Research Article

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Coupled measure of noncompactness and functional integral equations

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Abstract: The aim of this article is to study the results of the fixed-point in coupled and tripled measure of noncompactness (MNC). We will use the technique of MNC for coupled and tripled MNC. Also, we tend to prove some results of coupled and tripled MNC for the family of JS-contractive-type mappings. Moreover, an application with an example is provided to illustrate the results.

Keywords: fixed-point, JS-contraction-type mapping, Darbo's fixed-point theorem, measure of noncompactness, coupled measure of noncompactness, functional integral equation

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1 Introduction

In nonlinear analysis, one of the most important tools is the concept of measure of noncompactness (MNC) to address the problems in functional operator equations. This important concept in mathematical sciences has been defined by many authors in various ways (see [1–8]). In [9], Aghajani et al. established some generalizations of Darbo's fixed-point theorem and presented an application in functional integral equations.

In this paper, we investigate the fixed-point results that generalize Darbo's fixed-point theorem and many existing results in the literature by introducing the notion of coupled MNC. As an application, we prove the existence of solutions of a functional integral equation in Banach space $BC(\mathcal{R}_+)$. Finally, an example is supplied to illustrate the results.

Throughout this study, we consider E as a Banach space and briefly represent a measure of noncompactness with MNC, $B(\upsilon, r)$ represents a closed ball in Banach space E to center υ and radius r. Also, we use B_r to represent $B(\theta, r)$, where θ is the zero element, the family of all nonempty bounded subsets of E is represented with $\mathcal{B}_E$. To begin, we have the following preliminaries from [6,10,11].

Definition 1.1. [6]. Let $\mu : \mathcal{B}_E \to \mathcal{R}_+$ be a mapping. The family $\mathcal{B}_E$ is called MNC on Banach space E if the following conditions hold:

(1) For each $\mathcal{U}^1 \in \mathcal{B}_E$, $\mu(\mathcal{U}^1) = \theta$ iff $\mathcal{U}^1$ is a precompact set;

(2) For each pair $(\mathcal{U}^1, \mathcal{U}^2) \in \mathcal{B}_E \times \mathcal{B}_E$, we have

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$$\mathcal{U}^1 \subseteq \mathcal{U}^2 \text{ implies } \mu(\mathcal{U}^1) \leq \mu(\mathcal{U}^2);$$

(3) For each $\mathcal{U}^1 \in \mathcal{B}_E$,

$$\mu(\mathcal{U}^1) = \mu(\overline{\mathcal{U}^1}) = \mu(\text{conv } \mathcal{U}^1),$$

where $\overline{\mathcal{U}^1}$ represents the closure of $\mathcal{U}^1$ and conv $\mathcal{U}^1$ represents the convex hull of $\mathcal{U}^1$;

(4) $\mu(\lambda\mathcal{U}^1 + (1-\lambda)\mathcal{U}^2) \leq \lambda\mu(\mathcal{U}^1) + (1-\lambda)\mu(\mathcal{U}^2)$ for $\lambda \in [0,1]$;

(5) If $\{\upsilon_n\}_0^{+\infty} \in \mathcal{B}_E$ is a decreasing sequence of closed sets and $\lim_{n\to+\infty}\mu(\upsilon_n) = 0$, then $\mathcal{U}^1_{+\infty} = \bigcap_{n=0}^{+\infty}\mathcal{U}^1_n \neq \varnothing$.

In this part, we have the following theorems from [10–12].

Theorem 1.2. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : G \to G$ be a compact and continuous operator. Then, F has at least one fixed-point.*

Theorem 1.3. (Schauder) *Let G be a nonempty, closed, and convex subset of a normed space and F be a continuous operator from G into a compact subset of G. Then, F has a fixed-point.*

Theorem 1.4. (Darbo) *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : G \to G$ be a continuous operator. Suppose there is $\lambda \in [0,1)$ such that $\mu(F\mathcal{U}) \leq \lambda\mu(\mathcal{U})$ for each $\mathcal{U} \in G$. Then, F has a fixed-point.*

Theorem 1.5. (Brouwer) *Let G be a nonempty, compact, and convex subset of a finite-dimensional normed space and $F : G \to G$ be a continuous operator. Then, F has a fixed-point.*

Lemma 1.6. [9] *Let $\varphi_1 : \mathcal{R}_+ \to \mathcal{R}_+$ be an upper semicontinuous and nondecreasing function. In this case, the following conditions are equivalent:*

(1) $\lim_{n\to+\infty} \varphi_1^n(r) = 0$ *for every* $r > 0$;

(2) $\varphi_1(r) < r$ *for every* $r > 0$.

2 Coupled MNC

We start this section with the following concept, and then, we turn to the main subject.

Definition 2.1. Let E be a Banach space and $\mu : \mathcal{B}_E^2 \to \mathcal{R}_+$ be a mapping. We say that μ is a coupled MNC on E, if it has the following conditions:

(1) $\ker\mu = \{(\mathcal{U}^1, \mathcal{U}^2) \in \mathcal{B}_E^2 : \mu(\mathcal{U}^1, \mathcal{U}^2) = \theta\}$ is nonempty;

(2) For every $(\mathcal{U}^1, \mathcal{U}^2) \in \mathcal{B}_E^2$, $\mu(\mathcal{U}^1, \mathcal{U}^2) = \theta \Leftrightarrow (\mathcal{U}^1, \mathcal{U}^2)$ is a precompact set;

(3) For each $((\mathcal{U}^1, \mathcal{U}^2), (\mathcal{U}'^1, \mathcal{U}'^2)) \in \mathcal{B}_E^2 \times \mathcal{B}_E^2$ and $(\mathcal{U}^1, \mathcal{U}^2) \subseteq (\mathcal{U}'^1, \mathcal{U}'^2)$, where $\mathcal{U}^1 \subseteq \mathcal{U}'^1$ and $\mathcal{U}^2 \subseteq \mathcal{U}'^2$, we have

$$(\mathcal{U}^1, \mathcal{U}^2) \subseteq (\mathcal{U}'^1, \mathcal{U}'^2) \text{ implies } \mu(\mathcal{U}^1, \mathcal{U}^2) \leq \mu(\mathcal{U}'^1, \mathcal{U}'^2);$$

(4) For every $(\mathcal{U}^1, \mathcal{U}^2) \in \mathcal{B}_E^2$,

$$\mu(\overline{\mathcal{U}^1}, \overline{\mathcal{U}^2}) = \mu(\mathcal{U}^1, \mathcal{U}^2) = \mu(\text{conv}(\mathcal{U}^1, \mathcal{U}^2)),$$

where $\text{conv}(\mathcal{U}^1, \mathcal{U}^2)$ denotes the convex hull of $(\mathcal{U}^1, \mathcal{U}^2)$;

(5) $\mu(\lambda(\mathcal{U}^1, \mathcal{U}^2) + (1-\lambda)(\mathcal{U}'^1, \mathcal{U}'^2)) \leq \lambda\mu(\mathcal{U}^1, \mathcal{U}^2) + (1-\lambda)\mu(\mathcal{U}'^1, \mathcal{U}'^2)$ for $\lambda \in [0,1]$;

(6) If $\{\mathcal{U}^1_n\}_0^{+\infty}$, $\{\mathcal{U}^2_n\}_0^{+\infty}$ in $\mathcal{B}_E$ are decreasing sequences of closed sets and

$$\lim_{n\to+\infty} \mu\{(\mathcal{U}^1_n, \mathcal{U}^2_n)\}_0^{+\infty} = 0, \quad \text{then } (\mathcal{U}^1_{+\infty}, \mathcal{U}^2_{+\infty}) = \bigcap_{n=0}^{+\infty} (\mathcal{U}^1_n, \mathcal{U}^2_n) \neq \varnothing.$$

Theorem 2.2. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : G \to G$ be a continuous map such that*

$$\varphi_2(\mu(F\mathcal{U}^1, F\mathcal{U}^2)) \leq \varphi_2(\mu(\mathcal{U}^1, \mathcal{U}^2)) - \varphi_1(\mu(\mathcal{U}^1, \mathcal{U}^2)), \tag{2.1}$$

for each $\emptyset \neq \mathcal{U}^1 \subseteq G, \emptyset \neq \mathcal{U}^2 \subseteq G$, where μ is an arbitrary coupled MNC and $\varphi_1, \varphi_2 : \mathcal{R}_+ \to \mathcal{R}_+$ such that φ_2 is continuous and φ_1 is lower semicontinuous on $\mathcal{R}_+$. Furthermore, $\varphi_1(0) = 0$ and $\varphi_1(s) > 0$ for $s > 0$. Then, F has at least one fixed-point in G.

Proof. Taking $\mathcal{U}_0^1, \mathcal{U}_0^2 = G$, $\mathcal{U}_{n+1}^1 = \overline{\text{conv}(F\mathcal{U}_n^1)}$, $\mathcal{U}_{n+1}^2 = \overline{\text{conv}(F\mathcal{U}_n^2)}$, for $n = 0, 1, 2, \ldots,$ we obtain $\mathcal{U}_{n+1}^1 \subseteq \mathcal{U}_n^1$, $\mathcal{U}_{n+1}^2 \subseteq \mathcal{U}_n^2$ for $n = 0, 1, \ldots$. Therefore, $\{\mathcal{U}_n^1\}_0^{+\infty}, \{\mathcal{U}_n^2\}_0^{+\infty}$ are decreasing sequences of closed and convex sets. Moreover, from (2.1), we have

$$\begin{aligned}\varphi_2(\mu(\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2)) &= \varphi_2(\mu(\overline{\text{conv}(F\mathcal{U}_n^1)}, \overline{\text{conv}(F\mathcal{U}_n^2)}))\\ &= \varphi_2(\mu(F\mathcal{U}_n^1, F\mathcal{U}_n^2))\\ &\leq \varphi_2(\mu(\mathcal{U}_n^1, \mathcal{U}_n^2)) - \varphi_1(\mu(\mathcal{U}_n^1, \mathcal{U}_n^2)),\end{aligned} \tag{2.2}$$

for $n = 0, 1, 2, \ldots$. Since the sequence $\{\mu(\mathcal{U}_n^1, \mathcal{U}_n^2)\}$ is nonnegative and nonincreasing, we deduce that $\mu(\mathcal{U}_n^1, \mathcal{U}_n^2) \to m$ when n tends to infinity, where $m \geq 0$ is a real number. On the other hand, considering equation (2.2), we obtain

$$\limsup_{n\to+\infty} \varphi_2(\mu(\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2)) \leq \limsup_{n\to+\infty} \varphi_2(\mu(\mathcal{U}_n^1, \mathcal{U}_n^2)) - \liminf_{n\to+\infty} \varphi_1(\mu(\mathcal{U}_n^1, \mathcal{U}_n^2)). \tag{2.3}$$

This yields $\varphi_2(m) \leq \varphi_2(m) - \varphi_1(m)$. Consequently, $\varphi_1(m) = 0$ and so $m = 0$. Therefore, we infer $\mu(\mathcal{U}_n^1, \mathcal{U}_n^2) \to 0$ as $n \to +\infty$. Now, considering that $(\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2) \subseteq (\mathcal{U}_n^1, \mathcal{U}_n^2)$, by Definition 2.1 (6), $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2) = \bigcap_{n=0}^{+\infty}(\mathcal{U}_n^1, \mathcal{U}_n^2)$ is nonempty, closed, and convex. Furthermore, the set $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2)$ under the operator F is invariant and $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2) \in \ker\mu$. So, by applying Theorem 1.2, the proof is complete. □

Theorem 2.3. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : G \to G$ a continuous map such that*

$$\mu(F\mathcal{U}^1, F\mathcal{U}^2) \leq \varphi_1(\mu(\mathcal{U}^1, \mathcal{U}^2)), \tag{2.4}$$

for each $\emptyset \neq \mathcal{U}^1 \subseteq G$, $\emptyset \neq \mathcal{U}^2 \subseteq G$, where μ is an arbitrary coupled MNC and $\varphi_1 : \mathcal{R}_+ \to \mathcal{R}_+$ is a nondecreasing function with $\lim_{n\to+\infty}\varphi_1^n(s) = 0$ for every $s \geq 0$. Then, F has at least one fixed-point.

Proof. According to the proof of Theorem 2.2, we define the sequences $\{\mathcal{U}_n^1\}, \{\mathcal{U}_n^2\}$ by induction, where $\mathcal{U}_0^1, \mathcal{U}_0^2 = G$, $\mathcal{U}_{n+1}^1 = \overline{\text{conv}(F\mathcal{U}_n^1)}$, $\mathcal{U}_{n+1}^2 = \overline{\text{conv}(F\mathcal{U}_n^2)}$, for $n = 0, 1, \ldots$. Moreover, in the same as the previous method, we can assume $\mu(\mathcal{U}_n^1, \mathcal{U}_n^2) > 0$ for all $n = 1, 2, \ldots$. In addition, by given assumptions, we obtain

$$\begin{aligned}\mu(\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2) &= \mu(\overline{\text{conv}(F\mathcal{U}_n^1)}, \overline{\text{conv}(F\mathcal{U}_n^2)})\\ &= \mu(F\mathcal{U}_n^1, F\mathcal{U}_n^2)\\ &\leq \varphi_1(\mu(\mathcal{U}_n^1, \mathcal{U}_n^2))\\ &\leq \varphi_1^2(\mu(\mathcal{U}_{n-1}^1, \mathcal{U}_{n-1}^2))\\ &\ \ \vdots\\ &\leq \varphi_1^{n+1}(\mu(\mathcal{U}_0^1, \mathcal{U}_0^2)).\end{aligned} \tag{2.5}$$

This shows that $\mu(\mathcal{U}_n^1, \mathcal{U}_n^2) \to 0$ as $n \to +\infty$. Since the sequence $\{(\mathcal{U}_n^1, \mathcal{U}_n^2)\}$ is nested, by Definition 2.1 (6), $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2) = \bigcap_{n=0}^{+\infty}(\mathcal{U}_n^1, \mathcal{U}_n^2)$ is a nonempty, closed, and convex subset of $(\mathcal{U}^1, \mathcal{U}^2)$. Therefore, we obtain that $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2)$ is a member of $\ker\mu$. So, $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2)$ is compact. Next, note that F maps $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2)$ into itself, and considering Theorem 1.2, we deduce that F has fixed-point in $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2)$. So the proof is complete. □

Now, from the aforementioned theorem, we have the following.

Corollary 2.4. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : G \to G$ be an operator such that*

$$\|(Fv^1, Fv^2) - (Fv'^1, Fv'^2)\| \le \varphi_1(\|v^1v^2 - v'^1v'^2\|), \text{ for all } v^1, v'^1, v^2, v'^2 \in G, \tag{2.6}$$

where $\varphi_1 : \mathcal{R}_+ \to \mathcal{R}_+$ is a nondecreasing function with $\lim_{n\to+\infty}\varphi_1^n(s) = 0$ for any $s \ge 0$. Then, F has a fixed-point in G.

Proof. Let $\mu : \mathcal{B}_E^2 \to \mathcal{R}_+$ and

$$\mu(\mathcal{U}^1, \mathcal{U}^2) := \operatorname{diam}(\mathcal{U}^1, \mathcal{U}^2),$$

where $\operatorname{diam}(\mathcal{U}^1, \mathcal{U}^2) = \sup\{\|v^1v^2 - v'^1v'^2\| : v^1, v'^1 \in \mathcal{U}^1, v^2, v'^2 \in \mathcal{U}^2\}$. It can be easily seen that μ is coupled MNC in E by Definition 2.1. Furthermore, since φ_1 is nondecreasing, then in view of (2.6), we have

$$\begin{aligned}\sup_{v^1,v'^1\in\mathcal{U}^1,v^2,v'^2\in\mathcal{U}^2} \|(Fv^1, Fv^2) - (Fv'^1, Fv'^2)\| &\le \sup_{v^1,v'^1\in\mathcal{U}^1,v^2,v'^2\in\mathcal{U}^2} \varphi_1\|v^1v^2 - v'^1v'^2\| \\ &\le \varphi_1\left(\sup_{v^1,v'^1\in\mathcal{U}^1,v^2,v'^2\in\mathcal{U}^2} \|v^1v^2 - v'^1v'^2\|\right),\end{aligned}$$

which yields that

$$\mu(F\mathcal{U}^1, F\mathcal{U}^2) \le \varphi_1(\mu(\mathcal{U}^1, \mathcal{U}^2)).$$

By using Theorem 2.3, the proof is complete. □

3 Tripled MNC

In this section, as a result of Section 2, we define the notion of tripled MNC as follows.

Definition 3.1. Let E be a Banach space and $\mu : \mathcal{B}_E^3 \to \mathcal{R}_+$ be a mapping. We say that μ is a tripled MNC on E, if it has the following conditions:

(1) $\ker\mu = \{(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) \in \mathcal{B}_E^3 : \mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) = \theta\}$ is nonempty;

(2) For every $(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) \in \mathcal{B}_E^3$, $\mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) = \theta \Leftrightarrow (\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3)$ is a precompact set;

(3) For each $((\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3), (\mathcal{U}'^1, \mathcal{U}'^2, \mathcal{U}'^3)) \in \mathcal{B}_E^3 \times \mathcal{B}_E^3$, ($(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) \subseteq (\mathcal{U}'^1, \mathcal{U}'^2, \mathcal{U}'^3)$ yields $\mathcal{U}^1 \subseteq \mathcal{U}'^1$, $\mathcal{U}^2 \subseteq \mathcal{U}'^2$ and $\mathcal{U}^3 \subseteq \mathcal{U}'^3$), we have

$$(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) \subseteq (\mathcal{U}'^1, \mathcal{U}'^2, \mathcal{U}'^3) \text{ implies } \mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) \le \mu(\mathcal{U}'^1, \mathcal{U}'^2, \mathcal{U}'^3);$$

(4) For every $(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) \in \mathcal{B}_E^3$, one has

$$\mu(\overline{\mathcal{U}^1}, \overline{\mathcal{U}^2}, \overline{\mathcal{U}^3}) = \mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) = \mu(\operatorname{conv}(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3)),$$

where $\operatorname{conv}(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3)$ denotes the convex hull of $(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3)$;

(5) $\mu(\lambda(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) + (1-\lambda)(\mathcal{U}'^1, \mathcal{U}'^2, \mathcal{U}'^3)) \le \lambda\mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3) + (1-\lambda)\mu(\mathcal{U}'^1, \mathcal{U}'^2, \mathcal{U}'^3)$ for $\lambda \in [0, 1]$;

(6) If $\{\mathcal{U}_n^1\}_0^{+\infty}, \{\mathcal{U}_n^2\}_0^{+\infty}$ and $\{\mathcal{U}_n^3\}_0^{+\infty}$ in $\mathcal{B}_E$ are decreasing sequences of closed sets and $\lim_{n\to+\infty} \mu\{(\mathcal{U}_n^1, \mathcal{U}_n^2, \mathcal{U}_n^3)\}_0^{+\infty} = 0$, then $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2, \mathcal{U}_{+\infty}^3) = \bigcap_{n=0}^{+\infty}(\mathcal{U}_n^1, \mathcal{U}_n^2, \mathcal{U}_n^3) \ne \varnothing$.

Theorem 3.2. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : G \to G$ be a continuous map such that*

$$\varphi_2(\mu(F\mathcal{U}^1, F\mathcal{U}^2, F\mathcal{U}^3)) \le \varphi_2(\mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3)) - \varphi_1(\mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3)), \tag{3.1}$$

for each $\emptyset \neq \mathcal{U}^1 \subseteq G$, $\emptyset \neq \mathcal{U}^2 \subseteq G$, *and* $\emptyset \neq \mathcal{U}^3 \subseteq G$, *where* μ *is an arbitrary tripled MNC and* $\varphi_1, \varphi_2 : \mathcal{R}_+ \to \mathcal{R}_+$ *such that* φ_2 *is continuous and* φ_1 *is lower semicontinuous on* $\mathcal{R}_+$. *Furthermore,* $\varphi_1(0) = 0$ *and* $\varphi_1(s) > 0$ for $s > 0$. *Then, F has at least one fixed-point in G.*

Theorem 3.3. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and* $F : G \to G$ *be a continuous map such that*

$$\mu(F\mathcal{U}^1, F\mathcal{U}^2, F\mathcal{U}^3) \leq \varphi_1(\mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3)), \tag{3.2}$$

for each $\emptyset \neq \mathcal{U}^1 \subseteq G$, $\emptyset \neq \mathcal{U}^2 \subseteq G$, *and* $\emptyset \neq \mathcal{U}^3 \subseteq G$, *where* μ *is an arbitrary tripled MNC and* $\varphi_1 : \mathcal{R}_+ \to \mathcal{R}_+$ *is a nondecreasing function with* $\lim_{n\to+\infty} \varphi_1^n(s) = 0$ *for every* $s \geq 0$. *Then, F has at least one fixed-point.*

Corollary 3.4. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and* $F : G \to G$ *be an operator such that*

$$\|(F\upsilon^1, F\upsilon^2, F\upsilon^3) - (F\upsilon'^1, F\upsilon'^2, F\upsilon'^3)\| \leq \varphi_1(\|\upsilon^1\upsilon^2\upsilon^3 - \upsilon'^1\upsilon'^2\upsilon'^3\|), \tag{3.3}$$

for all $\upsilon^1, \upsilon'^1, \upsilon^2, \upsilon'^2, \upsilon^3, \upsilon'^3 \in G$, *where* $\varphi_1 : \mathcal{R}_+ \to \mathcal{R}_+$ *is a nondecreasing function with* $\lim_{n\to+\infty} \varphi_1^n(s) = 0$ *for any* $s \geq 0$. *Then, F has a fixed-point in G.*

4 MNC and JS-contraction

In this section, we tend to prove some results of MNC for the family of JS-contractive-type mappings. Also, we generalize Darbo's fixed-point theorem to coupled and tripled MNC through JS-contraction-type mappings.

Denote by Θ the set of all functions $\theta : (0, +\infty) \to (1, +\infty)$ so that:

(θ_1) θ is continuous and increasing;

(θ_2) $\lim_{n\to+\infty} t_n = 0$ iff $\lim_{n\to+\infty} \theta(t_n) = 1$ for all $\{t_n\} \subseteq (0, +\infty)$.

Theorem 4.1. [13] *Let* (G, ϱ) *be a complete metric space and* $F : G \to G$ *be a given mapping. Suppose that there exist* $\theta \in \Theta$ *and* $\nu \in (0, 1)$ *such that for all* $\iota, \varsigma \in G$,

$$\varrho(F\iota, F\varsigma) \neq 0 \Rightarrow \theta(\varrho(F\iota, F\varsigma)) \leq (\theta(\varrho(\iota, \varsigma)))^{\nu}. \tag{4.1}$$

Then, F has a unique fixed-point.

Theorem 4.2. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and* $F : G \to G$, *be a continuous map such that*

$$\theta(\varphi_2(\mu(F\mathcal{U}^1, F\mathcal{U}^2))) \leq \frac{\theta(\varphi_2(\mu(\mathcal{U}^1, \mathcal{U}^2)))}{\theta(\varphi_2(\varphi_1(\mu(\mathcal{U}^1, \mathcal{U}^2))))}, \tag{4.2}$$

for each $\emptyset \neq \mathcal{U}^1 \subseteq G$, $\emptyset \neq \mathcal{U}^2 \subseteq G$, *where* $\theta \in \Theta$, *and* μ *is an arbitrary coupled MNC and functions* $\varphi_1, \varphi_2 : \mathcal{R}_+ \to \mathcal{R}_+$, *such that* φ_2 *is continuous and* φ_1 *is lower semicontinuous on* $\mathcal{R}_+$. *Furthermore,* $\varphi_1(0) = 0$ *and* $\varphi_1(s) > 0$ *for* $s > 0$. *Then, F has at least one fixed-point in G.*

Proof. According to the proof of Theorem 2.2, we define the sequences $\{\mathcal{U}_n^1\}, \{\mathcal{U}_n^2\}$ by induction. Moreover, from (4.2), we obtain

$$\begin{aligned}\theta(\varphi_2(\mu(\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2))) &= \theta(\varphi_2(\mu(\overline{\text{conv}(F\mathcal{U}_n^1)}, \overline{\text{conv}(F\mathcal{U}_n^2)}))) \\ &= \theta(\varphi_2(\mu(F\mathcal{U}_n^1, F\mathcal{U}_n^2))) \\ &\leq \frac{\theta(\varphi_2(\mu(\mathcal{U}_n^1, \mathcal{U}_n^2)))}{\theta(\varphi_2(\varphi_1(\mu(\mathcal{U}_n^1, \mathcal{U}_n^2))))},\end{aligned} \tag{4.3}$$

for $n = 0, 1, 2, \ldots$. Since the sequence $\{\mu(\mathcal{U}_n^1, \mathcal{U}_n^2)\}$ is nonnegative and nonincreasing, we deduce that $\mu(\mathcal{U}_n^1, \mathcal{U}_n^2) \to m$ when n tends to infinity, where $m \geq 0$ is a real number. On the other hand, considering equation (4.3), we obtain

$$\limsup_{n\to+\infty} \theta(\varphi_2(\mu(\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2))) \leq \limsup_{n\to+\infty} \frac{\theta(\varphi_2(\mu(\mathcal{U}_n^1, \mathcal{U}_n^2)))}{\theta(\varphi_2(\varphi_1(\mu(\mathcal{U}_n^1, \mathcal{U}_n^2))))}, \tag{4.4}$$

which yields that $\theta(\varphi_2(m)) \leq \frac{\theta(\varphi_2(m))}{\theta(\varphi_2(\varphi_1(m)))}$. Consequently, $\theta(\varphi_2(\varphi_1(m))) = 1$, then $\varphi_2(\varphi_1(m)) = 0$ and $\varphi_1(m) = 0$ so $m = 0$. Therefore, we infer $\mu(\mathcal{U}_n^1, \mathcal{U}_n^2) \to 0$ as $n \to +\infty$. Now, considering that $(\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2) \subseteq (\mathcal{U}_n^1, \mathcal{U}_n^2)$, by Definition 2.1 (6), $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2) = \bigcap_{n=0}^{+\infty}(\mathcal{U}_n^1, \mathcal{U}_n^2)$ is nonempty, closed, and convex. Furthermore, the set $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2)$ under the operator F is invariant and $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2) \in ker\mu$. So, by applying Theorem 1.2, the proof is complete. □

Denote by Ψ the set of all functions $\varphi_2 : (1, +\infty) \to (1, +\infty)$ so that:

(φ_{2_1}) φ_2 is continuous and increasing;

(φ_{2_2}) $\lim_{n\to+\infty} \varphi_2^n(s) = 1$ for all $s \in (1, +\infty)$.

Theorem 4.3. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : G \to G$ a continuous map such that*

$$\theta(\mu(F\mathcal{U}^1, F\mathcal{U}^2)) \leq \varphi_2(\theta(\mu(\mathcal{U}^1, \mathcal{U}^2))), \tag{4.5}$$

for each $\emptyset \neq \mathcal{U}^1 \subseteq G$, $\emptyset \neq \mathcal{U}^2 \subseteq G$, where $\theta \in \Theta$, $\varphi_2 \in \Psi$, and μ is an arbitrary coupled MNC. Then, F has at least one fixed-point.

Proof. According to the proof of Theorem 2.2, we define the sequences $\{\mathcal{U}_n^1\}$, $\{\mathcal{U}_n^2\}$ by induction.

If for an integer $N \in \mathbb{N}$ one has $\mu(\mathcal{U}_N^1, \mathcal{U}_N^2) = 0$, then $(\mathcal{U}_N^1, \mathcal{U}_N^2)$ is a precompact set. So the Schauder theorem ensures the existence of a fixed-point for F. Therefore, we can assume $\mu(\mathcal{U}_n^1, \mathcal{U}_n^2) > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Obviously, $\{(\mathcal{U}_n^1, \mathcal{U}_n^2)\}_{n\in\mathbb{N}}$ is a sequence of nonempty, bounded, closed, and convex subsets such that

$$(\mathcal{U}_0^1, \mathcal{U}_0^2) \supseteq (\mathcal{U}_1^1, \mathcal{U}_1^2) \supseteq \cdots \supseteq (\mathcal{U}_n^1, \mathcal{U}_n^2) \supseteq (\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2).$$

On the other hand,

$$\begin{aligned}\theta(\mu(\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2)) &= \theta(\mu(F\mathcal{U}_n^1, F\mathcal{U}_n^2)) \\ &\leq \varphi_2(\theta(\mu(\mathcal{U}_n^1, \mathcal{U}_n^2))) \\ &\leq \\ &\vdots \\ &\leq \varphi_2^{n+1}(\theta(\mu(\mathcal{U}_0^1, \mathcal{U}_0^2))).\end{aligned} \tag{4.6}$$

Thus, $\{\mu(\mathcal{U}_n^1, \mathcal{U}_n^2)\}_{n\in\mathbb{N}}$ is a convergent sequence. Assume that

$$\lim_{n\to+\infty} \mu(\mathcal{U}_n^1, \mathcal{U}_n^2) = \eta.$$

By taking the limit from (4.6), $\lim_{n\to+\infty}\theta(\mu(\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2)) = 1$. By (θ_2), we obtain

$$\lim_{n\to+\infty} \mu(\mathcal{U}_{n+1}^1, \mathcal{U}_{n+1}^2) = 0.$$

From Definition 2.1 (6), $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2) = \bigcap_{n=0}^{+\infty}(\mathcal{U}_n^1, \mathcal{U}_n^2)$ is a nonempty, closed, and convex subset of $(\mathcal{U}^1, \mathcal{U}^2)$. Therefore, we obtain $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2)$ is a member of kerμ. So, $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2)$ is compact. Note that F maps $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2)$ into itself, and considering Theorem 1.2, we deduce that F has a fixed-point in $(\mathcal{U}_{+\infty}^1, \mathcal{U}_{+\infty}^2)$. So the proof is complete. □

Theorem 4.4. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : G \to G$ be a continuous map such that*

$$\theta(\varphi_2(\mu(F\mathcal{U}^1, F\mathcal{U}^2, F\mathcal{U}^3))) \leq \frac{\theta(\varphi_2(\mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3)))}{\theta(\varphi_2(\varphi_1(\mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3))))}, \tag{4.7}$$

for each $\emptyset \neq \mathcal{U}^1 \subseteq G$, $\emptyset \neq \mathcal{U}^2 \subseteq G$, *and* $\emptyset \neq \mathcal{U}^3 \subseteq G$, where $\theta \in \Theta$ and μ is an arbitrary tripled MNC and functions $\varphi_1, \varphi_2 : \mathcal{R}_+ \to \mathcal{R}_+$, such that φ_2 is continuous and φ_1 is lower semicontinuous on $\mathcal{R}_+$. Furthermore, $\varphi_1(0) = 0$ and $\varphi_1(s) > 0$ for $s > 0$. *Then, F has at least one fixed-point in G.*

Theorem 4.5. *Let G be a nonempty, bounded, closed, and convex subset of a Banach space E and $F : G \to G$ be a continuous map such that*

$$\theta(\mu(F\mathcal{U}^1, F\mathcal{U}^2, F\mathcal{U}^3)) \leq \varphi_2(\theta(\mu(\mathcal{U}^1, \mathcal{U}^2, \mathcal{U}^3))), \tag{4.8}$$

for each $\emptyset \neq \mathcal{U}^1 \subseteq G$, $\emptyset \neq \mathcal{U}^2 \subseteq G$, *and* $\emptyset \neq \mathcal{U}^3 \subseteq G$, *where* $\theta \in \Theta$, $\varphi_2 \Psi \in$, and μ *is an arbitrary tripled MNC. Then, F has at least one fixed-point.*

5 Application

We offer the applications of Theorem 2.3 to prove the existence of solutions of a functional integral equation in the Banach space $BC(\mathcal{R}_+)$ consisting of all real functions that are bounded and continuous on $\mathcal{R}_+$. This space is endowed with the supremum norm

$$\|v^1\| = \sup\{\|v^1(r)\| : r \in \mathcal{R}_+\}.$$

We choose nonempty bounded subsets $\mathcal{U}^1, \mathcal{U}^2$ of $BC(\mathcal{R}_+)$ and a number $L > 0$. For $\varepsilon > 0$, $v^1 \in \mathcal{U}^1$, and $v^2 \in \mathcal{U}^2$, we show the modulus of continuity of function (v^1, v^2) on the interval $[0, L]$ by $\omega^L((v^1, v^2), \varepsilon)$:

$$\omega^L((v^1, v^2), \varepsilon) = \sup\{|v^1v^2(r) - v^1v^2(s)| : r, s \in [0, L], |r - s| \leq \varepsilon\}.$$

Also,

$$\begin{aligned} \omega^L((\mathcal{U}^1, \mathcal{U}^2), \varepsilon) &= \sup\{\omega^L((v^1, v^2), \varepsilon) : v^1 \in \mathcal{U}^1, v^2 \in \mathcal{U}^2\}, \\ \omega_0^L(\mathcal{U}^1, \mathcal{U}^2) &= \lim_{\varepsilon \to 0} \omega^L((\mathcal{U}^1, \mathcal{U}^2), \varepsilon), \\ \omega_0(\mathcal{U}^1, \mathcal{U}^2) &= \lim_{L \to +\infty} \omega_0^L(\mathcal{U}^1, \mathcal{U}^2). \end{aligned}$$

Moreover, for $r \in \mathcal{R}_+$, we define

$$\begin{aligned} (\mathcal{U}^1, \mathcal{U}^2)(r) &= \{(v^1, v^2)(r) : v^1 \in \mathcal{U}^1, v^2 \in \mathcal{U}^2\}, \\ \mu(\mathcal{U}^1, \mathcal{U}^2) &= \omega_0(\mathcal{U}^1, \mathcal{U}^2) + \limsup_{r \to +\infty} \operatorname{diam}(\mathcal{U}^1, \mathcal{U}^2)(r), \\ \operatorname{diam}(\mathcal{U}^1, \mathcal{U}^2)(r) &= \sup\{|v^1v^2(r) - v'^1v'^2(r)| : v^1, v'^1 \in \mathcal{U}^1, v^2, v'^2 \in \mathcal{U}^2\}. \end{aligned}$$

Now, we consider the following hypotheses:

(i) Let $f_1 : \mathcal{R}_+ \times \mathcal{R}^2 \to \mathcal{R}$ be a continuous function. Furthermore, the function $r \to f_1(r, 0, 0)$ is a member of $BC(\mathcal{R}_+)$.

(ii) There is an upper semicontinuous function $\varphi_1 \in \phi$, where ϕ is family of all functions $\varphi_1 : \mathcal{R}_+ \to \mathcal{R}_+$, which φ_1 is a nondecreasing function such that $\lim_{n \to +\infty} \varphi_1^n(s) = 0$ for each $s \geq 0$, such that

$$|f_1(r, v^1, v^2) - f_1(r, v'^1, v'^2)| \leq \varphi_1(|v^1v^2 - v'^1v'^2|), \quad r \in \mathcal{R}_+ \quad v^1, v'^1, v^2, v'^2 \in \mathcal{R}.$$

In addition, we presume that $\varphi_1(s) + \varphi_1(s') \leq \varphi_1(s + s')$ for all $s, s' \in \mathcal{R}_+$.

(iii) Let $f_2 : \mathcal{R}_+^2 \times \mathcal{R}^2 \to \mathcal{R}$ be a continuous function, and there are continuous functions $u, v : \mathcal{R}_+ \to \mathcal{R}_+$ such that

$$\lim_{r\to+\infty} u(r)\int_0^r v(s)\mathrm{d}s = 0$$

and

$$|f_2(r, s, \upsilon^1, \upsilon^2)| \leq u(r)v(s)$$

for $r, s \in \mathcal{R}_+$ such that $s \leq r$ and for every $\upsilon^1, \upsilon^2 \in \mathcal{R}$.

(iv) There is a positive solution r_0^1 from

$$\varphi_1(r^1) + p \leq r^1,$$

where

$$p = \sup\{|f_1(r, 0, 0) + u(r)\int_0^r v(s)\mathrm{d}s : r \geq 0\}.$$

Now, we consider the integral equation

$$(\upsilon^1, \upsilon^2)(r) = f_1(r, \upsilon^1(r), \upsilon^2(r)) + \int_0^r f_2(r, s, \upsilon^1(s), \upsilon^2(s))\mathrm{d}s. \tag{5.1}$$

We define operator F on $\mathrm{BC}(\mathcal{R}_+)$ by

$$(F\upsilon^1, F\upsilon^2)(r) = f_1(r, \upsilon^1(r), \upsilon^2(r)) + \int_0^r f_2(r, s, \upsilon^1(s), \upsilon^2(s))\mathrm{d}s, \quad \text{for } r \in \mathcal{R}_+, \tag{5.2}$$

where the function $(F\upsilon^1, F\upsilon^2)$ is continuous on $\mathcal{R}_+$.

Theorem 5.1. *According to hypotheses* (*i*)–(*iv*), *relation* (5.1) *has at least one solution in* $\mathrm{BC}(\mathcal{R}_+)$.

Proof. For arbitrary $\upsilon^1, \upsilon^2 \in BC(\mathcal{R}_+)$, using the aforementioned hypotheses, we obtain

$$\begin{aligned}|(F\upsilon^1, F\upsilon^2)(r)| &\leq |f_1(r, \upsilon^1(r), \upsilon^2(r)) - f_1(r, 0, 0)| + |f_1(r, 0, 0)| + \int_0^r |f_2(r, s, \upsilon^1(s), \upsilon^2(s))|\mathrm{d}s\\ &\leq \varphi_1(|\upsilon^1\upsilon^2(r)|) + |f_1(r, 0, 0)| + u(r)\int_0^r v(s)\mathrm{d}s\\ &= \varphi_1(|\upsilon^1\upsilon^2(r)|) + |f_1(r, 0, 0)| + a(r),\end{aligned}$$

where

$$a(r) := u(r)\int_0^r v(s)\mathrm{d}s.$$

Since φ_1 is nondecreasing, in accordance with the fourth condition, we obtain

$$\|(F\upsilon^1, F\upsilon^2)\| \leq \varphi_1(\|\upsilon^1\upsilon^2\|) + p.$$

Thus, F is a self-mapping of $BC(\mathcal{R}_+)$. On the other hand, applying assumption (iv), we deduce that F is a self-mapping of the ball $B_{r_0^1}$. To show that F is continuous on $B_{r_0^1}$, take $\varepsilon > 0$ and $\upsilon^1, \upsilon'^1, \upsilon^2, \upsilon'^2 \in B_{r_0^1}$ such that $\|\upsilon^1\upsilon^2 - \upsilon'^1\upsilon'^2\| < \varepsilon$, we obtain

$$\begin{aligned}
&|(Fv^1, Fv^2)(r) - (Fv'^1, Fv'^2)(r)| \\
&\quad \le \varphi_1(|v^1v^2(r) - v'^1v'^2(r)|) + \int_0^r |f_2(r, s, v^1(s), v^2(s)) - f_2(r, s, v'^1(s), v'^2(s))| \mathrm{d}s \\
&\quad \le \varphi_1(|v^1v^2(r) - v'^1v'^2(r)|) + \int_0^r |f_2(r, s, v^1(s), v^2(s))| \mathrm{d}s + \int_0^r |f_2(r, s, v'^1(s), v'^2(s))| \mathrm{d}s \\
&\quad \le \varphi_1(\varepsilon) + 2a(r),
\end{aligned} \tag{5.3}$$

for any $r \in \mathcal{R}_+$. By assumption (iii), there is a number $L > 0$ such that

$$2u(r)\int_0^r v(s)\mathrm{d}s \le \varepsilon, \quad \text{for every } L \le r. \tag{5.4}$$

So, considering Lemma 1.6 and the similar evaluation mentioned earlier, for an arbitrary $L \le r$, we have

$$|(Fv^1, Fv^2)(r) - (Fv'^1, Fv'^2)(r)| \le 2\varepsilon. \tag{5.5}$$

Now, we define

$$\omega^L(f_2, \varepsilon) := \sup\{|f_2(r, s, v^1, v^2) - f_2(r, s, v'^1, v'^2)| : r, s \in [0, L], v^1, v'^1, v^2, v'^2 \in [-r_0^1, r_0^1], |v^1v^2 - v'^1v'^2| \le \varepsilon\}.$$

Due to the uniform continuity of $f_2(r, s, v^1, v^2)$ on $[0, L]^2 \times [-r_0^1, r_0^1]^2$, we infer that $\omega^L(f_2, \varepsilon) \to 0$ as $\varepsilon \to 0$. Now, considering the first part of equation (5.3), for the arbitrary constant $r \in [0, L]$, we obtain

$$|(Fv^1, Fv^2)(r) - (Fv'^1, Fv'^2)(r)| \le \varphi_1(\varepsilon) + \int_0^L \omega^L(f_2, \varepsilon)\mathrm{d}s = \varphi_1(\varepsilon) + L\omega^L(f_2, \varepsilon). \tag{5.6}$$

By combining (5.5) and (5.6) and based on the aforementioned fact about $\omega^L(f_2, \varepsilon)$, the operator F on the ball $B_{r_0^1}$, is continuous. Next, we can choose arbitrary nonempty subsets $\mathcal{U}^1, \mathcal{U}^2$ of the ball $B_{r_0^1}$. To do this, we consider constant numbers $L > 0$ and $\varepsilon > 0$. Also, take arbitrary numbers $r, r' \in [0, L]$ with $|r - r'| \le \varepsilon$. Without loss of generality, it can be assumed that $r' < r$. So, for $v^1 \in \mathcal{U}^1$ and $v^2 \in \mathcal{U}^2$, we obtain

$$\begin{aligned}
&|(Fv^1, Fv^2)(r) - (Fv^1, Fv^2)(r')| \\
&\quad \le |f_1(r, v^1(r), v^2(r)) - f_1(r', v^1(r'), v^2(r'))| + \left|\int_0^r f_2(r, s, v^1(s), v^2(s))\mathrm{d}s - \int_0^{r'} f_2(r', s, v^1(s), v^2(s))\mathrm{d}s\right| \\
&\quad \le |f_1(r, v^1(r), v^2(r)) - f_1(r', v^1(r), v^2(r))| + |f_1(r', v^1(r), v^2(r)) - f_1(r', v^1(r'), v^2(r'))| \\
&\qquad + \left|\int_0^r f_2(r, s, v^1(s), v^2(s))\mathrm{d}s - \int_0^r f_2(r', s, v^1(s), v^2(s))\mathrm{d}s\right| + \left|\int_0^r f_2(r', s, v^1(s), v^2(s))\mathrm{d}s \right. \\
&\qquad \left. - \int_0^{r'} f_2(r', s, v^1(s), v^2(s))\mathrm{d}s\right| \\
&\quad \le \omega_1^L(f_1, \varepsilon) + \varphi_1(|v^1v^2(r) - v^1v^2(r')|) + \int_0^r |f_2(r, s, v^1(s), v^2(s))\mathrm{d}s - f_2(r', s, v^1(s), v^2(s))|\mathrm{d}s \\
&\qquad + \int_{r'}^r |f_2(r', s, v^1(s), v^2(s))|\mathrm{d}s \\
&\quad \le \omega_1^L(f_1, \varepsilon) + \varphi_1(\omega^L((v^1, v^2), \varepsilon)) + \int_0^r \omega_1^L(f_2, \varepsilon)\mathrm{d}s + u(r')\int_{r'}^r v(s)\mathrm{d}s \\
&\quad \le \omega_1^L(f_1, \varepsilon) + \varphi_1(\omega^L((v^1, v^2), \varepsilon)) + L\omega_1^L(f_2, \varepsilon) + \varepsilon \sup\{u(r')v(r) : r, r' \in [0, L]\},
\end{aligned} \tag{5.7}$$

where

$$\omega_1^L(f_1,\varepsilon) := \sup\{|f_1(r,\upsilon^1,\upsilon^2) - f_1(r',\upsilon^1,\upsilon^2)| : \quad r,r' \in [0,L], \upsilon^1,\upsilon^2 \in [-r_0^1, r_0^1], |r-r'| \le \varepsilon\},$$
$$\omega_1^L(f_2,\varepsilon) := \sup\{|f_2(r,s,\upsilon^1,\upsilon^2) - f_2(r',s,\upsilon^1,\upsilon^2)| : \quad r,r',s \in [0,L], \upsilon^1,\upsilon^2 \in [-r_0^1, r_0^1], |r-r'| \le \varepsilon\}.$$

In addition, due to the uniform continuity of f_1 on $[0,L]\times[-r_0^1,r_0^1]^2$ and f_2 on $[0,L]^2\times[-r_0^1,r_0^1]^2$, we deduce that $\omega_1^L(f_1,\varepsilon)\to 0$ and $\omega_1^L(f_2,\varepsilon)\to 0$ as $\varepsilon\to 0$. Also, since $u=u(r)$ and $v=v(r)$ are continuous on $\mathcal{R}_+$, we have

$$\sup\{u(r')v(r) : r,r' \in [0,L]\} < +\infty.$$

Therefore, from (5.7), we obtain

$$\omega_0^L(F\mathcal{U}^1, F\mathcal{U}^2) \le \lim_{\varepsilon\to 0}\varphi_1(\omega^L((\mathcal{U}^1,\mathcal{U}^2),\varepsilon)).$$

As a result, given the upper semicontinuity of the function φ_1, we obtain

$$\omega_0^L(F\mathcal{U}^1, F\mathcal{U}^2) \le \varphi_1(\omega_0^L(\mathcal{U}^1,\mathcal{U}^2)),$$

and eventually

$$\omega_0(F\mathcal{U}^1, F\mathcal{U}^2) \le \varphi_1(\omega_0(\mathcal{U}^1,\mathcal{U}^2)). \tag{5.8}$$

Now, take arbitrary functions $\upsilon^1,\upsilon'^1 \in \mathcal{U}^1$ and $\upsilon^2,\upsilon'^2 \in \mathcal{U}^2$. Then, for $r\in\mathcal{R}$, we obtain

$$\begin{aligned}
&|(F\upsilon^1,F\upsilon^2)(r) - (F\upsilon'^1,F\upsilon'^2)(r)|\\
&\quad \le |f_1(r,\upsilon^1(r),\upsilon^2(r)) - f_1(r,\upsilon'^1(r),\upsilon'^2(r))| + \int_0^r |f_2(r,s,\upsilon^1(s),\upsilon^2(s))|\mathrm{d}s + \int_0^r |f_2(r,s,\upsilon'^1(s),\upsilon'^2(s))|\mathrm{d}s\\
&\quad \le \varphi_1(|\upsilon^1\upsilon^2(r) - \upsilon'^1\upsilon'^2(r)|) + 2u(r)\int_0^r v(s)\mathrm{d}s\\
&\quad = \varphi_1(|\upsilon^1\upsilon^2(r) - \upsilon'^1\upsilon'^2(r)|) + 2a(r).
\end{aligned}$$

Hence, from the aforementioned inequality, we have

$$\operatorname{diam}(F\mathcal{U}^1, F\mathcal{U}^2)(r) \le \varphi_1(\operatorname{diam}(\mathcal{U}^1,\mathcal{U}^2)(r)) + 2a(r).$$

As a result, given the upper semicontinuity of φ_1, we obtain

$$\limsup_{r\to+\infty}\operatorname{diam}(F\mathcal{U}^1, F\mathcal{U}^2)(r) \le \varphi_1(\limsup_{r\to+\infty}\operatorname{diam}(\mathcal{U}^1,\mathcal{U}^2)(r)). \tag{5.9}$$

By combining (5.8) and (5.9) and considering the superadditivity of φ_1, we obtain

$$\omega_0(F\mathcal{U}^1, F\mathcal{U}^2) + \limsup_{r\to+\infty}\operatorname{diam}(F\mathcal{U}^1, F\mathcal{U}^2)(r) \le \varphi_1(\omega_0(\mathcal{U}^1,\mathcal{U}^2) + \limsup_{r\to+\infty}\operatorname{diam}(\mathcal{U}^1,\mathcal{U}^2)(r)),$$

or equivalently,

$$\mu(F\mathcal{U}^1, F\mathcal{U}^2) \le \varphi_1(\mu(\mathcal{U}^1,\mathcal{U}^2)), \tag{5.10}$$

where μ is coupled MNC in $\mathrm{BC}(\mathcal{R}_+)$. So, from (5.10) and using Theorem 2.3, the result is obtained. □

Example 5.2. Let us define the functional integral equation as follows, which is a special mode of equation (5.1),

$$(\upsilon^1,\upsilon^2)(r) := \frac{r}{r+1}\ln(1+|\upsilon^1\upsilon^2(r)|) + \int_0^r \frac{e^{s-1-r}\cos\upsilon^1\upsilon^2(s)}{1+|\sin\upsilon^1\upsilon^2(s)|}\mathrm{d}s, \;\; (\text{for } r\in\mathcal{R}_+). \tag{5.11}$$

Here,

$$f_1(r, \upsilon^1, \upsilon^2) = \frac{r}{r+1}\ln(1 + |\upsilon^1\upsilon^2|),$$

$$f_2(r, s, \upsilon^1, \upsilon^2) = \frac{e^{s-1-r}\cos\upsilon^1\upsilon^2}{1 + |\sin\upsilon^1\upsilon^2|}.$$

In fact, if we take $\varphi_1(s) = ln(1+s)$, we see that $\varphi_1(s) < s$ for $s > 0$. Evidently, φ_1 is concave and increasing on $\mathcal{R}_+$. Moreover, for $\upsilon^1, \upsilon'^1, \upsilon^2, \upsilon'^2 \in \mathcal{R}$ with $|\upsilon^1\upsilon^2| \geq |\upsilon'^1\upsilon'^2|$ and for $r > 0$, we obtain

$$\begin{aligned}|f_1(r, \upsilon^1, \upsilon^2) - f_1(r, \upsilon'^1, \upsilon'^2)| &= \frac{r}{r+1}\ln\frac{1 + |\upsilon^1\upsilon^2|}{1 + |\upsilon'^1\upsilon'^2|}\\ &\leq \ln\left(1 + \frac{|\upsilon^1\upsilon^2| - |\upsilon'^1\upsilon'^2|}{1 + |\upsilon'^1\upsilon'^2|}\right)\\ &< \ln(1 + (|\upsilon^1\upsilon^2 - \upsilon'^1\upsilon'^2|))\\ &= \varphi_1(|\upsilon^1\upsilon^2 - \upsilon'^1\upsilon'^2|).\end{aligned}$$

In the case $|\upsilon'^1\upsilon'^2| \geq |\upsilon^1\upsilon^2|$, the same can be done. Therefore, we conclude that the function f_1 gives hypothesis (ii) and also (i). In addition, note that the function f_2 operates continuously from $\mathcal{R}_+^2 \times \mathcal{R}^2 \to \mathcal{R}$. Furthermore,

$$|f_2(r, s, \upsilon^1, \upsilon^2)| \leq e^{s-1-r}, \quad r, s \in \mathcal{R}_+, \ \upsilon^1, \upsilon^2 \in \mathcal{R}.$$

Then, if $u(r) \coloneqq e^{-1-r}$, $v(s) \coloneqq e^s$, we see that hypothesis (iii) holds. In fact,

$$\lim_{r\to+\infty} u(r)\int_0^r v(s)\mathrm{d}s = \lim_{r\to+\infty} e^{-1-r}\int_0^r e^s\mathrm{d}s = 0.$$

Now, we calculate p according to assumption (iv). Then,

$$p = \sup\{|f_1(r, 0, 0)| + u(r)\int_0^r v(s)\mathrm{d}s : r \geq 0\} = \sup\{e^{-1} : r \geq 0\} = e^{-1}.$$

In addition, we consider the hypothesis inequality (iv), we have

$$\ln(1 + r^1) + p \leq r^1.$$

It can be easily seen that every $r^1 \geq 1$ holds in the aforementioned inequality. Thus, as a number r_0^1, we can catch $r_0^1 = 1$. Therefore, we conclude that according to Theorem 5.1, equation (5.11) has at least one solution that is on the ball $B_{r_0^1} = B_1$, in $\mathrm{BC}(\mathcal{R}_+)$.

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