Research Article

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Multilevel MC method for weak approximation of stochastic differential equation with the exact coupling scheme

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Abstract: Davie's exact coupling technique for stochastic differential equations may be used to enhance the convergence of the multilevel Monte Carlo (MC) methodology. Giles developed the multilevel MC technique, which is based on executing the MC method several times with various time increments. It cuts computing costs significantly by executing most simulations at a low cost. The essential concept behind the multilevel MC approach with the exact coupling is discussed in this article. Numerical implementation reveals significant computational savings, which supports the analysis.

Keywords: exact coupling, stochastic differential equations, numerical solution of stochastic differential equations, weak convergence

MSC 2020: 11-xx, 60-xx, 65-xx, 68-xx

1 Introduction

A stochastic differential equation is a mixture of deterministic and probabilistic elements. It is fundamental in many fields of application, and therefore, we find it with great interest in financial and actuarial mathematics, engineering applications, and insurance companies. In recent years, specialists in stochastic differential equations have been trying to find approximate solutions to it, either by using strong approximations or weak approximations. Strong approximation methods have had the most significant use in many scientific papers with a clear difference in applications and conditions. For example, when a stochastic differential equation is one-dimensional, there are often no difficulties in its application. When the Wiener process is multidimensional, the task gets more difficult. The previous articles [1–3] explore methods for estimating double integrals in any dimension using Fourier expansion. For the interested reader to learn more about the accomplishment of the simulation of the stochastic differential equation, we refer to [4,5]. Due to its superior properties for many problems, solutions to fractional stochastic differential equations (FSDEs) driven by the Brownian motion have recently received much attention from scientific researchers, see [6,7]. In [8], the authors look at a stochastic viscoelastic wave equation with nonlinear damping and logarithmic nonlinear source terms that established a blow-up result. On the other hand, a strong approximation of an Ito process is not always needed. It is possible that you are only interested in a function of the value of the Ito process at a certain point in time T, one of the first two moments, for example. This method and the use of expectation for some functions give an excellent approximation to the probability distribution of random variables. As a result, the kind of approximation that is needed in this case is significantly weaker than that provided by the pathwise convergence method.

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In this paper, we will apply the process of weak approximations using the technique of coupling developed by Davie [9].

This paper is structured as follows: Section 2 summarizes the recent stochastic differential equation (SDE) studies and addresses Davie's methodology [9]. Section 3 discusses the integrated correct relation and the system of Euler. Section 4, offers a numerical example of convergence behavior. In this work, for the exact coupling, we use the multilevel Monte Carlo (MC) technique [10,11] on the two-dimensional SDEs.

2 Stochastic differential equations

Assume the SDE is as follows:

$$\mathrm{d}Y_i(t) = O_i(t, Y(t))\mathrm{d}t + \sum_{k=1}^{n} B_{ik}(t, Y(t))\mathrm{d}V_k(t), \quad Y_i(0) = Y_i^{(0)}, \tag{2.1}$$

where $i = 1, \ldots, n$ on $[0, T]$, $Y(t)$ is a n-dimensional vector, and $V(t)$ is a n-dimensional path. Further, the coefficients $B_{ik}(t, Y(t))$ satisfy a global Lipschitz condition $|O(t, Y) - O(t, y)| \le A|Y - y|$, and $|B(t, Y) - B(t, y)| \le A|Y - y|$, for all $t \in [t_0, T]$ and $Y, y \in \mathbb{R}$, with $A > 0$ is a constant. Now assuming A and B are continuously on t for each Y, then there is a unique solution $Y(t)$ to equation (2.1). To find an approximate solution on this interval $[0, T]$, we divided this interval into positive integer N equal length of intervals, i.e., $S = T/N$. Adding the following quadratic terms to the Euler scheme yields the Milstein scheme: $\sum_{k,l=1}^{n} \chi_{ikl}(jS, y^{(j)})P_{kl}^{(j)}$. This will give the following scheme:

$$y_i^{(j+1)} = y_i^{(j)} + A_i(jS, y^{(j)})S + \sum_{k=1}^{n} B_{ik}(jS, y^{(j)})\Delta V_k^{(j)} + \sum_{k,l=1}^{n} \chi_{ikl}(jS, y^{(j)})P_{kl}^{(j)}, \tag{2.2}$$

where $\Delta V_k^{(j)} = V_k((j+1)S) - V_k(jS)$, $P_{kl}^{(j)} = \int_{jS}^{(j+1)S} \{V_k(t) - V_k(jS)\}\mathrm{d}V_l(t)$, and $\chi_{ikl}(t, y) = \sum_{m=1}^{q} B_{mk}(t, y)\frac{\partial B_{il}}{\partial y_m}(t, y)$. If the commutativity condition

$$\chi_{ikl}(t, y) = \chi_{ilk}(t, y) \tag{2.3}$$

is satisfied for all $y \in \mathbb{R}^d$, $t \in [0, T]$, and all i, k, l, After that, the Milstein method is reduced to

$$y_i^{(j+1)} = y_i^{(j)} + A_i(jS, y^{(j)})S + \sum_{k=1}^{n} B_{ik}(jS, y^{(j)})\Delta V_k^{(j)} + \sum_{k,l=1}^{n} \chi_{ikl}(jS, y^{(j)})P_{kl}^{(j)}. \tag{2.4}$$

It should be noted that the previous strategy is solely dependent on the Brownian motion $\Delta V_k^{(j)}$ being implemented . We may use $\Delta V_k^{(j)}$ only and apply the special equations for the Milstein method. This can be implemented from: the observation that $P_{kl}^{(j)} + P_{lk}^{(j)} = 2F_{kl}^{(j)}$, where $F_{kl}^{(j)} = \frac{1}{2}\Delta V_k^{(j)}\Delta V_l^{(j)}$ for $k \neq l$ and $F_{kk}^{(j)} = \frac{1}{2}\{(\Delta V_k^{(j)})^2 - S\}$. Also note that scheme (2.4) has the order 1 for $n = 1$, but if $n > 1$, we obtain the order $\frac{1}{2}$. As delineated in the previous paper [9], we tend to modify to scheme (2.4), which can provide the order 1 with the invertible diffusion. As a result, we will have the exact coupling strategy as follows.

2.1 Exact coupling scheme

For the sake of brevity and simplicity, we consider scheme (2.7) with the explicit version and $B_{ik}(y)$ depend only on y, and moreover, set the drift term as zero, so

$$y_i^{(j+1)} = y_i^{(j)} + \sum_{k=1}^{n} B_{ik}(y^{(j)})Y_k^{(j)} + \sum_{k,l=1}^{n} \chi_{ikl}(y^{(j)})(Y_k^{(j)}Y_l^{(j)} - S\delta_{kl}). \tag{2.5}$$

In [12], we show the order of the exact coupling method, and we obtained the following scheme:

$$Y_i^{(r,j)} = Y_i^{(r+1,2j)} + Y_i^{(r+1,2j+1)} + \sum_{k,l=1}^{n} \tau_{ikl}(Y_k^{(r+1,2j+1)}Y_l^{(r+1,2j)} - Y_l^{(r+1,2j+1)}Y_k^{(r+1,2j)}) + O((S^{(r)})^{3/2}), \tag{2.6}$$

where $\tau_{ikl} = \frac{1}{2}\sum_j c_{ij}\chi_{ikl}$. and $S^{(r)} = \frac{T}{2^r}$ is the step size. Under a condition of nondegeneracy, the interpretation of the normal distribution that was generate in scheme (2.4) will be changed, which in turn will converge to the first order, see [9]. So we apply (2.4) with the increment $Y_k^{(j)}$ replacing $\Delta V_k^{(j)}$, obtaining the following approximate solution $y_i^{(j)}$,

$$y_i^{(j+1)} = y_i^{(j)} + \sum_{k=1}^{n} b_{ik}(y^{(j)})Y_k^{(j)} + \sum_{k,l=1}^{n} \rho_{ikl}(y^{(j)})(Y_k^{(j)}Y_l^{(j)} - S\delta_{kl}), \tag{2.7}$$

when the increment $Y_k^{(j)}$ are independent normal distribution, i.e., $N(0, S)$. Notice that (2.7) is the same as scheme (2.4) except $\Delta V_k^{(j)}$ is substituted by $Y_k^{(j)}$ and we do not assume $\Delta V_k^{(j)}$ as equal to $Y_k^{(j)}$. We now need

$$Z_i \coloneqq \sum_{k=1}^{n} b_{ik}(jS, x^{(j)})X_k^{(j)} + \sum_{k,l=1}^{n} \rho_{ikl}(jS, x^{(j)})(X_k^{(j)}X_l^{(j)} - S\delta_{kl}).$$

In [12], we demonstrate that the local error of the exact coupling scheme is $E|y^{(r,1)} - y^{(r+1,2)}|^2 \le D_2 a^2 S^3$, where a and D_2 are functions of $y^{(r,j)}$.

2.2 Weak order of convergence

We thus set the step size S, which splits the interval $[0, T]$ into equal steps. $Y(t)$ is the solution for the SDE, where C is a positive constant that is independent of S. If any smooth function f is used

$$|E(f(y_S)) - E(f(Y(T)))| \le CS^{\gamma}, \quad S \in (0, 1),$$

then, at time $T = NS$, a discrete time approximations y_S, with the step size S, converges to the solution $Y(t)$ with order γ.

3 Multilevel method for weak order of SDEs

Giles is developing the multilevel MC technique and based on using the MC technique many times with various time increments. So, let us assume that we have the SDE.

$$\mathrm{d}x_i(t) = \sum_{k=1}^{n} b_{ik}(x(t))\mathrm{d}W_k(t), \quad x_i(0) = x_i^{(0)} \tag{3.1}$$

where $i = 1, \ldots, q$ on $[0, T]$, $x(t)$ is a n-dimensional vector, and $W(t)$ is a n-dimensional Brownian path. We will estimate $Ef(x(T))$, by the mean of N successive estimates, i.e., $\frac{1}{N}\sum_{i=1}^{N} f(x_S^{(i)})$, where $x_S^{(i)}$ is a solution of (3.1) and f is globally Lipschitz. The multilevel approach has the benefit of reducing the estimation's computational load. In the multilevel MC simulations, we examine MC simulations with various time steps $S^{(r)} = \frac{T}{2^r}$, while the standard method depends on a fixed time step. Now, we assume that $\mu_r = Ef(x_{S^{(r)}})$. Then, we will estimate μ_n, where n is large enough.

From Giles [10], we have

$$\mu_n = \mu_0 + \mu_1 - \mu_0 + \mu_2 - \mu_1 \pm \cdots + \mu_{n-1} - \mu_{n-2} + \mu_n - \mu_{n-1} = \mu_0 + \sum_{k=1}^{n}(\mu_k - \mu_{k-1}). \tag{3.2}$$

The multilevel MC approach depends on approximating all terms in the right-hand side of (3.2) independently.

First, for μ_0, we have

$$\mu_0 = Ef\left(x_{S^0}\right) \tag{3.3}$$

and

$$Y_0 = \frac{1}{N_0}\sum_{i=1}^{N} f\left(x_{S^0}^{(i)}\right). \tag{3.4}$$

Next, for $\mu_k - \mu_{k-1}$, we have

$$\mu_k - \mu_{k-1} = \frac{1}{N_k}\sum_{i=1}^{N_k}\left(f\left(x_{S^{(k)}}^{(i)}\right) - f\left(x_{S^{(k-1)}}^{(i)}\right)\right) := \hat{Y}_k. \tag{3.5}$$

Thus, the right-hand side of (3.2) becomes

$$\frac{1}{N_0}\sum_{i=1}^{N} f\left(x_{S^0}^{(i)}\right) + \sum_{k=1}^{n}\frac{1}{N_k}\sum_{i=1}^{N_k}\left(f\left(x_{S^{(k)}}^{(i)}\right) - f\left(x_{S^{(k-1)}}^{(i)}\right)\right). \tag{3.6}$$

It is obvious that the quantity $(f(x_{S^{(k)}}) - f(x_{S^{(k-1)}}))$ is the outcome of two approximations with successive time steps $S^{(k)}$ and $S^{(k-1)}$, but the same Brownian path. The variance may be reduced by using the same Brownian path, so that a smaller value for Nk can be obtained.

Now, V_k represents the variance of a single sample $(f(x_{S^{(k)}}) - f(x_{S^{(k-1)}}))$. $\hat{Y}_k$ simple estimator's variance is expressed as follows:

$$\mathrm{Var}(\hat{Y}_k) = \mathrm{Var}\left(\frac{1}{N_k}\sum_{i=1}^{N_k}\left(f\left(x_{S^{(k)}}^{(i)}\right) - f\left(x_{S^{(k-1)}}^{(i)}\right)\right)\right) = \frac{V_k}{N_k}. \tag{3.7}$$

As a result, if the variance of the starting level is

$$\mathrm{Var}(Y_0) = \frac{1}{N_0}\sum_{i=1}^{N}\mathrm{Var}\left(f\left(x_{S^0}^{(i)}\right)\right) = \frac{V_0}{N_0},$$

then the variance of $\hat{Y} = \sum_{k=1}^{n}\hat{Y}_k$ is expressed as follows:

$$\mathrm{Var}(\hat{Y}) = \mathrm{Var}\left(\sum_{k=0}^{n}\hat{Y}_k\right) = \sum_{k=0}^{n}\mathrm{Var}(\hat{Y}_k) = \sum_{k=0}^{n}\frac{V_k}{N_k}. \tag{3.8}$$

Furthermore, the computational load being proportional to

$$\sum_{k=0}^{n}\frac{N_k}{S_k}.$$

Therefore, by choosing N_k to be proportional to $\sqrt{V_k S_k}$, the variance, for a fixed computational load, is minimized.

The following theorem presents the general application of the multilevel MC approach, see [10]. By applying this theorem to the exact coupling, we obtain the computational load results.

Theorem 3.1. *Let $f(x(T))$ be a functional of the solution of SDE* (3.1) *with a given Brownian path $W(t)$, and $f(x_{S^{(k)}})$ represents the corresponding approximation to $x(T)$, utilizing a time step numerical discretization $S_k = \frac{T}{M^k}$.*

If there exist independent estimators $\hat{Y}_k$ using N_k MC simulation, and the positive constants $\alpha \geq \frac{1}{2}$, β, b_1, b_2, b_3 such that

1. $$|E[f(x_{S^{(k)}}) - f(x(T))]| \le b_1 S_k^{\alpha}$$
2. $$E[\hat{Y}_k] = \begin{cases} E[f(x_{S^0})], & k = 0 \\ E[f(x_{S^{(k)}}) - f(x_{S^{(k)-1}})], & k > 0 \end{cases}$$
3. $$V[\hat{Y}_k] \le \frac{b_2 S_k^{\beta}}{N_k}$$

4. *B_k, the computational complexity of $\hat{Y}_k$, is bounded by*

$$C_k \le \frac{b_3 N_k}{S_k}.$$

Then there is a b_4 positive constant., s.t. for any $\varepsilon < e^{-1}$, there are certain values n and N_k for which the multi-level estimator:

$$\hat{Y} = \sum_{k=0}^{n} \hat{Y}_k$$

has a limit on the mean-square-error (MSE)

$$\text{MSE} = E[(\hat{Y} - E[f(x(T))])^2] < \varepsilon^2$$

having a computational cost B with bound

$$B \le \begin{cases} b_4 \varepsilon^{-2}, & \beta > 1, \\ b_4 \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ b_4 \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

Proof. Look at Giles [10]. □

The previous theorem may be used to obtain β, which is twice the scheme's strong order.

We'll examine the order of the variance in the following applications, i.e., $V_k = O(S_k^{\beta})$, and use Theorem 3.1 to calculate the computational load.

4 The multilevel MC method for the exact coupling method using scheme (2.7)

In [13], Giles and Szpruch have worked on the multilevel approach and obtained $O(\varepsilon^{-2})$ computational load although they employ a more stringent demand for the function's regularity f than Lipschitz. So, we showed in [14] that under nondegeneracy, the order of method (2.7) with the exact coupling is $O(S)$, and a single sample's variance is

$$V_k = \mathbb{V}(f(x_{S^{(k)}}) - f(x_{S^{(k-1)}})) = O(S_k^2). \tag{4.1}$$

As a result of (4.1) and because f is a Lipschitz function, the vector and matrix functions fulfill the Lipschitz conditions with uniformly bounded derivatives, and we have the weak order $O(S)$ for (2.7) with the invertibility condition for matrix $b_{ik}(t, X(t))$. So that

$$|E(f(x_{S^{(k)}}) - f(x(T))| = O(S_k).$$

Thus, the single sample variance V_k is $O(S_k^2)$ for the standard estimator (3.5). In addition, the best option for N_k is proportional to $(S_k^{3/2})$ asymptotically. As a result, since the variance is $\beta = 2 > 1$, the computational load will be $O(\varepsilon^{-2})$, according to Theorem 3.1.

As a consequence, the number of simulations N_k will drop rapidly as the variance of the multilevel MC for scheme (2.7) with the exact coupling decreases. So the main order of the convergence of this method will be obtained from the first level of the simulation, which will be the dominant term with the computational load $O(\varepsilon^{-2})$.

4.1 Numerical implementation

In this section, we will look at invertible two-dimensional SDE equations and demonstrate numerically how well the multilevel MC method performs for the exact coupling method (2.7). We will also look at how reducing the number of simulations affects the computing load. To see the outcomes, the following two-dimensional invertible SDEs will be used for the Matlab implementation.

$$\begin{aligned} dX_1(t) &= (\sin(X_2(t)))^2\,\mathrm{d}W_1(t) - \left(\frac{1}{1+X_1^2(t)}\right)\mathrm{d}W_2(t), \\ dX_2(t) &= \left(\frac{1}{1+X_2^4(t)}\right)\mathrm{d}W_1(t) + (\cos(X_1(t)))^2\,\mathrm{d}W_2(t), \\ &\text{for} \quad 0 \le t \le 1, \quad \text{with} \quad X_1(0) = 2 \quad \text{and} \quad X_2(0) = 0. \end{aligned} \tag{4.2}$$

$W_1(t)$ and $W_2(t)$ are two independent standard Brownian motions.

At each level of simulation, the Lipschitz function $f(x_{S^{(k)}}) = \sin(x_{S^{(k)}})$ will be used to estimate the following amount $(f(x_{S^{(k)}}) - f(x_{S^{(k-1)}}))$, where $x_{S^{(k)}}$ and $x_{S^{(k-1)}}$ are two discrete approximations with distinct time steps $S^{(k)}$ and $S^{(k-1)}$. However, they both follow the Brownian path. The number of simulations N_k, the step-size S, and the result of the simple estimator $\hat{Y}_k$ with its confidence interval are presented in Table 1 for each level.

Now we will implement a Matlab code to obtain the basic estimator's findings at each level, i.e., $\hat{Y}_k$ in (3.5) and the number of levels will be $M = 9$. For the estimate, the starting value of the number of simulations N_k will be 2^{22}, and also the same Brownian path will be used for each simulation. We calculate the 90% confidence interval of $\hat{Y}_k$, $(\mu - \psi, \mu + \psi)$, given its mean value μ in the numerical simulations, where

$$\psi = Z_{\alpha/2} \times \frac{\sigma}{\sqrt{N_k}}. \tag{4.3}$$

The sample size is N_k, the confidence coefficient is $Z_{\alpha/2}$, and the standard deviation is σ. We compute the standard deviation as follows, and we obtain a N_k estimate for the amount $(f(x_{h^{(k)}}) - f(x_{h^{(k-1)}}))$, for each level M. We might then express their values as a vector and compute their standard deviation σ using the Matlab function (Std).

We obtain two terms for the multilevel MC. The first comes from the estimate of the top level μ_0, which includes a step-size S^0, i.e.,

$$Ef(x_S) = \frac{1}{N}\sum_{i=1}^{N} f(x_S^{(i)}) \approx 0.6840.$$

Table 1: The multilevel MC with the exact coupling two-dimensional SDE

Level k	(N_k) The number of simulations	The step size $S = \frac{T}{2^{(n-1)}}$	$(\hat{Y}_k)$ The simple estimator	Confidence interval of $\hat{Y}_k$
1	4,194,304	1	0.0973	(0.0977, 0.0969)
2	1,482,910	0.5	0.0248	(0.0242, 0.249)
3	524,288	0.25	0.0074	(0.0072, 0.0077)
4	185,364	0.125	0.0033	(0.0031, 0.0037)
5	65,536	0.0625	0.0015	(0.0013, 0.0017)
6	23,170	0.03125	0.00075	(0.00058, 0.00090)
7	8,192	0.015625	0.00052	(0.00040, 0.00063)
8	2,896	0.0078125	0.00015	(0.000040, 0.00027)
9	1,024	0.00390625	0.000033	(−0.00006, 0.00013)

The remaining terms are derived from the sum of the simple estimator $\hat{Y}_k$, which will be

$$\sum_{k=1}^{M} \frac{1}{N_k} \sum_{i=1}^{N_k} (f\left(x_{S^{(k)}}^{(i)}\right) - f\left(x_{S^{(k-1)}}^{(i)}\right)) \approx 0.1364.$$

As a result, the estimate for the exact coupling multilevel MC technique is expressed as follows:

$$\mu_M = \frac{1}{N} \sum_{i=1}^{N} f(x_S^{(i)}) + \sum_{k=1}^{M} \frac{1}{N_k} \sum_{i=1}^{N_k} (f\left(x_{S^{(k)}}^{(i)}\right) - f\left(x_{S^{(k-1)}}^{(i)}\right)) \approx 0.6840 + 0.1364 = 0.8204. \tag{4.4}$$

5 Conclusion

We established a weak convergence method to numerically solve stochastic differential equations with a possibly degenerate diffusion coefficient. We used an exact coupling in a two-dimensional case, which has a condition that the SDE should be invertible.

Conflict of interest: The author states no conflict of interest.

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