Research Article

Shuzhen Luo and Xiaoquan Xu*

Relational representations of algebraic lattices and their applications

https://doi.org/10.1515/math-2022-0010
received July 15, 2021; accepted January 15, 2022

Abstract: In this paper, we define the concepts of strongly regular relation, finitely strongly regular relation, and generalized finitely strongly regular relation, and get the relational representations of strongly algebraic, hyperalgebraic, and quasi-hyperalgebraic lattices. The main results are as follows: (1) a binary relation $\rho : X \rightharpoonup Y$ is strongly regular if and only if the complete lattice $(\Phi_\rho(X), \subseteq)$ is a strongly algebraic lattice; (2) a binary relation $\rho : X \rightharpoonup Y$ is finitely strongly regular if and only if $(\Phi_\rho(X), \subseteq)$ is a hyperalgebraic lattice if and only if the finite extension of ρ is strongly regular; and (3) a binary relation $\rho : X \rightharpoonup Y$ is generalized finitely strongly regular if and only if $(\Phi_\rho(X), \subseteq)$ is a quasi-hyperalgebraic lattice if and only if $(\Phi_\rho(X), \subseteq)$ equipped with the interval topology is a Priestley space.

Keywords: strongly regular relation, finitely strongly regular relation, generalized finitely strongly regular relation, Priestley spaces

MSC 2020: 06B35, 54H10, 06A11

1 Introduction

The regularity of binary relations was first characterized by Zareckiĭ. In [1], he proved the following remarkable result: a binary relation ρ on a set X is regular if and only if the complete lattice $(\Phi_\rho(X), \subseteq)$ is completely distributive, where $\Phi_\rho(X) = \{\rho(A) : A \subseteq X\}$, $\rho(A) = \{y \in X : \exists a \in A \text{ with } (a, y) \in \rho\}$. Further criteria for regularity were given by Markowsky [2] and Schein [3] (see also [4] and [5]). Motivated by the fundamental works of Zareckiĭ on regular relations, in [6–8], Xu and Liu successfully generalized the regular relation to finitely regular relation and generalized finitely regular relation, got the relational representations of hypercontinuous lattices and quasi-hypercontinuous lattices, and proved that a binary relation ρ is finitely regular if and only if the complete lattice $(\Phi_\rho(X), \subseteq)$ is a hypercontinuous lattice and ρ is generalized finitely regular if and only if $(\Phi_\rho(X), \subseteq)$ is a quasi-hypercontinuous lattice if and only if the interval topology on $(\Phi_\rho(X), \subseteq)$ is T_2. Moreover, the discussion about the relation representation theory of lattices has also attracted a considerable deal of attention (see [9–18]).

As a common generalization of continuous lattices, algebraic lattices play a very important role in domain theory. The so-called strongly algebraic [19], hyperalgebraic [20,21], and quasi-hyperalgebraic lattices [21,22] are among the most successful such generalizations, which now have been familiar objects as algebraic lattices in domain theory. However, the relation representations of them are still lacking. The aim of this paper is to introduce and investigate some binary relations in order to establish the theory of relation representations of algebraic lattices.

*** Corresponding author: Xiaoquan Xu,** Fujian key Laboratory of Granular Computing and Applications, Minnan Normal University, Zhangzhou, Fujian 363000, China, e-mail: xiqxu2002@163.com
Shuzhen Luo: Faculty of Science, Jiangxi University of Science and Technology, Ganzhou, Jiangxi 341000, China, e-mail: shuzhenluo@163.com

In this paper, we define the concepts of strongly regular relation, finitely strongly regular relation, and generalized finitely strongly regular relation, and get the relational representations of strongly algebraic, hyperalgebraic, and quasi-hyperalgebraic lattices. Meanwhile, several equivalent characterizations of them are obtained. Let $(X, \delta, \leq)$ be a partially ordered topological space. $(X, \delta, \leq)$ is called totally order-disconnected if whenever $x \nleq y$ there is a clopen upper set U such that $x \in U$ and $y \notin U$. A compact totally order-disconnected space is called a Priestley space. The interest in these spaces is mainly due to the celebrated Priestley duality [23,24]. We also discuss this Priestley property about complete lattices with respect to the interval topology. In this paper, the main results are as follows: (1) a binary relation $\rho : X \rightharpoonup Y$ is strongly regular if and only if the complete lattice $(\Phi_\rho(X), \subseteq)$ is a strongly algebraic lattice; (2) a binary relation $\rho : X \rightharpoonup Y$ is finitely strongly regular if and only if $(\Phi_\rho(X), \subseteq)$ is a hyperalgebraic lattice if and only if the finite extension of ρ is strongly regular; and (3) a binary relation $\rho : X \rightharpoonup Y$ is generalized finitely strongly regular if and only if $(\Phi_\rho(X), \subseteq)$ is a quasi-hyperalgebraic lattice if and only if $(\Phi_\rho(X), \subseteq)$ equipped with the interval topology is a Priestley space.

2 Preliminaries

In this section, we recall some basic concepts needed in this paper; for other nonexplicitly stated elementary notions please refer to [6,20].

Let P be a poset with a partial order $\leq$. For all $x, y \in P$, $A \subseteq P$, let $\uparrow x = \{y \in P : x \leq y\}$ and $\uparrow A = \bigcup_{a \in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. A is called the *upper set* if $\uparrow A = A$ and the *lower set* defined dually. The least upper bound of A in P is written as $\vee A$ or $\sup A$; similarly, the greatest lower bound of A is written as $\wedge A$ or $\inf A$. Define an order $\leq^{op}$ by $x \leq^{op} y \Leftrightarrow y \leq x$, and write P^{op} as short for $(P, \leq^{op})$, called the *order dual* of P. A poset L is called the *complete lattice* if for any subset $A \subseteq L$, $\vee A$ and $\wedge A$ exist in L. For two complete lattices L_1 and L_2, the symbol $L_1 \cong L_2$ means that L_1 is *order isomorphic* to L_2.

For a poset P, $x \in P$, $A \subseteq P$. $P \backslash A = \{x \in P : x \notin A\}$. The topology generated by the collection of sets $P \backslash \downarrow x$ (as a subbase) is called the *upper topology* and is denoted by $\upsilon(P)$; the *lower topology* $\omega(P)$ on P is defined dually. The topology $\theta(P) = \upsilon(P) \vee \omega(P)$ is called the *interval topology* on P, i.e., the interval topology is the coarsest common refinement of the upper topology $\upsilon(P)$ and the lower topology $\omega(P)$. For any set X, let $X^{(<\omega)} = \{F \subseteq X : F \text{ is nonempty and finite}\}$. **Set** denotes the class of all sets. The class of all complete lattices is denoted by **Com**.

Let (X, δ) be a topology space and $A \subseteq X$. The interior of A in (X, δ) is denoted by $\mathrm{int}_\delta A$. A subset U of X is called δ-clopen if U is open and closed in (X, δ). The notation $(X, \delta, \leq)$ is used to denote a set X endowed with a topology δ and a partially order $\leq$. The space $(X, \delta, \leq)$ is called *totally order-disconnected*, if given $x, y \in X$ with $x \nleq y$, there exists a δ-clopen upper set U such that $x \in U$ and $y \notin U$. A compact totally order-disconnected space is called a *Priestley space*.

For two sets X and Y, we call $\rho : X \rightharpoonup Y$ a binary relation if $\rho \subseteq X \times Y$. When $X = Y$, ρ is usually called a binary relation on X. Let $\mathcal{B}(X)$ denote the set of all binary relations between X and Y, and $\mathbf{Rel} = \bigcup_{X \in \mathbf{Set}} \mathcal{B}(X)$.

Definition 2.1. [6] Let $\rho : X \rightharpoonup Y$, $\tau : Y \rightharpoonup Z$ be two binary relations. Define
(1) $\tau \circ \rho = \{(x, z) : \exists y \in Y, (x, y) \in \rho, (y, z) \in \tau\}$. The relation $\tau \circ \rho : X \rightharpoonup Z$ is called the *composition* of ρ and τ.
(2) $\rho^{-1} = \{(y, x) \in Y \times X : (x, y) \in \rho\}$.
(3) $\rho(A) = \{y \in Y : \exists x \in A \text{ with } (x, y) \in \rho\}$. We call it the image of A under a binary relation ρ. Instead of $\rho(\{x\})$, we write $\rho(x)$ for short.
(4) $\Phi_\rho(X) = \{\rho(A) : A \subseteq X\}$.

Clearly, $(\Phi_\rho(X), \subseteq)$ is a complete lattice, in which the join operation $\vee$ is the set union operator $\cup$, since $\bigvee_{i \in I} \rho(A_i) = \bigcup_{i \in I} \rho(A_i) = \rho(\bigcup_{i \in I} A_i)$ for any $A_i \subseteq X$. But the meet operation $\wedge$ in $(\Phi_\rho(X), \subseteq)$ is not the set union operation $\cap$ in general. For instance, let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$. Define a binary relation $\rho : X \rightharpoonup Y$

as follows: $\rho(x_1) = \{y_1, y_2\}$ and $\rho(x_2) = \{y_2, y_3\}$. Then $\rho(x_1) \cap \rho(x_2) = \{y_1, y_2\} \cap \{y_2, y_3\} = \{y_2\}$. But $\{y_2\} \notin \Phi_\rho(X)$ since for any $C \subseteq X$, $\rho(C) \neq \{y_2\}$. Thus, $\{y_2\}$ is not the infimum of $\rho(x_1)$ and $\rho(x_2)$ in $(\Phi_\rho(X), \subseteq)$. In fact, it is easy to see that $\rho(x_1) \wedge \rho(x_2) = \varnothing$ in $(\Phi_\rho(X), \subseteq)$.

Definition 2.2. [6] Suppose p is a property about a complete lattice and s is a property about a binary relation. Let $\mathbf{S} = \{\rho \in \mathbf{Rel} : \rho$ has the property $s\}$ and $\mathbf{P} = \{L \in \mathbf{Com} : L$ has the property $p\}$. We call the relation of s type is a representation of the complete lattices of p type, if the following two conditions are satisfied:
(i) $\forall \rho \subseteq X \times Y, \rho \in \mathbf{S} \Leftrightarrow (\Phi_\rho(X), \subseteq) \in \mathbf{P}$;
(ii) $\forall L \in \mathbf{P}, \exists \rho : X \rightharpoonup Y \in \mathbf{S}$ such that $L \cong (\Phi_\rho(X), \subseteq)$.

In order to make sense of the s type relational representation of p type lattices, the properties p and s need to be explicitly defined.

Definition 2.3. [1] A binary relation $\rho : X \rightharpoonup Y$ is called *regular*, if there is a binary relation $\sigma : Y \rightharpoonup X$ such that $\rho = \rho \circ \sigma \circ \rho$.

In [1], Zareckiĭ gave the regular relation representation of completely distributive lattices. Furthermore, an intrinsic characterization of regular relations was obtained as follows.

Theorem 2.4. [6] *For a binary relation $\rho : X \rightharpoonup X$, the following two conditions are equivalent*:
(1) *ρ is regular*;
(2) *$\forall (x, y) \in \rho, \exists u, v \in X$ such that*
  (a) $(x, v) \in \rho, (u, y) \in \rho$;
  (b) *For all $s, t \in X$, if $(s, v) \in \rho, (u, t) \in \rho$, then $(s, t) \in \rho$.*

Definition 2.5. [6] For a binary relation $\rho : X \rightharpoonup Y$, define a relation $\rho^{(<\omega)} : X^{(<\omega)} \rightharpoonup Y^{(<\omega)}$, *called the finite extension of ρ, by*

$$\forall (F, G) \in X^{(<\omega)} \times Y^{(<\omega)}, (F, G) \in \rho^{(<\omega)} \Leftrightarrow G \subseteq \rho(F).$$

For a complete lattice L, $x, y \in L$. Define $\lhd$ on L by $x \lhd y \Leftrightarrow$ for any subset $A \subseteq L$, $y \leq \vee A$ implies $x \leq a$ for some $a \in A$. Raney [14] proved that a complete lattice L is a completely distributive iff $x = \vee\{y \in L : y \lhd x\}$.

Definition 2.6. [19] A complete lattice L is called *strongly algebraic*, if $\forall x \in L$, $x = \vee\{y \in L : y \lhd y \leq x\}$.

Definition 2.7. [20, 22] Let L be a complete lattice and $x, y \in L$. We define $y \prec x$ if and only if $x \in \operatorname{int}_{\upsilon(L)} \uparrow y$. L is called *hyperalgebraic*, if for any $x \in L$, $x = \vee\{y \in L : y \prec y \leq x\}$.

In [22], Yang proved that a complete lattice L is *hyperalgebraic* if and only if for any $x \in L$ and $U \in \upsilon(L)$ with $x \in U$, there exists $u \in L$ such that $x \in \operatorname{int}_{\upsilon(L)} \uparrow u = \uparrow u \subseteq U$. As the generalization of hyperalgebraic lattices, Yang also gave the notion of quasi-hyperalgebraic lattices.

Definition 2.8. [22] Let L be a complete lattice. L is called *quasi-hyperalgebraic*, if for any $x \in L$ and $U \in \upsilon(L)$ with $x \in U$, there exists $F \in L^{(<\omega)}$ such that $x \in \operatorname{int}_{\upsilon(L)} \uparrow F = \uparrow F \subseteq U$.

3 Strongly regular relation

In this section, we define a concept of strongly regular relation and get the strongly regular relation representation of strongly algebraic lattices. Meanwhile, we also obtain some equivalent characterizations of strongly regular relation.

Definition 3.1. A binary relation $\rho : X \rightharpoonup Y$ is called *strongly regular*, if there is a binary relation $\sigma \subseteq \rho^{-1} : Y \rightharpoonup X$ such that $\rho = \rho \circ \sigma \circ \rho$.

By the definition of strongly regular relation, it is easy to obtain the following conclusion.

Remark 3.2.
(1) If a binary relation $\rho : X \rightharpoonup Y$ is strongly regular, then ρ is regular.
(2) If $\rho : X \rightharpoonup Y$ is strongly regular, then $\rho^{-1} : Y \rightharpoonup X$ is also.
(3) Let X, Y be two sets and f a function from X to Y. Then f is strongly regular. In fact, define a binary relation $\tau : Y \rightharpoonup X$ as $(y, x) \in \tau \Leftrightarrow y = f(x)$. Then $\tau \subseteq f^{-1}$ and $f \circ \tau \circ f = f$.
(4) Let X be a set and $\mathcal{P}(X)$ a power set of X. Then the binary relation $\in: X \rightharpoonup \mathcal{P}(X)$, $x \in A$ is strongly regular. In fact, define a binary relation $\sigma : \mathcal{P}(X) \rightharpoonup X$ as $(A, x) \in \sigma \Leftrightarrow A = \{x\}$. Then $\sigma \subseteq \in^{-1}$ and $\in \circ \sigma \circ \in = \in$.

Lemma 3.3. *Let $\rho : X \rightharpoonup Y$ be a binary relation and $M \in \Phi_\rho(X)$. Then for any $y \in M$, $\bigwedge_{y \in N \in \Phi_\rho(X)} N \lhd M$. Furthermore, if $z \in \bigwedge_{z \in N \in \Phi_\rho(X)} N$, then $\bigwedge_{z \in N \in \Phi_\rho(X)} N \lhd \bigwedge_{z \in N \in \Phi_\rho(X)} N$.*

Proof. Let $y \in M$ and $M \subseteq \bigcup_{i \in I} \rho(A_i)$. Then $\exists i \in I$ such that $y \in \rho(A_i)$, which implies $\bigwedge_{y \in N \in \Phi_\rho(X)} N \subseteq \rho(A_i)$. Thus, $\bigwedge_{y \in N \in \Phi_\rho(X)} N \lhd M$. Therefore, if $z \in \bigwedge_{z \in N \in \Phi_\rho(X)} N$, then $\bigwedge_{z \in N \in \Phi_\rho(X)} N \lhd \bigwedge_{z \in N \in \Phi_\rho(X)} N$. □

Now we give the strongly regular relation representation of strongly algebraic lattices.

Theorem 3.4. *For a binary relation $\rho : X \rightharpoonup Y$, the following three conditions are equivalent:*
(1) *ρ is strongly regular.*
(2) *$\forall (x, y) \in X \times Y$ with $(x, y) \in \rho$, $\exists (u, v) \in X \times Y$ such that*
  (i) *$(u, v) \in \rho$;*
  (ii) *$(x, v) \in \rho$, $(u, y) \in \rho$;*
  (iii) *$\forall (s, t) \in X \times Y$, if $(s, v) \in \rho$ and $(u, t) \in \rho$, then $(s, t) \in \rho$.*
(3) *$(\Phi_\rho(X), \subseteq)$ is a strongly algebraic lattice, i.e., $\forall A \subseteq X$, $\rho(A) = \bigcup\{G \in \Phi_\rho(X) : G \lhd G \subseteq \rho(A)\}$.*
(4) *$\forall M \in \Phi_\rho(X)$, $y \in M$, $\exists z \in M$ such that $z \in \bigwedge_{z \in N \in \Phi_\rho(X)} N$, $y \in \bigwedge_{z \in N \in \Phi_\rho(X)} N$.*
(5) *$\forall M \in \Phi_\rho(X)$, $M = \bigcup_{v \in M} \bigwedge_{v \in N \in \Phi_\rho(X)} N$, and $\forall y \in M$, $\exists z \in M$ such that $z \in \bigwedge_{z \in N \in \Phi_\rho(X)} N$, $y \in \bigwedge_{z \in N \in \Phi_\rho(X)} N$.*

Proof.

(1) ⇒ (2) Let ρ be a strongly regular relation. Then there exists $\sigma \subseteq \rho^{-1}$ such that $\rho = \rho \circ \sigma \circ \rho$. $\forall (x, y) \in X \times Y$ with $(x, y) \in \rho$, $\exists (v, u) \in Y \times X$ such that $(x, v) \in \rho$, $(v, u) \in \sigma \subseteq \rho^{-1}$, $(u, y) \in \rho$, so conditions (i) and (ii) hold. $\forall (s, t) \in X \times Y$, if $(s, v) \in \rho$ and $(u, t) \in \rho$, by $(v, u) \in \sigma$ and $\rho = \rho \circ \sigma \circ \rho$, we have $(s, t) \in \rho$.

(2) ⇒ (3) $\forall N = \rho(A) \in \Phi_\rho(X)$, $y \in N$, $\exists x \in A$ such that $(x, y) \in \rho$. By (2), there is a $(u, v) \in X \times Y$ such that
(i) $(u, v) \in \rho$;
(ii) $(x, v) \in \rho$, $(u, y) \in \rho$;
(iii) $\forall (s, t) \in X \times Y$, if $(s, v) \in \rho$ and $(u, t) \in \rho$, then $(s, t) \in \rho$.

Thus, $y \in \rho(u) \subseteq \rho(x)$. In fact, $\forall t \in \rho(u)$, since $(x, v) \in \rho$, $(u, t) \in \rho$ and (iii), we have that $(x, t) \in \rho$, i.e., $t \in \rho(x)$.

Now we only need to show $\rho(u) \lhd \rho(u)$. Let $\rho(u) \subseteq \bigcup_{i \in I} \rho(A_i)$. Then $\exists i \in I$ such that $v \in \rho(A_i)$ since $(u, v) \in \rho$, which implies that there exists $x_i \in A_i$ such that $v \in \rho(x_i)$, i.e., $(x_i, v) \in \rho$. For any $t \in \rho(u)$, i.e., $(u, t) \in \rho$, by (iii) $(x_i, t) \in \rho$, that is, $t \in \rho(x_i)$, so $\rho(u) \subseteq \rho(x_i) \subseteq \rho(A_i)$. Hence, $\rho(u) \lhd \rho(u)$. Therefore, $\rho(A) = \bigcup\{G \in \Phi_\rho(X) : G \lhd G \subseteq \rho(A)\}$.

(3) ⇒ (4) Let $M = \rho(A) \in \Phi_\rho(X)$ and $y \in M$. Since $(\Phi_\rho(X), \subseteq)$ is a strongly algebraic lattice, $\exists G \in \Phi_\rho(X)$ such that $y \in G \lhd G \subseteq M$. Let $G = \rho(B) = \bigcup_{b \in B} \rho(b)$. By the definition of $\lhd$, $\exists b \in B$ such that $G = \rho(b)$ and $\rho(b) \nsubseteq \bigcup(\Phi_\rho(X) \backslash \uparrow \rho(b)) = \bigcup\{H \in \Phi_\rho(X) : \rho(b) \nsubseteq H\}$. Thus, $\exists z \in \rho(b) \backslash \bigcup(\Phi_\rho(X) \backslash \uparrow \rho(b))$.

Now we show that $\rho(b) = \bigwedge_{z\in N\in\Phi_\rho(X)} N$. Let $N \in \Phi_\rho(X)$ with $z \in N$. Assume that $\rho(b) \nsubseteq N$, then $N \in \Phi_\rho(X)\backslash\uparrow\rho(b)$. Thus, $z \in N \subseteq \bigcup(\Phi_\rho(X)\backslash\uparrow\rho(b))$, which contradicts $z \notin \bigcup(\Phi_\rho(X)\backslash\uparrow\rho(b))$. Thus, $\rho(b) \subseteq N$, which implies $\rho(b) \subseteq \bigwedge_{z\in N\in\Phi_\rho(X)} N$. The converse inclusion is always true, since $z \in G = \rho(b)$. Hence, $M = \bigcup_{v\in M}\bigwedge_{v\in N\in\Phi_\rho(X)} N$, and for any $y \in M$, there is a $z \in M$ such that $z \in \bigwedge_{z\in N\in\Phi_\rho(X)} N$, $y \in \bigwedge_{z\in N\in\Phi_\rho(X)} N$.

(4) ⇒ (5) Obviously.

(5) ⇒ (1) Define a relation $\sigma : Y \rightharpoonup X$ by

$$(y, x) \in \sigma \Leftrightarrow y \in \rho(x) = \bigwedge_{y\in N\in\Phi_\rho(X)} N.$$

Obviously $\sigma \subseteq \rho^{-1}$.

Next we will prove that $\rho = \rho\circ\sigma\circ\rho$. $\forall(x, y) \in \rho$, $y \in \rho(x)$, by (5), $\rho(x) = \bigcup_{v\in M}\bigwedge_{v\in N\in\Phi_\rho(X)} N$, and $\exists z \in \rho(x)$ such that $z, y \in \bigwedge_{z\in N\in\Phi_\rho(X)} N$. By Lemma 3.3, $\bigwedge_{z\in N\in\Phi_\rho(X)} N \lhd \bigwedge_{z\in N\in\Phi_\rho(X)} N$. Let $\bigwedge_{z\in N\in\Phi_\rho(X)} N = \rho(C) = \bigcup_{c\in C}\rho(c)$. Then $\exists c \in C$ such that $\bigwedge_{z\in N\in\Phi_\rho(X)} N = \rho(c)$. Thus, $z \in \rho(c) = \bigwedge_{z\in N\in\Phi_\rho(X)} N$. Follows from the definition of σ, it is that $(z, c) \in \sigma$. Note that $(x, z) \in \rho$ and $(c, y) \in \rho$, we have that $(x, y) \in \rho\circ\sigma\circ\rho$. Hence, $\rho \subseteq \rho\circ\sigma\circ\rho$. $\forall(x, y) \in \rho\circ\sigma\circ\rho$, then $\exists(p, q) \in X\times Y$ such that $(x, q) \in \rho$, $(q, p) \in \sigma$ and $(p, y) \in \rho$. Since $(q, p) \in \sigma$, $q \in \rho(p) = \bigwedge_{q\in N\in\Phi_\rho(X)} N$. Thus, $y \in \rho(p) = \bigwedge_{q\in N\in\Phi_\rho(X)} N \subseteq \rho(x)$, i.e., $(x, y) \in \rho$. Therefore, ρ is strongly regular. □

Theorem 3.5. *Let L be a complete lattice. Then the following conditions are equivalent*:
(1) *L is a strongly algebraic lattice, i.e.*, $\forall x \in L$, $x = \vee\{z \in L : z \lhd z \leq x\}$.
(2) *There is a strongly regular relation $\rho : X \rightharpoonup X$ such that $L \cong (\Phi_\rho(X), \subseteq)$.*
(3) *The relation $\nleq$ on L is strongly regular, i.e.*, $\forall x, y \in L$ *with* $x \nleq y$, $\exists u, v \in L$ *such that*
  (*i*) $u \nleq v$;
  (*ii*) $u \nleq y$, $x \nleq v$;
  (*iii*) $\forall s, t \in L$, *if* $u \nleq t$, $s \nleq v$, *then* $s \nleq t$.
(4) $\forall x, y \in L$ *with* $x \nleq y$, $\exists u, v \in L$ *such that*
  (*i*) $u \nleq v$;
  (*ii*) $u \nleq y$, $x \nleq v$;
  (*iii*) $\forall z \in L$, $u \leq z$ *or* $z \leq v$.
(5) $\forall x, y \in L$ *with* $x \nleq y$, $\exists u, v \in L$ *such that*
  (*a*) $x \notin \downarrow v$, $y \notin \uparrow u$;
  (*b*) $\downarrow v \cup \uparrow u = L$, $\downarrow v \cap \uparrow u = \varnothing$.

Proof.

(1) ⇒ (2) Let $X = L$. Define a binary relation $\rho : X \rightharpoonup X$, $(x, y) \in \rho \Leftrightarrow y \lhd y \leq x$ and a relation $\sigma : X \rightharpoonup X$, $(u, v) \in \sigma \Leftrightarrow u \lhd v$ and $v = u$. Obviously, $\sigma \subseteq \rho^{-1}$. We claim that $\rho = \rho\circ\sigma\circ\rho$. In fact, for any $(x, y) \in \rho$, i.e., $y \lhd y \leq x$. Since L is strongly algebraic, $\exists z \in L$ such that $y \lhd y \leq z \lhd z \leq x$. By the definition of ρ and σ, we have $(x, z) \in \rho$, $(z, z) \in \sigma$, and $(z, y) \in \rho$. Hence, $(x, y) \in \rho\circ\sigma\circ\rho$. For any $(x, y) \in \rho\circ\sigma\circ\rho$, $\exists u \in L$ such that $(x, u) \in \rho$, $(u, u) \in \sigma$, and $(u, y) \in \rho$. Then $y \lhd y \leq u \lhd u \leq x$, which implies $y \lhd y \leq x$. So $(x, y) \in \rho$. All the above show that $\rho = \rho\circ\sigma\circ\rho$, i.e., ρ is strongly regular.

Now we only need to verify $L \cong (\Phi_\rho(X)\subseteq)$. $\forall\, \rho(A) \in \Phi_\rho(X)$, $\rho(A) = \{y \in L : \exists a \in A, (a, y) \in \rho\} = \{y \in L: \exists a \in A, y \lhd y \leq a\} = \bigcup_{a\in A}\{y \in L : y \lhd y \leq a\} = \{y \in L : y \lhd y \leq \vee A\}$. Define two functions $f : L \to (\Phi_\rho(X), \subseteq)$, $f(x) = \rho(x) = \{y \in L : y \lhd y \leq x\}$ and $g : (\Phi_\rho(X), \subseteq) \to L$, $g(\rho(A)) = \vee\rho(A) = \vee A$. Since L is strongly algebraic, it is easy to see that $f\circ g = id_{\Phi_\rho(X)}$ and $g\circ f = id_L$. Hence, $L \cong (\Phi_\rho(X), \subseteq)$.

(2) ⇒ (1) By Theorem 3.4.

(1) ⇒ (3) $\forall x, y \in L$ with $x \nleq y$. Since L is a strongly algebraic lattice, there is a $u \in L$ such that $u \lhd u \leq x$ with $u \nleq y$. Let $v = \sup(L\backslash\uparrow u)$. Then it is easy to see that $u \nleq v$, $u \nleq y$, and $x \nleq v$. $\forall s, t \in L$, if $u \nleq t$, $s \nleq v$, then $t \in L\backslash\uparrow u$. Thus, $t \leq v$ which implies $s \nleq t$ (otherwise, $s \leq v$, a contradiction). This proof shows that $\nleq$ is strongly regular.

(3) ⇒ (4) $\forall x, y \in L$ with $x \nleq y$. By (4), $\exists u, v \in L$ such that
(i) $u \nleq v$;
(ii) $u \nleq y$, $x \nleq v$;

(iii) $\forall s, t \in L$, if $u \nleq t$, $s \nleq v$, then $s \nleq t$.
$\forall z \in L$, let $s = t = z$. Then $u \leq z$ or $z \leq v$. Thus, condition (5) holds.
(4) ⇔ (5) ⇔ (1) see [25]. □

From the above theorem, we can obtain the following corollary.

Corollary 3.6. *Let L be a complete lattice. If L is strongly algebraic, then L^{op} is strongly algebraic.*

4 Finitely strongly regular relation

In this section, we define a concept of finitely strongly regular relation and obtain the finitely strongly regular relation representation of hyperalgebraic lattices, prove that ρ is finitely strongly regular if and only if $(\Phi_\rho(X), \subseteq)$ is a hyperalgebraic lattice if and only if the finite extension $\rho^{(<\omega)}$ of ρ is strongly regular.

Definition 4.1. A binary relation $\rho : X \rightharpoonup Y$ is called *finitely strongly regular*, if $\forall (x, y) \in X \times Y$ with $(x, y) \in \rho$, $\exists \{u_1, u_2, \dots, u_n\} \in X^{(<\omega)}$ and $\{v_1, v_2, \dots, v_m\} \in Y^{(<\omega)}$ such that

(i) $\{v_1, v_2, \dots, v_m\} \subseteq \rho(\{u_1, u_2, \dots, u_n\})$;

(ii) $y \in \rho(\{u_1, u_2, \dots, u_n\})$, and $(x, v_j) \in \rho$ $(j = 1, 2, \dots, m)$;

(iii) $\forall \{s_1, s_2, \dots, s_m\} \in X^{(<\omega)}$, $t \in Y$, if $t \in \rho(\{u_1, u_2, \dots, u_n\})$, $(s_j, v_j) \in \rho$ $(j = 1, 2, \dots, m)$, then $\exists k \in \{1, 2, \dots, m\}$ such that $(s_k, t) \in \rho$.

Remark 4.2.

(1) It is easy to observe that if the relation ρ is strongly regular, then ρ is finitely strongly regular.

(2) Condition (iii) of Definition 4.1 is equivalent to the following condition:

$$\text{(iii}'\text{)}\ \forall S \in X^{(<\omega)}, t \in Y, \quad \text{if } t \in \rho(\{u_1, u_2, \dots, u_n\}),\ \{v_1, v_2, \dots, v_m\} \subseteq \rho(S),\ \text{ then } t \in \rho(S).$$

(3) Let $\rho : X \rightharpoonup Y$ be a binary relation satisfying conditions (i) and (iii) of Definition 4.1. Then for any $S \subseteq X$,

$$\{v_1, v_2, \dots, v_m\} \subseteq \rho(S) \Leftrightarrow \rho(\{u_1, u_2, \dots, u_n\}) \subseteq \rho(S).$$

Let $U = \{u_1, u_2, \dots, u_n\}$, $V = \{v_1, v_2, \dots, v_m\}$. Then $U \in X^{(<\omega)}$, $V \in Y^{(<\omega)}$. By Definition 4.1, it is easy to obtain the following lemma.

Lemma 4.3. *Let $\rho : X \rightharpoonup Y$ be a binary relation. Then the following conditions are equivalent:*

(1) *$\rho : X \rightharpoonup Y$ is finitely strongly regular.*

(2) *$\forall (x, y) \in X \times Y$ with $(x, y) \in \rho$, $\exists (U, V) \in X^{(<\omega)} \times Y^{(<\omega)}$ such that*
(1°) *$V \subseteq \rho(U)$;*
(2°) *$y \in \rho(U)$, $V \subseteq \rho(x)$;*
(3°) *$\forall S \in X^{(<\omega)}$, $t \in Y$, if $t \in \rho(U)$, $V \subseteq \rho(S)$, then $t \in \rho(S)$.*

Now we give the finitely strongly regular relation representation of hyperalgebraic lattices.

Theorem 4.4. *For a binary relation $\rho : X \rightharpoonup Y$, then the following conditions are equivalent:*

(1) *ρ is finitely strongly regular.*

(2) *The finite extension $\rho^{(<\omega)}$ of $\rho : X^{(<\omega)} \rightharpoonup Y^{(<\omega)}$ is strongly regular.*

(3) $\forall(F, G) \in X^{(<\omega)} \times Y^{(<\omega)}$, $G \subseteq \rho(F) \Rightarrow \exists(U, V) \in X^{(<\omega)} \times Y^{(<\omega)}$ *such that*
 (i) $V \subseteq \rho(U)$;
 (ii) $G \subseteq \rho(U)$, $V \subseteq \rho(F)$;
 (iii) $\forall(S, T) \in X^{(<\omega)} \times Y^{(<\omega)}$, *if* $V \subseteq \rho(S)$ *and* $T \subseteq \rho(U)$, *then* $T \subseteq \rho(S)$.

(4) $(\{\bigcup_{F\in\mathcal{F}}\rho(F)^{(<\omega)} : \mathcal{F} \subseteq X^{(<\omega)}\}, \subseteq)$ *is a strongly algebraic lattice.*

(5) $(\Phi_\rho(X), \subseteq)$ *is a hyperalgebraic lattice.*

Proof.

(1) ⇒ (2) Define a relation $\delta : Y^{(<\omega)} \rightharpoonup X^{(<\omega)}$ by

$$(G, F) \in \delta \Leftrightarrow G \subseteq \rho(F),\ \text{and}\ \forall(S, T) \in X^{(<\omega)} \times Y^{(<\omega)},\quad \text{if}\ G \subseteq \rho(S), F \cap \rho^{-1}(T) \neq \varnothing,\ \text{then}\ T \cap \rho(S) \neq \varnothing.$$

Then

(a) $\delta \subseteq (\rho^{(<\omega)})^{-1}$.

(b) $\rho^{(<\omega)} \circ \delta \circ \rho^{(<\omega)} \subseteq \rho^{(<\omega)}$.

For any $(H, W) \in \rho^{(<\omega)} \circ \delta \circ \rho^{(<\omega)}$, there exists $(G, F) \in Y^{(<\omega)} \times X^{(<\omega)}$ such that $(H, G) \in \rho^{(<\omega)}$, $(G, F) \in \delta$, and $(F, W) \in \rho^{(<\omega)}$, that is, $G \subseteq \rho(H)$, $(G, F) \in \delta$, and $W \subseteq \rho(F)$. $\forall w \in W$, let $S = H$ and $T = \{w\}$. Then $G \subseteq \rho(S)$ and $F \cap \rho^{-1}(T) \neq \varnothing$. Then $T \cap \rho(S) \neq \varnothing$ since $(G, F) \in \delta$. Hence, $w \in \rho(S)$, it follows that $W \subseteq \rho(H)$, i.e., $(H, W) \in \rho^{(<\omega)}$.

(c) $\rho^{(<\omega)} \subseteq \rho^{(<\omega)} \circ \delta \circ \rho^{(<\omega)}$.

For any $(H, W) \in \rho^{(<\omega)}$ and $w \in W$, $\exists h(w) \in H$ such that $(h(w), w) \in \rho$. Since ρ is finitely strongly regular, by Lemma 4.3, $\exists U(w) \in X^{(<\omega)}$ and $V(w) \in Y^{(<\omega)}$ satisfy the following conditions:

(1°) $V(w) \subseteq \rho(U(w))$;

(2°) $w \in \rho(U(w))$, $V(w) \subseteq \rho(h(w))$;

(3°) $\forall S \in X^{(<\omega)}$, $t \in Y$, if $t \in \rho(U(w))$, $V(w) \subseteq \rho(S)$, then $t \in \rho(S)$.

Let $F = \bigcup_{w\in W}U(w)$, $G = \bigcup_{w\in W}V(w)$. Then $(F, G) \in X^{(<\omega)} \times Y^{(<\omega)}$, $G \subseteq \rho(F)$, $G \subseteq \bigcup_{w\in W}\rho(h_w) \subseteq \rho(H)$, and $W \subseteq \rho(F)$. By Definition 2.5, $(H, G) \in \rho^{(<\omega)}$ and $(F, W) \in \rho^{(<\omega)}$. Now we have to verify $(G, F) \in \delta$. Clearly, $(G, F) \in \rho^{(<\omega)}$. $\forall(S, T) \in X^{(<\omega)} \times Y^{(<\omega)}$, if $G \subseteq \rho(S)$, $F \cap \rho^{-1}(T) \neq \varnothing$, then there exist $w_0 \in W$ and $t_0 \in T$ such that $t_0 \in \rho(U(w_0))$, and $\forall w \in W, V(w) \subseteq \rho(S)$. Thus, $V(w_0) \subseteq \rho(S)$. By 3°, $t_0 \in \rho(S)$, which implies $T \cap \rho(S) \neq \varnothing$. Thus, $(G, F) \in \delta$. Hence, $(H, W) \in \rho^{(<\omega)} \circ \delta \circ \rho^{(<\omega)}$.

By (a), (b), and (c), we have that $\delta \subseteq (\rho^{(<\omega)})^{-1}$ and $\rho^{(<\omega)} = \rho^{(<\omega)} \circ \delta \circ \rho^{(<\omega)}$, i.e., $\rho^{(<\omega)}$ is strongly regular.

(2) ⇒ (3) By Theorem 3.4 and Definition 2.5.

(2) ⇔ (4) Let $\sigma = \rho^{(<\omega)}$. Then by Theorem 3.4, σ is strongly regular ⇔ $(\Phi_\sigma(X^{(<\omega)}), \subseteq) = (\{\bigcup_{F\in\mathcal{F}}\rho^{(<\omega)}(F): \mathcal{F} \subseteq X^{(<\omega)}\}, \subseteq) = (\{\bigcup_{F\in\mathcal{F}}\rho(F)^{(<\omega)} : \mathcal{F} \subseteq X^{(<\omega)}\}, \subseteq)$ is a strongly algebraic lattice.

(3) ⇒ (5) Let $L = (\Phi_\rho(X), \subseteq)$. For any $M = \rho(A) \in L$, $y \in M$, there is a $x \in A$ such that $y \in \rho(x)$, by (3), there exists $(U, V) \in X^{(<\omega)} \times Y^{(<\omega)}$ such that

(i) $V \subseteq \rho(U)$;

(ii) $y \in \rho(U)$, $V \subseteq \rho(x)$;

(iii) $\forall(S, T) \in X^{(<\omega)} \times Y^{(<\omega)}$, if $V \subseteq \rho(S)$ and $T \subseteq \rho(U)$, then $T \subseteq \rho(S)$.

Let $V = \{v_1, v_2, \ldots, v_m\}$ and $N_j = \bigcup\{N \in L : v_j \notin N\}$ $(j = 1, 2, \ldots, m)$ (N_j may be an empty set). Now we have to verify $\rho(U) \prec \rho(U) \subseteq M$. Since $V \subseteq \rho(x) \subseteq \rho(A) = M$ and $V \subseteq \rho(U)$, we have $M \in L\backslash{\downarrow}\{N_1, N_2, \ldots, N_m\}$ and $\rho(U) \in L\backslash{\downarrow}\{N_1, N_2, \ldots, N_m\}$. $\forall N = \rho(B) \in L\backslash{\downarrow}\{N_1, N_2, \ldots, N_m\}$, $V \subseteq N = \rho(B)$. Let $S = B$, $T = \rho(U)$, by the condition (iii), $\rho(U) \subseteq \rho(S) = \rho(B) = N$. Thus, $N \in {\uparrow}\rho(U)$. This proves that $\rho(U) \in L\backslash{\downarrow}\{N_1, N_2, \ldots, N_m\} \subseteq {\uparrow}\rho(U)$. Thus, $M \in \mathrm{int}_{\upsilon(L)}{\uparrow}\rho(U) = {\uparrow}\rho(U)$ with $y \in \rho(U)$. Hence, $M = \bigcup\{G \in L : G \prec G \subseteq M\} = \bigvee\{G \in L : G \prec G \subseteq M\}$, i.e., $L = (\Phi_\rho(X), \subseteq)$ is a hyperalgebraic lattice.

(5) ⇒ (1) Let $L = (\Phi_\rho(X), \subseteq)$. $\forall (x, y) \in \rho$, i.e., $y \in \rho(x)$, since L is a hyperalgebraic lattice, there is a $N = \rho(A) \in L$ such that $y \in N \prec N \subseteq \rho(x)$. Since $N \prec N$, $\exists M_1, M_2, \dots, M_m \in L^{(<\omega)}$ such that $y \in \rho(x) \in \uparrow N = L \backslash \downarrow \{M_1, M_2, \dots, M_m\}$. $\forall j \in \{1, 2, \dots, m\}$, by $N \nsubseteq M_j$, $\exists v_j \in N$ but $v_j \notin M_j$. It follows that there is a $u_j \in A$ such that $v_j \in \rho(u_j)$. Also, since $y \in \rho(A)$, $\exists u \in A$ such that $y \in \rho(u)$.

Let $U = \{u, u_1, u_2, \dots, u_m\} \in X^{(<\omega)}$, $V = \{v_1, v_2, \dots, v_m\} \in Y^{(<\omega)}$. Then it is easy to see that (U, V) satisfies the following conditions:
(a) $V \subseteq \rho(U)$;
(b) $y \in \rho(U)$, $V \subseteq \rho(x)$.

Now we check that (U, V) satisfies condition (c) of Lemma 4.3. $\forall S \in X^{(<\omega)}$, $t \in Y$, if $V \subseteq \rho(S)$, $t \in \rho(U)$, then $\forall j \in \{1, 2, \dots, m\}$, $\exists s_j \in S$ such that $v_j \in \rho(s_j)$. Let $S_0 = \{s_1, s_2, \dots, s_m\}$. Then $S_0 \subseteq S$ and $\rho(S_0) = \bigcup_{j=1}^{m} \rho(s_j) \in L \backslash \downarrow \{M_1, M_2, \dots, M_m\} = \uparrow N$. Note that $U \subseteq A$, thus $\rho(U) \subseteq \rho(A) = N \subseteq \rho(S_0)$. Since $t \in \rho(U)$, $\exists k \in \{1, 2, \dots, m\}$ such that $t \in \rho(s_k) \subseteq \rho(S_0) \subseteq \rho(S)$. Therefore, ρ is finitely strongly regular. □

Theorem 4.5. *Let L be a complete lattice. Then the following conditions are equivalent*:
(1) *L is a hyperalgebraic lattice, i.e., $\forall x \in L$, $x = \vee \{y \in L : y \prec y \leq x\}$.*
(2) *$\forall x, y \in L$ with $x \nleq y$, $\exists u \in L$, and $\{v_1, v_2, \dots, v_m\} \in L^{(<\omega)}$ such that*
  (i) *$u \nleq v_j$ $(j = 1, 2, \dots, m)$;*
  (ii) *$u \nleq y$, $x \nleq v_j$ $(j = 1, 2, \dots, m)$;*
  (iii) *$\forall \{s_1, s_2, \dots, s_m\} \in L^{(<\omega)}$ and $t \in L$, if $u \nleq t$, $s_j \nleq v_j$ $(j = 1, 2, \dots, m)$, then $\exists k \in \{1, 2, \dots, m\}$ such that $s_k \nleq t$.*
(3) *$\forall x, y \in L$ with $x \nleq y$, $\exists u \in L$, and $\{v_1, v_2, \dots, v_m\} \in L^{(<\omega)}$ such that*
  (i) *$u \nleq v_j$ $(j = 1, 2, \dots, m)$;*
  (ii) *$u \nleq y$, $x \nleq v_i$ $(i = 1, 2, \dots, m)$;*
  (iii) *$\forall z \in L$, $u \leq z$ or $z \leq v_k$ for some $k \in \{1, 2, \dots, m\}$.*
(4) *$\forall x, y \in L$ with $x \nleq y$, $\exists u \in L$, and $F \in L^{(<\omega)}$ such that*
  (a) *$x \notin \downarrow F$, $y \notin \uparrow u$;*
  (b) *$\downarrow F \cup \uparrow u = L$, $\downarrow F \cap \uparrow u = \emptyset$.*
(5) *$\forall x, y \in L$ with $x \nleq y$, $\exists u \in L$ such that $x \in \mathrm{int}_{\upsilon(L)} \uparrow u = \uparrow u \subseteq L \backslash \downarrow y$.*
(6) *$\forall x, y \in L$ with $x \nleq y$, $\exists \upsilon(L)$-closed subset C and $u \in L$ such that*
  (a) *$x \notin C$, $y \notin \uparrow u$;*
  (b) *$C \cup \uparrow u = L$, $C \cap \uparrow u = \emptyset$.*
(7) *The relation $\nleq$ on L is finitely strongly regular.*
(8) *The finite extension $\nleq^{(<\omega)}$ of $\nleq : L^{(<\omega)} \rightharpoonup L^{(<\omega)}$ is strongly regular.*
(9) *$(\{\bigcup_{F \in \mathcal{F}} (L \backslash \uparrow \vee F)^{(<\omega)} : \mathcal{F} \subseteq L^{(<\omega)}\}, \subseteq)$ is a strongly algebraic lattice.*
(10) *There is a finitely strongly regular relation $\rho : X \rightharpoonup X$ such that $L \cong (\Phi_\rho(X), \subseteq)$.*

Proof.
(1) ⇔ (4) ⇔ (5) See Lemma 2.2 of [22].

(1) ⇒ (2) $\forall x, y \in L$ with $x \nleq y$. Since L is a hyperalgebraic lattice, there exists $u \in L$ such that $u \prec u \leq x$ with $u \nleq y$. By the definition of $\prec$, we have that $x \in \mathrm{int}_{\upsilon(L)} \uparrow u = \uparrow u \subseteq L \backslash \downarrow y$. Hence, there exists a finite set $F = \{v_1, v_2, \dots, v_m\} \subseteq L$ such that $u \in L \backslash \downarrow F \subseteq \mathrm{int}_{\upsilon(L)} \uparrow u$, which implies $x \in L \backslash \downarrow F = \uparrow u \subseteq L \backslash \downarrow y$. So $u \nleq v_j$ $(j = 1, 2, \dots, m)$, $u \nleq y$, and $x \nleq v_j$ $(j = 1, 2, \dots, m)$.

Now we only need to show that u and $F = \{v_1, v_2, \dots, v_m\}$ satisfy condition (iii) of (2). Let $\{s_1, s_2, \dots, s_m\} \in L^{(<\omega)}$ and $t \in L$ with $u \nleq t$, $s_j \nleq v_j$ for all $j = 1, 2, \dots, m$. Assume that for any $j \in \{1, 2, \dots, m\}$, $s_j \leq t$, then $t \in L \backslash \downarrow F$; otherwise, there exists a $j_0 \in \{1, 2, \dots, m\}$ such that $t \leq v_{j_0}$, and then $s_{j_0} \leq v_{j_0}$, a contradiction. This implies $t \in L \backslash \downarrow F \subseteq \uparrow u$, a contradiction. Hence, there exists $k \in \{1, 2, \dots, m\}$ such that $s_k \nleq t$.

(2) ⇒ (3) We only need to show condition (iii) of (3). $\forall z \in L$. Let $t = z$ and $s_j = z$ $(j = 1, 2, \dots, m)$. By condition (iii) of (2), $u \leq t = z$ or $z \leq v_k$ for some $k \in \{1, 2, \dots, m\}$.

(3) ⇒ (1) $\forall x \in L$, if $y \in L$ with $x \nleq y$, by (3), $\exists u \in L$ and $\{v_1, v_2, \dots, v_m\} \in L^{(<\omega)}$ such that

(i) $u \nleq v_j$ $(j = 1, 2, \dots, m)$;
(ii) $u \nleq y$, $x \nleq v_i$ $(i = 1, 2, \dots, m)$;
(iii) $\forall z \in L$, $u \leq z$, or $z \leq v_k$ for some $k \in \{1, 2, \dots, m\}$.

Let $F = \{v_1, v_2, \dots, v_m\} \in L^{(<\omega)}$. Then it is easy to see that $x \in \uparrow u = L \backslash \downarrow F$. Thus, $u \prec u \leq x$ and $u \nleq y$. Hence, $x = \{y \in L : y \prec y \leq x\}$, i.e., L is a hyperalgebraic lattice.

(5) ⇒ (6) Let $C = L \backslash \mathrm{int}_{\upsilon(L)} \uparrow u$. Then C is $\upsilon(L)$-closed and satisfies condition (6) with u.

(6) ⇒ (5) $\forall x, y \in L$ with $x \nleq y$, by (6), $\exists$ $\upsilon(L)$-closed subset C and $u \in L$ such that
(a) $x \notin C$, $y \notin \uparrow u$;
(b) $C \cup \uparrow u = L$, $C \cap \uparrow u = \varnothing$.

By condition (b), $\uparrow u = L \backslash C \in \upsilon(L)$, and it follows from condition (a) that $x \in L \backslash C = \mathrm{int}_{\upsilon(L)} \uparrow u = \uparrow u \subseteq L \backslash \downarrow y$.

(2) ⇒ (7) $\forall x, y \in L$ with $x \nleq y$, by (2), there exist $u \in L$ and $\{v_1, v_2, \dots, v_m\} \in L^{(<\omega)}$ such that
(i) $u \nleq v_j$ $(j = 1, 2, \dots, m)$;
(ii) $u \nleq y$, $x \nleq v_j$ $(j = 1, 2, \dots, m)$;
(iii) $\forall \{s_1, s_2, \dots, s_m\} \in L^{(<\omega)}$ and $t \in L$, if $u \nleq t$, $s_j \nleq v_j$ $(j = 1, 2, \dots, m)$, then $\exists k \in \{1, 2, \dots, m\}$ such that $s_k \nleq t$.

Let $u_i = u$ $(i = 1, 2 \dots, n)$. Then it is easy to see that $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_m\}$ satisfy condition (7).

(7) ⇔ (8) ⇔ (9) By Theorem 4.4.

(7) ⇒ (10) Let $X = L$ and $\rho = \nleq : X \rightharpoonup X$. Then $L \cong (\Phi_{\nleq}(X), \subseteq)$.

(10) ⇒ (1) Using Theorem 4.4. □

5 Generalized finitely strongly regular relation

In this section, we define a concept of generalized finitely strongly regular relation and get the generalized finitely strongly regular representation of quasi-hyperalgebraic lattices, prove that ρ is generalized finitely strongly regular if and only if $(\Phi_\rho(X), \subseteq)$ is a quasi-hyperalgebraic lattice if and only if $(\Phi_\rho(X), \subseteq)$ with respect to the interval topology is a Priestley space. Meanwhile, some equivalent characterizations of quasi-hyperalgebraic lattices are obtained.

Definition 5.1. A binary relation $\rho : X \rightharpoonup Y$ is called *generalized finitely strongly regular*, if $\forall (x, y) \in \rho$, there exist $\{F_1, F_2, \dots, F_n\} \in (X^{(<\omega)})^{(<\omega)}$ and $\{v_{ij} : i = 1, 2, \dots, n;\ j = 1, 2, \dots, m\} \in Y^{(<\omega)}$ such that
(i) $\forall i \in \{1, 2, \dots, n\}$, $\{v_{i1}, v_{i2}, \dots, v_{im}\} \subseteq \rho(F_i)$;
(ii) $\forall i \in \{1, 2, \dots, n\}$, $y \in \rho(F_i)$, and $\exists \xi \in \prod_{j=1}^{m}\{1, 2, \dots, n\}$ such that $\{(x, v_{\xi(1)1}), (x, v_{\xi(2)2}), \dots, (x, v_{\xi(m)m})\} \subseteq \rho$;
(iii) $\forall \{s_1, s_2, \dots, s_m\} \in X^{(<\omega)}$, $\{t_1, t_2, \dots, t_n\} \in Y^{(<\omega)}$, and $\varphi \in \prod_{j=1}^{m}\{1, 2, \dots, n\}$, if $t_i \in \rho(F_i)$ $(i = 1, 2, \dots, n)$ and $\{(s_1, v_{\varphi(1)1}), (s_2, v_{\varphi(2)2}), \dots, (s_m, v_{\varphi(m)m})\} \subseteq \rho$, then $\exists (k, l) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ such that $(s_k, t_l) \in \rho$.

Obviously, the condition $\exists \xi \in \prod_{j=1}^{m}\{1, 2, \dots, n\}$ such that $\{(x, v_{\xi(1)1}), (x, v_{\xi(2)2}), \dots, (x, v_{\xi(m)m})\} \subseteq \rho$ is equivalent to $x \in \rho^{-1}(\{v_{1j}, v_{2j}, \dots, v_{nj}\})$ for any $j \in \{1, 2, \dots, m\}$. So we obtain the following equivalent characterizations of generalized finitely strongly regular relations.

Proposition 5.2. *Let $\rho : X \rightharpoonup Y$ be a binary relation. Then the following conditions are equivalent*:
(1) *ρ is generalized finitely strongly regular.*
(2) *$\forall (x, y) \in \rho$, there exist $\{F_1, F_2, \dots, F_n\} \in (X^{(<\omega)})^{(<\omega)}$ and $\{v_{ij} : i = 1, 2, \dots, n;\ j = 1, 2, \dots, m\} \in Y^{(<\omega)}$ such that*
(a) *$\forall i \in \{1, 2, \dots, n\}$, $\{v_{i1}, v_{i2}, \dots, v_{im}\} \subseteq \rho(F_i)$;*
(b) *$\forall i \in \{1, 2, \dots, n\}$, $y \in \rho(F_i)$, and $\exists k \in \{1, 2, \dots, n\}$ such that $\rho(F_k) \subseteq \rho(x)$;*

(c) $\forall\{s_1, s_2, \ldots, s_m\} \in X^{(<\omega)}$, $\{t_1, t_2, \ldots, t_n\} \in Y^{(<\omega)}$, *if* $t_i \in \rho(F_i)$ $(i = 1, 2, \ldots, n)$ *and* $s_j \in \rho^{-1}(\{v_{1j}, v_{2j}, \ldots, v_{nj}\})$ $(j = 1, 2, \ldots, m)$, *then* $\exists(k, l) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ *such that* $(s_k, t_l) \in \rho$.

(3) $\forall(x, y) \in \rho$, *there exist* $\{F_1, F_2, \ldots, F_n\} \in (X^{(<\omega)})^{(<\omega)}$ *and* $\{v_{ij} : i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m\} \in Y^{(<\omega)}$ *such that*

(a) $\forall i \in \{1, 2, \ldots, n\}$, $\{v_{i1}, v_{i2}, \ldots, v_{im}\} \subseteq \rho(F_i)$;

(b) $\forall i \in \{1, 2, \ldots, n\}$, $y \in \rho(F_i)$, *and* $\exists k \in \{1, 2, \ldots, n\}$ *such that* $\{v_{k1}, v_{k2}, \ldots, v_{km}\} \subseteq \rho(x)$;

(c) $\forall\{s_1, s_2, \ldots, s_m\} \in X^{(<\omega)}$, $\{t_1, t_2, \ldots, t_n\} \in Y^{(<\omega)}$, *if* $t_i \in \rho(F_i)$ $(i = 1, 2, \ldots, n)$ *and* $s_j \in \rho^{-1}(\{v_{1j}, v_{2j}, \ldots, v_{nj}\})$ $(j = 1, 2, \ldots, m)$, *then* $\exists(k, l) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ *such that* $(s_k, t_l) \in \rho$.

(4) $\forall(x, y) \in \rho$, *there exist* $\{F_1, F_2, \ldots, F_n\} \in (X^{(<\omega)})^{(<\omega)}$ *and* $\{G_1, G_2, \ldots, G_m\} \in (Y^{(<\omega)})^{(<\omega)}$ *such that*

(a) $\forall(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$, $\rho(F_i) \bigcap G_j \neq \varnothing$;

(b) $\forall i \in \{1, 2, \ldots, n\}$, $y \in \rho(F_i)$, *and* $x \in \rho^{-1}(G_j)$ $(j = 1, 2, \ldots, m)$;

(c) $\forall\{s_1, s_2, \ldots, s_m\} \in X^{(<\omega)}$, $\{t_1, t_2, \ldots, t_n\} \in Y^{(<\omega)}$, *if* $t_i \in \rho(F_i)$ $(i = 1, 2, \ldots, n)$ *and* $s_j \in \rho^{-1}(G_j)$ $(j = 1, 2, \ldots, m)$, *then* $\exists(k, l) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ *such that* $(s_k, t_l) \in \rho$.

Proof.

(1) ⇒ (2) We only need to prove condition (b). Assume that for any $i \in \{1, 2, \ldots, n\}$, $\rho(F_i) \nsubseteq \rho(x)$, then there is a $t_i \in \rho(F_i)\backslash\rho(x)$. By condition (ii) of Definition 5.1, $\exists\xi \in \prod_{j=1}^{m}\{1, 2, \ldots, n\}$ such that $\{(x, v_{\xi(1)1}), (x, v_{\xi(2)2}), \ldots, (x, v_{\xi(m)m})\} \subseteq \rho$. Let $s_1 = s_2 = \ldots = s_m = x$. From condition (iii) of Definition 5.1, it is follows that $\exists l \in \{1, 2, \ldots, n\}$ such that $(x, t_l) \in \rho$, i.e., $t_l \in \rho(x)$, contradicting to the assumption that $t_i \notin \rho(x)$ for any $i \in \{1, 2, \ldots, n\}$. Hence, condition (2) holds.

(2) ⇒ (3) Obviously.

(3) ⇒ (4) $\forall j \in \{1, 2, \ldots, m\}$, let $G_j = \{v_{1j}, v_{2j}, \ldots, v_{nj}\}$. Then $v_{ij} \in \rho(F_i) \cap G_j$. Thus, condition (a) of (4) holds. It follows from condition (b) of (3) that $x \in \rho^{-1}(v_{kj})$ $(j = 1, 2, \ldots, m)$. Hence, condition (4) holds.

(4) ⇒ (1) By condition (a) of (4), $\forall(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$, $\rho(F_i) \bigcap G_j \neq \varnothing$. Then $\exists v_{ij} \in \rho(F_i) \bigcap G_j$, i.e, $\forall i \in \{1, 2, \ldots, n\}$, $\{v_{i1}, v_{i2}, \ldots, v_{im}\} \subseteq \rho(F_i)$. Meanwhile, $\forall j \in \{1, 2, \ldots, m\}$, let $t_1 = v_{1j} \in \rho(F_1)$, $t_2 = v_{2j} \in \rho(F_2), \ldots$, $t_n = v_{nj} \in \rho(F_n)$. By condition (b) of (4), $x \in \rho^{-1}(G_j)\{j = 1, 2, \ldots, m\}$. Let $s_1 = s_2 = \ldots = s_m = x$. Then by condition (c) of (4), $\exists(k, l) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ such that $(s_k, t_l) \in \rho$, that is, $x \in \rho^{-1}(t_l) = \rho^{-1}(v_{lj}) \subseteq \rho^{-1}(v_{1j}, v_{2j}, \ldots, v_{nj})$. Hence, condition (ii) of Definition 5.1 holds.

Finally, we verify condition (iii) of Definition 5.1. $\forall\{s_1, s_2, \ldots, s_m\} \in X^{(<\omega)}$, $\{t_1, t_2, \ldots, t_n\} \in Y^{(<\omega)}$, if $t_i \in \rho(F_i)$ $(i = 1, 2, \ldots, n)$ and $s_j \in \rho^{-1}\{v_{1j}, v_{2j}, \ldots, v_{nj}\}$ $(j = 1, 2, \ldots, m)$. Note that $v_{ij} \in G_j$, we have that $\{v_{1j}, v_{2j}, \ldots, v_{nj}\} \subseteq G_j$. Hence, $s_j \in \rho^{-1}(G_j)$ $(j = 1, 2, \ldots, m)$. By condition (c) of (4), $\exists(k, l) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ such that $(s_k, t_l) \in \rho$. Therefore, ρ is generalized finitely strongly regular. □

Now we give the generalized finitely strongly regular relation representation of quasi-hyperalgebraic lattices.

Theorem 5.3. *For a binary relation* $\rho : X \rightharpoonup Y$, *the following conditions are equivalent*:

(1) ρ *is generalized finitely strongly regular.*

(2) $(\Phi_\rho(X), \subseteq)$ *is a quasi-hyperalgebraic lattice.*

(3) $(\Phi_\rho(X), \subseteq)$ *equipped with the interval topology is a Priestley space.*

(4) $(\Phi_\rho(X), \subseteq)$ *equipped with the interval topology is totally order-disconnected.*

Proof. Let $L = (\Phi_\rho(X), \subseteq)$.

(1) ⇒ (2) $\forall\rho(A) \in L$, $\mathcal{U} \in \upsilon(L)$, if $\rho(A) \in \mathcal{U}$, then $\exists\{\rho(A_1), \rho(A_2), \ldots, \rho(A_n)\} \in L^{(<\omega)}$ such that $\rho(A) \in L\backslash\downarrow\{\rho(A_1), \rho(A_2), \ldots, \rho(A_n)\} \subseteq \mathcal{U}$, that is, $\forall i \in \{1, 2, \ldots, n\}$, $\rho(A) \nsubseteq \rho(A_i)$. It follows that there exist $x_i \in A$ and $y_i \in Y$ such that $(x_i, y_i) \in \rho$, $y_i \notin \rho(A_i)$. Since ρ is generalized finitely strongly regular, by Proposition 5.2 (4), $\exists\{F_{i1}, F_{i2}, \ldots, F_{in(i)}\} \in (X^{(<\omega)})^{(<\omega)}$, $\{G_{i1}, G_{i2}, \ldots, G_{im(i)}\} \in (Y^{(<\omega)})^{(<\omega)}$ such that

(a) $\forall(k, l) \in \{1, 2, \ldots, n(i)\} \times \{1, 2, \ldots, m(i)\}$, $\rho(F_{ik}) \bigcap G_{il} \neq \varnothing$;

(b) $\forall k \in \{1, 2, \ldots, n(i)\}$, $y_i \in \rho(F_{ik})$, and $x_i \in \rho^{-1}(G_{il})(l = 1, 2, \ldots, m(i))$;

(c) $\forall\{s_1, s_2, \ldots, s_{m(i)}\} \in X^{(<\omega)}$, $\{t_1, t_2, \ldots, t_{n(i)}\} \in Y^{(<\omega)}$, if $t_k \in \rho(F_{ik})(k = 1, 2, \ldots, n(i))$ and $s_l \in \rho^{-1}(G_{il})(l = 1, 2, \ldots, m(i))$, then $\exists(g, h) \in \{1, 2, \ldots, m(i)\} \times \{1, 2, \ldots, n(i)\}$ such that $(s_g, t_h) \in \rho$.

$\forall i \in \{1, 2, \ldots, n\}$, $l \in \{1, 2, \ldots, m(i)\}$, let $N_{il} = \bigcup\{M \in L : G_{il} \cap M = \varnothing\}$ (N_{il} may be an empty set). Then for any $i \in \{1, 2, \cdots, n\}$, we have the following results:

1° $\rho(A) \in L\backslash\downarrow\{N_{il} : l = 1, 2, \ldots, m(i)\}$. $\forall l \in \{1, 2, \ldots, m(i)\}$, since $x_i \in A$ and $x_i \in \rho^{-1}(G_{il})$, $\rho(A) \cap G_{il} \neq \varnothing$. Hence, $\rho(A) \nsubseteq N_{il}$, that is, $\rho(A) \in L\backslash\downarrow\{N_{il} : l = 1, 2, \ldots, m(i)\}$.

2° $L\backslash\downarrow\{N_{il} : l = 1, 2, \ldots, m(i)\} \subseteq \uparrow\{\rho(F_{ik}) : k = 1, 2, \ldots, n(i)\}$. Assumed that there exists $\rho(B) \in L\backslash\downarrow\{N_{il} : l = 1, 2, \cdots, m(i)\}$ such that $\rho(B) \notin \uparrow\{\rho(F_{ik}) : k = 1, 2, \cdots, n(i)\}$, then for each $k \in \{1, 2, \ldots, n(i)\}$, choose $t_k \in \rho(F_{ik})\backslash\rho(B)$. $\forall l \in \{1, 2 \ldots, m(i)\}$, by the definition of N_{il} and $\rho(B) \in L\backslash\downarrow\{N_{il} : l = 1, 2, \cdots, m(i)\}$, we have that $G_{il} \cap \rho(B) \neq \varnothing$. Thus, there exist $w_l \in G_{il} \cap \rho(B)$ and $s_l \in B$ such that $(s_l, w_l) \in \rho$, i.e., $s_l \in \rho^{-1}(G_{il})$. By the above condition (c), $\exists(g, h) \in \{1, 2, \ldots, m(i)\} \times \{1, 2, \ldots, n(i)\}$ such that $(s_g, t_h) \in \rho$. Hence, $t_h \in \rho(s_g) \subseteq \rho(B)$, which contradicts $t_h \notin \rho(B)$. Therefore, $L\backslash\downarrow\{N_{il} : l = 1, 2, \ldots, m(i)\} \subseteq \uparrow\{\rho(F_{ik}) : k = 1, 2, \ldots, n(i)\}$.

3° $\uparrow\{\rho(F_{ik}) : k = 1, 2, \ldots, n(i)\} \subseteq L\backslash\downarrow\{N_{il} : l = 1, 2, \ldots, m(i)\}$. For any $\rho(C) \in \uparrow\{\rho(F_{ik}) : k = 1, 2, \ldots, n(i)\}$, there exists a $h \in \{1, 2, \ldots, n(i)\}$ such that $\rho(F_{ih}) \subseteq \rho(C)$. By condition (c) of above, $\forall l \in \{1, 2, \ldots, m(i)\}$, $\varnothing \neq \rho(F_{ih}) \cap G_{il} \subseteq \rho(C) \cap G_{il}$. By the definition of N_{il}, $\rho(C) \in L\backslash\downarrow\{N_{il} : l = 1, 2, \ldots, m(i)\}$,

4° $\uparrow\{\rho(F_{ik}) : k = 1, 2, \ldots, n(i)\} \subseteq L\backslash\downarrow \rho(A_i)$. $\forall\rho(D) \in \uparrow\{\rho(F_{ik}) : k = 1, 2, \ldots, n(i)\}$, $\exists k_0 \in \{1, 2, \ldots, n(i)\}$ such that $\rho(F_{ik_0}) \subseteq \rho(D)$. By condition (b) of above, $y_i \in \rho(F_{ik_0}) \subseteq \rho(D)$. Note that $y_i \notin \rho(A_i)$, so $\rho(D) \in L\backslash\downarrow \rho(A_i)$. Hence, $\uparrow\{\rho(F_{ik}) : k = 1, 2, \ldots, n(i)\} \subseteq L\downarrow \rho(A_i)$.

From the above 1°, 2°, 3°, and 4°, it follows that $\rho(A) \in \bigcap_{i=1}^{n}(L\backslash\downarrow\{N_{il} : l = 1, 2, \ldots, m(i)\}) = \bigcap_{i=1}^{n}\uparrow\{\rho(F_{ik}) : k = 1, 2, \ldots, n(i)\} \subseteq \bigcap_{i=1}^{n}(L\backslash\downarrow \rho(A_i)) = L\backslash\downarrow\{\rho(A_1), \rho(A_2), \ldots, \rho(A_k)\} \subseteq \mathcal{U}$.

Let $\mathcal{F} = \{\bigcup_{i=1}^{n}\rho(F_{i\varphi(i)}) = \rho(\bigcup_{i=1}^{n}\rho(F_{i\varphi(i)})) : \varphi \in \prod_{i=1}^{n}\{1, 2, \ldots, n(i)\}\}$ and $\mathcal{V} = \bigcap_{i=1}^{n}(L\backslash\downarrow\{N_{il} : l = 1, 2, \ldots, n(i)\}) = L\backslash\downarrow\{N_{il} : i = 1, 2, \ldots, n;\ l = 1, 2, \ldots, n(i)\}$. Then $\mathcal{F} \in L^{(<\omega)}$, $\mathcal{V} \in \upsilon(L)$ and $\rho(A) \in \mathcal{V} = \uparrow\mathcal{F} \subseteq \mathcal{U}$. Hence, $\rho(A) \in \operatorname{int}_{\upsilon(L)}\uparrow\mathcal{F} = \uparrow\mathcal{F} \subseteq \mathcal{U}$. Therefore, $L = (\Phi_\rho(X), \subseteq)$ is a quasi-hyperalgebraic lattice.

(2) ⇔ (3) See Theorem 2.1 of [22].

(3) ⇒ (4) Obviously.

(4) ⇒ (1) $\forall(x, y) \in \rho$, i.e., $y \in \rho(x)$, let $M_y = \bigcup\{N \in L : y \notin N\}$ (M_y may be an empty set). Then $\rho(x) \nsubseteq M_y$. Since $\theta(L)$ is totally order-disconnected, there exist $\{\rho(A_1), \rho(A_2), \ldots, \rho(A_m)\}$, $\{\rho(B_1), \rho(B_2), \ldots, \rho(B_n)\} \in L^{(<\omega)}$ such that $\rho(x) \in L\backslash\downarrow\{\rho(A_1), \rho(A_2), \ldots, \rho(A_m)\} = \uparrow\{\rho(B_1), \rho(B_2), \ldots, \rho(B_n)\} \subseteq L\backslash\downarrow M_y$. Hence, $\forall(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$, $\rho(B_i) \nsubseteq \rho(A_j)$, which implies $\exists v_{ij} \in \rho(B_i)\backslash\rho(A_j)$. $\forall i \in \{1, 2, \ldots, n\}$, since $\{v_{i1}, v_{i2}, \ldots, v_{im}\} \subseteq \rho(B_i)$, there exists $F_i \in B_i^{(<\omega)}$ such that $\{v_{i1}, v_{i2}, \ldots, v_{im}\} \subseteq \rho(F_i)$. Hence, condition (a) of Proposition 5.2(2) holds.

Now we show condition (b) of Proposition 5.2 (2). Since $\rho(x) \in \uparrow\{\rho(B_1), \rho(B_2), \ldots, \rho(B_n)\}$, $\exists k \in \{1, 2, \ldots, n\}$ such that $\rho(B_k) \subseteq \rho(x)$. Thus, $\rho(F_k) \subseteq \rho(x)$. $\forall i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, from $v_{ij} \in \rho(F_i)$ and $v_{ij} \notin \rho(A_j)$, it follows that $\rho(F_i) \nsubseteq \rho(A_j)$. Hence, $\rho(F_i) \in L\backslash\downarrow\{\rho(A_1), \rho(A_2), \ldots, \rho(A_m)\} = \uparrow\{\rho(B_1), \rho(B_2), \ldots, \rho(B_n)\} \subseteq L\backslash\downarrow M_y$. By the definition of M_y, $y \in \rho(F_i)$ for any $i \in \{1, 2, \ldots, n\}$.

Finally, we show condition (c) of Proposition 5.2 (2). $\forall\{s_1, s_2, \ldots, s_m\} \in X^{(<\omega)}$, $\{t_1, t_2, \ldots, t_n\} \in Y^{(<\omega)}$, let $t_i \in \rho(F_i)$ $(i = 1, 2, \ldots, n)$ and $s_j \in \rho^{-1}(\{v_{1j}, v_{2j}, \ldots, v_{nj}\})$ $(j = 1, 2, \ldots, m)$. Then $\forall j \in \{1, 2, \ldots, m\}$, $\varnothing \neq \{v_{1j}, v_{2j}, \ldots, v_{nj}\} \cap \rho(s_j) \subseteq \{v_{1j}, v_{2j}, \ldots, v_{nj}\} \cap \rho(\{s_1, s_2, \ldots, s_m\})$. Note that for any $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, $v_{ij} \notin \rho(A_j)$, so $\rho(\{s_1, s_2, \ldots, s_m\}) \in L\backslash\downarrow\{\rho(A_1), \rho(A_2), \ldots, \rho(A_m)\} = \uparrow\{\rho(B_1), \rho(B_2), \ldots, \rho(B_n)\}$. Hence, $\exists l \in \{1, 2, \ldots, n\}$ such that $\rho(B_l) \subseteq \rho(\{s_1, s_2, \ldots, s_m\})$, which implies $t_l \in \rho(F_l) \subseteq \rho(B_l) \subseteq \rho(\{s_1, s_2, \ldots, s_m\})$. Therefore, $\exists k \in \{1, 2, \ldots, m\}$ such that $(s_k, t_l) \in \rho$.

By Proposition 5.2, ρ is generalized finitely strongly regular. □

Lemma 5.4. *Let L be a complete lattice and $F \in L^{(<\omega)}$. If $\operatorname{int}_{\upsilon(L)}\uparrow F = \uparrow F$, then $\exists G \in L^{(<\omega)}$ such that $\uparrow F = L\backslash\downarrow G$.*

Proof. Since $\operatorname{int}_{\upsilon(L)}\uparrow F = \uparrow F$ and F is a finite set, there exists $\{G_j : j = 1, 2, \ldots, m\} \subseteq L^{(<\omega)}$ such that $\uparrow F = \bigcup_{j=1}^{m}(L\backslash\downarrow G_j)$. Let $G = \{\wedge\varphi(\{1, 2, \ldots, m\}) : \varphi \in \prod_{j=1}^{m}G_j\}$. Then $\bigcup_{j=1}^{m}(L\backslash\downarrow G_j) = L\backslash\bigcap_{j=1}^{m}\downarrow G_j = L\backslash\downarrow G$. Hence, $\uparrow F = L\backslash\downarrow G$. □

The following theorem gives some equivalent characterizations of complete lattices that are Priestley spaces with respect to the interval topology.

Theorem 5.5. *Let L be a complete lattice. Then the following conditions are equivalent*:

(1) *L is a quasi-hyperalgebraic lattice, i.e.,* $\forall x \in L$, $U \in \upsilon(L)$ *with* $x \in U$, $\exists F \in L^{(<\omega)}$ *such that* $x \in \mathrm{int}_{\upsilon(L)} \uparrow F = \uparrow F \subseteq U$.

(2) $(\upsilon(L), \subseteq)$ *is a hyperalgebraic lattice.*

(3) $\forall (x, y) \in L$ with $x \nleq y$, $\exists \{u_1, u_2, \dots, u_n\}, \{v_1, v_2, \dots, v_m\} \in L^{(<\omega)}$ *satisfying the following conditions*:
  - (i) $\forall (i, j) \in \{1, 2, \dots, n\} \times \{1, 2, \dots, m\}$, $u_i \nleq v_j$;
  - (ii) $u_i \nleq y$ $(i = 1, 2, \dots, n)$, $x \nleq v_j$ $(j = 1, 2, \dots, m)$;
  - (iii) $\forall \{s_1, s_2, \dots, s_m\}, \{t_1, t_2, \dots, t_n\} \in L^{(<\omega)}$, *if* $u_i \nleq t_j (i, j \in \{1, 2, \dots, n\})$, $s_i \nleq v_j (i, j \in \{1, 2, \dots, m\})$, *then* $\exists (k, l) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ *such that* $s_k \nleq t_l$.

(4) $\forall (x, y) \in L$ *with* $x \nleq y$, $\exists \{u_1, u_2, \dots, u_n\}, \{v_1, v_2, \dots, v_m\} \in L^{(<\omega)}$ *satisfying the following conditions*:
  - (i) $\forall (i, j) \in \{1, 2, \dots, n\} \times \{1, 2, \dots, m\}$, $u_i \nleq v_j$;
  - (ii) $u_i \nleq y$ $(i = 1, 2, \dots, n)$, $x \nleq v_j$ $(j = 1, 2, \dots, m)$;
  - (iii) $\forall z \in L$, $u_k \leq z$ *for some* $k \in \{1, 2, \dots, n\}$ *or* $z \leq v_l$ *for some* $l \in \{1, 2, \dots, m\}$.

(5) $\forall x, y \in L$ *with* $x \nleq y$, $\exists F, G \in L^{(<\omega)}$ *such that*
  - (i) $x \notin \downarrow G$, $y \notin \uparrow F$;
  - (ii) $\downarrow G \cup \uparrow F = L$ *and* $\downarrow G \cap \uparrow F = \emptyset$.

(6) $\forall x, y \in L$ *with* $x \nleq y$, $\exists \upsilon(L)$*-closed subset* C *and* $F \in L^{(<\omega)}$ *such that*
  - (i) $x \notin C$, $y \notin \uparrow F$;
  - (ii) $C \cup \uparrow F = L$ *and* $C \cap \uparrow F = \emptyset$.

(7) $\forall x, y \in L$ *with* $x \nleq y$, $\exists F \in L^{(<\omega)}$ *such that* $x \in \mathrm{int}_{\upsilon(L)} \uparrow F = \uparrow F \subseteq L \setminus \downarrow y$.

(8) $\forall x, y \in L$ *with* $x \nleq y$, $\exists G \in L^{(<\omega)}$ *and* $\omega(L)$*-closed subset* B *such that*
  - (i) $x \notin \downarrow G$, $y \notin B$;
  - (ii) $\downarrow G \cup B = L$ *and* $\downarrow G \cap B = \emptyset$.

(9) $\forall x, y \in L$ *with* $x \nleq y$, $\exists G \in L^{(<\omega)}$ *such that* $y \in \mathrm{int}_{\omega(L)} \downarrow G = \downarrow G \subseteq L \setminus \uparrow x$.

(10) $\forall x, y \in L$ *with* $x \nleq y$, $\exists \upsilon(L)$*-closed subset* C *and* $\omega(L)$*-closed subset* B *such that*
  - (i) $x \notin C$, $y \notin B$;
  - (ii) $C \cup B = L$ *and* $C \cap B = \emptyset$.

(11) *The relation* $\nleq$ *on* L *is generalized finitely strongly regular, i.e.,* $\forall (x, y) \in L$ *with* $x \nleq y$, $\exists \{F_1, F_2, \dots, F_n\} \in (X^{(<\omega)})^{(<\omega)}$ *and* $\{v_{ij} : i = 1, 2, \dots, n; j = 1, 2, \dots, m\} \in Y^{(<\omega)}$ *such that*
  - (a) $\forall i \in \{1, 2, \dots, n\}$, $\{v_{i1}, v_{i2}, \dots, v_{im}\} \subseteq\ \nleq (F_i) = \bigcup_{u \in F_i} (L \setminus \uparrow u) = L \setminus \bigcap_{u \in F_i} \uparrow u$;
  - (b) $\forall i \in \{1, 2, \dots, n\}$, $y \in\ \nleq (F_i) = L \setminus \bigcap_{u \in F_i} \uparrow u$, *and* $\exists k \in \{1, 2, \dots, n\}$ *such that* $\nleq (F_k) \subseteq\ \nleq (x) = L \setminus \uparrow x$;
  - (c) $\forall \{s_1, s_2, \dots, s_m\}, \{t_1, t_2, \dots, t_n\} \in Y^{(<\omega)}$, *if* $t_i \in\ \nleq (F_j) (i, j \in \{1, 2, \dots, n\})$ *and* $s_i \in\ \nleq^{-1}(\{v_{1j}, v_{2j}, \dots, v_{nj}\}) (i, j \in \{1, 2, \dots, m\})$, *then* $\exists (k, l) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ *such that* $s_k \nleq t_l$.

(12) $(L, \theta(L))$ *is a Priestley space.*

(13) *There is a generalized finitely strongly regular relation* $\rho : X \rightharpoonup X$ *such that* $L \cong (\Phi_\rho(X), \subseteq)$.

Proof.

(1) ⇔ (2), (1) ⇔ (12) see [22].

(1) ⇒ (3) $\forall x, y \in L$ with $x \nleq y$, that is, $x \in L \setminus \downarrow y \in \upsilon(L)$. Since L is quasi-hyperalgebraic, there exists $F = \{u_1, u_2, \dots, u_n\} \in L^{(<\omega)}$ such that $x \in \mathrm{int}_{\upsilon(L)} \uparrow F = \uparrow F \subseteq L \setminus \downarrow y$. Since $F \in L^{(<\omega)}$ and $\mathrm{int}_{\upsilon(L)} \uparrow F = \uparrow F$, by Lemma 5.4, $\exists G = \{v_1, v_2, \dots, v_m\} \in L^{(<\omega)}$ such that $\uparrow F = L \setminus \downarrow G$. Hence, $x \in L \setminus \downarrow G = \uparrow F \subseteq L \setminus \downarrow y$.

Now we show that $F = \{u_1, u_2, \dots, u_n\}$ and $G = \{v_1, v_2, \dots, v_m\}$ satisfy condition (3). Obviously, $\forall (i, j) \in \{1, 2, \dots, n\} \times \{1, 2, \dots, m\}$, $u_i \nleq v_j$, $u_i \nleq y$ and $x \nleq v_j$. $\forall \{s_1, s_2, \dots, s_m\} \in L^{(<\omega)}$, $\{t_1, t_2, \dots, t_n\} \in L^{(<\omega)}$, if $u_i \nleq t_j (i, j \in \{1, 2, \dots, n\})$, $s_i \nleq v_j (i, j \in \{1, 2, \dots, m\})$, then we have $t_i \notin \uparrow F$ $(i = 1, 2, \dots, n)$ and $s_j \notin \downarrow G$ $(j = 1, 2, \dots, m)$. Thus, for any $j \in \{1, 2, \dots, m\}$, $s_j \in L \setminus \downarrow G$. Assume that $\forall (i, j) \in \{1, 2, \dots, n\} \times \{1, 2, \dots, m\}$, $s_j \leq t_i$, then $t_i \in L \setminus \downarrow G = \uparrow F$, which contradicts $t_i \notin \uparrow F$ $(i = 1, 2, \dots, n)$. Hence, $\exists (k, l) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ such that $s_k \nleq t_l$.

(3) ⇒ (4) We only need to verify condition (iii) of (4). $\forall z \in L$, let $s_j = z = t_i$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$. By (3)(iii), it is easy to see that condition (iii) of (4) holds.

(4) ⇒ (5) Let $F = \{u_1, u_2, \ldots, u_n\}$ and $G = \{v_1, v_2, \ldots, v_m\}$. Then $F, G \in L^{(<\omega)}$. From (4), we can obtain that F, G satisfy condition (5).

(5) ⇒ (6) Let $C = \downarrow G$. Then $\downarrow G$ is $\upsilon(L)$-closed, and satisfies condition (6) with $\uparrow F$.

(6) ⇒ (7) Obviously.

(7) ⇒ (8) $\forall x, y \in$ with $x \nleq y$, by (7), $\exists F \in L^{(<\omega)}$ such that $x \in \mathrm{int}_{\upsilon(L)} \uparrow F = \uparrow F \subseteq L \backslash \downarrow y$. Let $B = \uparrow F$. Then B is $\omega(L)$-closed. Since $F \in L^{(<\omega)}$ and $\mathrm{int}_{\upsilon(L)} \uparrow F = \uparrow F$, by Lemma 5.4, $\exists G \in L^{(<\omega)}$ such that $\uparrow F = L \backslash \downarrow G$. Hence, $B = \uparrow F$ and G satisfy condition (8).

(8)⇒ (9) Trivial.

(9) ⇒ (10) Let $C = \downarrow G$, $B = L \backslash \downarrow G$. Then C is $\upsilon(L)$-closed and B is $\omega(L)$-closed satisfying condition (10).

(10) ⇒ (1) $\forall x \in L$, $U \in \upsilon(L)$ with $x \in U$, $\exists \{y_1, y_2, \ldots, y_n\} \in L^{(<\omega)}$ such that $x \in L \backslash \downarrow \{y_1, y_2, \ldots, y_n\} \subseteq U$. Then $x \nleq y_i$ for any $i \in \{1, 2, \ldots, n\}$. By (10), $\exists$ $\upsilon(L)$-closed subset C_i and $\omega(L)$-closed set B_i such that $x \in L \backslash C_i = B_i \subseteq L \backslash \downarrow y_i$. Then $\bigcap_{i=1}^n (L \backslash C_i) = L \backslash \bigcup_{i=1}^n C_i = \bigcap_{i=1}^n B_i \subseteq \bigcap_{i=1}^n (L \backslash \downarrow y_i) = L \backslash \downarrow \{y_1, y_2, \ldots, y_n\} \subseteq U$. Since B_i is a $\theta(L)$-clopen upper set, there exist $\{G_1, G_2, \ldots, G_n\} \in L^{(<\omega)}$ such that $B_i = \uparrow G_i$ for all $i \in \{1, 2 \ldots, n\}$. Let $F = \{\bigvee_{i=1}^n a_i : a_i \in G_i\}$. Then $F \in L^{(<\omega)}$ and $x \in \uparrow F = \bigcap_{i=1}^n B_i = L \backslash \bigcup_{i=1}^n C_i \in \upsilon(L)$. Hence, $x \in \mathrm{int}_{\upsilon(L)} \uparrow F = \uparrow F \subseteq L \backslash \downarrow \{y_1, y_2, \ldots, y_n\} \subseteq U$. Therefore, L is a quasi-hyperalgebraic lattice.

(3) ⇒ (11) $\forall x, y \in L$ with $x \nleq y$, by (3), $\exists \{u_1, u_2, \ldots, u_n\}, \{v_1, v_2, \ldots, v_m\} \in L^{(<\omega)}$ satisfying

(i) $\forall (i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$, $u_i \nleq v_j$;

(ii) $u_i \nleq y$ $(i = 1, 2, \ldots, n)$, $x \nleq v_j$ $(j = 1, 2, \ldots, m)$;

(iii) $\forall \{s_1, s_2, \ldots, s_m\}, \{t_1, t_2, \ldots, t_n\} \in L^{(<\omega)}$, if $u_i \nleq t_j (i, j \in \{1, 2, \ldots, n\})$, $s_i \nleq v_j (i, j \in \{1, 2, \ldots, m\})$, then $\exists (k, l) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ such that $s_k \nleq t_l$.

$\forall i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$, let $F_i = \{u_i\}$, $v_{1j} = v_{2j} = \ldots = v_{nj} = v_j$. By (i), $\forall i \in \{1, 2, \ldots, n\}$, $\{v_{i1}, v_{i2}, \ldots, v_{im}\} = \{v_1, v_2, \ldots, v_m\} \subseteq L \backslash \uparrow u_i = \nleq (F_i)$. Now we verify condition (b) of (11). By (ii), $\forall i \in \{1, 2, \ldots, n\}, y \in L \backslash \uparrow u_i = \nleq (F_i)$. Let $s_1 = s_2 = \ldots = s_m = x$, $t_1 = t_2 = \ldots = t_n = x$. Then $\forall (k, l) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$, $s_k \le t_l$. Note that $s_j = x \nleq v_j$ $(j = 1, 2, \ldots, m)$, by (iii), $\exists k \in \{1, 2, \ldots, n\}$ such that $u_k \le t_k = x$, that is, $\nleq (F_k) = L \backslash \uparrow u_k \subseteq L \backslash \uparrow x = \nleq (x)$. Obviously, condition (c) of (11) holds. Therefore, condition (11) holds.

(11) ⇒ (3) $\forall x, y \in L$ with $x \nleq y$, by (11), $\exists \{F_1, F_2, \ldots, F_n\} \in (X^{(<\omega)})^{(<\omega)}$ and $\{v_{ij} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\} \in Y^{(<\omega)}$ satisfying conditions (a),(b) and (c). Let $u_i = \vee F_i$, $v_j = \bigwedge_{i=1}^n v_{ij}$. Then it is easy to see that $\forall (i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$, $u_i \nleq v_j$, and $u_i \nleq y$ $(i = 1, 2, \ldots, n)$. By (b), $\exists k \in \{1, 2, \ldots, n\}$ such that $u_k \le x$. Thus, $x \nleq v_j$ $(j = 1, 2, \ldots, m)$ (otherwise, $u_k \le v_{j_0}$ for some $j_0 \in \{1, 2, \ldots, m\}$, a contradiction). Finally, we show that condition (iii) of (3). $\forall \{s_1, s_2, \ldots, s_m\}, \{t_1, t_2 \ldots, t_n\} \in L^{(<\omega)}$, if $u_i \nleq t_j (i, j \in \{1, 2, \ldots, n\})$, $s_i \nleq v_j (i, j \in \{1, 2, \ldots, m\})$. Then $t_j \in \nleq (F_i) = L \backslash \uparrow u_i (i, j \in \{1, 2, \ldots, n\})$ and $s_i \in \nleq^{-1}(\{v_{1j}, v_{2j}, \ldots, v_{nj}\}) = L \backslash \bigcap_{i=1}^n \downarrow v_{ij} = L \backslash \downarrow v_j (i, j \in \{1, 2, \ldots, m\})$. By (c), $\exists (k, l) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ such that $s_k \nleq t_l$. Therefore, condition (3) holds.

(11) ⇒ (13) Let $X = L$ and $\rho = \nleq$ on L. Then $L \cong (\Phi_{\nleq}(X), \subseteq)$. By Theorem 5.3, ρ is generalized finitely strongly regular.

(13) ⇒ (1) Using Theorem 5.3. □

Acknowledgements: The authors would like to thank the referees for their very careful reading of the manuscript and valuable comments.

Funding information: This research was supported by the National Natural Science Foundation of China (Nos 12071199, 11661057 and 12161043), and the Foundation of Education Department of Jiangxi Province (No. GJJ200838).

Conflict of interest: Authors state no conflict of interest.

References

[1] A. Zareckiĭ, *The semigroup of binary relations*, Mat. Sb. **61** (1963), 291–305.
[2] G. Markowsky, *Idempotents and product representations with applications to the semigroup of binary relations*, Semigroup Forum **5** (1972), 95–119.
[3] B. M. Schein, *Regular elements of the semigroup of all binary relations*, Semigroup Forum **13** (1976), 95–102.
[4] H. J. Bandelt, *Regularity and complete distributivity*, Semigroup Forum **19** (1980), 123–126.
[5] H. J. Bandelt, *On regularity classes of binary relations*, in: Universal Algebra and Applications, vol. 9, Banach Center Publications, PWN, Warsaw, 1982, pp. 329–333.
[6] X. Q. Xu and Y. M. Liu, *Relational representations of hypercontinuous lattices*, in: Domain Theory, Logic, and Computation, Kluwer Academic, Dordrecht, 2003, pp. 65–74.
[7] X. Q. Xu and Y. M. Liu, *Regular relations and completely regular spaces*, Chinese Ann. Math. **29A** (2008), 819–828.
[8] X. Q. Xu and M. K. Luo, *Regular relations and normal spaces*, Acta Math. Sinica (Chinese Ser.) **52** (2009), 393–402.
[9] G. H. Jiang and L. S. Xu, *Conjugative relations and applications*, Semigroup Forum **1** (2010), 85–91.
[10] G. H. Jiang, L. S. Xu, C. Cai, and G. W. Han, *Normal relations on sets and applications*, Int. J. Contemp. Math. Sci. **6** (2011), 721–726.
[11] G. H. Jiang and L. S. Xu, *Dually normal relations on sets*, Semigroup Forum **85** (2012), 75–80.
[12] M. Vinčić and D. A. Romano, *Finitely bi-quasiregular relations*, Sarajevo J. Math. **10** (2014), 21–26.
[13] M. Vinčić and D. A. Romano, *Finitely bi-normal relations*, Gulf J. Math. **3** (2015), 101–105.
[14] G. N. Raney, *A subdirect-union representation for completely distributive complete lattices*, Proc. Amer. Math. Soc. **4** (1953), 518–522.
[15] D. A. Romano, *Finitely bi-conjugative relations*, Rom. J. Math. Comput. Sci. **6** (2016), 134–138.
[16] D. A. Romano, *Finitely dual quasi-normal relations*, MAT-KOL **XXIII** (2017), 21–25.
[17] D. A. Romano, *Some specific classes of relations: a review*, MAT-KOL **XXIV** (2018), 19–22.
[18] S. Z. Luo and X. Q. Xu, *Split Hausdorff internal topologies on posets*, Open Math. **17** (2019), 1756–1763.
[19] X. Q. Xu, *Strongly algebraic lattices and conditions of mininal mapping preserving infs*, Chinese Ann. Math. **15B** (1994), 105–114.
[20] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003.
[21] X. Q. Xu, *Relational Representations of Complete Lattices and Their Applications*, PhD thesis, Sichuan University, 2004.
[22] J. B. Yang and M. K. Luo, *Priestley spaces,quasi-hyperalgebraic lattices and Smyth powerdomains*, Acta Math. Sin. (Engl. Ser.) **22** (2006), 951–958.
[23] H. A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. **2** (1970), 186–190.
[24] H. A. Priestley, *Ordered topological spaces and the represesentation of distributive lattices*, Proc. Lond. Math. Soc. (3) **24** (1972), 507–530.
[25] J. B. Yang and M. K. Luo, *Z-quasialgebraic domain*, J. Shanghai Univ. Nat. Sci. **42** (2005), 234–239.