Rapid Communication

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Discussion of foundation of mathematics and quantum theory

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Abstract: Following the results of our recently published book [F. Lev, *Finite Mathematics as the Foundation of Classical Mathematics and Quantum Theory. With Applications to Gravity and Particle Theory*, Springer, 2020, ISBN 978-3-030-61101-9], we discuss different aspects of classical and finite mathematics and explain why finite mathematics based on a finite ring of characteristic p is more general (fundamental) than classical mathematics: the former does not have foundational problems, and the latter is a special degenerate case of the former in the formal limit $p \to \infty$. In particular, quantum theory based on a finite ring of characteristic p is more general than standard quantum theory because the latter is a special degenerate case of the former in the formal limit $p \to \infty$.

Keywords: finite mathematics, classical mathematics, finite quantum theory

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1 Problem statement

The title of the famous Wigner's paper [1] is: "The unreasonable effectiveness of mathematics in the natural sciences," and the paper is concluded as follows:

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.

In view of these remarks, a problem arises whether there exists an approach to mathematics, which can be treated as more adequate than other approaches. Probably, the most common approaches are Hilbert's approach and the approach from the point of view of physics.

In Hilbert's approach, it is not posed a question whether mathematics should correctly describe nature. The goal of the approach is to find a complete and consistent set of axioms, which will make it possible to conclude whether any mathematical statement is true or false. This problem is also formulated as the Entscheidungs problem, which asks for algorithms that consider statements and answers "Yes" or "No" according to whether the statements are universally valid, i.e., valid in every structure satisfying the axioms.

In mathematical logic, one can pose a problem what kind of mathematics is more general (fundamental). However, since in Hilbert's approach, mathematics is treated as an abstract science, one cannot pose a problem what kind of mathematics is more adequate for applications. By definition, *classical mathematics involves infinitesimals and limits while finite mathematics involves only a finite number of*

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numbers. Those kinds of mathematics considerably differ from each other, but, in Hilbert's approach, the problem which mathematics is more adequate for applications does not arise.

In the framework of Hilbert's approach, the problem of the foundation of mathematics is very difficult. This problem has been considered by many great mathematicians. Gödel's incompleteness theorems state that mathematics involving standard arithmetic of natural numbers is incomplete and cannot demonstrate its own consistency. The problem widely discussed in the literature is whether the problems posed by the theorems can be circumvented by nonstandard approaches to natural numbers, e.g., by treating them in the framework of Peano arithmetic, Robinson arithmetic, finitistic arithmetic, transfinite numbers, etc. However, the results obtained by Tarski, Turing, and others show that, in Hilbert's approach, the problem of the foundation of mathematics remains. In the present paper, we do not consider this problem.

Although, as noted in [1], people do not understand the reason why mathematics is so effective in the natural sciences, in this paper, we treat mathematics not as an abstract science but as a tool for describing nature. Then, it is possible to pose a problem what mathematics is more pertinent for this goal.

In [2], we have proposed the following:

Definition. Let theory A contain a finite nonzero parameter and theory B be obtained from theory A in the formal limit when the parameter goes to zero or infinity. Suppose that, with any desired accuracy, A can reproduce any result of B by choosing a value of the parameter. On the contrary, when, the limit is already taken, one cannot return to A and B cannot reproduce all results of A. Then, A is more general than B and B is a special degenerate case of A.

Known examples are that:

(1) Nonrelativistic theory (NT) is a special degenerate case of relativistic theory (RT) in the formal limit $c \to \infty$ (where c is the speed of light);
(2) Classical (i.e., non-quantum) theory is a special degenerate case of quantum theory (QT) in the formal limit $\hbar \to 0$ (where $\hbar$ is the Planck constant);
(3) RT is a special degenerate case of de Sitter (dS) and anti-de Sitter (AdS) invariant theories in the formal limit $R \to \infty$, where R is the parameter of contraction from the dS or AdS groups or Lie algebras to the Poincare group or Lie algebra, respectively.

In the literature, those facts are explained from physical considerations but, as shown in the famous Dyson's paper "Missed Opportunities" [3], (1) follows from the pure mathematical fact that the Galilei group can be obtained from the Poincare one by contraction $c \to \infty$, and (3) follows from the pure mathematical fact that the Poincare group can be obtained from the dS or AdS groups by contraction $R \to \infty$. At the same time, since the dS and AdS groups are semisimple, they cannot be obtained from more symmetric groups by contraction.

However, as argued in [2], on quantum level, symmetry should be defined not by groups but by the corresponding Lie algebras. Then, the statements (1)–(3) follow from the facts that the Galilei Lie algebra can be obtained from the Poincare one by contraction $c \to \infty$, classical Lie algebra can be obtained from the quantum one by contraction $\hbar \to 0$, and the Poincare Lie algebra can be obtained from the dS or AdS Lie algebras by contraction $R \to \infty$. So, in general, theory B is a special degenerate case of theory A if the symmetry algebra for theory B can be obtained from the symmetry algebra for theory A by contraction. The main goal of this paper is to explain in the framework of *Definition* that:

Statement: Classical mathematics is a special degenerate case of finite one in the formal limit $p \to \infty$, where p is the characteristic of the ring in finite mathematics.

As explained later, a consequence of this **Statement** is that, *for describing nature at the most fundamental level, the concepts of infinitesimals, limits, continuity, etc. are not needed; they are needed only for describing nature approximately*.

The organization of this paper is clear from the titles of the sections.

2 Problems with describing nature by classical mathematics

Mathematical education at physics departments develops a belief that classical mathematics is the most fundamental mathematics, while finite mathematics is something inferior what is used only in special applications. Many mathematicians have a similar belief.

Historically, it happened so because more than 300 years ago, Newton and Leibniz proposed the calculus of infinitesimals, and, since that time, a titanic work has been done on the foundation of classical mathematics. As noted in Section 1, this problem has not been solved till the present time, but for most of physicists and many mathematicians, the most important thing is not whether a rigorous foundation exists but that in many cases standard mathematics works with very high accuracy.

The idea of infinitesimals was in the spirit of existed experience that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts, and, even in the nineteenth century, people did not know about atoms and elementary particles. But now we know that when we reach the level of atoms and elementary particles, then standard division loses its usual meaning, and in nature, there are no arbitrarily small parts and no continuity.

For example, typical energies of electrons in modern accelerators are millions of times greater than the electron rest energy, and such electrons experience many collisions with different particles. If it were possible to break the electron into parts, then it would have been noticed long ago.

Another example is that if we draw a line on a sheet of paper and look at this line by a microscope, then we will see that the line is strongly discontinuous because it consists of atoms. That is why standard geometry (the concepts of continuous lines and surfaces) can work well only in the approximation when sizes of atoms are neglected, and standard macroscopic theory can work well only in this approximation and so on.

Of course, when we consider water in the ocean and describe it by differential equations of hydrodynamics, this works well but this is only an approximation since water consists of atoms. However, it seems unnatural that even quantum theory is based on continuous mathematics. Even the name "quantum theory" reflects a belief that nature is quantized, i.e., discrete, and this name has arisen because in quantum theory, some quantities have discrete spectrum (i.e., the spectrum of the angular momentum operator, the energy spectrum of the hydrogen atom, etc.). But this discrete spectrum has appeared in the framework of classical mathematics.

I asked physicists and mathematicians whether, in their opinion, the indivisibility of the electron shows that in nature there are no infinitesimals, and standard division does not work always. Some mathematicians say that sooner or later the electron will be divided. On the other hand, as a rule, physicists agree that the electron is indivisible, and in nature, there are no infinitesimals. They say that, for example, $\mathrm{d}x/\mathrm{d}t$ should be understood as $\Delta x/\Delta t$, where Δx and Δt are small but not infinitesimal. I ask them: but you work with $\mathrm{d}x/\mathrm{d}t$, not $\Delta x/\Delta t$. They reply that since mathematics with derivatives works well, then there is no need to philosophize and develop something else (and they are not familiar with finite mathematics).

One of the key problems of modern quantum theory is the problem of infinities: the theory gives divergent expressions for the S-matrix in perturbation theory. In renormalized theories, the divergencies are eliminated by the renormalization procedure where finite observable quantities are formally expressed as products of singularities. Although this procedure is not well substantiated mathematically, in some cases, it results in excellent agreement with the experiment. Probably, the most famous case is that the results for the electron and muon magnetic moments obtained at the end of the 40th agree with the experiment at least with the accuracy of eight decimal digits (see, however, a discussion in [4]). In view of this and other successes of quantum theory, most physicists believe that agreement with the data is much more important than the rigorous mathematical substantiation.

At the same time, in nonrenormalized theories, infinities cannot be eliminated by the renormalization procedure, and this is a great obstacle for constructing quantum gravity based on quantum field theory (QFT). As the famous physicist and the Nobel Prize laureate Steven Weinberg writes in his book [5]: "Disappointingly this problem appeared with even greater severity in the early days of quantum theory, and although greatly ameliorated by subsequent improvements in the theory, it remains with us to the present day." The title of Weinberg's paper [6] is "Living with infinities."

In view of efforts to describe discrete nature by continuous mathematics, my friend told me the following joke: “A group of monkeys is ordered to reach the Moon. For solving this problem each monkey climbs a tree. The monkey who has reached the highest point believes that he has made the greatest progress and is closer to the goal than the other monkeys.” Is it reasonable to treat this joke as a hint on some aspects of the modern science? Indeed, people invented continuity and infinitesimals, which do not exist in nature, created problems for themselves and now apply titanic efforts for solving those problems. Below it will be explained on a popular level (and the rigorous proof is given in [2]) that classical mathematics is a special degenerate case of finite mathematics.

The founders of quantum theory and scientists who essentially contributed to it were highly educated. But they used only classical mathematics, and even now finite mathematics is not a part of standard education for physicists. The development of quantum theory has shown that the theory contains anomalies and divergences. Most physicists considering those problems worked in the framework of classical mathematics and did not acknowledge that they arise just because this mathematics was applied.

Several well-known physicists, including the Nobel Prize laureates Gross, Nambu and Schwinger discussed approaches when quantum theory involves finite mathematics (see, e.g., [7]). A detailed discussion of these approaches has been given in the book [8], where they are characterized as hybrid quantum systems. The reason is that here coordinates and/or momenta belong to a finite ring or field, but wave functions are elements of standard complex Hilbert spaces. Then, the problem of the foundation of quantum theory is related to the problem of the foundation of classical mathematics. On the other hand, in [9,10], we have proposed an approach called finite quantum theory (FQT), where physical quantities also belong to a finite ring or field, but wave functions are elements of a space over a finite ring or a field. As explained in Section 5, FQT is more general (fundamental) than standard quantum theory.

3 Why finite mathematics is more natural than classical one

We will now discuss *whether it is justified to use mathematics with infinitesimals although in nature there are no infinitesimals*. As noted in Section 1, a typical situation in physics is that there are two theories, A and B, and the problem arises when B can be treated as a special degenerate case of A. It has also been noted that this problem can be considered in the framework of *Definition*, and several examples have been mentioned.

Let us consider that NT is a special degenerate case of RT in the special case $c \to \infty$. According to *Definition*, this implies that RT can reproduce any result of NT with any accuracy if c is chosen to be sufficiently large. However, NT cannot reproduce all results of RT because RT also describes phenomena where it is important that c is finite. From the naive point of view, one might think that NT is more general than RT because NT corresponds to the case $c = \infty$, i.e., one might think that NT describes more cases than RT, where c is finite. However, NT gives the same results as RT only when speeds are much less than c, but when they are comparable to c, then NT does not work.

Since in many cases speeds are much less than c, then, for describing those cases, NT works with very high accuracy and there is no need to apply RT: although in principle RT describes those cases, typically describing them by RT involves unnecessary complications. In particular, there is no need to apply RT for describing everyday life. At the same time, when speeds are comparable to c, it is important that c is not infinitely large but finite, and only RT can be applied.

Let us consider, for example, the following problem. Suppose that some reference frame moves relative to us with the speed $V = 0.6c$, and in this frame, a body moves in the same direction with the same speed. Then, the speed of the body relative to us is not $v = 1.2c$, as one might think from naive considerations, but $v \approx 0.882c$, and if, for example, $V = 0.99c$, then $v \approx 0.9999495c$, i.e., there is no way to get $v > c$. The lesson of this example is that it is not always correct to make judgments proceeding from “common sense.”

Analogously, for describing almost all phenomena on a macroscopic level, there is no need to apply QT. In particular, there is no need to describe the motion of the Moon by the Schrödinger equation. In principle, this is possible but results in unnecessary complications. At the same time, microscopic phenomena can be correctly described only in the framework of QT.

In view of those examples, the following problem arises: Is it justified to always use mathematics with infinitesimals for describing nature in which infinitesimals do not exist? There is no doubt that the technique of classical mathematics is very powerful and in many cases describes physical phenomena with a very high accuracy. However, a problem arises whether there are phenomena, which cannot be correctly described by mathematics involving infinitesimals.

Some facts of classical mathematics seem to be unnatural from the point of view of common sense. For example, $tg(x)$ is one-to-one reflection of $(-\pi/2, \pi/2)$ onto $(-\infty, \infty)$, i.e., the impression might arise that the both intervals have the same numbers of elements although the first interval is a nontrivial part of the second one. Another example is the Hilbert paradox with an infinite hotel. But mathematicians even treat those facts as pretty ones. For example, Hilbert said: "No one shall expel us from the paradise that Cantor has created for us."

From the point of view of Hilbert's approach to mathematics (see Section 1), it is not important whether the aforementioned statements are natural, since the goal of the approach is to find a complete and consistent set of axioms. In the framework of this approach, the problem of the foundation of classical mathematics has been investigated by many great mathematicians (e.g., Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo, and others). Their philosophy was based on macroscopic experience in which the concepts of infinitesimals, continuity and standard division are natural. However, as noted earlier, those concepts contradict the existence of elementary particles and are not natural in quantum theory. The illusion of continuity arises when one neglects the discrete structure of matter.

The fact that in Hilbert's approach there exist foundational problems follows, in particular, from Gödel's incompleteness theorems that state that no system of axioms can ensure that all facts about natural numbers can be proved, and the system of axioms in classical mathematics cannot demonstrate its own consistency. The theorems are written in highly technical terms of mathematical logics. As noted in Section 1, in this paper, we do not consider Hilbert's approach to mathematics. However, simple arguments in [2] show that, if mathematics is treated as a tool for describing nature, then foundational problems of classical mathematics follow from simple considerations, and below, we give those arguments.

In the 20s of the twentieth century, the Viennese circle of philosophers under the leadership of Schlick developed an approach called logical positivism, which contains the verification principle: *A proposition is only cognitively meaningful if it can be definitively and conclusively determined to be either true or false* (see, e.g., [11,12]). However, this principle does not work if classical mathematics is treated as a tool for describing nature. For example, in Hilbert's approach, one of the axioms is that $a + b = b + a$ for all natural numbers a and b, and a question of whether this is true or false does not arise. However, in the approach when mathematics is treated as a tool for describing nature, it cannot be determined whether this statement is true or false.

As noted by Grayling [13], "The general laws of science are not, even in principle, verifiable, if verifying means furnishing conclusive proof of their truth. They can be strongly supported by repeated experiments and accumulated evidence but they cannot be verified completely." So, from the point of view of classical mathematics and classical physics, verification principle is too strong.

Popper proposed the concept of falsificationism [14]: *If no cases where a claim is false can be found, then the hypothesis is accepted as provisionally true*. In particular, the statement that $a + b = b + a$ for all natural numbers a and b can be treated as provisionally true until one has found some numbers a and b for which $a + b \neq b + a$.

Before discussing the foundation of mathematics and physics in greater details, let us make several remarks about problems in accepting new theories. Probably, the main problem is the following. Our experience is based on generally acknowledged theories and everything not in the spirit of this experience is treated as contradicting common sense. A known example is that, from the point of view of classical mechanics, it seems unreasonable that the speed $0.999c$ is possible while the speed $1.001c$ is not. The reason for this judgment is that the experience based on everyday life works only for speeds that are much less than c, and extrapolating this experience to cases where speeds are comparable to c is not correct.

Another example is the paradox of twins in the theory of relativity: one of the brothers flew to a distant star, and when he returned being 10 years older, he realized that 1,000 years had passed on Earth. From the point of view of "common sense," this seems meaningless, but this seems so because our experience based on everyday life is extrapolated to the case of speeds comparable to c, and this experience does not work there.

One more example is the following. If we accept that physics in our world is described by finite mathematics with characteristics p, then this can be treated as the statement that p is the greatest possible number in nature. The argument attributed to Euclid is that there can be no greatest number, because if p is such a number, then $p + 1$ is greater than p. This is again an example where our experience based on rather small numbers is extrapolated to numbers where it does not work.

According to the philosophy of quantum theory, *in contrast to Hilbert's approach to mathematics*, there should be no statements accepted without proof and based on belief in their correctness (i.e., axioms). The theory should contain only those statements that can be verified, where by "verified" physicists mean an experiment involving only a finite number of steps. This philosophy is the result of the fact that quantum theory describes phenomena that, from the point of view of "common sense," seem meaningless but they have been experimentally verified. So, the philosophy of quantum theory is similar to verificationism, not falsificationism. Note that Popper was a strong opponent of quantum theory and supported Einstein in his dispute with Bohr.

From the point of view of verificationism and the philosophy of quantum theory, classical mathematics is not well defined because it contains an infinite number of numbers. Consider, for example, whether the rules of standard arithmetic can be justified.

We can verify that $10 + 10 = 20$ and $100 + 100 = 200$, but can we verify that, $10^{1000000} + 10^{1000000} = 2 \cdot 10^{1000000}$? One might think that this is obvious, and in Hilbert's approach, this follows from main axioms. But, if mathematics is treated as a tool for describing nature, then this is only a belief based on extrapolating our everyday experience to numbers where it is not clear whether the experience still works. According to the principles of quantum theory, the statement that $10^{1000000} + 10^{1000000} = 2 \cdot 10^{1000000}$ is true or false depends on whether this statement can be verified. Is there a computer that can verify this statement? Any computing device can operate only with a finite number of resources and can perform calculations only modulo some number p. If our universe is finite and contains only N elementary particles, then there is no way to verify that $N + N = 2N$. So, if, for example, our universe is finite, then in principle, it is not possible to verify that standard rules of arithmetic are valid for any numbers.

That is why the statement $a + b = c$ is ambiguous because it does not contain information on the computing device which will verify this statement. For example, let us pose a problem whether $10 + 20$ equals 30. If our computing device is such that $p = 40$, then the experiment will confirm that $10 + 20 = 30$, while if $p = 25$, then we will get that $10 + 20 = 5$.

So, the statements that $10 + 20 = 30$ *and even that* $2 \cdot 2 = 4$ *are ambiguous because they do not contain information on how they should be verified*. On the other hand, the statements

$$10 + 20 = 30 \pmod{40}, \quad 10 + 20 = 5 \pmod{25},$$
$$2 \cdot 2 = 4 \pmod{5}, \quad 2 \cdot 2 = 2 \pmod{2}$$

are well defined because they do contain such information. So only operations modulo a number are well defined.

I believe the following observation is very important: Although classical mathematics (including its constructive version) is a part of our everyday life, people typically do not realize that *classical mathematics is implicitly based on the assumption that one can have any desired number of resources*. So, classical mathematics is based on the implicit assumption that we can consider an idealized case when a computing device can operate with an infinite number of resources. Typically, people do not realize that *standard operations with natural numbers are implicitly treated as limits of operations modulo* p *when* $p \to \infty$. For example, if (a, b, c, d) are natural numbers, then the statements

$$a + b = c, \quad a \cdot b = d$$

are implicitly treated as follows:

$$\lim_{p\to\infty}[(a+b)\pmod p]=c,\quad \lim_{p\to\infty}[(a\cdot b)\pmod p]=d.$$

As a rule, every limit in mathematics is thoroughly investigated, but, in the case of standard operations with natural numbers, it is not even mentioned that those operations are limits of operations modulo p. In real life, such limits even might not exist if, for example, the universe contains a finite number of elementary particles.

4 A sketch of the proof that finite mathematics is more general than classical one

In the standard *technique* of classical mathematics, there is no number ∞, infinity is understood only as a limit (i.e., as a potential infinity) and, as a rule, legitimacy of every limit is thoroughly investigated. However, the *basis* of classical mathematics involves actual infinity from the very beginning. For example, the ring of integers Z is involved from the very beginning and, even in standard textbooks, it is not even posed a problem whether Z should be treated as a limit of finite rings. Moreover, Z is the starting point for constructing the sets of rational, real and complex numbers and the sets with greater and greater cardinalities.

As noted in Section 1, for solving the problem of infinities, different kinds of arithmetic have been proposed. However, finite mathematics rejects infinities from the beginning. This mathematics starts from the ring $R_p = (0, 1, 2, \ldots p-1)$, where addition, subtraction and multiplication are performed as usual but modulo p, and p is called the characteristic of the ring. In the literature, the ring R_p is usually denoted as $Z/(pZ)$. In my opinion, this notation is not adequate because finite mathematics should not involve infinite sets. The notation may give a wrong impression that finite mathematics starts from the infinite set Z and that Z is more general than R_p. However, although Z has more elements than R_p, Z cannot be more general than R_p because Z does not contain operations modulo a number. We will see below that the concept of R_p is more general than the concept of Z, and Z is a special degenerate case of R_p in the formal limit $p \to \infty$.

In the aforementioned discussion of the relation between NT and RT, we noted that those theories give close results when speeds are much less than c, but the results are considerably different when speeds are comparable to c, and in RT, it is not possible to get $v > c$. Analogously, the results in finite and classical mathematics are the same if the numbers in question are much less than p but, since in finite mathematics all operations are modulo p, it is not possible to get a result greater than p. Physicists might think that calculations modulo a number are nonphysical, but, as noted earlier, just such calculations are more physical than calculations in classical mathematics.

One can prove [2]

Statement 1: For any $p_0 > 0$, there exists a set S belonging to all sets R_p with $p \geq p_0$, and a natural number n such that for any $m \leq n$, the result of any m operations of summation, subtraction or multiplication of elements from S is the same for any $p \geq p_0$ and the same as in Z, and that cardinality of S and the number n formally go to infinity when $p_0 \to \infty$.

The proof is analogous to the standard proof that a sequence of natural numbers (a_n) goes to infinity if $\forall M > 0\ \exists n_0$, such that $a_n \geq M\ \forall n \geq n_0$. In particular, the proof involves only potential infinity but not actual one. This means that: (a) for the set S and the number n there is no manifestation of operations modulo p, i.e., the results of any $m \leq n$ operations of elements from S are formally the same in R_p and Z; (b) when p_0 increases, the set S also increases, and in the formal limit $p_0 \to \infty$, S becomes Z.

That is why Z can be treated as a limit of R_p when $p \to \infty$. This result looks natural from the following considerations. Since all operations in R_p are modulo p, then R_p can be treated as a set $(-(p-1)/2, \ldots -1, 0, 1, \ldots (p-1)/2)$ if p is odd and as a set $(-p/2+1, \ldots -1, 0, 1, \ldots p/2)$ if p is even. In this representation, for relatively small sets of numbers with the absolute values much less than p, the results of addition, subtraction and multiplication are the same in R_p and in Z, i.e., for such sets of numbers, it is not manifested that in R_p, operations are modulo p. Then, since in theory with R_p, there exist operations modulo p, which do not exist in theory with Z, it follows from *Statement 1* and *Definition* that:

Statement 2: Theory with Z is a special degenerate case of theory with R_p in the formal limit $p \to \infty$.

This result is natural from the following graphical representation of the sets Z and R_p. If elements of Z are depicted as integer points on the x axis of the xy plane, then, if p is odd, the elements of R_p can be depicted as points of the circle in Figure 1 and analogously if p is even.

The analogy between R_p and the circle follows from the following observations. If we take an element of R_p and successively add 1 to it, then after p steps, we will return to the original element because addition in R_p is modulo p. This is analogous to the fact that if we are moving along the circle in same direction, then, sooner or later, we will arrive to the initial point.

Let us also note that Figure 1 is analogous to the figure illustrating stereographic projection. In this case, every point on the circle, except the northern pole, is projected to a certain point on the x axis, but the projection of the northern pole is not defined, and it is not clear whether this point corresponds to $+\infty$ or $-\infty$. Analogously, Figure 1 shows, that the closest points to the northern pole are $(p-1)/2$ and $-(p-1)/2$, i.e., very large positive and negative numbers when p is very large. In R_p, those points are close to each other because $((p-1)/2+1)\ \mathrm{mod}(p) = -(p-1)/2$, i.e., when we add 1 to a large positive number $(p-1)/2$, we get a large negative number $-(p-1)/2$.

Figure 1 is also natural from the following historical analogy. For many years, people believed that the Earth was flat and infinite, and only after a long period of time, they realized that it was finite and curved. It is difficult to notice the curvature when we deal only with distances much less than the radius of the curvature. Analogously, when we deal with numbers the modulus of which is much less than p, the results are the same in Z and R_p, i.e., we do not notice the "curvature" of R_p. This "curvature" is manifested only when we deal with numbers the modulus of which is comparable to p.

As noted earlier, Dyson's idea [3] is that theory A is more general than theory B if the symmetry in B can be obtained from the symmetry in A by contraction. It is clear from Figure 1 that R_p has a higher symmetry than Z. Mathematically this follows from the following facts. As noted earlier, when we take an element $a \in R_p$ and successively add 1 to it, then after p steps, we will get all elements of R_p. However, all elements of Z can be obtained from an element $a \in Z$ only in two infinite stages when the first stage is successively adding 1 to a and the second stage is successively adding -1 to a.

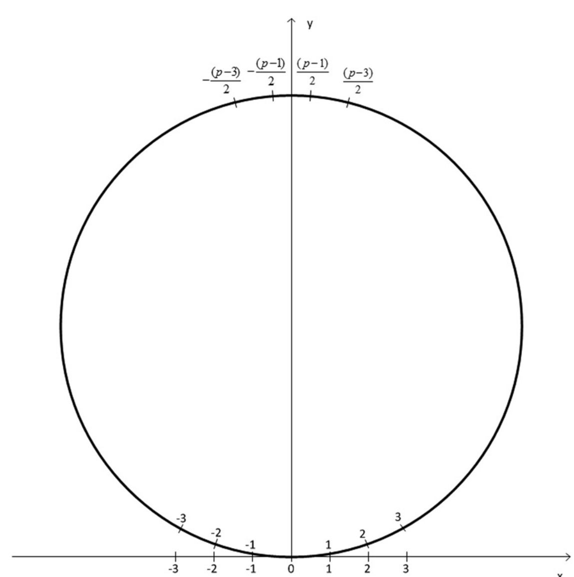

Figure 1: Relation between R_p and Z.

As already mentioned, in R_p, the elements $(p-1)/2$ and $-(p-1)/2$ are close to each other. The set Z can be treated as obtained from R_p as follows. First, we break the circle in Figure 1 at the top and move the points $(p-1)/2$ and $-(p-1)/2$ a great distance from each other. Then, in the formal limit $p \to \infty$, the part $(1, 2, \ldots (p-1)/2)$ of the circle becomes the part $(1, 2, \ldots \infty)$ of the straight line, and the part $(-1, -2, \ldots -(p-1)/2)$ of the circle becomes the part $(-1, -2, \ldots -\infty)$ of the straight line. Finally, by adding 0, we obtain the set Z.

This observation can be treated as an illustration of Dyson's idea because it becomes clear why R_p has a higher symmetry than Z. In Z, it is not possible to reproduce all results in R_p since in Z there are no operations modulo a number. The validity of *Statement 2* takes place although R_p contains less elements than Z. This situation is analogous to that discussed earlier that RT is more general than NT and to other cases discussed earlier when theory A is more general than theory B.

The fact that the theory with R_p is more general than the theory with Z implies that *even from purely mathematical point of view, the concept of infinity is not fundamental since, when we introduce infinity, we get the degenerate theory where all operations modulo a number disappear*.

The fact that $R_p \to Z$ when $p \to \infty$ can be proved in the framework of the theory of ultraproducts described in a vast literature. As pointed out to me by Zelmanov, infinite fields of zero characteristic (and Z) can be embedded in ultraproducts of finite fields. This approach is in the spirit of belief of many mathematicians that sets of characteristic 0 are more general that finite sets, and for investigating infinite sets, it might be convenient to use properties of simpler sets of positive characteristics.

The theory of ultraproducts is essentially based on classical results on infinite sets involving actual infinity. In particular, the theory is based on Łoŝ' theorem involving the axiom of choice. Therefore, the theory of ultraproducts cannot be used in proving that finite mathematics is more general than the classical one.

Let us also note that standard terminology that Z and the fields constructed from Z (e.g., the fields of rational, real and complex numbers) are sets of characteristic 0 reflects the usual spirit that classical mathematics is more fundamental than finite one. I think that it is natural to say that Z is the ring of characteristic ∞ because Z is a limit of rings of characteristic p when $p \to \infty$. The characteristic of the ring p is understood such that all operations in the ring are modulo p, but operations modulo 0 are meaningless. Usually, the characteristic n of the ring is defined as the smallest positive number n such that the sum of n units $1 + 1 + 1 \ldots$ in the ring equals zero if such a number exists and 0 otherwise. However, this sum can be written as $1 \cdot n$ and the equality $1 \cdot 0 = 0$ takes place in any ring.

5 Why finite quantum theory is more general than standard one

Consider now the following question. Does the fact that R_p is more general than Z mean that in applications finite mathematics is more general (fundamental) than classical one? Indeed, in applications, not only rings are used but also fields which contain division. For example, if p is prime, then R_p becomes the Galois field F_p in which division is defined as usual but modulo p.

As noted earlier, for numbers with the absolute values much less than p, the results of summation, subtraction and multiplication are the same in R_p and Z. That is why if an experiment deals only with such numbers, and the theory describing this experiment involves only sums, subtractions and multiplications, then the results of the experiment cannot answer the question what mathematics is more adequate for describing this experiment: classical or finite. However, in the case of division, the difference is essential. For example, $1/2$ in F_p equals $(p+1)/2$, i.e., a very large number if p is large. That is why an impression may arise that finite mathematics is not adequate for describing experimental data. Let us consider this problem in more detail.

Now it is accepted that the most general physical theory is quantum one, i.e., any classical theory is a special case of quantum one. This fact has been already mentioned earlier. Therefore, the problem arises whether quantum theory based on real and complex numbers containing division (and also quantum theories based on their generalizations, e.g., p-adic numbers or quaternions) can be a special case of a quantum theory based on finite mathematics.

In standard quantum theory (SQT), a state of a system is described by the wave function $\Psi = c_1e_1 + c_2e_2 + \dots$, where the e_j ($j = 1, 2, \dots \infty$) are the elements of the basis of the Hilbert space, and the c_j are complex coefficients. Usually, basis elements are normalized to one: $\|e_j\| = 1$, and then, the probability for a system to be in the state e_j is $|c_j|^2$. However, normalization to one is only the question of convention but not the question of principle. The matter is that not the probability itself but only relative probabilities of different events have a physical meaning. That is why spaces in quantum theory are projective: const $\cdot\, \Psi$ and Ψ describe the same state if const $\neq 0$.

Hence, one can choose the basis where all the $\|e_j\|$ are positive integers. Then, we use the theorem proved in standard textbooks on Hilbert spaces: any element of the Hilbert space can be approximated with any desired accuracy by a finite linear combination $\Psi = c_1e_1 + c_2e_2 + \dots + c_ne_n$, where the coefficients are rational numbers. Finally, by using the fact that spaces in quantum theory are projective, one can multiply Ψ by the common denominator of all the coefficients and get the case when all the complex coefficients $c_j = a_j + ib_j$ are such that all the numbers a_j and b_j are integers.

Therefore, although formally Hilbert spaces in quantum theory are complex, with any required accuracy, any state can be described by a set of coefficients, which are elements of $Z + iZ$. Hence, the description of states by means of Hilbert spaces is not optimal since such a description contains a big redundancy of elements, which are not needed for a full description.

Now we use *Statement 2* and describe quantum states not by elements of Hilbert spaces but by elements of spaces over a finite ring $R_p + iR_p$, i.e., now all the a_j, b_j and $\|e_j\|$ are elements of R_p. As noted earlier, we call this theory FQT. As mentioned earlier, FQT is more general than SQT: when the absolute values of all the a_j, b_j and $\|e_j\|$ are much less than p, then both theories give the same results, but if the absolute values of some of those quantities are comparable to p, then the descriptions are different because in SQT there are no operations modulo p. We conclude that if mathematics is treated as a tool for describing nature, then **Statement** in Section 1 is valid.

6 Examples when finite mathematics can solve problems that classical mathematics cannot

In [2], we considered phenomena where it is important that p is finite. They cannot be described by SQT, and this is analogous to the fact that NT cannot describe phenomena in which it is important that c is finite. Later, we describe several such phenomena.

Example 1. Gravity. Since quantum theory is treated as more general than the classical one, any result of classical theory should be a special case of a result obtained in quantum theory. However, the Newton gravitational law cannot be derived in QFT because the theory is nonrenormalizable. But in our approach, the universal law of gravitation can be derived as a consequence of FQT in semiclassical approximation [2]. In this case, the gravitational constant G is not a constant taken from the outside but a function of p, which depends on p as $1/\ln(p)$. By comparing the result with the experimental value, one gets that $\ln(p)$ is of the order of 10^{80} or more, and therefore, p is a huge number of the order of $\exp(10^{80})$ or more. One might think that since p is so huge, then in practice, p can be treated as an infinite number. However, since G depends on p as $1/\ln(p)$, and $\ln(p)$ is "only" of the order of 10^{80}, gravity is observable. In the formal limit $p \to \infty$, G becomes zero and gravity disappears. Therefore, in our approach, gravity is a consequence of finiteness of nature.

Example 2. Dirac vacuum energy problem. In quantum electrodynamics, the vacuum energy should be zero, but in the standard theory, the sum for this energy diverges, and this problem was posed by Dirac. To get the zero value for the vacuum energy, the artificial requirement that the operators should be written in the normal order is imposed, but this requirement does not follow the construction of the theory. In Section 8.8 of [2], I take the standard expression for this sum; then, I explicitly calculate this sum in finite mathematics without any assumptions, and, since all the calculations are modulo p, I obtain zero as it should be.

Example 3. Equality of masses of particles and their antiparticles. This is a very interesting example demonstrating the power of finite mathematics. Historically, the concept of particle–antiparticle has arisen because the Dirac equations have solutions with positive and negative energies. However, the probabilistic interpretation of the Dirac spinor is valid only in the approximation $1/c^2$, and particles and their antiparticles should have the same energy sign.

The fact that standard treatment of covariant field equations (Klein-Gordon equation, Dirac equations, Rarita-Schwinger equations and others) is not quite consistent, follows from the following observation. Since the equations are linear, any superposition of two solutions also is a solution. However, superpositions of two solutions with positive and negative energies are prohibited by superselection rules because solutions with positive energy are treated such that they describe particles, solutions with negative energy are treated such that they describe antiparticles, and their superposition contradicts conservation of electric charge, baryon quantum number, etc.

A detailed discussion in [2] shows that, on the quantum level, a particle and its antiparticle should be considered only from the point of view of irreducible representations (IRs) of the symmetry algebra. In SQT, the algebras are such that their IRs contain either only positive energies or only negative energies. In the first case, the objects described by IRs are called particles, and in the second one – antiparticles. Then, the energies of antiparticles become positive after second quantization. As noted in Section 1, dS and AdS symmetries are more general than Poincare symmetry. As explained in [2], in dS and AdS theories, the mass of the particle is dimensionless. For definiteness, we compare the results of the AdS theory in SQT and FQT.

In SQT, the spectrum of positive energies contains the values $(m_1, m_1+1, m_1+2, \ldots \infty)$, and for negative energies — the values $(-m_2, -m_2-1, -m_2-2, \ldots -\infty)$, where $m_1 > 0,\ \ m_2 > 0$, m_1 is called the mass of a particle and m_2 is called the mass of the corresponding antiparticle. Experimentally, $m_1 = m_2$, but in SQT, IRs with positive and negative energies are fully independent of each other. The usual statement is that $m_1 = m_2$ follows from the fact that local covariant equations (e.g., the Dirac equation) are CPT invariant. However, as discussed in detail in [2], the argument x in local quantized fields does not have a physical meaning because it is not associated with any operator. So, in fact, standard theory cannot explain why $m_1 = m_2$.

For understanding this problem, the following observation from particle theory may be helpful. In the formal case when electromagnetic and weak interactions are absent, isotopic invariance is exact, and the proton and the neutron have equal masses simply because they are different states in the same IR of the isotopic algebra. Therefore, the equality of the masses has nothing to do with locality.

Consider now what happens in FQT. For definiteness, we consider the case when p is odd, and the case when p is even can be considered analogously. One starts constructing the IR as usual with the value m_1, and, by acting on the states by raising operators, one gets the values $m_1+1, m_1+2, \ldots$. However, now we are moving not along the straight line but along the circle in Figure 1. When we reach the value $(p-1)/2$, then, as explained earlier, the next value is $-(p-1)/2$, i.e., one can say that by adding 1 to a large positive number $(p-1)/2$, one gets a large negative number $-(p-1)/2$. By continuing this process, one gets the numbers $-(p-1)/2+1 = -(p-3)/2$, $-(p-3)/2+1 = -(p-5)/2$, etc. The explicit calculation shows that the procedure ends when the value $-m_1$ is reached.

Therefore, finite mathematics gives a clear proof of the experimental fact that $m_1 = m_2$, and this is analogous to the aforementioned observation that two states have equal masses if they belong to the same IR of the symmetry algebra. In addition, finite mathematics shows that, instead of two independent IRs in classical mathematics, one gets only one IR describing both a particle and its antiparticle. The case described by classical mathematics can be called degenerate because, in the formal limit $p \to \infty$, one IR in finite mathematics splits into two IRs in classical mathematics. So, in the case $p \to \infty$, we get symmetry breaking. This example is a beautiful illustration of Dyson's idea [3] that theory A is more general than theory B if the symmetry in B can be obtained from the symmetry in A by contraction. The example is fully in the spirit of this idea because it shows that classical mathematical can be obtained from finite one by contraction of the symmetry in the formal limit $p \to \infty$. This example also shows that the standard concept of particle–antiparticle is only approximate and is approximately valid only when p is very large. Consequently, constructing complete quantum theory based on finite mathematics will be difficult because the construction should be based on new principles.

Example 4. The problem of baryon asymmetry of the universe. This problem is formulated as follows. According to the modern particle and cosmological theories, the numbers of baryons and antibaryons in the early stages of the universe were the same. Then, since the baryon number is the conserved quantum number, those numbers should be the same at the present stage. However, at this stage, the number of baryons is much greater than the number of antibaryons.

For understanding this problem, one should understand the concept of particle–antiparticle. As explained earlier, in SQT, this concept takes place because IRs describing particles and antiparticles are such that energies in them can be either only positive or only negative but cannot have both signs. However, as explained in **Example 3**, IRs in FQT necessarily contain both, positive and negative energies, and in the formal limit $p \to \infty$, one IR in FQT splits into two IRs in SQT with positive and negative energies.

If the laws of physics are described in finite mathematics with some p, then a question arises whether there are reasons for p to be as is or the value of p is a result of pure random circumstances. As noted earlier, every computing device can perform mathematical operations only modulo some number p, which is defined by the number of bits that this device can operate with. It is reasonable to believe that finite mathematics describing physics in our universe is characterized by a characteristic p, which depends on the current state of the universe, i.e., the universe can be treated as a computer. Therefore, it is reasonable to believe that the number p is different at different stages of the universe.

As noted in **Example 1**, at the present stage of the universe, the number p is huge, and therefore, the concepts of particles and antiparticles have a physical meaning. However, arguments given in [2] indicate that in early stages of the universe, the value of p was much less than now. Then, in general, each object described by IR is a superposition of particle and antiparticle (in SQT, such a situation is prohibited), and the electric charge and baryon quantum number are not conserved. Therefore, in early stages of the universe, SQT does not work, and the statement that at such stages the numbers of baryons and antibaryons were the same does not have a physical meaning. Therefore, the problem of baryon asymmetry of the universe does not arise.

Example 5. FQT gives arguments [2] that only Dirac's singletons [15] can be true elementary particles.

Example 6. FQT gives arguments [2] that the ultimate quantum theory will be based on a ring, not on a field, i.e., only addition, subtraction and multiplication are fundamental mathematical operations, while division is not.

The aforementioned examples demonstrate that there are phenomena that can be explained only in finite mathematics, because for them, it is important that p is finite and not infinitely large. So, we have an analogy with the case that RT can explain phenomena, where c is finite while NT cannot explain such phenomena.

7 Conclusion

In the literature, an idea is discussed that space and time should be quantized. However, as discussed in detail in [2], the concept of space–time has a physical meaning only on the classical level, i.e., when first FQT is approximated by SQT in the formal limit $p \to \infty$, and then, SQT is approximated by classical theory in the formal limit $\hbar \to 0$.

The problem of time is one of the most fundamental problems of quantum theory. Every physical quantity should be described by a selfadjoined operator but, as noted by Pauli, the existence of the time operator is a problem (see, e.g., the discussion in [2]). One of the principles of physics is that the definition of a physical quantity is a description how this quantity should be measured, and it is not correct to say that some quantity exists but cannot be measured. The present definition of a second is the time during which 9,192,631,770 transitions in a caesium-133 atom occur. The time cannot be measured with the absolute accuracy because the number of transitions cannot be infinite. With this definition, one second is defined with the accuracy $10^{-15}s$, and, e.g., [16] describes efforts to measure time with the accuracy $10^{-19}s$. However, it is not clear how to define time in early stages of the universe when atoms did not exist. So, treating time t as a continuous quantity belonging to the interval $(-\infty, +\infty)$ can be only an approximation, which works at some conditions. In [2], a conjecture that standard classical time t manifests itself because the value of p changes, i.e., t is a function of p has been discussed. We do not say that p changes over time because classical time t cannot be present in quantum theory; we say that we feel t because p changes. As noted in **Example 4** of the preceding section, and will be discussed in more details in a separate publication, with such an approach, the known problem of baryon asymmetry of the universe does not arise.

Let us note that in FQT there are no infinities in principle and that is why divergences are absent in principle. In addition, probabilistic interpretation of FQT is only approximate: It applies only to states described by the numbers a_j, b_j and $\|e_j\|$, which are much less than p.

This situation is a good illustration of the famous Kronecker's expression: "God made the natural numbers, all else is the work of man." In view of the aforementioned discussion, I propose to reformulate this expression as follows: "God made only finite sets of natural numbers, all else is the work of man." For illustration, consider a case when some experiment is conducted N times, the first event happens n_1 times, the second one – n_2 times, etc., such that $n_1 + n_2 + \ldots = N$. Then, the experiment is fully described by a finite set of natural numbers. But people introduce rational numbers $w_i = w_i(N) = n_i/N$ and introduce the concept of limit and define probabilities as limits of the quantities $w_i(N)$ when $N \to \infty$.

As noted in Section 1, when classical and finite mathematics are considered in the framework of Hilbert's approach (i.e., only as abstract sciences), then the question what mathematics is more adequate in applications does arise. However, the aforementioned discussion shows that, if mathematics is treated as a way of describing nature, then finite mathematics is more general (fundamental) than the classical one. In addition, in finite mathematics there are no foundational problems because every statement can be explicitly verified by a finite number of steps. The conclusion from the aforementioned consideration can be formulated as follows:

Mathematics describing nature at the most fundamental level involves only a finite number of numbers, while the concepts of limit, infinitesimals and continuity are needed only in calculations describing nature approximately.

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References

[1] E. Wigner, *The unreasonable effectiveness of mathematics in the natural sciences*, Commun. Pure Appl. Anal. **13** (1960), no. 1, 1–14, DOI: https://doi.org/10.1002/cpa.3160130102.
[2] F. Lev, *Finite Mathematics as the Foundation of Classical Mathematics and Quantum Theory. With Applications to Gravity and Particle Theory*, Springer, Cham, Switzerland, 2020, ISBN 978-3-030-61101-9.
[3] F. G. Dyson, *Missed opportunities*, Bull. Amer. Math. Soc. **78** (1972), no. 5, 635–652.
[4] O. Consa, *Something is wrong in the state of QED*, arXiv preprint (2021), https://arxiv.org/abs/2110.02078.
[5] S. Weinberg, *The Quantum Theory of Fields*, vol. I, Cambridge University Press, Cambridge, UK, 1999.
[6] S. Weinberg, *Living with Infinities*, arXiv preprint (2009), https://arxiv.org/abs/0903.0568.
[7] Y. Nambu, *Field theory of Galois fields*, in: Quantum Field Theory and Quantum Statistics, I. A. Batalin et al. editor, Adam Hilger, Bristol, GB, 1987, pp. 625–636.
[8] A. Vourdas, *Finite and profinite quantum systems*, in: Series “Quantum Science and Technology,” Springer Nature, Berlin/Heidelberg, Germany, 2017.
[9] F. Lev, *Modular representations as a possible basis of finite physics*, J. Math. Phys. **30** (1989), no. 9, 1985–1998, DOI: https://doi.org/10.1063/1.528235.
[10] F. Lev, *Finiteness of physics and its possible consequences*, J. Math. Phys. **34** (1993), no. 2, 490–527, DOI: https://doi.org/10.1063/1.530257.
[11] C. J. Misak, *Verificationism: Its History and Prospects*, Routledge, New York, USA, 1995.
[12] G. William, *Lycan’s Philosophy of Language: A Contemporary Introduction*, Routledge, New York, USA, 2000.
[13] A. C. Grayling, *Ideas That Matter*, Basic Books, New York, USA, 2012.
[14] J. Agassi, *Popper and His Popular Critics: Thomas Kuhn, Paul Feyerabend and Imre Lakatos*, Springer, Berlin/Heidelberg, Germany, 2014.
[15] P. A. M. Dirac, *A remarkable representation of the* $3 + 2$ *de Sitter group*, J. Math. Phys. **4** (1963), no. 7, 901–909, DOI: https://doi.org/10.1063/1.1704016.
[16] I. Ushijima, M. Takamoto, and H. Katori, *Operational magic intensity for Sr optical lattice clocks*, Phys. Rev. Lett. **121** (2018), no. 26–28, 263202, DOI: https://doi.org/10.1103/PhysRevLett.121.263202.
[17] M. Burgin and M. Czachor, *Non-Diophantine Arithmetic in Mathematics, Physics and Psychology*, World Scientific, Singapore, 2020, ISBN: 978-981-121-432-5.