Research Article

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Well posedness of magnetohydrodynamic equations in 3D mixed-norm Lebesgue space

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Abstract: In this paper, we introduce a new metric space called the mixed-norm Lebesgue space, which allows its norm decay to zero with different rates as $|x| \to \infty$ in different spatial directions. Then we study the well posedness for the system of magnetohydrodynamic equations in 3D mixed-norm Lebesgue spaces. By using some fundamental analysis theories in mixed-norm Lebesgue space such as Young's inequality, time decaying of solutions for heat equations, and the boundedness of the Helmholtz-Leray projection, we prove local well posedness and global well posedness of the solutions.

Keywords: magnetohydrodynamic equations, local well posedness, global well posedness

MSC 2020: Primary 35D30, Secondary 35Q30, 76W05, 76D06

1 Introduction

Unlike a usual Lebesgue space [1–3], the mixed-norm Lebesgue space allows its norm decay to zero with different rates as $|x| \to \infty$ in different spatial directions [4]. On the basis of the mixed-norm Lebesgue space feature, we investigate the following magnetohydrodynamic (MHD) equations in $\mathbb{R}^3$ [1]:

$$\begin{cases} \partial_t u - \dfrac{1}{\mathrm{Re}}\Delta u + u\cdot\nabla u - S(\nabla\times b)\times b + \nabla\tilde{p} = 0, & \text{in } \mathbb{R}^3\times(0,+\infty),\\ \partial_t b - \nabla\times(u\times b) + \dfrac{1}{\mathrm{Rm}}\nabla\times(\nabla\times b) = 0, & \text{in } \mathbb{R}^3\times(0,+\infty),\\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where u, b, u_0, and b_0 satisfy

$$\operatorname{div} u(x,t) = \operatorname{div} b(x,t) = 0, \quad \operatorname{div} u_0 = \operatorname{div} b_0 = 0, \tag{1.2}$$

$S = \frac{M^2}{\mathrm{ReRm}}$, $\mathrm{Re} > 0$ is the Reynolds number, $\mathrm{Rm} > 0$ is the magnetic Reynolds number, M is the Hartman number. $u : \mathbb{R}^3\times(0,+\infty) \to \mathbb{R}^3$ denote the velocity of the fluid, $b : \mathbb{R}^3\times(0,+\infty)\to\mathbb{R}^3$ denote the magnetic field, and $\tilde{p} = \tilde{p}(x,t) \in \mathbb{R}$ denote the pressure.

The main purpose of this paper is to study the well posedness of the solution for the equations (1.1) in 3D mixed-norm Lebesgue spaces. First, we review some of the relevant work of the MHD equations. If $b = 0$, then the equations (1.1) can be reduced to the incompressible Navier-Stokes equations:

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$$\begin{cases} \partial_t u - \dfrac{1}{\mathrm{Re}}\Delta u + u \cdot \nabla u + \nabla \tilde{p} = 0, & \text{in } \mathbb{R}^3 \times (0, +\infty), \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}^3. \end{cases}$$

The mathematical theory of the Navier-Stokes equation has been much studied in recent years. For example, Leray first introduced the weak solution [5], and later, Hopf gained the existence of global weak solutions with $u_0 \in H^s(\mathbb{R}^N)$ [6], and Fujita and Kato demonstrated that the well posedness of the Cauchy problem when $u_0 \in H^s(\mathbb{R}^N)(N \geq 2)$ [7]. In addition, there are many monographs that study the Navier-Stokes equation, for example, Temam [8], Lions [9], and Constantin and Foias [10]. The mild and self-similar solutions in $\mathbb{R}^3$ are obtained in references [11,12]. Especially, Jia and Šverák derived that the homogeneous classical Cauchy problem with initial value has a global scale invariant solution, and the solution is smooth in positive time [13].

For the MHD system, the coupling between u and b makes the situation more complicated. Because it describes abundant natural phenomena, as well as physical importance and mathematical challenges, the MHD system has become the subject of the study by physicists and mathematicians. Duraut and Lions derived the global weak solution and the local strong solution of the initial boundary value problem of equations (1.1) and derived the existence of the global strong solution in the case of the small initial value [14]. Nevertheless, it is still a challenging open problem whether a unique local solution of the exists globally when the initial value is large. Furthermore, Sermange and Temam derived the regularity of the weak solution $(u, b) \in L^\infty([0, T]; H^1(\mathbb{R}^3))$ [3]. Kozono derived the existence of classical solutions in bounded domain $\Omega \subset \mathbb{R}^3$ for equations (1.1) [15]. For the appropriately weak solution, He and Xin gained different local regularity results [16]. Cao and Wu obtained the global well posedness of the MHD system for any initial value in $H^2(\mathbb{R}^2)$, but it needs a condition of mixed partial dissipation and additional magnetic diffusion in $\mathbb{R}^2$ [2].

Since the coefficients in the equations have no critical influence on the subsequent analysis, we can simply take Re = Rm = S = 1. Futhermore, we can obtain the following equations [1]:

$$\begin{aligned} (\nabla \times b) \times b &= (b \cdot \nabla)b - b \cdot (\nabla b), \\ \nabla \times \nabla \times b \ &= \nabla \mathrm{div}\, b - \Delta b, \\ \nabla \times (u \times b) &= (b \cdot \nabla)u - (u \cdot \nabla)b + u \,\mathrm{div}\, b - b \,\mathrm{div}\, u. \end{aligned}$$

Then, (1.1) can be rewritten as follows:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + b \cdot (\nabla b) + \nabla \tilde{p}, & \text{in } \mathbb{R}^3 \times (0, +\infty), \\ \partial_t b - \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0, & \text{in } \mathbb{R}^3 \times (0, +\infty), \\ u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x), & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

In 2D and 3D cases, Ai et al. applied the semi-Galerkin approximation method to obtain the existence of weak solutions [1]. In 2D case, Ai et al. proved the global existence of strong solutions to (1.1), the continuous dependence of initial-boundary data, and the uniqueness of weak-strong solutions. On this basis, they also proved the existence of a uniform attractor for (1.1). Inspired by [1,4], the main purpose of this paper is to study the well posedness of the solution to (1.1) in 3D mixed-norm Lebesgue spaces. Specifically, in Section 2, we state some mixed-norm legesgue spaces and related properties. In Section 3, we prove the existence, uniqueness, and stability of the solution.

2 Preliminary

We define the 3D mixed-norm Lebesgue spaces as follows [4]:

$$L_{p_1p_2p_3}(\mathbb{R}^3) = \left\{ f \,\middle|\, f : \mathbb{R}^3 \to \mathbb{R},\ \|f\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} = \left(\int \left(\int \left(\int |f|^{p_1} \mathrm{d}x_1 \right)^{\frac{p_2}{p_1}} \mathrm{d}x_2 \right)^{\frac{p_3}{p_2}} \mathrm{d}x_3 \right)^{\frac{1}{p_3}} < +\infty \right\},$$

where $p_1, p_2, p_3 \in [1, +\infty)$.

Let $\mathrm{PL}_{p_1p_2p_3}(\mathbb{R}^3)$ as follows:

$$\mathrm{PL}_{p_1p_2p_3}(\mathbb{R}^3) = \{f \in L_{p_1p_2p_3}(\mathbb{R}^3) : \operatorname{div} f = 0\}.$$

All the spaces that appear in this paper are invariant with respect to the scaling $f(\cdot) \to \lambda f(\lambda\,\cdot)$, $\lambda > 0$. The mixed-norm space $L_{p_1p_2p_3}(\mathbb{R}^3)$ is invariant if and only if $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ [4]. For given $T \in (0, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = \delta \in (0, 1)$, $p_k \in (1, +\infty)$, $q_k \in (p_k, +\infty)$, $k = 1, 2, 3$, and $f : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ denote the measurable vector field functions, we denote $\mathcal{X}_{p,q,T}$ as follows [4]:

$$\mathcal{X}_{p,q,T} = \Big\{f : g(x, t) := t^{\frac{1-\delta}{2}} f(x, t), \quad \tilde{g}(x, t) := t^{\frac{1}{2}} D_x f(x, t), \quad (x, t) \in \mathbb{R}^3 \times (0, T)\Big\},$$

then

$$g \in C\big([0, T], \mathrm{PL}_{q_1q_2q_3}(\mathbb{R}^3)\big), \quad \tilde{g} \in C\big([0, T], \mathrm{PL}_{p_1p_2p_3}(\mathbb{R}^3)\big).$$

Moreover, $g(x, 0) = 0$, $\tilde{g}(x, 0) = 0$, and the norm

$$\|f\|_{\mathcal{X}_{p,q,T}} = \sup_{t\in(0,T)} [\|g(\cdot, t)\|_{L_{q_1q_2q_3}(\mathbb{R}^3)} + \|\tilde{g}\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}] < +\infty.$$

We denote $\mathcal{Y}_{p,T}$ as follows [4]:

$$\mathcal{Y}_{p,T} = \Big\{f : f \in C\big([0, T], \mathrm{PL}_{p_1p_2p_3}(\mathbb{R}^3)\big), \quad t^{\frac{1}{2}} D_x f \in C\big([0, T], \mathrm{PL}_{p_1p_2p_3}(\mathbb{R}^3)\big)\Big\},$$

and the norm

$$\|f\|_{\mathcal{Y}_{p,T}} = \sup_{t\in(0,T)} [\|f(t)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} + t^{1/2}\, \|D_x f(t)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}] < +\infty.$$

We state the following result on Young's inequality in mixed-norm Lebesgue spaces [4].

Lemma 2.1. [4] *Let p_k, r_k, and q_k be given numbers in $[1, +\infty]$ that satisfy*

$$\frac{1}{p_k} + 1 = \frac{1}{q_k} + \frac{1}{r_k}, \quad k = 1, 2, 3.$$

Then,

$$\|f * g\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} = \|f\|_{L_{q_1q_2q_3}(\mathbb{R}^3)} \|g\|_{L_{r_1r_2r_3}(\mathbb{R}^3)}, \tag{2.1}$$

for all $f \in L_{q_1q_2q_3}(\mathbb{R}^3)$ and $g \in L_{r_1r_2r_3}(\mathbb{R}^3)$.

We state the following results on heat equations in mixed-norm Lebesgue spaces. First, we see the Cauchy problem for the heat equations:

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \mathbb{R}^3 \times (0, +\infty), \\ b_t - \Delta b = 0, & \text{in } \mathbb{R}^3 \times (0, +\infty), \\ u(x, 0) = u_0(x), b(x, 0) = b_0(x), & \text{in } \mathbb{R}^3. \end{cases} \tag{2.2}$$

We see that (2.2) can be written as follows:

$$\begin{cases} U_t - \Delta U = 0, & \text{in } \mathbb{R}^3 \times (0, +\infty), \\ U(x, 0) = U_0(x), & \text{in } \mathbb{R}^3, \end{cases} \tag{2.3}$$

where $U = \begin{pmatrix} u \\ b \end{pmatrix}$, $U_0 = \begin{pmatrix} u_0 \\ b_0 \end{pmatrix}$. It is well known that a solution of (2.3) is

$$U(x, t) = e^{\Delta t} U_0(x) = (G_t {*} U_0)(x, t), \quad (x, t) \in \mathbb{R}^3 \times (0, +\infty), \tag{2.4}$$

where

$$G_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad (x, t) \in \mathbb{R}^3 \times (0, +\infty).$$

Next, we state the following fundamental results of the solution of the heat equation in the mixed-norm Lebesgue space.

Lemma 2.2. [4] *Let $1 \le p_k \le q_k \le +\infty$. There exists a positive constant N depending only on p_1, p_2, p_3, q_1, q_2, and q_3 such that for every solution $U(x,t) = e^{\Delta t}U_0(x)$ defined in* (2.4) *of the Cauchy problem* (2.3) *with $U_0 \in L_{q_1q_2q_3}(\mathbb{R}^3)$, then for $t > 0$*

$$\|U(x,t)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \le Nt^{-\frac{1}{2}\sum\limits_{k=1}^{n=3}\left(\frac{1}{q_k}-\frac{1}{p_k}\right)} \|U_0\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}, \tag{2.5}$$

$$\|D_xU(x,t)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \le Nt^{-\frac{1}{2}-\frac{1}{2}\sum\limits_{k=1}^{n=3}\left(\frac{1}{q_k}-\frac{1}{p_k}\right)} \|U_0\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}. \tag{2.6}$$

Lemma 2.3. [4] *Let $p_1, p_2, p_3 \in (1,\infty)$, and $U_0 \in L_{p_1p_2p_3}(\mathbb{R}^3)$. Let $U(x,t) = e^{\Delta t}U_0(x)$ be the solution of the heat equation* (2.3) *defined in* (2.4). *Then, $U \in C([0,\infty), L_{p_1p_2p_3}(\mathbb{R}^3))$ and*

$$\lim_{t\to 0^+} \|U(x,t) - U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} = 0. \tag{2.7}$$

The following consequence illustrates the boundedness of Helmholtz-Leray projection $\mathbb{P}$ in mixed-norm Lebesgue spaces.

Lemma 2.4. [4] *Let $\mathbb{P} = \mathrm{Id} - \nabla\Delta^{-1}\nabla\cdot$ be the Helmholtz-Leray projection onto the divergence-free vector fields. Let $p_1, p_2, p_3 \in (1,\infty)$. Then, one has*

$$\|\mathbb{P}(f)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \le N\,\|f\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}, \tag{2.8}$$

for all $f \in (L_{p_1p_2p_3}(\mathbb{R}^3))^3$, where $N = N(p_1,p_2,p_3)$ is a positive constant.

3 MHD equations in 3D mixed-norm Lebesgue space

We apply $\mathbb{P}$ on the system (1.3), and then (1.3) can be expressed as follows:

$$\begin{cases} U_t + \mathcal{A}U + F(U,U) = 0, & \text{in } \mathbb{R}^3 \times (0,+\infty), \\ U(x,0) = U_0(x), & \text{in } \mathbb{R}^3, \end{cases} \tag{3.1}$$

as $\mathbb{P}\nabla\tilde{p} = 0$, where

$$U = \begin{pmatrix} u \\ b \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ b_0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} -\mathbb{P}\Delta & 0 \\ 0 & -\mathbb{P}\Delta \end{pmatrix},$$

$$F(U,U) = \begin{pmatrix} \mathbb{P}(u\cdot\nabla)u - \mathbb{P}(b\cdot\nabla)b + \mathbb{P}b\cdot(\nabla b) \\ \mathbb{P}(u\cdot\nabla)b - \mathbb{P}(b\cdot\nabla)u \end{pmatrix}.$$

By the Duhamel's principle, the system (3.1) can be transformed into the following integral equations:

$$U = U_1 + G(U,U), \tag{3.2}$$

where

$$U_1 = e^{-\mathcal{A}t}U_0(x) = \begin{pmatrix} e^{\Delta t}u_0(x) \\ e^{\Delta t}b_0(x) \end{pmatrix}$$

and

$$G(U,U) = -\int_0^t e^{-(t-s)\mathcal{A}}F(U,U)\mathrm{d}s \tag{3.3}$$

$$= \begin{pmatrix} -\int\limits_0^t e^{(t-s)\Delta}(\mathbb{P}(u\cdot\nabla)u - \mathbb{P}(b\cdot\nabla)b + \mathbb{P}b\cdot(\nabla b))\mathrm{d}s \\ -\int\limits_0^t e^{(t-s)\Delta}(\mathbb{P}(u\cdot\nabla)b - \mathbb{P}(b\cdot\nabla)u)\mathrm{d}s \end{pmatrix}.$$

The main results in the paper are as follows.

Theorem 3.1. *Let* $p_k \in (1, +\infty)$, $q_k \in [p_k, +\infty)$, $k = 1, 2, 3$, *and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = \delta \in (0, 1).$$

Then, there are a sufficiently small constant $\lambda_0 > 0$ *and a number* $N > 0$, *depending on* p_1, p_2, p_3, q_1, q_2, *and* q_3, *such that the following results hold.*

(*i*) *For all* $U_0 \in L_{p_1p_2p_3}(\mathbb{R}^3)$ *with* $\operatorname{div} U_0 = 0$, *if* $\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \le \lambda_0$, *then* (3.1) *has an unique global time solution* $U \in \mathcal{X}_{p,q,\infty} \cap \mathcal{Y}_{p,\infty}$ *with*

$$\|U\|_{\mathcal{X}_{p,q,\infty}} \le N\,\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)},$$
$$\|U\|_{\mathcal{Y}_{p,\infty}} \le N(\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} + \|U_0\|^2_{L_{p_1p_2p_3}(\mathbb{R}^3)}).$$

(*ii*) *For all* $U_0 \in L_{p_1p_2p_3}(\mathbb{R}^3)$ *with* $\operatorname{div} U_0 = 0$, *there is a sufficiently small* $T_0 > 0$ *depending on* p_1, p_2, p_3, q_1, q_2, *and* q_3 *such that* (3.1) *has an unique local time solution* $U \in \mathcal{X}_{p,q,T_0} \cap \mathcal{Y}_{p,T_0}$ *with*

$$\|U\|_{\mathcal{X}_{p,q,T_0}} \le N\,\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)},$$
$$\|U\|_{\mathcal{Y}_{p,T_0}} \le N\Big(\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} + \|U_0\|^2_{L_{p_1p_2p_3}(\mathbb{R}^3)}\Big).$$

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.1. [4] *Let* p_1, p_2, p_3, q_1, q_2, *and* q_3 *be given numbers and* $1 < p_k \le q_k < \infty$. *Also, let* $\sigma \ge 0$ *be defined by* $\sigma = \sum_{k=1}^{n=3}\Big(\frac{1}{p_k} - \frac{1}{q_k}\Big)$.

(*i*) *There exists a number* N *depending only on* p_1, p_2, p_3, q_1, q_2, *and* q_3 *such that*

$$\|e^{-\mathcal{A}t}\mathbb{P}f\|_{L_{q_1q_1q_3}(\mathbb{R}^3)} \le Nt^{-\frac{\sigma}{2}}\|f\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}, \tag{3.4}$$

$$\|D_x e^{-\mathcal{A}t}\mathbb{P}f\|_{L_{q_1q_1q_3}(\mathbb{R}^3)} \le Nt^{-\frac{1}{2}(1+\sigma)}\|f\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}, \tag{3.5}$$

for all $f \in L_{p_1p_2p_3}(\mathbb{R}^3)^3$.

(*ii*) *For all* $f \in L_{p_1p_2p_3}(\mathbb{R}^3)^3$, *the following assertions hold: if* $\sigma > 0$, *then*

$$\lim_{t\to 0^+} t^{\frac{\sigma}{2}}\,\|e^{-\mathcal{A}t}\mathbb{P}f\|_{L_{q_1q_1q_3}(\mathbb{R}^3)} = 0, \tag{3.6}$$

$$\lim_{t\to 0^+} \|[e^{-\mathcal{A}t}\mathbb{P}f] - \mathbb{P}f\|_{L_{p_1p_1p_3}(\mathbb{R}^3)} = 0, \tag{3.7}$$

$$\lim_{t\to 0^+} t^{-\frac{1}{2}(1+\sigma)}\,\|D_x e^{-\mathcal{A}t}\mathbb{P}f\|_{L_{q_1q_1q_3}(\mathbb{R}^3)} = 0. \tag{3.8}$$

Lemma 3.2. *Let* $p_k \in (1, \infty)$, $\alpha_k, \beta_k, \gamma_k \in (0, 1]$ *be given numbers satisfying* $\gamma_k \le \alpha_k + \beta_k < p_k$, $k = 1, 2, 3$. *Let*

$$\alpha = \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} + \frac{\alpha_2}{p_2}, \quad \beta = \frac{\beta_1}{p_1} + \frac{\beta_2}{p_2} + \frac{\beta_2}{p_2}, \quad \gamma = \frac{\gamma_1}{p_1} + \frac{\gamma_2}{p_2} + \frac{\gamma_2}{p_2}.$$

Then,

$$
\begin{aligned}
&\|G(U,U)\|_{L_{\frac{p_1}{\gamma_1}\frac{p_2}{\gamma_2}\frac{p_3}{\gamma_3}}(\mathbb{R}^3)}\\
&\quad\le N\int_0^t (t-s)^{-\frac{\alpha+\beta-\gamma}{2}}\Big(\|u\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x u\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)}+2\|b\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x b\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)}\\
&\qquad+\|u\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x b\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)}+\|b\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x u\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)}\Big)\mathrm{d}s,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&\|D_xG(U,U)\|_{L_{\frac{p_1}{\gamma_1}\frac{p_2}{\gamma_2}\frac{p_3}{\gamma_3}}(\mathbb{R}^3)}\\
&\quad\le N\int_0^t (t-s)^{-\frac{1+\alpha+\beta-\gamma}{2}}\Big(\|u\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x u\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)}+2\|b\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x b\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)}\\
&\qquad+\|u\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x b\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)}+\|b\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x u\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)}\Big)\mathrm{d}s,
\end{aligned}
\tag{3.10}
$$

where $N>0$ is a constant depending on $p_k,\alpha_k,\beta_k,\gamma_k$, $k=1,2,3$.

Proof. We first prove (3.9) in Lemma 3.2. For $\gamma_k\le\alpha_k+\beta_k<p_k$, we can gain

$$
\frac{p_k}{\gamma_k}\ge\frac{p_k}{\alpha_k+\beta_k},\qquad \sum_{k=1}^{n=3}\Big(\frac{\alpha_k+\beta_k}{p_k}-\frac{\gamma_k}{p_k}\Big)=\alpha+\beta-\gamma.
$$

By (3.4), we can obtain

$$
\|G(U,U)\|_{L_{\frac{p_1}{\gamma_1}\frac{p_2}{\gamma_2}\frac{p_3}{\gamma_3}}(\mathbb{R}^3)}\le N\int_0^t (t-s)^{-\frac{\alpha+\beta-\gamma}{2}}\|F_1(U,U)\|_{L_{\frac{p_1}{\alpha_1+\beta_1}\frac{p_2}{\alpha_2+\beta_2}\frac{p_3}{\alpha_3+\beta_{13}}}(\mathbb{R}^3)}\mathrm{d}s,
\tag{3.11}
$$

where $F_1(U,U)=\begin{pmatrix}(u\cdot\nabla)u-(b\cdot\nabla)b+b\cdot(\nabla b)\\(u\cdot\nabla)b-(b\cdot\nabla)u\end{pmatrix}$.

By using Hölder's inequality repeatedly, we can find that

$$
\begin{aligned}
&\|(u\cdot\nabla)b\|_{L_{\frac{p_1}{\alpha_1+\beta_1}\frac{p_2}{\alpha_2+\beta_2}\frac{p_3}{\alpha_3+\beta_3}}(\mathbb{R}^3)}\\
&\quad=\left[\int\left[\int\left[\int|u\cdot(\nabla b)|^{\frac{p_1}{\alpha_1+\beta_1}}\mathrm{d}x_1\right]^{\frac{p_2(\alpha_1+\beta_1)}{p_1(\alpha_2+\beta_2)}}\mathrm{d}x_2\right]^{\frac{p_3(\alpha_2+\beta_2)}{p_2(\alpha_3+\beta_3)}}\mathrm{d}x_3\right]^{\frac{(\alpha_3+\beta_3)}{p_3}}\\
&\quad\le\left[\int\left[\int\left[\Big(\int|u|^{\frac{p_1}{\alpha_1}}\mathrm{d}x_1\Big)^{\frac{\alpha_1}{p_1}}\Big(\int|D_xb|^{\frac{p_1}{\beta_1}}\mathrm{d}x_1\Big)^{\frac{\beta_1}{p_1}}\right]^{\frac{p_2}{(\alpha_2+\beta_2)}}\mathrm{d}x_2\right]^{\frac{p_3(\alpha_2+\beta_2)}{p_2(\alpha_3+\beta_3)}}\mathrm{d}x_3\right]^{\frac{(\alpha_3+\beta_3)}{p_3}}\\
&\quad\le\left[\int\left[\left[\int\left[\int|u|^{\frac{p_1}{\alpha_1}}\mathrm{d}x_1\right]^{\frac{p_2\alpha_1}{p_1\alpha_2}}\mathrm{d}x_2\right]^{\frac{\alpha_2}{p_2}}\left[\int\left[\int|D_xb|^{\frac{p_1}{\beta_1}}\mathrm{d}x_1\right]^{\frac{p_2\beta_1}{p_1\beta_2}}\mathrm{d}x_2\right]^{\frac{\beta_2}{p_2}}\right]^{\frac{p_3}{(\alpha_3+\beta_3)}}\mathrm{d}x_3\right]^{\frac{(\alpha_3+\beta_3)}{p_3}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \left[\int\left[\int\left[\int |u|^{\frac{p_1}{\alpha_1}}\mathrm{d}x_1\right]^{\frac{p_2\alpha_1}{p_1\alpha_2}}\mathrm{d}x_2\right]^{\frac{p_3\alpha_2}{p_2\alpha_3}}\mathrm{d}x_3\right]^{\frac{\alpha_3}{p_3}}\left[\int\left[\int\left[\int |D_x b|^{\frac{p_1}{\beta_1}}\mathrm{d}x_1\right]^{\frac{p_2\beta_1}{p_1\beta_2}}\mathrm{d}x_2\right]^{\frac{p_3\beta_2}{p_2\beta_3}}\mathrm{d}x_3\right]^{\frac{\beta_3}{p_3}} \\
&= \|u\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x b\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\|F_1(U,U)\|_{L_{\frac{p_1}{\alpha_1+\beta_1}\frac{p_2}{\alpha_2+\beta_2}\frac{p_3}{\alpha_3+\beta_{13}}}(\mathbb{R}^3)} \\
&\le \||(u\cdot\nabla)u| + |(b\cdot\nabla)b| + |b\cdot(\nabla b)| + |(u\cdot\nabla)b| + |(b\cdot\nabla)u|\|_{L_{\frac{p_1}{\alpha_1+\beta_1}\frac{p_2}{\alpha_2+\beta_2}\frac{p_3}{\alpha_3+\beta_{13}}}(\mathbb{R}^3)} \\
&\le \|(u\cdot\nabla)u\|_{L_{\frac{p_1}{\alpha_1+\beta_1}\frac{p_2}{\alpha_2+\beta_2}\frac{p_3}{\alpha_3+\beta_{13}}}(\mathbb{R}^3)} + \|(b\cdot\nabla)b\|_{L_{\frac{p_1}{\alpha_1+\beta_1}\frac{p_2}{\alpha_2+\beta_2}\frac{p_3}{\alpha_3+\beta_{13}}}(\mathbb{R}^3)} \\
&\quad + \|b\cdot(\nabla b)\|_{L_{\frac{p_1}{\alpha_1+\beta_1}\frac{p_2}{\alpha_2+\beta_2}\frac{p_3}{\alpha_3+\beta_{13}}}(\mathbb{R}^3)} + \|(u\cdot\nabla)b\|_{L_{\frac{p_1}{\alpha_1+\beta_1}\frac{p_2}{\alpha_2+\beta_2}\frac{p_3}{\alpha_3+\beta_{13}}}(\mathbb{R}^3)} + \|(b\cdot\nabla)u\|_{L_{\frac{p_1}{\alpha_1+\beta_1}\frac{p_2}{\alpha_2+\beta_2}\frac{p_3}{\alpha_3+\beta_{13}}}(\mathbb{R}^3)} \\
&\le \|u\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x u\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)} + 2\|b\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x b\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)} \\
&\quad + \|u\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x b\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)} + \|b\|_{L_{\frac{p_1}{\alpha_1}\frac{p_2}{\alpha_2}\frac{p_3}{\alpha_3}}(\mathbb{R}^3)}\|D_x u\|_{L_{\frac{p_1}{\beta_1}\frac{p_2}{\beta_2}\frac{p_3}{\beta_3}}(\mathbb{R}^3)}.
\end{aligned}
$$

By substituting the aforementioned formula into (3.11), we can obtain (3.9). Similarly, (3.10) can be proved by (3.5). □

Lemma 3.3. [4] *Let X be a Banach space with norm $\|\cdot\|_X$. Let $G : X\times X \to X$ be a bilinear map such that there is $N_0 > 0$ so that*

$$\|G(U,V)\|_X \le N_0\|U\|_X\|V\|_X, \quad \forall U, V \in X.$$

Then, for all $U_1 \in X$ with $4N_0\|U_1\|_X < 1$, the equation

$$U = U_1 + G(U,U)$$

has an unique solution U with

$$\|U\|_X \le 2\|U_1\|_X.$$

Proof of Theorem 3.1. We now prove (i). First, we start from the proof that $U \in \chi_{p,q,\infty}$. From Lemma 2.2 and $\sigma = \sum_{k=1}^{n=3}\left(\frac{1}{p_k} - \frac{1}{q_k}\right) = 1-\delta$, we have

$$\|U_1\|_{L_{q_1q_1q_3}(\mathbb{R}^3)} \le N_1 t^{-\frac{1-\delta}{2}}\|U_0\|_{L_{p_1P_2p_3}(\mathbb{R}^3)},$$
$$\|D_xU_1\|_{L_{q_1q_2q_3}(\mathbb{R}^3)} \le N_1 t^{-\frac{1}{2}}\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)},$$

where $N_1 > 0$ is a constant depending on p_1, p_2, p_3, q_1, q_2, *and* q_3.

Furthermore, according to Lemma 3.1, we know that when $t \to 0$, $t^{-\frac{1-\sigma}{2}}e^{-\mathcal{A}t}\mathbb{P} \to 0$, and when $t = 0$, $t^{-\frac{1-\sigma}{2}}U_1 = 0$. Hence,

$$t^{-\frac{1-\sigma}{2}}e^{-\mathcal{A}t}\mathbb{P} : L_{p_1p_2p_3}(\mathbb{R}^3) \to PL_{q_1q_2q_3}(\mathbb{R}^3)$$

is uniformly bounded. Similarly, when $t \to 0$, $t^{-\frac{1}{2}}D_xe^{-\mathcal{A}t}\mathbb{P} \to 0$, and when $t = 0$, $t^{-\frac{1}{2}}D_xU_1 = 0$. Hence,

$$t^{-\frac{1-\sigma}{2}}D_xe^{-\mathcal{A}t}\mathbb{P} : L_{p_1p_2p_3}(\mathbb{R}^3) \to PL_{p_1p_2p_3}(\mathbb{R}^3)$$

is also uniformly bounded. Hence, we have $U_0 \in \chi_{p,q,\infty}$ and

$$\|U_1\|_{\chi_{p,q,\infty}} \le N_1\,\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}. \tag{3.12}$$

Now let's prove that bilinear $G : \chi_{p,q,\infty}\times\chi_{p,q,\infty} \to \chi_{p,q,\infty}$ is bounded.

Let $\beta_k = 1$, $\gamma_k = \alpha_k = \frac{p_k}{q_k} \in (0, 1]$, we obtain $\frac{p_k}{\gamma_k} = q_k$,

$$-\frac{\alpha+\beta-\gamma}{2} = -\frac{1}{2}\sum_{k=1}^{n=3}\left(\frac{\alpha_k}{p_k} + \frac{\beta_k}{p_k} - \frac{\gamma_k}{p_k}\right) = -\frac{1}{2}\sum_{k=1}^{n=3}\left(\frac{1}{q_k} + \frac{1}{p_k} - \frac{1}{q_k}\right) = -\frac{1}{2}\sum_{k=1}^{n=3}\left(\frac{1}{p_k}\right) = -\frac{1}{2}.$$

By using $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = \delta \in (0, 1)$, (3.9) and the definition of $\mathcal{X}_{p,q,T}$, and applying (3.3), we can obtain

$$\begin{aligned}
\|G(U,U)\|_{L_{q_1q_2q_3}(\mathbb{R}^3)} &\le N\int_0^t (t-s)^{-\frac{1}{2}}(\|u\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xu\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} + 2\|b\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xb\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \\
&\quad + \|u\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xb\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} + \|b\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xu\|_{L_{p_1p_2p_3}(\mathbb{R}^3)})\mathrm{d}s \\
&\le N\int_0^t (t-s)^{-\frac{1}{2}}\Big(s^{\frac{1-\delta}{2}}\|u\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}s^{\frac{1}{2}}\ \|D_xu\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}s^{-\frac{1-\delta}{2}}s^{-\frac{1}{2}} \\
&\quad + 2s^{\frac{1-\delta}{2}}\|b\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}s^{\frac{1}{2}}\ \|D_xb\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}s^{-\frac{1-\delta}{2}}s^{-\frac{1}{2}} + s^{\frac{1-\delta}{2}}\|u\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}s^{\frac{1}{2}} \\
&\quad \times\ \|D_xb\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}s^{-\frac{1-\delta}{2}}s^{-\frac{1}{2}} + s^{\frac{1-\delta}{2}}\|b\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}s^{\frac{1}{2}}\ \|D_xu\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}s^{-\frac{1-\delta}{2}}s^{-\frac{1}{2}}\Big)\mathrm{d}s \\
&\le N(\|u\|^2_{\chi_{p,q,\infty}} + \|b\|^2_{\chi_{p,q,\infty}} + 2\|u\|_{\chi_{p,q,\infty}}\|b\|_{\chi_{p,q,\infty}})\int_0^t (t-s)^{-\frac{1}{2}}s^{-1+\frac{\delta}{2}}\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t (t-s)^{-\frac{1}{2}}s^{-1+\frac{\delta}{2}}\mathrm{d}s &= \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}}s^{-1+\frac{\delta}{2}}\mathrm{d}s + \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}}s^{-1+\frac{\delta}{2}}\mathrm{d}s \\
&\le \left(\frac{t}{2}\right)^{-\frac{1}{2}}\int_0^{\frac{t}{2}} s^{-1+\frac{\delta}{2}}\mathrm{d}s + \left(\frac{t}{2}\right)^{-1+\frac{\delta}{2}}\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}}\mathrm{d}s \\
&\le N\left(\frac{t}{2}\right)^{-\frac{1}{2}}\left(\frac{t}{2}\right)^{\frac{\delta}{2}} + N\left(\frac{t}{2}\right)^{-1+\frac{\delta}{2}}\left(\frac{t}{2}\right)^{\frac{1}{2}} \\
&\le Nt^{-\frac{1-\delta}{2}}.
\end{aligned}$$

So, we have

$$\|G(U,U)\|_{L_{q_1q_2q_3}(\mathbb{R}^3)} \le Nt^{-\frac{1-\delta}{2}}(\|u\|^2_{\chi_{p,q,\infty}} + 2\ \|b\|^2_{\chi_{p,q,\infty}} + 2\|u\|_{\chi_{p,q,\infty}}\|b\|_{\chi_{p,q,\infty}}). \tag{3.13}$$

Similarly, let $\beta_k = \gamma_k = 1$, $\alpha_k = \frac{p_k}{q_k} \in (0, 1]$, and $\frac{p_k}{\gamma_k} = q_k$,

$$-\frac{1+\alpha+\beta-\gamma}{2} = -\frac{1}{2}\left(1 + \sum_{k=1}^{n=3}\left(\frac{\alpha_k}{p_k} + \frac{\beta_k}{p_k} - \frac{\gamma_k}{p_k}\right)\right) = -\frac{1}{2}\left(1 + \sum_{k=1}^{n=3}\left(\frac{1}{q_k} + \frac{1}{p_k} - \frac{1}{p_k}\right)\right) = -\frac{1}{2}\left(1 + \sum_{k=1}^{n=3}\frac{1}{q_k}\right) = -\frac{1+\delta}{2}.$$

By using $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = \delta \in (0, 1)$, and (3.10), we also obtain

$$\begin{aligned}
&\|D_xG(U,U)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \\
&\le\ N\int_0^t (t-s)^{-\frac{1+\delta}{2}}(\|u\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xu\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} + 2\|b\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xb\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \\
&\quad + \|u\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xb\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} + \|b\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xu\|_{L_{p_1p_2p_3}(\mathbb{R}^3)})\mathrm{d}s \\
&\le\ N(\|u\|^2_{\chi_{p,q,\infty}} + \|b\|^2_{\chi_{p,q,\infty}} + 2\|u\|_{\chi_{p,q,\infty}}\|b\|_{\chi_{p,q,\infty}})\int_0^t (t-s)^{-\frac{1+\delta}{2}}s^{-1+\frac{\delta}{2}}\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t (t-s)^{-\frac{1+\delta}{2}} s^{-1+\frac{\delta}{2}} ds \\
&= \int_0^{\frac{t}{2}} (t-s)^{-\frac{1+\delta}{2}} s^{-1+\frac{\delta}{2}} ds + \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1+\delta}{2}} s^{-1+\frac{\delta}{2}} ds \\
&\le \left(\frac{t}{2}\right)^{-\frac{1+\delta}{2}} \int_0^{\frac{t}{2}} s^{-1+\frac{\delta}{2}} ds + \left(\frac{t}{2}\right)^{-1+\frac{\delta}{2}} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1+\delta}{2}} ds \\
&\le N\left(\frac{t}{2}\right)^{-\frac{1+\delta}{2}} \left(\frac{t}{2}\right)^{\frac{\delta}{2}} + N\left(\frac{t}{2}\right)^{-1+\frac{\delta}{2}} \left(\frac{t}{2}\right)^{\frac{1+\delta}{2}} \\
&\le Nt^{-\frac{1}{2}}.
\end{aligned}$$

So, we have

$$\|D_x G(U,U)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \le Nt^{-\frac{1}{2}}(\|u\|^2_{\chi_{p,q,\infty}} + 2\,\|b\|^2_{\chi_{p,q,\infty}} + 2\|u\|_{\chi_{p,q,\infty}}\|b\|_{\chi_{p,q,\infty}}). \tag{3.14}$$

From the estimates (3.13), (3.14), and Lemma 3.1, we can prove that $t^{\frac{1-\delta}{2}}G(U,U):[0,\infty)\to \mathbb{P}L_{q_1q_2q_3}(\mathbb{R}^3)$ is continuous and when $t\to 0$, $t^{\frac{1-\delta}{2}}G(U,U)\to 0$. Similarly, we can also prove that $t^{\frac{1}{2}}D_xG(U,U):[0,\infty)\to \mathbb{P}L_{p_1p_2p_3}(\mathbb{R}^3)$ is continuous and when $t\to 0$, $t^{\frac{1}{2}}D_xG(U,U)\to 0$. Therefore, we obtain $G(U,U)\in\chi_{p,q,\infty}$ and for all $U\in\chi_{p,q,\infty}$:

$$\begin{aligned}
\|G(U,U)\|\chi_{p,q,\infty} &\le N(\|u\|^2_{\chi_{p,q,\infty}} + 2\,\|b\|^2_{\chi_{p,q,\infty}} + 2\|u\|_{\chi_{p,q,\infty}}\|b\|_{\chi_{p,q,\infty}}) \\
&\le N\|U\|_{\chi_{p,q,\infty}}\|U\|_{\chi_{p,q,\infty}} \\
&\le N_2\,\|U\|^2_{\chi_{p,q,\infty}}, \quad \forall U\in\chi_{p,q,\infty},
\end{aligned} \tag{3.15}$$

where N_2 is a constant depending on p_1, p_2, p_3, q_1, q_2, and q_3. That is, the bilinear $G:\chi_{p,q,\infty}\times\chi_{p,q,\infty}\to\chi_{p,q,\infty}$ is bounded.

Next, let us choose a sufficiently small constant λ_0 so that

$$4N_1N_2\lambda_0 < 1,$$

where N_1 is defined in (3.12) and N_2 is defined in (3.15). If $\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}\le\lambda_0$, then it can be obtained by (3.12)

$$4N_2\|U_1\|_{\chi_{p,q,\infty}} \le 4N_1N_2\,\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \le 4N_1N_2\lambda_0 < 1.$$

By this and Lemma 3.3, we can gain a unique time solution $U\in\chi_{p,q,\infty}$ of (3.2) such that

$$\|U\|_{\chi_{p,q,\infty}} \le 2\,\|U_1\|_{\chi_{p,q,\infty}} \le 2N_1\,\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}. \tag{3.16}$$

Now, we need to prove that $U\in\mathcal{Y}_{p,\infty}$. As $U=U_1+G(U,U)$, we have

$$\|U\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \le \|U_1\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} + \|G(U,U)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}, \tag{3.17}$$

$$\|D_xU\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \le \|D_xU_1\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} + \|D_xG(U,U)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}. \tag{3.18}$$

Then, by Lemma 3.1, we see that

$$\|U_1\|_{L_{p_1p_1p_3}(\mathbb{R}^3)} \le N\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}, \tag{3.19}$$

$$\|D_xU_1\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} \le Nt^{-\frac{1}{2}}\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}. \tag{3.20}$$

Let $\beta_k=\gamma_k=1$, $\alpha_k=\frac{p_k}{q_k}\in(0,1]$, and $\frac{p_k}{\gamma_k}=p_k$,

$$-\frac{\alpha+\beta-\gamma}{2}=-\frac{1}{2}\sum_{k=1}^{n=3}\left(\frac{\alpha_k}{p_k}+\frac{\beta_k}{p_k}-\frac{\gamma_k}{p_k}\right)=-\frac{1}{2}\sum_{k=1}^{n=3}\left(\frac{1}{q_k}+\frac{1}{p_k}-\frac{1}{p_k}\right)=-\frac{1}{2}\sum_{k=1}^{n=3}\left(\frac{1}{q_k}\right)=-\frac{\delta}{2}.$$

By using (3.9), we can infer that

$$\begin{aligned}
&\|G(U,U)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}\\
&\leq N\int_0^t (t-s)^{-\frac{\delta}{2}}(\|u\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xu\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}+2\|b\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xb\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}\\
&\quad+\|u\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xb\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}+\|b\|_{L_{q_1q_2q_3}(\mathbb{R}^3)}\|D_xu\|_{L_{p_1p_2p_3}(\mathbb{R}^3)})\mathrm{d}s\\
&\leq N(\|u\|^2_{\chi_{p,q,\infty}}+\|b\|^2_{\chi_{p,q,\infty}}+2\|u\|_{\chi_{p,q,\infty}}\|b\|_{\chi_{p,q,\infty}})\int_0^t (t-s)^{-\frac{\delta}{2}}s^{-1+\frac{\delta}{2}}\mathrm{d}s.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^t (t-s)^{-\frac{\delta}{2}}s^{-1+\frac{\delta}{2}}\mathrm{d}s&=\int_0^{\frac{t}{2}}(t-s)^{-\frac{\delta}{2}}s^{-1+\frac{\delta}{2}}\mathrm{d}s+\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{\delta}{2}}s^{-1+\frac{\delta}{2}}\mathrm{d}s\\
&\leq\left(\frac{t}{2}\right)^{-\frac{\delta}{2}}\int_0^{\frac{t}{2}}s^{-1+\frac{\delta}{2}}\mathrm{d}s+\left(\frac{t}{2}\right)^{-1+\frac{\delta}{2}}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{\delta}{2}}\mathrm{d}s\\
&\leq N_1\left(\frac{t}{2}\right)^{-\frac{\delta}{2}}\left(\frac{t}{2}\right)^{\frac{\delta}{2}}+N_2\left(\frac{t}{2}\right)^{-1+\frac{\delta}{2}}\left(\frac{t}{2}\right)^{-\frac{\delta}{2}+1}\\
&\leq N,
\end{aligned}$$

by using (3.16), we obtain that

$$\begin{aligned}
\|G(U,U)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}&\leq N(\|u\|^2_{\chi_{p,q,\infty}}+2\,\|b\|^2_{\chi_{p,q,\infty}}+2\|u\|_{\chi_{p,q,\infty}}\|b\|_{\chi_{p,q,\infty}})\\
&\leq N\,\|U\|^2_{\chi_{p,q,\infty}}\leq N\,\|U_0\|^2_{L_{p_1p_2p_3}(\mathbb{R}^3)}.
\end{aligned}\tag{3.21}$$

Similarly, by using (3.14) and (3.16), we obtain that

$$\begin{aligned}
\|D_xG(U,U)\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}&\leq Nt^{-\frac{1}{2}}(\|u\|^2_{\chi_{p,q,\infty}}+2\,\|b\|^2_{\chi_{p,q,\infty}}+2\|u\|_{\chi_{p,q,\infty}}\|b\|_{\chi_{p,q,\infty}})\\
&\leq Nt^{-\frac{1}{2}}\,\|U\|^2_{\chi_{p,q,\infty}}\leq Nt^{-\frac{1}{2}}\,\|U_0\|^2_{L_{p_1p_2p_3}(\mathbb{R}^3)}.
\end{aligned}\tag{3.22}$$

From estimates (3.17), (3.19), and (3.21), we see that

$$\|U\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}\leq N(\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}+\|U_0\|^2_{L_{p_1p_2p_3}(\mathbb{R}^3)}).$$

From the estimates (3.18), (3.20), and (3.22), we see that

$$\|D_xU\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}\leq Nt^{-\frac{1}{2}}(\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}+\|U_0\|^2_{L_{p_1p_2p_3}(\mathbb{R}^3)}).$$

It is known that $\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}$ is sufficiently small, we can gain

$$\|U\|_{\mathcal{Y}_{p,\infty}}\leq N_0(\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}+\|U_0\|^2_{L_{p_1p_2p_3}(\mathbb{R}^3)}).$$

Now, we have to prove (ii). Similar to the proof of (3.12), we found that $U_1\in\chi_{p,q,\infty}$, $\chi_{p,q,\infty}$ is continuous, and when $t\to 0$, $t^{\frac{1-\delta}{2}}U_1\to 0$, $t^{\frac{1}{2}}D_xU_1\to 0$. There is a sufficiently small constant $T_0>0$ depending on p_1, p_2, p_3, q_1, q_2, q_3, and U_0 such that

$$\|U_1\|_{\chi_{p,q,T_0}}\leq\lambda_0.$$

Furthermore, similar to the proof of (3.15), we also found that the bilinear G: $\chi_{p,q,T_0} \times \chi_{p,q,T_0} \to \chi_{p,q,T_0}$ is bounded and

$$\|G(U,U)\|_{\chi_{p,q,T_0}} \leq N\|U\|_{\chi_{p,q,T_0}}\|U\|_{\chi_{p,q,T_0}}, \quad \forall U \in \chi_{p,q,T_0}.$$

Then, by Lemma 3.3, we can obtain a unique local time solution $U \in \chi_{p,q,T_0}$ of (3.2) with

$$\|U\|_{\chi_{p,q,T_0}} \leq 2N_1 \|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)}.$$

As in proof (i), we can gain $U \in \mathcal{Y}_{p,T_0}$ and

$$\|U\|_{\mathcal{Y}_{p,T_0}} \leq N(\|U_0\|_{L_{p_1p_2p_3}(\mathbb{R}^3)} + \|U_0\|^2_{L_{p_1p_2p_3}(\mathbb{R}^3)}). \qquad \square$$

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References

[1] C. Ai, Z. Tan, and J. Zhou, *Global well posedness and existence of uniformattractor for magnetohydrodynamic equations*, Math. Methods Appl. Sci. **43** (2020), no. 12, 7045–7069.
[2] C. Cao and J. Wu, *Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion*, Adv. Math. **226** (2011), no. 2, 1803–1822.
[3] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations*, Commun. Pure Appl. Math. **36** (1983), no. 5, 635–664.
[4] T. Phan, *Well-posedness for the Navier–Stokes equations in critical mixed-norm Lebesgue spaces*, J. Evol. Equ. **20** (2020), 553–576.
[5] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), no. 1, 193–248.
[6] E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1951), 213–231.
[7] H. Fujita and T. Kato, *On the Navier–Stokes initial value problem. I*, Arch. Ration. Mech. Anal. **16** (1964), no. 4, 269–315.
[8] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1997.
[9] P. L. Lions, *Mathematical Topics in Fluid Mechanics*, vol. 1, Oxford Science Publications, USA, 1996.
[10] P. Constantin and C. Foias, *Navier–Stokes Equations*, University of Chicago Press, Chicago, 1988.
[11] M. Cannone, *Ondelettes, paraproduits et Navier–Stokes*, Diderot Editeur, Paris, 1995.
[12] Y. Meyer, *Wavelets, paraproducts and Navier–Stokes equations*, Current Develop. Math. **1996** (1996), no. 1, 105–212.
[13] H. Jia and V. Šverák, *Local-in-space estimates near initial time for weak solutions of the Navier–Stokes equations and forward self-similar solutions*, Invent. Math. **196** (2014), no. 1, 233–265.
[14] G. Duraut and J. Lions, *Inéquations en thermoélasticité et magnéto-hydrodynamique*, Arch. Ration. Mech. Anal. **46** (1972), no. 4, 241–279.
[15] H. Kozono, *Weak and classical solutions of the two-dimensional magnetohydrodynamic equations*, Tohoku Math. J. **41** (1989), no. 3, 471–488.
[16] C. He and Z. Xin, *Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations*, J. Funct. Anal. **227** (2005), no. 1, 113–152.