Research Article

Ingrid Carbone*

A random von Neumann theorem for uniformly distributed sequences of partitions

https://doi.org/10.1515/math-2022-0016
received January 1, 2022; accepted February 7, 2022

Abstract: In this paper, we prove a theorem that confirms, under a supplementary condition, a conjecture concerning random permutations of sequences of partitions of the unit interval.

Keywords: number theory, discrepancy, uniformly distributed sequences of partitions, probability theory

MSC 2020: 11-xx, 40-xx, 60-xx

1 Introduction

The general study of uniformly distributed sequences of partitions was initiated in [1], inspired by a beautiful construction and result from the study by Kakutani [2]. The subject is closely related to the theory of uniformly distributed sequences, initiated in [3]. There are two classical references for the subject: [4] and [5].

Kakutani took the interval $I = [0, 1]$, a number $\alpha \in]0, 1[$ and divided the interval in proportion $\alpha : 1 - \alpha$. Then, he divided the longest interval of this partition in the same proportion and iterated the procedure dividing always the longest interval of the nth partition, so as to obtain a sequence of partitions of $]0, 1[$. If at a certain step there were two or more intervals of maximal length, they were divided simultaneously.

Kakutani proved that this sequence of partitions (denote it by $\{\alpha^n I\}$) is *uniformly distributed*, which means that if $\alpha^k I = \{0 < t_1^k < t_2^k < \cdots < t_{N_k}^n < 1\}$ is the kth partition, then

$$\lim_{k\to\infty} \frac{1}{N_k}\sum_{i=1}^{N_k} f(t_i^k) = \int_0^1 f(t)\mathrm{d}t,$$

for every continuous function f.

In other words, the discrete measure concentrated in the points t_i^k converges weakly to the Lebesgue measure on $[0, 1]$.

The construction has been generalized in [1]. Let ρ be any non trivial finite partition of I.

In the first step, the longest interval(s) of ρ is subdivided positively homothetically to ρ. The partition obtained in this manner is denoted by $\rho^2 I$. In the second step, the same procedure is repeated on the longest interval(s), operating with ρ on $\rho^2 I$. Iteration of this procedure leads to a sequence of partitions $\{\rho^n I\}$.

If $\rho = \{[0, \alpha[, [\alpha, 1[\}$, one gets Kakutani's sequence.

The following theorem includes the results of the study by Kakutani ([1], Theorem 2.7]).

* **Corresponding author: Ingrid Carbone,** Department of Mathematics and Computer Science, Università della Calabria, Ponte Bucci, Cubo 30B, 87036, Arcavacata di Rende (Cosenza), Italy, e-mail: ingrid.carbone@unical.it

Open Access. © 2022 Ingrid Carbone, published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 International License.

Theorem 1. *The sequence* $\{\rho^n I\}$ *is uniformly distributed.*

There are interesting connections between the theory of u.d. sequences of partitions and u.d. sequences of points. This connection is far reaching in the construction of a significant subclass of ρ-refinements, the so-called LS-sequences. The subject was initiated by the present author in [6], and it is connected with the van der Corput sequences of points.

LS-sequences are constructed starting from the partition ρ_{LS} made of $L + S$ intervals (L and S are positive integers) of length β and β^2, respectively, where β is the positive solution of the equation $L\beta + S\beta^2 = 1$.

The present paper is concerned with a result in the domain of uniformly distributed sequences of partitions, related to a proposition by von Neumann for uniformly sequences of points [7].

Theorem 2. *If* $\{x_n\}$ *is a dense sequence of points in* $[0, 1]$, *then there exists a rearrangement of these points*, $\{x_{n_k}\}$, *which is uniformly distributed.*

One of the consequences of von Neumann's result is that there are many u.d. sequences of points.

We will now introduce the definitions we need.

Definitions. Given a partition $\pi = \{[t_{i-1}, t_i], 1 \le i \le N\}$, we denote by $l_i = t_i - t_{i-1}$ the length of its ith interval.

The *diameter* of π, denoted by L, is equal to $\max_{1\le i\le N} l_i$.

Given a sequence of partitions $\{\pi_k\}$, we say that it is *dense* if, denoted by L_k the diameter of π_k, $\lim_{k\to\infty} L_k = 0$.

If $\pi = \{[t_{i-1}, t_i], 1 \le i \le N\}$ is a partition, its *random permutation* is a partition $\pi' = \{[s_{h-1}, s_h], 1 \le h \le N\}$ defined by the points $s_h = \sum_{j=0}^{h} l_{i_j}$, for $0 \le h \le N$, where $s_0 = 0$ and the indices $\{i_j\}$, are successively taken at random, with probability $\frac{1}{N}$, from the set $\{i_s : 1 \le s \le N\}$.

We will denote by $\pi\,!$ the set of all the $N!$ permutations of π.

2 Main results

In a previous paper [8], we proved the following result.

Proposition 3. *If* $\{\pi_n\}$ *is a dense sequence of partitions, then there exists a sequence of partitions* $\{\sigma_n\}$, *with* $\sigma_n \in \pi_n\,!$, *which is uniformly distributed.*

In the same paper, we made the following conjecture.

Conjecture. *If* $\{\pi_n\}$ *is a dense sequence of partitions and we select at random a partition* $\sigma_k \in \pi_n\,!$, *then* $\{\sigma_k\}$ *is uniformly distributed with probability* 1.

We need some preliminary calculations.

Let $q \in]0, 1[$ and denote by $N_k(q)$ the integer such that

$$\frac{N_k(q)}{N_k} \le q < \frac{N_k(q) + 1}{N_k}.$$

Consider a sequence $\{q_m\}$ of points, which is dense in $[0, 1]$. For later convenience, we will denote by $N_k(m)$ the integer $N_k(q_m)$.

Select at random from the N_k intervals of π_k, with probability $\frac{1}{N_k}$, $N_k(m)$ intervals. Denote by ξ_i^k the length of the interval selected in the ith draw ($1 \le i \le N_k(m)$) and consider the random variable

$$\eta_k^m = \sum_{i=1}^{N_k(m)} \xi_i^k.$$

Obviously,

$$E(\eta_k^m) = \sum_{i=1}^{N_k(m)} E(\xi_i^k) = \sum_{i=1}^{N_k(m)} \frac{1}{N_k} = \frac{N_k(m)}{N_k},$$

hence,

$$|E(\eta_k^m) - q_m| \le \frac{1}{N_k}.$$

It is easy to see that the second moment of η_k^m is uniformly bounded for any sequence of partitions.

This, together with the independence of the η_k^m's (for $k \in \mathbb{N}$), would allow us to apply the strong law of large numbers and to conclude that, when k tends to infinity, the sequence η_k^m tends to q_m in the Cesàro mean (and nothing more, at least following this line of thought).

But this is not what we were looking for.

If we want to identify a class for which the conjecture is true, we have to make some assumptions on the sequence $\{\pi_k\}$.

A simple sufficient assumption is expressed as follows:

$$\sum_{i=1}^{\infty} L_k^2 < \infty.$$

Theorem 4. *If the series of squares of diameters of $\{\pi_k\}$ is convergent, then its random permutations σ_k are uniformly distributed with probability* 1.

Proof. We have

$$\begin{aligned}
\operatorname{Var}(\eta_k^m) &= E\left(\left(\sum_{i=1}^{N_k(m)} \xi_i^k - \sum_{i=1}^{N_k(m)} \frac{1}{N_k}\right)^2\right) \\
&= E\left(\sum_{i=1}^{N_k(m)} \left(\xi_i^k - \frac{1}{N_k}\right)^2\right) \\
&= E\left(\sum_{i=1}^{N_k(m)} (\xi_i^k)^2 - 2\sum_{i=1}^{N_k(m)} \xi_i^k \frac{1}{N_k} + \sum_{i=1}^{N_k(m)} \frac{1}{N_k^2}\right) \\
&= E\left(\sum_{i=1}^{N_k(m)} (\xi_i^k)^2\right) - 2\frac{1}{N_k} E\left(\sum_{i=1}^{N_k(m)} \xi_i^k\right) + E\left(\sum_{i=1}^{N_k(m)} \frac{1}{N_k^2}\right) \\
&\le E\left(\sum_{i=1}^{N_k} (\xi_i^k)^2\right) - \frac{2}{N_k} E\left(\sum_{i=1}^{N_k} \xi_i^k\right) + \sum_{i=1}^{N_k} \frac{1}{N_k} \\
&= E\left(\sum_{i=1}^{N_k} (\xi_i^k)^2\right) - \sum_{i=1}^{N_k} \frac{1}{N_k} \le \sum_{i=1}^{N_k} L_k^2 - 1 < \sum_{i=1}^{N_k} L_k^2.
\end{aligned}$$

Apply now the Čebišëv inequality. By our assumption, we have, for every $\varepsilon > 0$ (and every m),

$$\sum_{k=1}^{\infty} P(|\eta_k^m - E(\eta_k^m)| > \varepsilon) \le \sum_{n=1}^{\infty} \frac{\operatorname{Var}(\eta_k^m)}{\varepsilon^2} < \infty.$$

Recalling that $E(\eta_k^m)$ tends to q_m and applying the Borel-Cantelli lemma [9, Theorem 4.2.1], we obtain that

$$\lim_{k\to\infty} \eta_k^m = q_m$$

almost surely for every $m \in \mathbb{N}$.

The set $\{q_m\}$ is countable; therefore, the aforementioned limit holds almost surely for all the values of m simultaneously.

Observe now that $\lim_{k\to\infty} \frac{N_k(q)}{N_k}$ is an increasing function of q. Therefore, it follows that, almost surely,

$$\lim_{k\to\infty} \frac{N_k(q)}{N_k} = q$$

for every $q \in [0, 1]$.

In other words, the empirical distribution function F_k of σ_k tends almost surely to the distribution function of the random variable U uniformly distributed on $[0, 1]$.

On the other hand, convergence in distribution is known to be equivalent to weak convergence, so the desired conclusion follows. □

Acknowledgements: The author wishes to thank the referees for their useful comments.

Conflict of interest: The author states no conflict of interest.

References

[1] A. Volčič, *A generalization of Kakutani's splitting procedure*, Ann. Mat. Pura Appl. **190** (2011), no. 1, 45–54,
[2] S. Kakutani, *A problem on equidistribution on the unit interval* [0,1], Measure Theory (Proc. Conf., Oberwolfach, 1975), Lecture Notes in Mathematics, vol. 541, Springer, Berlin, 1976, pp. 369–375.
[3] H. Weyl, *Über ein Problem aus dem Gebiete der diophantischen Approximationen*, Nach. Ges. Wiss. Göttingen, Math.-Phys. Kl. **21** (1914), 234–244.
[4] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Pure and Applied Mathematics, Wiley-Interscience, New York-London-Sidney, 1974.
[5] M. Drmota and R. F. Tichy, *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics, vol. 1651, Springer Verlag, Berlin, 1997.
[6] I. Carbone, *Discrepancy of LS-sequences of partitions and points*, Ann. Mat. Pura Appl. **191** (2012), no. 4, 819–844.
[7] J. von Neumann, *Gleichmässig dichte Zahlenfolgen*, Mat. Fiz. Lapok **32** (1925), 32–40.
[8] I. Carbone and A. Volčič, *A von Neumann theorem for uniformly distributed sequences of partitions*, Rend. Circ. Mat. Palermo **60** (2011), no. 1–2, 83–88.
[9] K. L. Chung, *A Coure in Probability Theory*, third edn, Academic Press, San Diego, 2001.