taking the place $P^*$ as axis, if $vR = v$, $\forall v \in P^*$, and $vR \in P$, $\forall v \in P$. Furthermore, we call $R$ a *hyperbolic rotation* if $R \in O_{2v}^+(F_q, S)$.

**Lemma 2.10.** ([6], Section 9.3 Theorem 1) *Let $v \geq 1$. In $O_{2v}(F_q, S)$, each hyperbolic motion $R$ is similar to one of the following forms*:

$$R_1 = \left(\begin{array}{cc|cc} a & & & \\ & I^{(\nu-1)} & & \\ \hline & & a^{-1} & \\ & & & I^{(\nu-1)} \end{array}\right) \quad or \quad R_2 = \left(\begin{array}{cc|cc} 0 & & a & \\ & I^{(\nu-1)} & & 0^{(\nu-1)} \\ \hline a^{-1} & & 0 & \\ & 0^{(\nu-1)} & & I^{(\nu-1)} \end{array}\right), \quad a \in F_q^*.$$

*If $R$ is a hyperbolic rotation, then $R$ must be similar to $R_1$.*

**Lemma 2.11.** *If $R$ is a hyperbolic rotation, then $|R| \neq p$, i.e., the order of $R$ is not $p$.*

**Proof.** By Lemma 2.10, $|R| = |R_1| = |a|$. Since $a \in F_q^*$, it implies that $|a| \neq p$. □

**Lemma 2.12.** *Let $R \in O_{2\nu}(F_q, S)$. Then $R$ is a hyperbolic rotation if and only if the invariant subspace $V^R$ is a type II of 2-codim subspace.*

**Proof.** The following proof is similar to Lemma 2.8, thus we just give the main idea. Suppose that $R$ is a hyperbolic rotation. Let $R_1 = \begin{pmatrix} a & & & \\ & I^{(\nu-1)} & & \\ & & a^{-1} & \\ & & & I^{(\nu-1)} \end{pmatrix}$, $1 \neq a \in F_q^*$. We have that $V^{R_1} = \{k_2e_2 + \cdots + k_\nu e_\nu + k_{\nu+2}e_{\nu+2} + \cdots + k_{2\nu}e_{2\nu} | k_i \in F_q\}$ and $\dim V^{R_1} = 2\nu - 2$. Then $V^{R_1}$ is a type II of 2-codim subspace. By Lemma 2.10, $ARA^{-1} = R_1$ for some $A \in O_{2\nu}(F_q, S)$, hence $V^R = V^{R_1}A$ is also a type II of 2-codim subspace by Lemma 2.4.

Conversely, suppose that $V^R$ is a type II of 2-codim subspace. Then there exists $A \in O_{2\nu}(F_q, S)$ such that $V^{R_1} = V^RA$ by Lemma 2.4. Let $R_3 = A^{-1}RA$. We have that $V^{R_1} = V^{R_3}$, $R_3 \in O_{2\nu}(F_q, S)$, and the elements $e_2, \ldots, e_\nu, e_{\nu+2}, \ldots, e_{2\nu}$ are invariant under the action of $R_3$. Hence, consider the equations

$$R_3SR_3' = S \quad \text{and} \quad e_iR_3 = e_i, \quad i = 2, \ldots, \nu, \nu + 2, \ldots, 2\nu,$$

we obtain that

$$R_3 = \left(\begin{array}{cc|cc} a_{11} & & & \\ & I^{(\nu-1)} & & \\ \hline & & a_{11}^{-1} & \\ & & & I^{(\nu-1)} \end{array}\right), \quad a_{11} \neq 0,$$

and so $R_3$ is a hyperbolic rotation. Consequently, $R = AR_3A^{-1}$ is also a hyperbolic rotation. □

Now, we shall consider the cases of the type III and type IV of 2-codim subspaces.

Let

$$\mathbf{H} = \left\{ H = \left(\begin{array}{ccc|ccc} 1 & 0 & & 0 & -b & \\ a & 1 & & b & -ab & \\ & & I^{(\nu-2)} & & & 0^{(\nu-2)} \\ \hline & & & 1 & -a & \\ & & & 0 & 1 & \\ & & & & & I^{(\nu-2)} \end{array}\right) \;\middle|\; a, b \in F_q^* \right\}$$

be a set. The construction of this set is motivated by combining two 2-transvections.

**Definition 2.13.** An element $A \in \mathrm{GL}_{2\nu}(F_q)$ is called a $H_1$-type (resp. $H_z$-type) matrix if there exists a $B \in \mathrm{GL}_{2\nu}(F_q)$ such that $BAB^{-1} \in \mathbf{H}$ and $-2ab^{-1}$ corresponding with $BAB^{-1}$ is a square (resp. nonsquare) element in $F_q^*$.

**Lemma 2.14.**
(1) $\mathbf{H} \subseteq O_{2\nu}(F_q, S)$.
(2) *If* $H \in \mathbf{H}$, *then* $|H| = p$.
(3) $H$ *is a* $H_1$*-type* (*resp.* $H_z$*-type*) *matrix if and only if the invariant subspace* $V^H$ *is a type III* (*resp. type IV*) *of* 2*-codim subspace.*

**Proof.** It is easy to check (1) and (2). The proof of (3) is similar to Lemma 2.8, so we just give the main idea. Suppose that $H$ is an $H_1$-type (resp. $H_z$-type) matrix. Then we have that $V^H = \{k_1(-ab^{-1}e_1 + e_{\nu+1}) + k_3e_3 + \cdots + k_\nu e_\nu + k_{\nu+2}e_{\nu+2} + \cdots + k_{2\nu}e_{2\nu} \mid k_i \in F_q\}$ and $\dim V^H = 2\nu - 2$. Thus, the type of the matrix corresponding with $V^H$ is $\begin{pmatrix} & I^{(\nu-2)} & & \\ I^{(\nu-2)} & & & \\ & & -2ab^{-1} & 0 \\ & & 0 & 0 \end{pmatrix}$. Consequently, if $-2ab^{-1}$ is a square (resp. nonsquare) element in $F_q^*$, then $V^H$ is a type III (resp. type IV) of 2-codim subspace.

Conversely, we give only the proof for the type III of 2-codim subspace. Suppose that the invariant subspace $V^A$ is a type III of 2-codim subspace. Then there exists an element $B \in O_{2\nu}(F_q, S)$ such that $V^H = V^AB = V^{B^{-1}AB} = V^C$ by Lemma 2.4. Let $C = B^{-1}AB$. We have that $V^C = V^H$, $C \in O_{2\nu}(F_q, S)$, and $\dim(V^C) = 2\nu - 2$. Moreover, the elements $e_3, \ldots, e_\nu, e_{\nu+2}, \ldots, e_{2\nu}$, and $-ab^{-1}e_1 + e_{\nu+1}$ are invariants under the action of $C$. Hence, consider the equations

$$\begin{gathered} CSC' = S, \\ e_iC = e_i,\ i = 3, \ldots, \nu, \nu+2, \ldots, 2\nu, \\ (-ab^{-1}e_1 + e_{\nu+1})C = -ab^{-1}e_1 + e_{\nu+1}, \end{gathered}$$

we obtain that

$$C = \left(\begin{array}{ccc|ccc} 1 & 0 & & 0 & -a_{2\nu+1} & \\ a_{21} & 1 & & a_{2\nu+1} & -a_{21}a_{2\nu+1} & \\ & & I^{(\nu-2)} & & & 0^{(\nu-2)} \\ \hline & & & 1 & -a_{21} & \\ & & & 0 & 1 & \\ & & & & & I^{(\nu-2)} \end{array}\right).$$

Next, we claim that $a_{21} \neq 0$ and $a_{2\nu+1} \neq 0$. If $a_{21} = a_{2\nu+1} = 0$, then $C = I^{(2\nu)}$, contradicting $\dim V^C = 2\nu - 2$. If $a_{21} = 0$ and $a_{2\nu+1} \neq 0$, then $C$ is a 2-transvection. By Lemma 2.8, $V^C$ is a type I of 2-codim subspace, which contradicts that $V^C = V^H$ is a type III of 2-codim subspace. If $a_{21} \neq 0$ and $a_{2\nu+1} = 0$, then $C$ is also a 2-transvection, a contradiction. Therefore, $a_{21} \neq 0$ and $a_{2\nu+1} \neq 0$, thus $C \in \mathbf{H}$. Since $V^C = V^H$ is a type III of 2-codim subspace, it follows that $-2a_{21}a_{2\nu+1}^{-1}$ is a square element in $F_q^*$. Hence, $A = BCB^{-1}$ is a $H_1$-type matrix. □

Finally, we shall show the structures of the type V of 2-codim subspaces.

**Lemma 2.15.** *If the order of the matrix* $\bar{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is not* $p$, *where* $A, B, C, D$ *are* $n \times n$ *matrices over* $F_q$, *then the order of matrix* $\bar{\bar{A}} = \left(\begin{array}{cc|cc} A & * & B & * \\ 0 & I & 0 & 0 \\ \hline C & * & D & * \\ 0 & 0 & 0 & I \end{array}\right)$ *is not* $p$ *either, where* $I$ *is an* $m \times m$ *identity matrix, and each* $*$ *is an arbitrary* $n \times m$ *matrix over* $F_q$.

**Proof.** If $\bar{A}^i = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$, then $\bar{\bar{A}}^i = \left(\begin{array}{cc|cc} A_i & * & B_i & * \\ 0 & I & 0 & 0 \\ \hline C_i & * & D_i & * \\ 0 & 0 & 0 & I \end{array}\right)$. Therefore, if $\bar{A}^p \neq I^{(n)}$, then we conclude that $\bar{\bar{A}}^p \neq I^{(n+m)}$. □

Let

$$Q = \left(\begin{array}{ccc|ccc} 0 & 0 & & a & 0 & \\ 0 & 0 & & 0 & b & \\ & & I^{(\nu-2)} & & & 0^{(\nu-2)} \\ \hline a^{-1} & 0 & & 0 & 0 & \\ 0 & b^{-1} & & 0 & 0 & \\ & & 0^{(\nu-2)} & & & I^{(\nu-2)} \end{array}\right),$$

where $a, b \in F_q^*$, $2a$ is a square element, and $2b = -z$. The construction of this element is motivated by combining two hyperbolic motions. It is easily seen that $Q \in O_{2\nu}(F_q, S)$, and

$$V^Q = \{k_1(e_1 + ae_{\nu+1}) + k_2(e_2 + be_{\nu+2}) + k_3e_3 + \cdots + k_\nu e_\nu + k_{\nu+3}e_{\nu+3} + \cdots + k_{2\nu}e_{2\nu} \mid k_i \in F_q\}.$$

Then the type of the matrix corresponding with the invariant subspace $V^Q$ is $\begin{pmatrix} & I^{(\nu-2)} & & \\ I^{(\nu-2)} & & & \\ & & 1 & 0 \\ & & 0 & -z \end{pmatrix}$. Hence, $V^Q$ is a type V of 2-codim subspace.

**Lemma 2.16.** *If $A \in O_{2\nu}(F_q, S)$ and the invariant subspace $V^A$ is a type V of 2-codim subspace, then $|A| \neq p$.*

**Proof.** Since the invariant subspaces $V^A$ and $V^Q$ are both the type V of 2-codim subspaces, there exists an element $B \in O_{2\nu}(F_q, S)$ such that $V^Q = V^A B$ by Lemma 2.4. Let $C = B^{-1}AB$. Then $C \in O_{2\nu}(F_q, S)$ and the elements $e_3, \ldots, e_\nu, e_{\nu+3}, \ldots, e_{2\nu}$, $e_1 + ae_{\nu+1}$, and $e_2 + be_{\nu+2}$ are all invariants under the action of $C$. Therefore, we may assume that

$$C = \left(\begin{array}{ccc|ccc} a_{11} & a_{12} & H_{11} & a_{1\nu+1} & a_{1\nu+2} & H_{12} \\ a_{21} & a_{22} & H_{21} & a_{2\nu+1} & a_{2\nu+2} & H_{22} \\ 0 & 0 & I^{(\nu-2)} & 0 & 0 & 0^{(\nu-2)} \\ \hline a_{\nu+1\,1} & a_{\nu+1\,2} & H_{31} & a_{\nu+1\,\nu+1} & a_{\nu+1\,\nu+2} & H_{32} \\ a_{\nu+2\,1} & a_{\nu+2\,2} & H_{41} & a_{\nu+2\,\nu+1} & a_{\nu+2\,\nu+2} & H_{42} \\ 0 & 0 & 0^{(\nu-2)} & 0 & 0 & I^{(\nu-2)} \end{array}\right),$$

where $H_{ij}$, $i = 1, 2, 3, 4$, $j = 1, 2$, are $1 \times (\nu - 2)$ matrices over $F_q$.

Now we consider the sub-block of matrix $C$, i.e.,

$$\bar{C} = \left(\begin{array}{cc|cc} a_{11} & a_{12} & a_{1\nu+1} & a_{1\nu+2} \\ a_{21} & a_{22} & a_{2\nu+1} & a_{2\nu+2} \\ \hline a_{\nu+1\,1} & a_{\nu+1\,2} & a_{\nu+1\,\nu+1} & a_{\nu+1\,\nu+2} \\ a_{\nu+2\,1} & a_{\nu+2\,2} & a_{\nu+2\,\nu+1} & a_{\nu+2\,\nu+2}. \end{array}\right).$$

Since $e_1 + ae_{\nu+1}$ and $e_2 + be_{\nu+2}$ are invariants under the action of $C$, we have

$$\begin{cases} a_{11} = 1 - aa_{\nu+1\,1} \\ a_{12} = -aa_{\nu+1\,2} \\ a_{1\nu+1} = a - aa_{\nu+1\,\nu+1} \\ a_{1\nu+2} = -aa_{\nu+1\,\nu+2} \\ a_{21} = -ba_{\nu+2\,1} \\ a_{22} = 1 - ba_{\nu+2\,2} \\ a_{2\nu+1} = -ba_{\nu+2\,\nu+1} \\ a_{2\nu+2} = b - ba_{\nu+2\,\nu+2} \end{cases} \tag{2.2}$$

Since $C \in O_{2\nu}(F_q, S)$, it must satisfy $CSC' = S$, then $\bar{C}$ satisfies

$$\bar{C}\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)\bar{C}' = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

And adding (2.2), we can obtain the following equations:

$$a_{v+1\,v+1} = 1 - aa_{v+1\,1} \tag{2.3}$$

$$a_{v+1\,v+2} = -ba_{v+1\,2} \tag{2.4}$$

$$a_{v+2\,v+1} = -aa_{v+2\,1} \tag{2.5}$$

$$a_{v+2\,v+2} = 1 - ba_{v+2\,2} \tag{2.6}$$

$$a_{v+1\,1}(1 - aa_{v+1\,1}) = ba_{v+1\,2}^2 \tag{2.7}$$

$$a_{v+2\,2}(1 - ba_{v+2\,2}) = aa_{v+2\,1}^2 \tag{2.8}$$

$$a_{v+2\,1}(2aa_{v+1\,1} - 1) + a_{v+1\,2}(2ba_{v+2\,2} - 1) = 0. \tag{2.9}$$

Next, we have to consider the following situations: Whether or not $a_{v+1\,1}$, $a_{v+1\,2}$, $a_{v+2\,1}$, and $a_{v+2\,2}$ are 0, respectively. We only give the proof of the most difficult case when $a_{v+2\,1} \neq 0$ and $a_{v+2\,2} \neq 2^{-1}b^{-1}$.

Since $a_{v+2\,2} \neq 2^{-1}b^{-1}$, $1 - 2ba_{v+2\,2} \neq 0$. So by (2.9), we obtain

$$\frac{1 - 2aa_{v+1\,1}}{1 - 2ba_{v+2\,2}} = -\frac{a_{v+1\,2}}{a_{v+2\,1}}. \tag{2.10}$$

Combining (2.7) with (2.8), we have

$$\frac{a_{v+1\,1}(1 - aa_{v+1\,1})}{a_{v+2\,2}(1 - ba_{v+2\,2})} = \frac{ba_{v+1\,2}^2}{aa_{v+2\,1}^2}. \tag{2.11}$$

Combining (2.11) with the square of (2.10), we conclude that

$$\frac{aa_{v+1\,1}(1 - aa_{v+1\,1})}{ba_{v+2\,2}(1 - ba_{v+2\,2})} = 1 = \frac{a_{v+1\,2}^2}{a_{v+2\,1}^2}, \tag{2.12}$$

then $a_{v+1\,2} = \pm a_{v+2\,1}$.

If $a_{v+1\,2} = -a_{v+2\,1}$, then it follows that $a_{v+1\,1} = a^{-1}ba_{v+2\,2}$ according to 2.9. Thus, adding (2.2)–(2.6), we have

$$\bar{C} = \left(\begin{array}{cc|cc} 1 - ba_{v+2\,2} & aa_{v+2\,1} & aba_{v+2\,2} & -aba_{v+2\,1} \\ -ba_{v+2\,1} & 1 - ba_{v+2\,2} & aba_{v+2\,1} & b^2a_{v+2\,2} \\ \hline a^{-1}ba_{v+2\,2} & -a_{v+2\,1} & 1 - ba_{v+2\,2} & ba_{v+2\,1} \\ a_{v+2\,1} & a_{v+2\,2} & -aa_{v+2\,1} & 1 - ba_{v+2\,2} \end{array}\right).$$

Let

$$P_1 \triangleq \left(\begin{array}{cc|cc} 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \\ \hline 0 & 0 & aa_{v+2\,1} & -ba_{v+2\,2} \\ 0 & 0 & 0 & 1 \end{array}\right)$$

and

$$P_2 \triangleq P_1\bar{C}P_1^{-1} - I = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & -a_{v+2\,2} & 0 & 2ba_{v+2\,2} \\ a_{v+2\,1} & a_{v+2\,2} & -2 & -4ba_{v+2\,2} \end{array}\right).$$

If we denote $E = \begin{pmatrix} 0 & -a_{v+2\,2} \\ a_{v+2\,1} & a_{v+2\,2} \end{pmatrix}$, $F = \begin{pmatrix} 0 & 2ba_{v+2\,2} \\ -2 & -4ba_{v+2\,2} \end{pmatrix}$, then $P_2^i = \begin{pmatrix} 0 & 0 \\ F^{i-1}E & F^i \end{pmatrix}$. Since $\det(F) = 4ba_{v+2\,2}$, $a_{v+2\,2} \neq 0$ by (2.8), we have $\det(F) \neq 0$, thus $P_2^p \neq (0)$. Therefore, $|P_1\bar{C}P_1^{-1}| \neq p$, $|\bar{C}| \neq p$.

If $a_{v+1\,2} = a_{v+2\,1}$, then it follows that $|\bar{C}| = 2 \neq p$ according to (2.2)–(2.9).

Using the same method, we finally prove that $|\bar{C}| \neq p$ for all situations. Hence, $|C| \neq p$ by Lemma 2.15. Since $A = BCB^{-1}$, it follows that $|A| \neq p$. □

We have totally described the structures of all types of 2-codimensional invariant subspaces. We summarize the results so far obtained as follows:

**Theorem 2.17.** *Let $A \in O_{2\nu}(F_q, S)$, $|A| = p$, and the codimension of the invariant subspace $V^A$ be 2. Then the invariant subspace $V^A$ is a type I of 2-codim, type III of 2-codim, or type IV of 2-codim subspace, but neither a type II of 2-codim nor a type V of 2-codim subspace.*

**Proof.** By Lemma 2.2, the 2-codimensional subspaces of orthogonal space $V$ have five types. According to Lemmas 2.11 and 2.12, $V^A$ cannot be a type II of 2-codim subspace since $|A| = p$. Also, $V^A$ cannot be a type V of 2-codim subspace by Lemma 2.16. Consequently, $V^A$ is a type I of 2-codim, type III of 2-codim, or type IV of 2-codim subspace. □

**Remark 2.18.** By Lemmas 2.8 and 2.14, the invariant subspaces whose types are type I of 2-codim, type III of 2-codim, and type IV of 2-codim are not empty. And the type of an invariant subspace $V^A$ and the type of an element $A$ can be determined by each other.

# 3 Embedding of 3-codimensional subspaces

In this section, we shall consider the invariant subspaces under the action of the elements of order $p$ in $O_{2\nu}(F_q, S)$, i.e., the set $E = \{V^A | A \in O_{2\nu}(F_q, S),\ \ |A| = p\}$. First, if $A \in O_{2\nu}(F_q, S)$ and the codimension of the invariant subspace $V^A$ is 1, then, by Section 9 of Chapter 7 in [6], $A$ is a quasi-symmetry transformation and its order is $2 \neq p$. So we know that the set $E$ cannot contain any 1-codimensional invariant subspace. Furthermore, the invariant subspaces $V^A$ whose codimensions are 2 have been studied in Section 2 and every $m$-codimensional subspace with $m \geq 4$ can be embedded into a 3-codimensional subspace. Hence, the remainder work is to study the 3-codimensional subspaces of the orthogonal space $V$.

**Proposition 3.1.** *Let $W$ be a 3-codimensional subspace of the orthogonal space $V$. Then $W \subseteq U$, where $U$ is a type I of 2-codim, type III of 2-codim, or type IV of 2-codim subspace.*

**Proof.** According to Lemma 2.1, there are seven types of the 3-codimensional subspaces in the orthogonal space $V$. Now we shall choose the most complex type whose corresponding matrix is

$$M = \left(\begin{array}{cc|ccc} & I^{(\nu-3)} & & & \\ I^{(\nu-3)} & & & & \\ \hline & & 1 & 0 & 0 \\ & & 0 & -z & 0 \\ & & 0 & 0 & 0 \end{array}\right)$$

to prove. Let $P$ be the corresponding matrix of subspace $W$. Suppose that $PSP' = M$.

If $-z$ is a square (resp. nonsquare) element, then $-z$ is cogredient to 1 (resp. $z$). Let $W_1$ be a type III (resp. type IV) of 2-codim subspace. The corresponding matrix $P_1$ of subspace $W_1$ satisfies

$$P_1SP_1' = \left(\begin{array}{cc|cccc} & I^{(\nu-3)} & & & & \\ I^{(\nu-3)} & & & & & \\ \hline & & 0 & 1 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & 0 & 0 & -z & 0 \\ & & 0 & 0 & 0 & 0 \end{array}\right).$$

Let $A_1 = \left(\begin{array}{cc|cccc} I^{(\nu-3)} & & & & & \\ & I^{(\nu-3)} & & & & \\ \hline & & 1 & \frac{1}{2} & 0 & 0 \\ & & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 \\ & & 0 & 1 & 0 & 0 \end{array}\right)$. Then $A_1P_1SP_1'A_1' = \left(\begin{array}{cc|cccc} & I^{(\nu-3)} & & & & \\ I^{(\nu-3)} & & & & & \\ \hline & & 1 & 0 & 0 & 1 \\ & & 0 & -z & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \end{array}\right)$.

Suppose that $A_1P_1 = \begin{pmatrix} v_1' \\ \vdots \\ v_{2\nu-3}' \\ v_{2\nu-2}' \end{pmatrix}$. Let $\overline{A_1P_1} = \begin{pmatrix} v_1' \\ \vdots \\ v_{2\nu-3}' \end{pmatrix}$ be the matrix obtained by deleting the last row vector of $A_1P_1$. Then the corresponding subspace of $\overline{A_1P_1}$ is embedded in the corresponding subspace of $A_1P_1$, and $\overline{A_1P_1}S\overline{A_1P_1}' = M$. By Lemma 2.5, there exists an $A \in O_{2\nu}(F_q, S)$ such that $P = \overline{A_1P_1}A$, thus, the subspace $W$ is embedded in the corresponding subspace of $A_1P_1$. Since $A_1$ is an invertible matrix, the corresponding subspace of $A_1P_1$ is the corresponding subspace of $P_1$, i.e., subspace $W_1$. Hence, the subspace $W$ is embedded in the type III (resp. type IV) of 2-codim subspace $W_1$.

The proofs of the other six types are similar to the above type and are omitted. We summarize all situations that if $W$ be a 3-codimensional subspace, then $W$ is embedded in a type I of 2-codim, type III of 2-codim, or type IV of 2-codim subspace. □

# 4 Transfer ideal

In this section, we shall determine the structures of the transfer variety and transfer ideal. First, we recall some notations. If $J$ is an ideal of $F[x_1, \ldots, x_n]$, then

$$\mathbb{V}(J) = \{(a_1, \ldots, a_n) \in F^n \mid f(a_1, \ldots, a_n) = 0, \quad \forall f \in J\}$$

is called the variety defined by an ideal $J$. Consider a collection, $S$, of points of the affine space $F^n$. We define the set $I(S)$ of polynomials in $F[x_1, \ldots, x_n]$ by

$$I(S) = \{f \in F[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0, \quad \forall (a_1, \ldots, a_n) \in S\}.$$

It is easy to verify that the set $I(S)$ is an ideal in $F[x_1, \ldots, x_n]$ and $\mathbb{V}(I(S)) = S$.

**Lemma 4.1.** ([7, Hilbert Nullstellensatz]) *Let $F$ be an algebraically closed field. If $J$ is an ideal of $F[x_1, \ldots, x_n]$, then $I(\mathbb{V}(J)) = \sqrt{J}$.*

Let $\rho : G \hookrightarrow GL(n, F)$ be a faithful representation of a finite group over the field $F$. The **transfer variety**, denoted by $\Omega_G$, is defined by ([4, Section 6.4])

$$\Omega_G = \{v \in V | \mathrm{Tr}^G(f)(v) = 0, \quad \forall f \in \ \mathrm{Tot}(F[V])\}.$$

Since $\mathrm{Im}(\mathrm{Tr}^G)$ is an ideal of $F[V]^G$, $F[V]^G \subseteq F[V]$ is a ring extension, we have

$$\Omega_G = \{v \in V | f(v) = 0, \quad \forall f \in (\mathrm{Im}(\mathrm{Tr}^G))^e\} = \mathbb{V}((\mathrm{Im}(\mathrm{Tr}^G))^e),$$

where $(\mathrm{Im}(\mathrm{Tr}^G))^e$ denotes the extension ideal of $\mathrm{Im}(\mathrm{Tr}^G)$ in $F[V]$.

**Lemma 4.2.** ([Corollary 2.6] and [4, Corollary 6.4.6]) *Let $\rho : G \hookrightarrow GL(n, F)$ be a representation of a finite group over the field $F$ of characteristic $p$. Then*

$$\Omega_G = \bigcup_{g \in G, |g| = p} V^g,$$

*i.e., transfer variety is the union of the fixed-point sets of the elements in $G$ of order $p$.*

By Theorem 6.4.7 in [4] and its proof, we have

**Lemma 4.3.** [4] *Let $\rho : G \hookrightarrow GL(n, F)$ be a representation of a finite group over the field $F$ of characteristic $p$. Then*

$$\sqrt{\mathrm{Im}(\mathrm{Tr}^G)} = \sqrt{(\mathrm{Im}(\mathrm{Tr}^G))^e} \cap F[V]^G,$$

*and* $\mathrm{ht}(\mathrm{Im}(\mathrm{Tr}^G)) = \mathrm{ht}(\sqrt{\mathrm{Im}(\mathrm{Tr}^G)}) = n - \max\{\dim_F(V^g) \mid g \in G \text{ and } |g| = p\} < n.$

Now we can obtain the main results for the transfer variety.

**Theorem 4.4.** *The transfer variety $\Omega_{O_{2\nu}(F_q,S)}$ of the orthogonal group $O_{2\nu}(F_q, S)$ is*

$$\Omega_{O_{2\nu}(F_q,S)} = \left( \bigcup_{\substack{U_1 \text{ is type I} \\ \text{of 2-codim}}} U_1 \right) \cup \left( \bigcup_{\substack{U_2 \text{ is type III} \\ \text{of 2-codim}}} U_2 \right) \cup \left( \bigcup_{\substack{U_3 \text{ is type IV} \\ \text{of 2-codim}}} U_3 \right),$$

*i.e., $\Omega_{O_{2\nu}(F_q,S)}$ is the union of all type I of 2-codim, type III of 2-codim, and type IV of 2-codim subspaces (the same notations in* Remark 2.3).

**Proof.** Let $U$ be the right side of the above equality. For each $U_1$, let $T \in O_{2\nu}(F_q, S)$ be a 2-transvection. By Lemmas 2.8 and 2.4, there exists an element $A_1 \in O_{2\nu}(F_q, S)$ such that $U_1 = V^T A_1 = V^{A_1^{-1}TA_1}$ and $|A_1^{-1}TA_1| = |T| = p$. For each $U_2$ (resp. $U_3$), let $\bar{H}_1$ (resp. $\bar{H}_z$) be a $H_1$-type (resp. $H_z$-type) matrix. By Lemmas 2.14 and 2.4, there exists an element $A_2$ (resp. $A_3$) $\in O_{2\nu}(F_q, S)$ such that $U_2 = V^{\bar{H}_1} A_2 = V^{A_2^{-1}\bar{H}_1 A_2}$ and $|A_2^{-1}\bar{H}_1 A_2| = |\bar{H}_1| = p$ (resp. $U_3 = V^{\bar{H}_z} A_3 = V^{A_3^{-1}\bar{H}_z A_3}$ and $|A_3^{-1}\bar{H}_z A_3| = |\bar{H}_z| = p$). So we have $U \subseteq \bigcup_{\substack{A \in O_{2\nu}(F_q,S), \\ |A|=p}} V^A$. On the other hand, according to Theorem 2.17 and Section 3, it follows that $\bigcup_{\substack{A \in O_{2\nu}(F_q,S), \\ |A|=p}} V^A \subseteq U$. Hence, $U = \bigcup_{\substack{A \in O_{2\nu}(F_q,S), \\ |A|=p}} V^A$. Then $\Omega_{O_{2\nu}(F_q,S)} = U$ by Lemma 4.2. □

**Remark 4.5.** By the discussions in Section 3, we deduce that the whole space $V$ is contained in transfer variety. Then $\Omega_{O_{2\nu}(F_q,S)} = V$ over $F_q$. Let $\overline{F_q}$ be the algebraic closure of $F_q$ and $\overline{\Omega}_{O_{2\nu}(F_q,S)}$ be the transfer variety over $\overline{F_q}$. By the similar argument, we have that

$$\overline{\Omega}_{O_{2\nu}(F_q,S)} = \left( \bigcup_{\substack{U_1 \text{ is type I} \\ \text{of 2-codim}}} U_1 \otimes_{F_q} \overline{F_q} \right) \cup \left( \bigcup_{\substack{U_2 \text{ is type III} \\ \text{of 2-codim}}} U_2 \otimes_{F_q} \overline{F_q} \right) \cup \left( \bigcup_{\substack{U_3 \text{ is type IV} \\ \text{of 2-codim}}} U_3 \otimes_{F_q} \overline{F_q} \right) \neq V \otimes_{F_q} \overline{F_q}.$$

Now, we shall determine the structures of radical ideal of transfer. Let $T = \left( \begin{array}{c|c} I^{(\nu)} & K \\ \hline & I^{(\nu)} \end{array} \right)$, where $K = \left( \begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & \\ \hline & & 0^{(\nu-2)} \end{array} \right)$. By the proof of Lemma 2.8, $T$ is a 2-transvection. Moreover, the invariant subspace $V^T$ is a type I of 2-codim subspace and

$$V^T = \{k_3 e_3 + \cdots + k_{2\nu} e_{2\nu} \mid k_i \in F_q\}.$$

Thus, the ideal

$$I(V^T) = \langle x_1, x_2 \rangle_{F_q[V]},$$

where $\langle x_1, x_2 \rangle_{F_q[V]}$ is the ideal generated by $x_1$ and $x_2$ in the polynomial ring $F_q[V]$.

Let
$$H=\left(\begin{array}{ccc|ccc} 1 & 0 & & 0 & -b & \\ a & 1 & & b & -ab & \\ & & I^{(\nu-2)} & & & 0^{(\nu-2)} \\ \hline & & & 1 & -a & \\ & & & 0 & 1 & \\ & & & & & I^{(\nu-2)} \end{array}\right).$$

By the proof of Lemma 2.14, we see that

$$V^H=\{k_1(-ab^{-1}e_1+e_{\nu+1})+k_3e_3+\cdots+k_\nu e_\nu+k_{\nu+2}e_{\nu+2}+\cdots+k_{2\nu}e_{2\nu}\mid k_i\in F_q\}.$$

When $a=1$, $b=-2$, let $\bar{H}_1=H$. Then the invariant subspace $V^{\bar{H}_1}$ is a type III of 2-codim subspace and the ideal

$$I(V^{\bar{H}_1})=\langle x_2,2x_1-x_{\nu+1}\rangle_{F_q[V]}.$$

When $a=1$, $b=-2z$, let $\bar{H}_z=H$. Then the invariant subspace $V^{\bar{H}_z}$ is a type IV of 2-codim subspace and the ideal

$$I(V^{\bar{H}_z})=\langle x_2,2zx_1-x_{\nu+1}\rangle_{F_q[V]}.$$

**Remark 4.6.** For convenience, we denote

$$\begin{aligned}
J_0&=\langle x_2\rangle_{F_q[V]}\cap F_q[V]^{O_{2\nu}(F_q,S)};\\
J_1&=\langle x_1,x_2\rangle_{F_q[V]}\cap F_q[V]^{O_{2\nu}(F_q,S)};\\
J_2&=\langle x_2,2x_1-x_{\nu+1}\rangle_{F_q[V]}\cap F_q[V]^{O_{2\nu}(F_q,S)};\\
J_3&=\langle x_2,2zx_1-x_{\nu+1}\rangle_{F_q[V]}\cap F_q[V]^{O_{2\nu}(F_q,S)}.
\end{aligned}$$

**Theorem 4.7.** *The radical ideal of transfer is*

$$\sqrt{\mathrm{Im}(\mathrm{Tr}^{O_{2\nu}(F_q,S)})}=J_1\cap J_2\cap J_3.$$

*Moreover, it is a primary decomposition of* $\sqrt{\mathrm{Im}(\mathrm{Tr}^{O_{2\nu}(F_q,S)})}$ *and* $J_i$, $i=1,2,3$ *are all prime ideals.*

**Proof.** From Lemma 2.4, we know that $O_{2\nu}(F_q,S)$ acts transitively on each set of subspaces of the same type. Thus, by Remark 4.5,

$$\begin{aligned}
\overline{\Omega}_{O_{2\nu}(F_q,S)}&=\Big(\bigcup_{A\in O_{2\nu}(F_q,S)}V^TA\otimes_{F_q}\overline{F_q}\Big)\cup\Big(\bigcup_{A\in O_{2\nu}(F_q,S)}V^{\bar{H}_1}A\otimes_{F_q}\overline{F_q}\Big)\cup\Big(\bigcup_{A\in O_{2\nu}(F_q,S)}V^{\bar{H}_z}A\otimes_{F_q}\overline{F_q}\Big)\\
&=\mathbb{V}\Big(\bigcap_{A\in O_{2\nu}(F_q,S)}A\langle x_1,x_2\rangle_{F_q[V]}\otimes_{F_q}1\Big)\cup\mathbb{V}\Big(\bigcap_{A\in O_{2\nu}(F_q,S)}A\langle x_2,2x_1-x_{\nu+1}\rangle_{F_q[V]}\otimes_{F_q}1\Big)\\
&\quad\cup\mathbb{V}\Big(\bigcap_{A\in O_{2\nu}(F_q,S)}A\langle x_2,2zx_1-x_{\nu+1}\rangle_{F_q[V]}\otimes_{F_q}1\Big).
\end{aligned}$$

Therefore, by Hilbert's Nullstellensatz, it follows that

$$\begin{aligned}
\sqrt{(\mathrm{Im}(\mathrm{Tr}^{O_{2\nu}(F_q,S)}))^e}&=\Big(\bigcap_{A\in O_{2\nu}(F_q,S)}A\langle x_1,x_2\rangle_{F_q[V]}\Big)\cap\Big(\bigcap_{A\in O_{2\nu}(F_q,S)}A\langle x_2,2x_1-x_{\nu+1}\rangle_{F_q[V]}\Big)\\
&\quad\cap\Big(\bigcap_{A\in O_{2\nu}(F_q,S)}A\langle x_2,2zx_1-x_{\nu+1}\rangle_{F_q[V]}\Big)\text{ in }\overline{F_q}[\overline{V}].
\end{aligned}$$

By flat base change [4, p. 276], we deduce that the above equation is also held in $F_q[V]$, since each $x_i$ is defined over $F_q$. Hence, by Lemma 4.3, we see that

$$\begin{aligned}\sqrt{\operatorname{Im}(\operatorname{Tr}^{O_{2v}(F_q,S)})} &= \sqrt{(\operatorname{Im}(\operatorname{Tr}^{O_{2v}(F_q,S)}))^e} \cap F_q[V]^{O_{2v}(F_q,S)}\\ &= \left(\bigcap_{A\in O_{2v}(F_q,S)} A\langle x_1, x_2\rangle_{F_q[V]}\right) \cap \left(\bigcap_{A\in O_{2v}(F_q,S)} A\langle x_2, 2x_1 - x_{v+1}\rangle_{F_q[V]}\right)\\ &\quad \cap \left(\bigcap_{A\in O_{2v}(F_q,S)} A\langle x_2, 2zx_1 - x_{v+1}\rangle_{F_q[V]}\right) \cap F_q[V]^{O_{2v}(F_q,S)}\\ &= \langle x_1, x_2\rangle_{F_q[V]} \cap \langle x_2, 2x_1 - x_{v+1}\rangle_{F_q[V]} \cap \langle x_2, 2zx_1 - x_{v+1}\rangle_{F_q[V]} \cap F_q[V]^{O_{2v}(F_q,S)}\\ &= J_1 \cap J_2 \cap J_3.\end{aligned}$$

The third equation is satisfied because

$$\left(\bigcap_{A\in O_{2v}(F_q,S)} A\langle l_1, l_2\rangle_{F_q[V]}\right) \cap F_q[V]^{O_{2v}(F_q,S)} = \langle l_1, l_2\rangle_{F_q[V]} \cap F_q[V]^{O_{2v}(F_q,S)}. \tag{4.1}$$

Consider the embedded mapping $\varphi : F_q[V]^{O_{2v}(F_q,S)} \hookrightarrow F_q[V]$. We have that the limit ideal $\langle l_1, l_2\rangle_{F_q[V]} \cap F_q[V]^{O_{2v}(F_q,S)}$ is also a prime ideal in the ring of invariant $F_q[V]^{O_{2v}(F_q,S)}$, since $\langle l_1, l_2\rangle_{F_q[V]}$ is a prime ideal in $F_q[V]$. Thus, $J_1, J_2$, and $J_3$ are prime ideals in $F_q[V]^{O_{2v}(F_q,S)}$. Consequently, $\sqrt{\operatorname{Im}(\operatorname{Tr}^{O_{2v}(F_q,S)})} = J_1 \cap J_2 \cap J_3$ is a primary decomposition. □

**Theorem 4.8.** $\operatorname{ht}(\operatorname{Im}(\operatorname{Tr}^{O_{2v}(F_q,S)})) = \operatorname{ht}(\sqrt{\operatorname{Im}(\operatorname{Tr}^{O_{2v}(F_q,S)})}) = 2$. *And*

$$(0) \subseteq J_0 \subseteq J_1, \quad (0) \subseteq J_0 \subseteq J_2, \quad (0) \subseteq J_0 \subseteq J_3$$

*are all prime ideal chains of height* 2 ($J_i$ *the same notations in* Remark 4.6).

**Proof.** In Section 3, we know that the set $E = \{V^A \mid A \in O_{2v}(F_q, S), |A| = p\}$ cannot contain any 1-codimensional invariant subspace. From Section 2, we have that there exist the 2-codimensional invariant subspaces in set $E$. Then combining Lemma 4.3, it follows that

$$\operatorname{ht}\Big(\operatorname{Im}\Big(\operatorname{Tr}^{O_{2v}(F_q,S)}\Big)\Big) = \operatorname{ht}\Big(\sqrt{\operatorname{Im}\Big(\operatorname{Tr}^{O_{2v}(F_q,S)}\Big)}\Big) = 2v - (2v - 2) = 2.$$

By the proof of Theorem 4.7, it is obvious that $(0) \subseteq J_0 \subseteq J_i,\ i = 1, 2, 3$ are all prime ideal chains of height 2. □

# 5 Example

Suppose $q = 3, v = 2$ and consider the orthogonal group $O_4(F_3, S)$ over the field $F_3$ with respect to the symmetric matrix $S = \begin{pmatrix} & & 1 & 0\\ & & 0 & 1\\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}$. We shall give the structures of the radical ideal $\sqrt{\operatorname{Im}(\operatorname{Tr}^{O_4(F_3,S)})}$ and the prime ideal chains of the transfer ideal $\operatorname{Im}(\operatorname{Tr}^{O_4(F_3,S)})$.

First, by Lemma 2.4, we have these orbits

$$\begin{aligned} o[x_1] = o[x_2] &= \{k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 | k_1x_3 + k_2x_4 = 0\},\\ o[x_1 - x_3] &= \{k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 | k_1x_3 + k_2x_4 = -1\},\\ o[-x_1 - x_3] &= \{k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 | k_1x_3 + k_2x_4 = 1\}.\end{aligned}$$

Second, using MACAULAY, which is a kind of computer algebra systems, to compute the intersection of the ideals in the orbit $o[\langle x_1, x_2\rangle_{F_3[V]}]$, we have

$$I_1 = \bigcap_{A\in O_4(F_3,S)} A\langle x_1, x_2\rangle_{F_3[V]} = \langle x_1x_3 + x_2x_4, x_2x_4(x_1^2 - x_2^2 + x_3^2 - x_4^2)\rangle_{F_3[V]}.$$

The intersection of the ideals in the orbit $o[\langle x_2, x_1 - x_3\rangle_{F_3[V]}]$ is

$$\begin{aligned} I_2 &= \bigcap_{A\in O_4(F_3,S)} A\langle x_2, x_1 - x_3\rangle \\ &= \langle x_1^3x_3 + x_1^2x_3^2 + x_1x_3^3 + x_2^3x_4 - x_1x_2x_3x_4 + x_2^2x_4^2 + x_2x_4^3, x_1^3x_2 - x_1x_2^3 + x_1^2x_2x_3 + x_2^3x_3 - x_1x_2x_3^2 \\ &\quad - x_2x_3^3 - x_1^3x_4 - x_1x_2^4x_4 - x_1^2x_3x_4 + x_2^2x_1x_4 + x_1x_3^2x_4 + x_3^3x_4 + x_1x_2x_4^2 - x_2x_3x_4^2 + x_1x_4^3 - x_3x_4^3\rangle_{F_3[V]}. \end{aligned}$$

The intersection of the ideals in the orbit $o[\langle x_2, -x_1 - x_3\rangle_{F_3[V]}]$ is

$$\begin{aligned} I_3 &= \bigcap_{A\in O_4(F_3,S)} A\langle x_2, -x_1 - x_3\rangle \\ &= \langle x_1^3x_3 - x_1^2x_3^2 + x_1x_3^3 + x_2^3x_4 + x_1x_2x_3x_4 - x_2^2x_4^2 + x_2x_4^3, x_1^3x_2 - x_1x_2^3 - x_1^2x_2x_3 - x_2^3x_3 - x_1x_2x_3^2 \\ &\quad + x_2x_3^3 + x_1^3x_4 + x_1x_2^4x_4 - x_1^2x_3x_4 + x_2^2x_1x_4 - x_1x_3^2x_4 + x_3^3x_4 + x_1x_2x_4^2 + x_2x_3x_4^2 - x_1x_4^3 - x_3x_4^3\rangle_{F_3[V]}. \end{aligned}$$

And

$$\begin{aligned} I_0 &= \bigcap_{A\in O_4(F_3,S)} A\langle x_2\rangle \\ &= \langle x_1x_2x_3x_4(x_1^2 - x_2^2)(x_1^2 - x_4^2)(x_2^2 - x_3^2)(x_3^2 - x_4^2) \\ &\quad (x_1 + x_2 + x_3 - x_4)(x_1 + x_2 - x_3 + x_4)(x_1 - x_2 + x_3 + x_4)(-x_1 + x_2 + x_3 + x_4)\rangle_{F_3[V]}. \end{aligned}$$

In [8], Chu found the polynomial invariants of 4-dimensional orthogonal group with the nondegenerate quadratic form $Q_4^+ = x_1^2 - x_2^2 + x_3^2 - x_4^2$. By the variable substitution, $x_1 \to -x_1 + x_3, x_2 \to -x_1 - x_3, x_3 \to -x_2 + x_4, x_4 \to x_2 + x_4$, we can obtain the polynomial invariants of $O_4(F_3, S)$ with respect to the symmetric matrix $S$,

$$F_3[V]^{O_4(F_3,S)} = F_3\left[Q_{40}, Q_{41}, Q_{42}, \frac{G_1(Q_{40}, Q_{41}, Q_{42}, Q_{43})}{F(Q_{40}, Q_{41}, Q_{42})}, \frac{G_2(Q_{40}, Q_{41}, Q_{42}, Q_{43})}{F(Q_{40}, Q_{41}, Q_{42})}\right],$$

where

$$\begin{aligned} Q_{40} &= -x_1x_3 - x_2x_4, \quad Q_{41} = x_1^3x_3 + x_1x_3^3 + x_2^3x_4 + x_2x_4^3, \\ Q_{42} &= x_1^9x_3 + x_1x_3^9 + x_2^9x_4 + x_2x_4^9, \quad Q_{43} = x_1^{27}x_3 + x_1x_3^{27} + x_2^{27}x_4 + x_2x_4^{27}, \\ G_1(Q_{40}, Q_{41}, Q_{42}, Q_{43}) &= Q_{43}Q_{40}^3 - Q_{41}Q_{42}^3 - (Q_{42}^2 - Q_{40}^{10})(Q_{41}Q_{42} - Q_{40}Q_{41}^3), \\ G_2(Q_{40}, Q_{41}, Q_{42}, Q_{43}) &= Q_{43}Q_{41}^3 - Q_{41}^{10} - (Q_{42}^2 - Q_{40}^{10})^2, \\ F(Q_{40}, Q_{41}, Q_{42}) &= (Q_{42}Q_{40}^3 - Q_{41}^4) - (Q_{41}^2 - Q_{40}^4)^2. \end{aligned}$$

Then, using MACAULAY, we have

$$\begin{aligned} J_0 &= I_0 \cap F_3[V]^{O_4(F_3,S)} = \langle Q_{40}^8 + Q_{40}^4Q_{41}^2 - Q_{41}^4 - Q_{40}^3Q_{42}\rangle_{F_3[V]^{O_4(F_3,S)}}, \\ J_1 &= I_1 \cap F_3[V]^{O_4(F_3,S)} = \langle Q_{40}, Q_{41}, Q_{42}\rangle_{F_3[V]^{O_4(F_3,S)}}, \\ J_2 &= I_2 \cap F_3[V]^{O_4(F_3,S)} = \left\langle Q_{40}^2 - Q_{41}, Q_{40}Q_{41}^2 - Q_{42}, Q_{40}Q_{41}\frac{G_1}{F} - \frac{G_2}{F}\right\rangle_{F_3[V]^{O_4(F_3,S)}}, \\ J_3 &= I_3 \cap F_3[V]^{O_4(F_3,S)} = \left\langle Q_{40}^2 + Q_{41}, Q_{40}Q_{41}^2 - Q_{42}, Q_{40}Q_{41}\frac{G_1}{F} - \frac{G_2}{F}\right\rangle_{F_3[V]^{O_4(F_3,S)}}, \end{aligned}$$

and by Theorem 4.6, we have

$$\begin{aligned} \sqrt{\operatorname{Im}(\operatorname{Tr}^{O_4(F_3,S)})} &= J_1 \cap J_2 \cap J_3 \\ &= \left\langle Q_{40}Q_{41}^2 - Q_{42}, Q_{40}^4 + Q_{41}^2, Q_{42}\frac{G_1}{F} - Q_{41}\frac{G_2}{F}, Q_{40}^2Q_{41}\frac{G_1}{F} - Q_{40}\frac{G_2}{F}\right\rangle_{F_3[V]^{O_4(F_3,S)}}. \end{aligned}$$

Moreover,

$$(0) \subseteq J_0 \subseteq J_1, \quad (0) \subseteq J_0 \subseteq J_2, \quad (0) \subseteq J_0 \subseteq J_3$$

are prime ideal chains of height 2.

**Acknowledgments:** The author thanks the referees for their careful reading and helpful suggestions that helped improve this paper.

**Funding information**: This work was supported by National Natural Science Foundation of China (Program No. 11701449) and the Scientific Research Program Funded by Shaanxi Provincial Education Department (No. 16JK1789).

**Author contributions**: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

**Conflict of interest**: Authors state no conflict of interest.

**Data availability statement**: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

# References

[1] R. J. Shank and D. L. Wehlau, *The transfer in modular invariant theory*, J. Pure Appl. Algebra **142** (1999), 63–77.
[2] M. D. Neusel, *The transfer in the invariant theory of modular permutation representations*, Pacific J. Math. **199** (2001), no. 1, 121–135.
[3] M. D. Neusel, *The transfer in the invariant theory of modular permutation representations II*, Canad. Math. Bull. **45** (2002), no. 2, 272–283, DOI: https://doi.org/10.4153/CMB-2002-031-7.
[4] M. D. Neusel and L. Smith, *Invariant Theory of Finite Groups*, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2002.
[5] Z. X. Wan, *Geometry of Classical Groups Over Finite Fields*, Science Press, Beijing, New York, 2002.
[6] L. G. Hua and Z. X. Wan, *Classical Groups*, Shanghai Science and Technology Press, Shanghai, 1963.
[7] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969.
[8] H. Chu, *Polynomial invariants of four-dimensional orthogonal groups*, Comm. Algebra **29** (2001), no. 3, 1153–1164, DOI: https://doi.org/10.1081/AGB-100001673.