Research Article

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On second-order fuzzy discrete population model

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Abstract: This work is concerned with dynamical behavior of a second-order fuzzy discrete population model:

$$x_n = \frac{Ax_{n-1}}{1 + x_{n-1} + Bx_{n-2}}, \quad n = 1, 2, \ldots,$$

where A, B are positive fuzzy numbers. x_n is a positive fuzzy number and represents the population size at the observation instant n. According to a generalization of division (g-division) of fuzzy number, we study the dynamical behaviors including boundedness, global asymptotical stability, and persistence of positive fuzzy solution. Finally, two examples are given to demonstrate the effectiveness of the results obtained.

Keywords: fuzzy discrete population model, g-division, boundedness, global asymptotic behavior

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1 Introduction

The discrete time population model is the most appropriate mathematical description of life histories of organism. These models are used widely in fisheries and many organisms [1]. The Beverton-Holt model also known as the Skellam equation [2] is one of the classic population model that has been studied

$$x_n = \frac{\beta x_{n-1}}{1 + \delta x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1}$$

where x_n is population at the nth generation, β represents a productivity parameter, and δ controls the level of density dependence. Since then, many results on the model and the generation of the model have been widely obtained by some researchers [3–5].

In model (1), population is assumed to respond instantly to size variations. But in fact, there is a lag between the variations of external conditions and response of the population to these variations. Therefore, population dynamics is indeed described by delay models. For example, Pielou [6] studied the difference equation with delay:

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$$x_n = \frac{\alpha x_{n-1}}{1 + \beta x_{n-k}}, \quad n = 1, 2, \ldots, \tag{2}$$

where $\alpha > 1, \beta > 0$ and $k \in \{1, 2, \ldots\}$.

Also the generalization of (2) with many delays

$$x_n = \frac{\alpha x_{n-1}}{1 + \sum_{i=1}^{s} \beta_i x_{n-k_i}}, \quad n = 1, 2, \ldots, \tag{3}$$

where $\alpha > 1$ and $\beta_i > 0$ is studied in [7].

The generalization of (2) with infinite memory

$$x_n = \frac{\alpha x_{n-1}}{1 + x_{n-1} + \beta \sum_{j=1}^{\infty} c_j x_{n-j}}, \quad n = 1, 2, \ldots, \tag{4}$$

where $\alpha > 1, \beta > 0$ and $\sum_{j=1}^{\infty} c_j = 1$, is studied in [8].

In fact, the identification of the population dynamics model is usually based on the statistical method, starting from data experimentally obtained and on the choice of some method adapted to the identification. These models, even the classic deterministic approach, are subjected to inaccuracies (fuzzy uncertainty) that can be caused by either the nature of the state variables or by parameters as model coefficients.

In our real life, we have learned to deal with uncertainty. Scientists also accept the fact that uncertainty is a very important factor in most applications. Modeling the real life problems in such cases usually involves vagueness or uncertainty. The concept of fuzzy set and system was introduced by Zadeh [9], and its development has been growing rapidly to various situations of theory and application including fuzzy differential and fuzzy difference equations. It is well known that fuzzy difference equation is a difference equation whose parameters or the state variable are fuzzy numbers, and its solutions are sequences of fuzzy numbers. It has been used to model a dynamical system under possibility uncertainty. Due to the applicability of fuzzy difference equation for the analysis of phenomena where imprecision is inherent, this class of difference equation is a very important topic from theoretical point of view and also its applications. Recently, there has been an increasing interest in the study of fuzzy difference equations [10–29].

Inspired with the previous publication, by virtue of the theory of fuzzy difference equation, in this work, we consider the following discrete population model with fuzzy state variable:

$$x_n = \frac{A x_{n-1}}{1 + x_{n-1} + B x_{n-2}}, \quad n = 1, 2, \ldots, \tag{5}$$

where x_n is the population size at the observation instant nth generation and x_n is a fuzzy number. Parameter A is regarded as the natural growth coefficient and $A_{l,\alpha} > 1, \alpha \in (0, 1]$. The variation in the distributive coefficient B, which is a positive fuzzy number, defines the response of the environment to population growth, depending on the age structure and prehistory of the population.

The main aim of this work is to study the existence of positive solutions of the population dynamics model (5). Furthermore, according to a generation of division (g-division) of fuzzy numbers, we derive some conditions so that every positive solution of population dynamics model (5) is bounded and persistent. Finally, under some conditions, we prove that the population dynamics model (5) has a unique positive equilibrium x and every positive solution tends to x as $n \to \infty$.

2 Preliminary and definitions

First, we provide the following definitions.

Definition 2.1. [30] $u : R \to [0, 1]$ is said to be a fuzzy number if it satisfies conditions (i)–(iv) as follows:

(i) u is normal, i.e., there exists an $x \in R$ such that $u(x) = 1$;

(ii) u is fuzzy convex, i.e., for all $t \in [0,1]$ and $x_1, x_2 \in R$ such that

$$u(tx_1 + (1-t)x_2) \geq \min\{u(x_1), u(x_2)\};$$

(iii) u is upper semicontinuous;
(iv) The support of u, $\operatorname{supp} u = \overline{\bigcup_{\alpha \in (0,1]} [u]^\alpha} = \overline{\{x : u(x) > 0\}}$ is compact.

For $\alpha \in (0,1]$, the α-cuts of fuzzy number u is $[u]^\alpha = \{x \in R : u(x) \geq \alpha\}$, and for $\alpha = 0$, the support of u is defined as $\operatorname{supp} u = [u]^0 = \overline{\{x \in R | u(x) > 0\}}$.

Definition 2.2. Fuzzy number (parametric form) [30] A fuzzy number u in a parametric form is a pair $(\underline{u}, \bar{u})$ of functions $\underline{u}(r), \bar{u}(r)$, $0 \leq r \leq 1$, which satisfies the following requirements:
(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
(2) $\bar{u}(r)$ is a bounded monotonic decreasing left continuous function,
(3) $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.

A crisp (real) number x is simply represented by $(\underline{u}(r), \bar{u}(r)) = (x, x)$, $0 \leq r \leq 1$. The fuzzy number space $\{(\underline{u}(r), \bar{u}(r))\}$ becomes a convex cone E^1, which could be embedded isomorphically and isometrically into a Banach space [30].

Definition 2.3. [30] The distance between two arbitrary fuzzy numbers u and v is defined as follows:

$$D(u, v) = \sup_{\alpha \in [0,1]} \max\{|u_{l,\alpha} - v_{l,\alpha}|, |u_{r,\alpha} - v_{r,\alpha}|\}. \tag{6}$$

It is clear that (E^1, D) is a complete metric space.

Definition 2.4. [30] Let $u = (\underline{u}(r), \bar{u}(r))$, $v = (\underline{v}(r), \bar{v}(r)) \in E^1$, $0 \leq r \leq 1$, and arbitrary $k \in R$. Then,
(i) $u = v$ iff $\underline{u}(r) = \underline{v}(r)$, $\bar{u}(r) = \bar{v}(r)$,
(ii) $u + v = (\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r))$,
(iii) $u - v = (\underline{u}(r) - \bar{v}(r), \bar{u}(r) - \underline{v}(r))$,
(iv) $ku = \begin{cases} (k\underline{u}(r), k\bar{u}(r)), & k \geq 0; \\ (k\bar{u}(r), k\underline{u}(r)), & k < 0, \end{cases}$
(v) $uv = (\min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\bar{v}(r), \bar{u}(r)\underline{v}(r), \bar{u}(r)\bar{v}(r)\})$.

Definition 2.5. (Triangular fuzzy number) [30] A triangular fuzzy number (TFN) denoted by A is defined as (a, b, c), where the membership function:

$$A(x) = \begin{cases} 0, & x \leq a; \\ \dfrac{x-a}{b-a}, & a \leq x \leq b; \\ 1, & x = b; \\ \dfrac{c-x}{c-b}, & b \leq x \leq c; \\ 0, & x \geq c. \end{cases}$$

The α-cuts of $A = (a, b, c)$ are described by $[A]^\alpha = \{x \in R : A(x) \geq \alpha\} = [a + \alpha(b-a), c - \alpha(c-b)] = [A_{l,\alpha}, A_{r,\alpha}]$, $\alpha \in [0,1]$, and it is clear that $[A]^\alpha$ are a closed interval. A fuzzy number A is positive if $\operatorname{supp} A \subset (0, \infty)$.

The following proposition is fundamental since it characterizes a fuzzy set through the α-levels.

Proposition 2.1. [30] *If $\{A^\alpha : \alpha \in [0,1]\}$ is a compact, convex, and not empty subset family of R^n such that*
(i) $\overline{\bigcup A^\alpha} \subset A^0$.
(ii) $A^{\alpha_2} \subset A^{\alpha_1}$ *if* $\alpha_1 \leq \alpha_2$.
(iii) $A^\alpha = \bigcap_{k \geq 1} A^{\alpha_k}$ *if* $\alpha_k \uparrow \alpha > 0$.

Then, there is $u \in E^n$ *(*E^n *denotes n dimensional fuzzy number space) such that* $[u]^\alpha = A^\alpha$ *for all* $\alpha \in (0,1]$ *and* $[u]^0 = \overline{\bigcup_{0<\alpha\leq 1} A^\alpha} \subset A^0$.

Definition 2.6. [31] Suppose that $A, B \in E^1$ have α-cuts $[A]^\alpha = [A_{l,\alpha}, A_{r,\alpha}]$, $[B]^\alpha = [B_{l,\alpha}, B_{r,\alpha}]$, with $0 \notin [B]^\alpha$, $\forall\alpha \in [0,1]$. The g-division $\div_g$ is the operation that calculates the fuzzy number $C = A \div_g B$ having level cuts $[C]^\alpha = [C_{l,\alpha}, C_{r,\alpha}]$ (here $[A]^{\alpha-1} = [1/A_{r,\alpha}, 1/A_{l,\alpha}]$) defined by

$$[C]^\alpha = [A]^\alpha \div_g [B]^\alpha \Leftrightarrow \begin{cases} (i) \quad [A]^\alpha = [B]^\alpha [C]^\alpha, \\ \text{or} \\ (ii) \quad [B]^\alpha = [A]^\alpha [C]^{\alpha-1} \end{cases} \tag{7}$$

provided that C is a proper fuzzy number ($C_{l,\alpha}$ is nondecreasing, $C_{r,\alpha}$ is nonincreasing, $C_{l,1} \leq C_{r,1}$).

Remark 2.1. According to [31], in this paper, the fuzzy number is positive, if $A \div_g B = C \in E^1$ exists, then the following two cases are possible:

Case I. If $A_{l,\alpha}B_{r,\alpha} \leq A_{r,\alpha}B_{l,\alpha}$, $\forall\alpha \in [0,1]$, then $C_{l,\alpha} = \frac{A_{l,\alpha}}{B_{l,\alpha}}$, $C_{r,\alpha} = \frac{A_{r,\alpha}}{Br,\alpha}$,

Case II. If $A_{l,\alpha}B_{r,\alpha} \geq A_{r,\alpha}B_{l,\alpha}$, $\forall\alpha \in [0,1]$, then $C_{l,\alpha} = \frac{A_{r,\alpha}}{B_{r,\alpha}}$, $C_{r,\alpha} = \frac{A_{l,\alpha}}{Bl,\alpha}$.

The fuzzy analog of the boundedness and persistence (see [22,23]) is as follows:

Definition 2.7. A sequence of positive fuzzy numbers (x_n) is persistence (resp. bounded) if there exists a positive real number M (resp. N) such that

$$\text{supp } x_n \subset [M, \infty)(\text{resp. supp } x_n \subset (0, N]), \quad n = 1, 2, \ldots,$$

A sequence of positive fuzzy numbers (x_n) is bounded and persistence if there exist positive real numbers $M, N > 0$ such that

$$\text{supp } x_n \subset [M, N], \quad n = 1, 2, \ldots$$

A sequence of positive fuzzy numbers (x_n), $n = 1, 2, \ldots$, is an unbounded if the norm $\|x_n\|$, $n = 1, 2, \ldots$, is an unbounded sequence.

Definition 2.8. x_n is said to be a positive solution of (5) if (x_n) is a sequence of positive fuzzy numbers, which satisfy (5). A positive fuzzy number x is called a positive equilibrium of (5) if

$$x = \frac{Ax}{1 + x + Bx}.$$

Let (x_n) be a sequence of positive fuzzy numbers and x is a positive fuzzy number, $x_n \to x$ as $n \to \infty$ if $\lim_{n\to\infty} D(x_n, x) = 0$.

3 Main results

3.1 Existence of positive solution of population dynamics model (5)

First, we study the existence of positive solutions of the population dynamics model (5). We need the following lemma.

Lemma 3.1. [30] *Let* $f : R^+ \times R^+ \to R^+$ *be continuous, and,* A, B *are fuzzy numbers. Then,*

$$[f(A,B)]^\alpha = f([A]^\alpha, [B]^\alpha) \quad \alpha \in (0,1]. \tag{8}$$

Theorem 3.1. *If A, B, and initial values x_{-2}, x_{-1} of population dynamics model* (5) *are positive fuzzy numbers, then, there exists a unique positive solution x_n of the population dynamics model* (5).

Proof. Suppose that there exists a sequence of fuzzy numbers (x_n) satisfying (5) with initial condition x_{-1}, x_{-2}. Consider the α-cuts, $\alpha \in (0,1]$,

$$[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad n = 0, 1, 2, \dots \tag{9}$$

It follows from (5), (9), and Lemma 3.1 that

$$\begin{aligned}
[x_n]^\alpha &= [L_{n,\alpha}, R_{n,\alpha}] \\
&= \left[\frac{Ax_{n-1}}{1 + x_{n-1} + Bx_{n-2}}\right]^\alpha \\
&= \frac{[A]^\alpha \times [x_{n-1}]^\alpha}{1 + [x_{n-1}]^\alpha + [B]^\alpha \times [x_{n-2}]^\alpha} \\
&= \frac{[A_{l,\alpha}L_{n-1,\alpha}, A_{r,\alpha}R_{n-1,\alpha}]}{[1 + L_{n-1,\alpha} + B_{l,\alpha}L_{n-2,\alpha}, 1 + R_{n-1,\alpha} + B_{r,\alpha}R_{n-2,\alpha}]}.
\end{aligned}$$

Noting Remark 2.1, one of the following two cases holds:
Case I:

$$[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}] = \left[\frac{A_{l,\alpha}L_{n-1,\alpha}}{1 + L_{n-1,\alpha} + B_{l,\alpha}L_{n-2,\alpha}}, \frac{A_{r,\alpha}R_{n-1,\alpha}}{1 + R_{n-1,\alpha} + B_{r,\alpha}R_{n-2,\alpha}}\right]. \tag{10}$$

Case II:

$$[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}] = \left[\frac{A_{r,\alpha}R_{n-1,\alpha}}{1 + R_{n-1,\alpha} + B_{r,\alpha}R_{n-2,\alpha}}, \frac{A_{l,\alpha}L_{n-1,\alpha}}{1 + L_{n-1,\alpha} + B_{l,\alpha}L_{n-2,\alpha}}\right]. \tag{11}$$

If Case I holds true, it follows that for $n \in \{0, 1, 2, \dots\}$, $\alpha \in (0, 1]$:

$$L_{n,\alpha} = \frac{A_{l,\alpha}L_{n-1,\alpha}}{1 + L_{n-1,\alpha} + B_{l,\alpha}L_{n-2,\alpha}}, \quad R_{n,\alpha} = \frac{A_{r,\alpha}R_{n-1,\alpha}}{1 + R_{n-1,\alpha} + B_{r,\alpha}R_{n-2,\alpha}}. \tag{12}$$

□

Next, the proof is similar to those of Theorem 3.1 [10]. We omit it.

Remark 3.1. From theoretical point of view, the existence of solution for fuzzy difference equation is very important with initial condition. Therefore, in this sense, the existence of positive fuzzy solution for discrete population dynamics model is of vital importance and practical significance. In fact, according to Theorem 3.1, the positive solution of discrete population dynamics model with fuzzy state is a sequence of positive fuzzy numbers, which can describe the fuzzy uncertainty of the dynamics model.

3.2 Dynamics of discrete population model (5)

To study the dynamical behavior of the positive solutions of the discrete population model (5), according to Definition 2.6, we consider two cases.

First, if Case I holds true, the following lemma is essential to the proof of next theorem.

Lemma 3.2. *Consider the difference equation*:

$$y_n = \frac{ky_{n-1}}{1 + y_{n-1} + \beta y_{n-2}}, \quad n = 1, 2, \dots, \tag{13}$$

where $y_{-2}, y_{-1} \in (0, +\infty)$, if $k > 1$, $0 \le \beta < 1$, then the following statements are true.

(*i*) *The system exists a trivial equilibrium point* $y^* = 0$, *which is unstable.*
(*ii*) *The system exists a unique positive equilibrium* $y^* = \frac{k-1}{\beta+1}$, *which is globally asymptotically stable.*
(*iii*) *Every positive solution* y_n *of* (13) *is bounded and persistent.*

Proof.
(i) Let y^* be an equilibrium point of (13). It is easy to obtain that there exist two equilibrium points:

$$y^* = 0, \quad \text{and} \quad y^* = \frac{k-1}{\beta+1}.$$

It is clear that the trivial equilibrium point y^* is unstable. So we omit it.

(ii) The linearized equation associated with (13) at equilibrium expressed as follows: $y^* = \frac{k-1}{\beta+1}$ is expressed as follows:

$$y_n - \frac{k\beta+1}{k(\beta+1)}y_{n-1} + \frac{\beta(k-1)}{k(\beta+1)}y_{n-2} = 0.$$

Since $k > 1$, $0 \le \beta < 1$, it is easy to obtain

$$\frac{k\beta+1}{k(\beta+1)} + \frac{\beta(k-1)}{k(\beta+1)} < 1. \tag{14}$$

By virtue of Theorem 1.3.7 in [7], we have that the system is locally asymptotically stable.

On the other hand, it is similar to the proof of Theorem 1 in [32], and we can show that $\lim_{n\to\infty} y_n = \frac{k-1}{\beta+1}$. Thus, the unique positive equilibrium $y^* = \frac{k-1}{\beta+1}$ is globally asymptotically stable.

(iii) Let y_n be a solution of (13). We consider the following difference equation:

$$u_n = \frac{ku_{n-1}}{1+u_{n-1}}, \quad n = 1, 2, 3, \ldots, \tag{15}$$

and the initial values of (15) are satisfied with

$$y_0 \le u_0. \tag{16}$$

It follows from (13), (15), and (16) that

$$y_n = \frac{ky_{n-1}}{1+y_{n-1}+\beta y_{n-2}} \le \frac{ky_{n-1}}{1+y_{n-1}} \le \frac{ku_{n-1}}{1+u_{n-1}} = u_n, \quad n \ge 2. \tag{17}$$

It is clear that every solution u_n of (15) converges to equilibrium $k-1$, i.e., $\lim_{n\to\infty} u_n = k-1$. Therefore, it follows from (17) that

$$y_n \le \limsup_{n\to\infty} y_n \le \limsup_{n\to\infty} u_n = \lim_{n\to\infty} u_n = k-1. \tag{18}$$

On the other hand, we consider the family of sequences $\{u_n(\varepsilon)\}$, where

$$u_n(\varepsilon) = \frac{ku_{n-1}(\varepsilon)}{1+u_{n-1}(\varepsilon)+\beta(k-1+\varepsilon)}, \quad n = 2, 3, \ldots. \tag{19}$$

Without loss of generality, we take $0 < \varepsilon < \frac{(k-1)(1-\beta)}{\beta}$. For every fixed ε, difference equation (19) has a stationary trajectory such that $u(\varepsilon) = \lim_{n\to\infty} u_n(\varepsilon) = (k-1)(1-\beta) - \beta\varepsilon$. From (ii), there exists an $n_0 \in N$ such that $y_n \le k-1+\varepsilon$ for $n \ge n_0$.

Let the initial conditions of (19) be positive and satisfy the conditions:

$$u_{n_0+1}(\varepsilon) \le y_{n_0+1}. \tag{20}$$

Then, we obtain that

$$y_n = \frac{ky_{n-1}}{1 + y_{n-1} + \beta y_{n-2}} \geq \frac{ky_{n-1}}{1 + y_{n-1} + \beta(k-1+\varepsilon)} \geq \frac{ku_{n-1}(\varepsilon)}{1 + u_{n-1}(\varepsilon) + \beta(k-1+\varepsilon)} = u_n(\varepsilon). \tag{21}$$

Therefore, $y_n \geq u_n(\varepsilon)$, $\lim_{n\to\infty} \inf y_n \geq \lim_{n\to\infty} \inf u_n(\varepsilon) = \lim_{n\to\infty} u_n(\varepsilon) = (k-1)(1-\beta)$.

So $y_n \geq (k-1)(1-\beta)$ for $n \geq n_0$. This completes the proof. □

Theorem 3.2. *Consider the discrete population model* (5), *where parameters A and B and the initial conditions x_{-1}, x_0 are positive fuzzy numbers. If*

$$A_{l,\alpha} > 1, \ B_{r,\alpha} < 1, \ \alpha \in (0,1], \tag{22}$$

then the following statements are true.

(*i*) *Every positive solution x_n of discrete fuzzy population model* (5) *is bounded and persistent.*

(*ii*) *Every positive solution x_n of discrete fuzzy population model* (5) *tends to the positive equilibrium point x as $n \to \infty$.*

Proof.

(i) Since A, B and the initial value x_{-1}, x_0 are positive fuzzy numbers, there exist positive real numbers $M_A, N_A, M_B, N_B, M_0, N_0, M_{-1}, N_{-1}$ such that, for all $\alpha \in (0,1]$,

$$[A_{l,\alpha}, A_{r,\alpha}] \subset [M_A, N_A], \quad [B_{l,\alpha}, B_{r,\alpha}] \subset [M_B, N_B], \quad [L_{0,\alpha}, R_{0,\alpha}] \subset [M_0, N_0], \quad [L_{-l,\alpha}, R_{-1,\alpha}] \subset [M_{-1}, N_{-1}]. \tag{23}$$

Let x_n be a positive solution of discrete fuzzy population model (5), from (22), (23), and Lemma 3.2, we obtain

$$L_{n,\alpha} > (A_{l,\alpha} - 1)(1 - B_{l,\alpha}) > (M_A - 1)(1 - N_B) := M, \quad R_{n,\alpha} < A_{r,\alpha} - 1 < N_A - 1 := N \tag{24}$$

From which, we get for $n \geq 2$, $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}] \subset [M, N]$, and so $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subseteq [M, N]$. Thus, the positive solution is bounded and persistent.

(ii) Suppose that there exists a fuzzy number x such that

$$x = \frac{Ax}{1 + x + Bx}, \quad [x]^\alpha = [L_\alpha, R_\alpha], \quad \alpha \in (0,1]. \tag{25}$$

where $L_\alpha, R_\alpha \geq 0$. Then, from (25), we can prove that

$$L_\alpha = \frac{A_{l,\alpha} L_\alpha}{1 + L_\alpha + B_{l,\alpha} L_\alpha}, \quad R_\alpha = \frac{A_{r,\alpha} R_\alpha}{1 + R_\alpha + B_{r,\alpha} R_\alpha}. \tag{26}$$

Hence, from (26), we have

$$L_\alpha = \frac{A_{l,\alpha} - 1}{B_{l,\alpha} + 1}, \quad R_\alpha = \frac{A_{r,\alpha} - 1}{B_{r,\alpha} + 1}. \tag{27}$$

Let x_n be a positive solution of discrete fuzzy population model (5). Since (22) holds true, we can apply Lemma 3.2 to system (13), and so we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \quad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha. \tag{28}$$

Therefore, from (28), we have

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \{\max\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\}\} = 0.$$

This completes the proof of the theorem. □

Second, if Case II holds true, it follows that for $n \in \{0, 1, 2, \ldots\}$, $\alpha \in (0,1]$

$$L_{n,\alpha} = \frac{A_{r,\alpha} R_{n-1,\alpha}}{1 + R_{n-1,\alpha} + B_{r,\alpha} R_{n-2,\alpha}}, \quad R_{n,\alpha} = \frac{A_{l,\alpha} L_{n-1,\alpha}}{1 + L_{n-1,\alpha} + B_{l,\alpha} R_{n-2,\alpha}}. \tag{29}$$

We need the following lemmas.

Lemma 3.3. *Consider the system of difference equations*:

$$y_n = \frac{az_{n-1}}{1 + z_{n-1} + cz_{n-2}}, \quad z_n = \frac{ay_{n-1}}{1 + y_{n-1} + cy_{n-2}}, \quad n = 1, 2, \dots, \tag{30}$$

where $a, c, y_{-1}, y_0, z_{-1}, z_0 \in (0, +\infty)$. *If*

$$a > 1, \tag{31}$$

then the following statements are true.

(*i*) *The system exists a trivial equilibrium point* $(0, 0)$, *which is unstable.*

(*ii*) *The system exists a unique positive equilibrium point* $(y^*, z^*) = \left(\frac{a-1}{c+1}, \frac{a-1}{c+1}\right)$, *which is locally asymptotically stable.*

Proof. (i) The linearized equation of system (30) about $(0, 0)$ is expressed as follows:

$$\Psi_n = G\Psi_{n-1}, \tag{32}$$

where

$$\Psi_{n-1} = \begin{pmatrix} y_{n-1} \\ y_{n-2} \\ z_{n-1} \\ z_{n-2} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 & a & 0 \\ 1 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of (32) is expressed as follows:

$$\lambda^2(\lambda^2 - a^2) = 0. \tag{33}$$

It follows from (31) that there are two roots of characteristic equation outside the unit disk. So the trivial equilibrium $(0, 0)$ is unstable.

(ii) The linearized equation of system (30) about (y^*, z^*) is expressed as follows:

$$\Psi_n = H\Psi_{n-1}, \tag{34}$$

where

$$H = \begin{pmatrix} 0 & 0 & \frac{a + acz^*}{(1 + z^* + cz^*)^2} & -\frac{acz^*}{(1 + z^* + cz^*)^2} \\ 1 & 0 & 0 & 0 \\ \frac{a + acy^*}{(1 + y^* + cy^*)^2} & -\frac{acy^*}{(1 + y^* + cy^*)^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ denote the eigenvalues of matrix H, let $K = \mathrm{diag}(m_1, m_2, m_3, m_4)$ be a diagonal matrix, where $m_1 = m_3 = 1$, $m_k = 1 - k\varepsilon (k = 2, 4)$, and

$$0 < \varepsilon < \frac{1}{4}\left(1 - \frac{acz^*}{(1 + z^* + cz^*)^2 - a - acz^*}\right). \tag{35}$$

Clearly, K is invertible. Computing matrix KHK^{-1}, we obtain that

$$KHK^{-1} = \begin{pmatrix} 0 & 0 & \frac{(a + acz^*)m_1m_3^{-1}}{(1 + z^* + cz^*)^2} & -\frac{acz^*m_1m_4^{-1}}{(1 + z^* + cz^*)^2} \\ m_2m_1^{-1} & 0 & 0 & 0 \\ \frac{(a + acy^*)m_3m_1^{-1}}{(1 + y^* + cy^*)^2} & -\frac{acy^*m_3m_2^{-1}}{(1 + y^* + cy^*)^2} & 0 & 0 \\ 0 & 0 & m_4m_3^{-1} & 0 \end{pmatrix}. \tag{36}$$

It is obtained from (36) that

$$\frac{a + acz^*}{(1 + z^* + cz^*)^2} + \frac{acz^*m_4^{-1}}{(1 + z^* + cz^*)^2} = \frac{a + acz^*\left(1 - \frac{1}{1-4\varepsilon}\right)}{(1 + z^* + cz^*)^2} < 1$$

$$\frac{a + acy^*}{(1 + y^* + cy^*)^2} + \frac{acy^*m_2^{-1}}{(1 + y^* + cy^*)^2} = \frac{a + acy^*\left(1 - \frac{1}{1-2\varepsilon}\right)}{(1 + y^* + cy^*)^2} < 1.$$

It is well known that H has the same eigenvalues as KHK^{-1}, and we have

$$\max_{1\le i\le 4}|\lambda_i| \le \|KHK^{-1}\|_\infty = \max\left\{m_2m_1^{-1}, m_4m_3^{-1}, \frac{a + acz^* + acz^*m_1m_4^{-1}}{(1 + z^* + cz^*)^2}, \frac{a + acy^* + acy^*m_3m_2^{-1}}{(1 + y^* + cy^*)^2}\right\} < 1. \tag{37}$$

This implies that the equilibrium (y^*, z^*) of (30) is locally asymptotically stable. □

Theorem 3.3. *Suppose that parameters A and B of discrete fuzzy population model* (5) *are positive trivial fuzzy numbers* (*positive real numbers*) *and* $A > 1$, $0 < B < 1$. *If the following condition are satisfied,*

$$\frac{1 + BL_{n-2,\alpha}}{1 + BR_{n-2,\alpha}} \le \frac{L_{n-1,\alpha}}{R_{n-1,\alpha}}, \quad n = 1, 2, \ldots, \forall\alpha \in (0, 1]. \tag{38}$$

Then, the following statements are true.

(*i*) *Every positive solution* x_n *of discrete fuzzy population model* (5) *is bounded and persistence.*

(*ii*) *Every positive solution* x_n *of discrete fuzzy population model* (5) *tends to the positive equilibrium point* x *as* $n \to +\infty$.

Proof.

(i) The proof is similar to those of (i) in Theorem 3.2. Let x_n be a positive solution of discrete fuzzy population model (5). It is easy to get that

$$L_{n,\alpha} \ge (A - 1)(1 - B), \quad R_{n,\alpha} \le A - 1, \quad \forall\alpha \in (0, 1].$$

From which, we obtain for $n \ge 2$, $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}] \subset [(A - 1)(1 - B), A - 1]$, and so $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subseteq [(A - 1)(1 - B), A - 1]$. Thus, the positive solution is bounded and persistent.

(ii) Suppose that there exists a positive fuzzy number x satisfying (25). Then from (25), we have

$$L_\alpha = \frac{AR_\alpha}{1 + R_\alpha + BR_\alpha}, \quad R_\alpha = \frac{AL_\alpha}{1 + L_\alpha + BL_\alpha}, \quad \forall\alpha \in (0, 1]. \tag{39}$$

From (39), we have

$$L_\alpha = R_\alpha = \frac{A - 1}{B + 1}.$$

Let x_n be a positive solution of discrete fuzzy population model (5) such that Case II holds. Namely,

$$L_{n,\alpha} = \frac{AR_{n-1,\alpha}}{1 + R_{n-1,\alpha} + +BR_{n-2,\alpha}}, \quad R_{n,\alpha} = \frac{AL_{n-1,\alpha}}{1 + L_{n-1,\alpha} + BL_{n-2,\alpha}}. \tag{40}$$

Since $A > 1$, $0 < B < 1$ and (38) is satisfied, we can apply Lemma 3.3 to system (40), and so we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \quad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha. \tag{41}$$

Therefore, from (41), we have

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty}\sup_{\alpha\in(0,1]}\{\max\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\}\} = 0.$$

This completes the proof of Theorem 3.3. □

Remark 3.2. In the population dynamical model, the parameters of model are derived from statistic data with vagueness or uncertainty. It corresponds to reality to use fuzzy parameters in the population dynamical model. In contrast with the classic population model, the solution of fuzzy population model is within a range of value (approximate value), which are taken into account fuzzy uncertainties. Furthermore, the global asymptotic behavior of the discrete second-order population model are obtained in fuzzy context.

4 Numerical examples

Example 4.1. Consider the following second-order fuzzy discrete population model:

$$x_n = \frac{Ax_{n-1}}{1 + x_{n-1} + Bx_{n-2}}, \quad n = 1, 2, \ldots, \tag{42}$$

and we take A, B, and the initial values x_{-1}, x_0 such that

$$A(x) = \begin{cases} 2x - 3, & 1.5 \le x \le 2 \\ -2x + 5, & 2 \le x \le 2.5 \end{cases}, \quad x_{-1}(x) = \begin{cases} 2x - 1, & 0.5 \le x \le 1 \\ -5x + 6, & 1 \le x \le 1.2 \end{cases} \tag{43}$$

$$B(x) = \begin{cases} 5x - 2, & 0.4 \le x \le 0.6 \\ -5x + 4, & 0.6 \le x \le 0.8 \end{cases}, \quad x_0(x) = \begin{cases} x - 3, & 3 \le x \le 4 \\ -x + 5, & 4 \le x \le 5. \end{cases} \tag{44}$$

From (43), we obtain

$$[A]^\alpha = \left[1.5 + \frac{1}{2}\alpha, 2.5 - \frac{1}{2}\alpha\right], \quad [x_{-1}]^\alpha = \left[0.5 + \frac{1}{2}\alpha, 1.2 - \frac{1}{5}\alpha\right], \quad \alpha \in (0, 1]. \tag{45}$$

From (44), we obtain

$$[B]^\alpha = \left[0.4 + \frac{1}{5}\alpha, 0.8 - \frac{1}{5}\alpha\right], \quad [x_0]^\alpha = [3 + \alpha, 5 - \alpha], \quad \alpha \in (0, 1]. \tag{46}$$

Therefore, it follows that

$$\overline{\bigcup_{\alpha\in(0,1]} [A]^\alpha} = [1.5, 2.5], \quad \overline{\bigcup_{\alpha\in(0,1]} [x_{-1}]^\alpha} = [0.5, 1.2], \quad \overline{\bigcup_{\alpha\in(0,1]} [B]^\alpha} = [0.4, 0.8], \quad \overline{\bigcup_{\alpha\in(0,1]} [x_0]^\alpha} = [3, 5]. \tag{47}$$

From (42), it results in a coupled system of difference equations with parameter α,

$$L_{n,\alpha} = \frac{A_{l,\alpha}L_{n-1,\alpha}}{1 + L_{n-1,\alpha} + B_{l,\alpha}L_{n-1,\alpha}}, \quad R_{n,\alpha} = \frac{A_{r,\alpha}R_{n-1,\alpha}}{1 + R_{n-1,\alpha} + B_{r,\alpha}R_{n-1,\alpha}}, \quad \alpha \in (0, 1]. \tag{48}$$

Therefore, $A_{l,\alpha} > 1$, $B_{r,\alpha} < 1$, $\forall\alpha \in (0, 1]$, and initial values x_0 are positive fuzzy numbers, so from Theorem 3.2, we have that every positive solution x_n of equation (42) is bounded and persistence. In addition, from Theorem 3.2, equation (42) has a unique positive equilibrium $\bar{x} = (0.357, 0.625, 0.833)$. Moreover, every positive solution x_n of equation (42) converges the unique equilibrium $\bar{x}$ with respect to D as $n \to \infty$ (see Figures 1–3).

Example 4.2. Consider the second-order fuzzy discrete population model (42), where $A = 1.5, B = 0.6$, and the initial values x_0, x_{-1} are satisfied

$$x_{-1}(x) = \begin{cases} 0.5x - 0.5, & 1 \le x \le 3 \\ -0.5x + 2.5, & 3 \le x \le 5 \end{cases}, \quad x_0(x) = \begin{cases} 2x - 6, & 3 \le x \le 3.5 \\ -2x + 8, & 3.5 \le x \le 4. \end{cases} \tag{49}$$

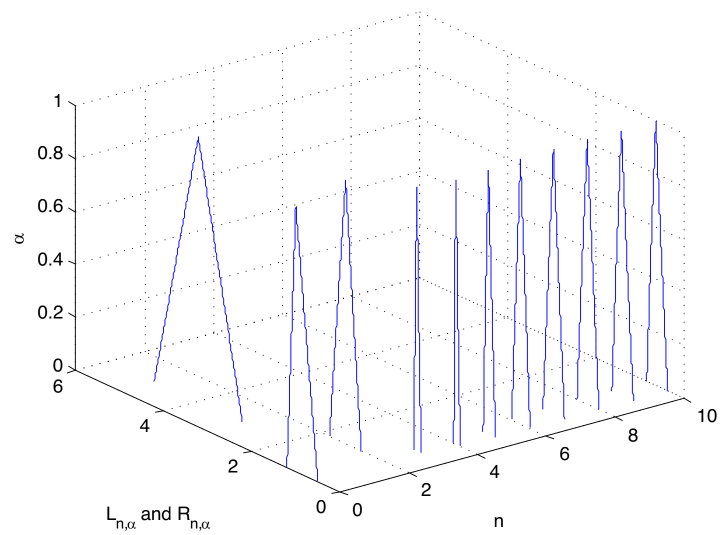

Figure 1: The dynamics of system (42).

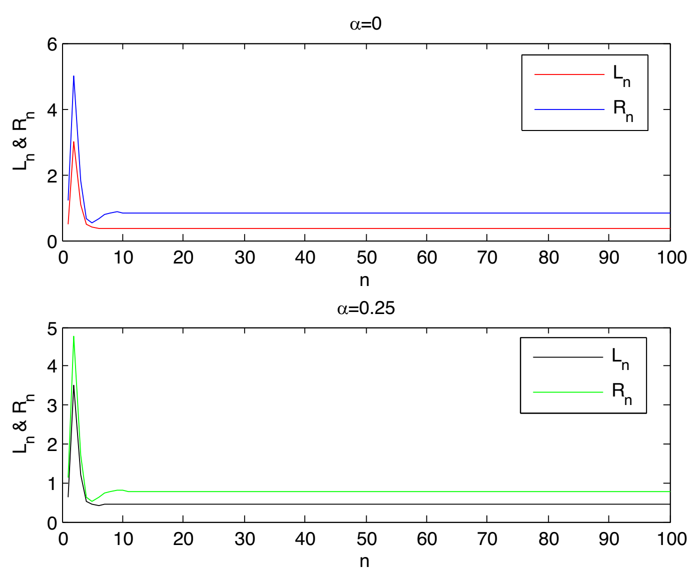

Figure 2: The solution of system (48) at $\alpha = 0$ and $\alpha = 0.25$.

From (48), we obtain

$$[x_{-1}]^\alpha = [1 + 2\alpha, 5 - 2\alpha], \quad [x_0]^\alpha = \left[3 + \frac{1}{2}\alpha, 4 - \frac{1}{2}\alpha\right], \quad \alpha \in (0, 1]. \tag{50}$$

Therefore, it follows that

$$\overline{\bigcup_{\alpha\in(0,1]} [x_{-1}]^\alpha} = [1, 5], \quad \overline{\bigcup_{\alpha\in(0,1]} [x_0]^\alpha} = [3, 4]. \tag{51}$$

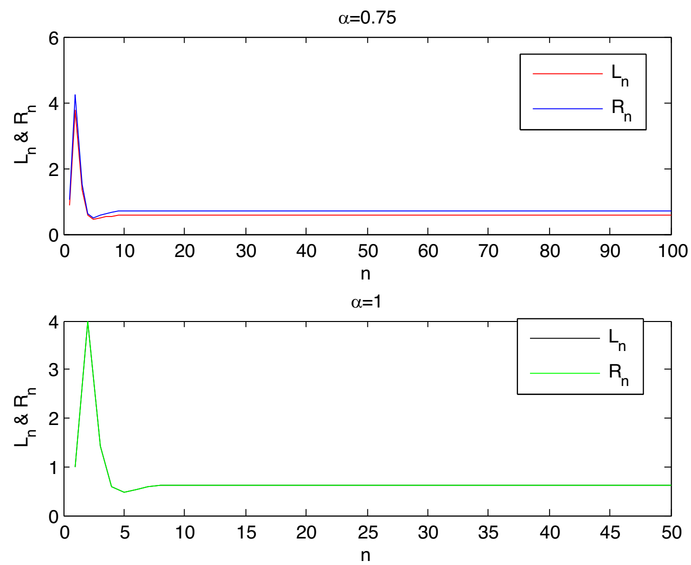

Figure 3: The solution of system (48) at $\alpha = 0.75$ and $\alpha = 1$.

From (42), it results in a coupled system of difference equation with parameter α,

$$L_{n,\alpha} = \frac{AR_{n-1,\alpha}}{1 + R_{n-1,\alpha} + BR_{n-2,\alpha}}, \quad R_{n,\alpha} = \frac{AL_{n-1,\alpha}}{1 + L_{n-1,\alpha} + BL_{n-2,\alpha}}, \quad \alpha \in (0, 1]. \tag{52}$$

It is clear that (38) is satisfied and initial values x_{-1}, x_0 are positive fuzzy numbers, so from Theorem 3.3, equation (42) has a unique positive equilibrium $\bar{x} = 0.3125$. Moreover, every positive solution x_n of equation (42) converges the unique equilibrium $\bar{x}$ with respect to D as $n \to \infty$ (see Figures 4–6).

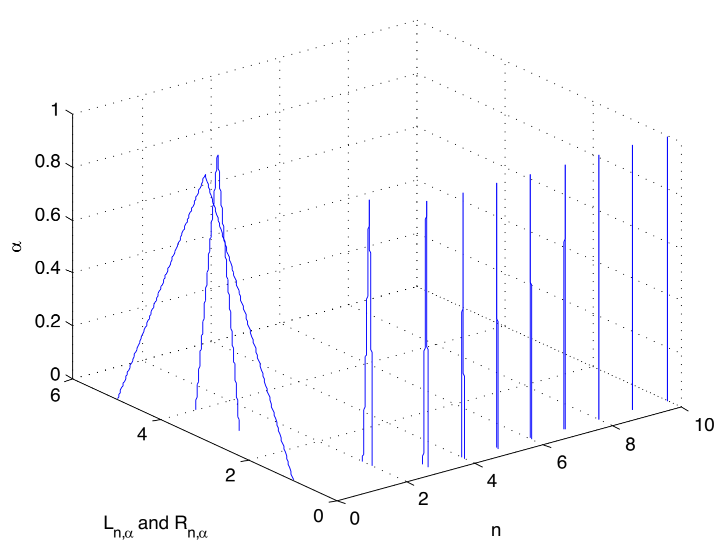

Figure 4: The dynamics of system (42).

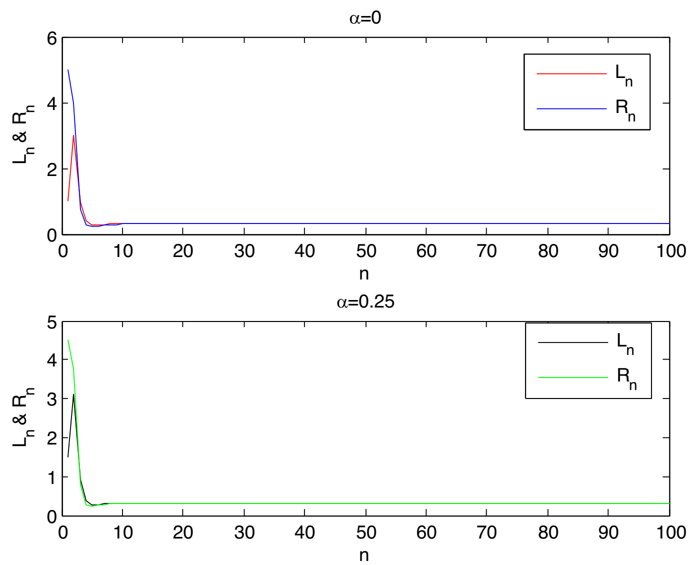

Figure 5: The solution of system (48) at $\alpha = 0$ and $\alpha = 0.25$.

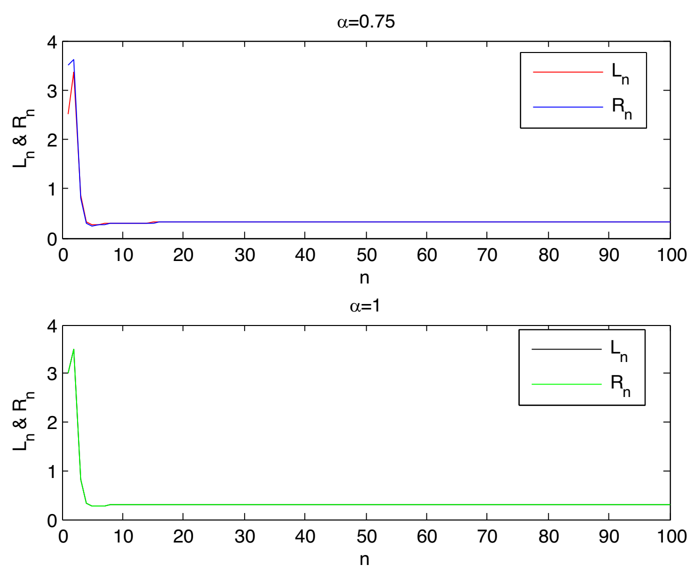

Figure 6: The solution of system (48) at $\alpha = 0.75$ and $\alpha = 1$.

5 Conclusion

In this work, according to a generalization of division (g-division) of fuzzy number, we study the second-order fuzzy discrete population model $x_n = \frac{Ax_{n-1}}{1+x_{n-1}+Bx_{n-2}}$. The existence of the positive solution and the qualitative behavior to (5) are investigated. The main results are as follows:

(1) Under Case I, the positive solution is bounded and persistent if $A_{l,\alpha} > 1$, $B_{r,\alpha} < 1$, $\alpha \in (0, 1]$. Every positive solution x_n tends to the unique equilibrium x as $n \to \infty$.

(2) Under Case II, the positive solution is bounded and persistent if A, B are positive trivial fuzzy numbers and $A > 1$, $B < 1$, $\frac{1+BL_{n-2,\alpha}}{1+BR_{n-2,\alpha}} \leq \frac{L_{n-1,\alpha}}{R_{n-1,\alpha}}$, $n = 1, 2, \ldots, \forall \alpha \in (0, 1]$. Every positive solution x_n tends to the unique equilibrium x as $n \to \infty$.

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