Research Article

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Explicit construction of mock modular forms from weakly holomorphic Hecke eigenforms

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Abstract: Extending our previous work we construct weakly holomorphic Hecke eigenforms whose period polynomials correspond to elements in a basis consisting of odd and even Hecke eigenpolynomials induced by only cusp forms. As an application of our results, we give an explicit construction of the holomorphic parts of harmonic weak Maass forms that are good for Hecke eigenforms. Moreover, we give an explicit construction of the Hecke-equivariant map between the space of weakly holomorphic cusp forms and two copies of the spaces of cusp forms, and show that the map is compatible with the corresponding map on the spaces of period polynomials.

Keywords: mock modular forms, Hecke operators, harmonic Maass forms, weakly holomorphic modular forms

MSC 2020: Primary 11F11, 11F67 Secondary 11F25

1 Introduction and statement of results

Let p be one or a prime and $\Gamma_0^+(p)$ be the group generated by the congruence subgroup $\Gamma_0(p)$ and the Fricke involution $W_p = \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}$. For any even integer k, let $M_k^!(p)$ be the space of weakly holomorphic (i.e., meromorphic with poles only at the cusps) modular forms f of weight k for $\Gamma_0(p)$. For $\varepsilon \in \{\pm 1\}$, let $M_k^{!,\varepsilon}(p)$ be the subspace of $M_k^!(p)$ with $f|_k W_p = \varepsilon f$. Each $f \in M_k^{!,\varepsilon}(p)$ has a Fourier development of the form

$$f(z) = \sum_{n \geq n_0} a_f(n) q^n,$$

where the parameter q stands for $\exp(2\pi i z)$, as usual. We set $\mathrm{ord}_\infty f = n_0$ if $a_f(n_0) \neq 0$. Let $\mathfrak{S}$ be the set consisting of values of p for which the genus of $\Gamma_0^+(p)$ is zero, that is,

$$\mathfrak{S} = \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

When $p \in \mathfrak{S}$, the space $M_k^{!,\varepsilon}(p)$ has a canonical basis: see [1–4]. We define

$$m_k^\varepsilon := \max\{\mathrm{ord}_\infty f | f \neq 0 \in M_k^{!,\varepsilon}(p)\}.$$

When $k > 2$, we note that m_k^ε is given by $\dim S_k^\varepsilon(p)$. Indeed, for every integer m with $-m \leq m_k^\varepsilon$, there exists a unique weakly holomorphic modular form $\mathfrak{f}_{k,m}^\varepsilon \in M_k^{!,\varepsilon}(p)$ with Fourier expansion of the form

$$\mathfrak{f}_{k,m}^\varepsilon(\tau) = q^{-m} + \sum_{n > m_k^\varepsilon} a_k^\varepsilon(m, n) q^n \tag{1}$$

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and together they form a basis for $M_k^{!,\varepsilon}(p)$. Indeed the basis element $\mathfrak{f}_{k,m}^{\varepsilon}$ can be given explicitly in the form $\mathfrak{f}_{k,m}^{\varepsilon} = \mathfrak{f}_{k,-m_k^{\varepsilon}}^{\varepsilon} F_{k,m+m_k^{\varepsilon}}(j_p^+)$, where j_p^+ is the Hauptmodul for $\Gamma_0^+(p)$ and $F_{k,D}(x)$ is a monic polynomial in x of degree D. Moreover, the Fourier coefficients $a_k^{\varepsilon}(m,n)$ of q^n in $\mathfrak{f}_{k,m}^{\varepsilon}$ turn out to be rational integers. Throughout this paper, we simply write $\mathfrak{f}_{k,m} := \mathfrak{f}_{k,m}^{+}$.

As usual, we denote by $S_k^+(p)$ the space of holomorphic cusp forms of weight k for $\Gamma_0^+(p)$. Let $S_k^{!,+}(p)$ be the subspace of $M_k^{!,+}(p)$ consisting of weakly holomorphic modular forms for $\Gamma_0^+(p)$ with zero constant term in the Fourier expansion. We now consider Hecke operators on $S_k^+(p)$. For each positive integer n relatively prime to p, the usual Hecke operator T_n on $S_k^+(p)$ extends to $S_k^{!,+}(p)$. In particular, for prime indices $l(\neq p)$, the Hecke operators T_ℓ on $S_k^{!,+}(p)$ are given as follows: for $f \in S_k^{!,+}(p)$,

$$T_\ell f = l^{k/2-1} \sum_{\substack{ad=l \\ b(\mathrm{mod}\ d)}} f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \sum_n (a_f(ln) + l^{k-1} a_f(n/l)) q^n. \tag{2}$$

Common eigenforms of all Hecke operators T_n on $S_k^+(p)$ with n coprime to p are called *Hecke eigenforms*. For later use, we let $t = \dim S_k^+(p)$ and

$$\left\{ f_n = \sum_{m>0} \lambda(n,m) q^m \mid n = 1, 2, \ldots, t \right\}$$

be a basis of $S_k^+(p)$ consisting of normalized Hecke eigenforms. Following [5,6] we call $f \in S_k^{!,+}(p)$ a *weakly holomorphic Hecke eigenform* with respect to $S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p))$ if for every Hecke operator T_n with $(n,p)=1$ there is a complex number λ_n for which

$$T_n f - \lambda_n f \in D^{k-1}(M_{2-k}^{!,+}(p)),$$

where D stands for the differential operator $\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}z}$.

In the work of Bringmann et al. [5, Theorem 1.5] weakly holomorphic Hecke eigenforms are constructed in the level 1 case by making use of harmonic weak Maass forms which are preimages of Hecke eigenforms under the differential operator $\xi_k := 2iy^k \overline{\frac{\partial}{\partial \bar{z}}}$ originated from [7]. Thus, their construction is not explicit. In [6, Theorem 1.2], we extended it to higher level cases to the primes for which $\Gamma_0^+(p)$ has genus zero (primes up to 71 excluding 37, 43, 61, and 67). The construction was given explicitly in terms of weakly holomorphic modular forms without relying on the theory of harmonic weak Maass forms, and we gave an explicit description of the "polar" eigenform h_n in terms of a linear combination of cuspidal eigenforms and the dual form f_n^*, as in (**EF1**)–(**EF3**). Here the duality is with respect to a certain pairing of functions introduced by Guerzhoy as follows.

Following [1,5,6,8,9], for $f, g \in M_k^{!,+}(p)$, we define a pairing $\{f, g\}$ originated from Bruinier and Funke [7] by

$$\{f,g\} := \sum_{n\in\mathbb{Z}, n\neq 0} \frac{a_f(-n) a_g(n)}{n^{k-1}}. \tag{3}$$

It is antisymmetric (since k is even), bilinear, and Hecke equivariant. Specifically, for any prime $\ell(\neq p)$

$$\{T_\ell f, g\} = \{f, T_\ell g\}.$$

Moreover, $D^{k-1}(M_{2-k}^{!,+}(p))$ is perpendicular to all of $S_k^{!,+}(p)$, they are the only weakly holomorphic modular forms with that property (by the Serre duality theorem), and two elements of $S_k^+(p)$ are perpendicular to one another. Let

$$f_n^* = \sum_{m\geq -t} \mu(m,n) q^m \tag{4}$$

with $\mu(m,n) \in \mathbb{C}$ be a linear combination of $\mathfrak{f}_{k,1}, \ldots, \mathfrak{f}_{k,t}$, which is dual to f_n with respect to the pairing, i.e., $\{f_m^*, f_n\} = \delta_{mn}$, where δ_{mn} is the Kronecker delta function. Then such functions f_n^* are unique.

Moreover, it follows from [1,9] that $T_\ell f_n^*$ and $\lambda(n,\ell)f_n^*$ represent the same coset in $S_k^{!,+}(p)/(D^{k-1}(M_{2-k}^{!,+}(p))\oplus S_k^+(p))$. Thus, we can write

$$T_\ell f_n^* = \lambda(n,\ell)f_n^* + D^{k-1}g_{n,\ell} + \sum_{j=1}^{t} a_{jn}(\ell)f_j \tag{5}$$

for some $g_{n,\ell}\in M_{2-k}^{!,+}(p)$ and $a_{jn}(\ell)\in\mathbb{C}$.

Let $p\in\mathfrak{S}$, $t=\dim S_k^+(p)$, and ℓ be a prime different from p. We then obtain from [6, Theorem 1.2] the following assertions.

(EF1) Let $i,n\in\{1,\dots,t\}$ with $i\neq n$. Let r be a prime ($\neq p$) such that $\lambda(i,r)\neq\lambda(n,r)$ and put

$$x_i(n) := \frac{a_{ni}(r)}{\lambda(i,r)-\lambda(n,r)}.$$

Then $x_i(n)$ is independent of the choice of r and the quantity $a_{ni}(r)$ is given by

$$a_{ni}(r) = \{f_n^*, T_r f_i^*\}.$$

(EF2) For each n with $1\le n\le t$, let

$$h_n := \sum_{\substack{i=1\\ i\neq n}}^{t} x_i(n)f_i + f_n^*.$$

Then h_n is a Hecke eigenform with respect to $S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p))$ having the same eigenvalues as those of f_n. More explicitly one has

$$T_\ell(h_n) = \lambda(n,\ell)h_n + D^{k-1}(g_{n,\ell}),$$

where $g_{n,\ell}$ is the modular form defined in (5) and computed as

$$g_{n,\ell} = -\sum_{\substack{1\le s\le t\\ s\ell>t}} \frac{\mu(-s,n)}{s^{k-1}}\mathfrak{f}_{2-k,s\ell},$$

where $\mu(\cdot,n)$ is the Fourier coefficient of f_n^*, defined in (4).

(EF3) The set

$$\{[f_1],\dots,[f_t],[h_1],\dots,[h_t]\}$$

forms a basis for $S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p))$, where $[f]$ stands for the class of f.

For $\Gamma\in\{\Gamma_0(p),\Gamma_0^+(p)\}$, let $H_{2-k}(\Gamma)$ be the space of harmonic weak Maass forms of weight $2-k$ for Γ. Following [10,11], we say that $\mathfrak{F}(z)\in H_{2-k}(\Gamma_0(p))$ is "good" for the Hecke eigenform $f^c(z):=\overline{f(-\bar z)}\in S_k(p)$ if it satisfies the following:

(1) The principal part of $\mathfrak{F}$ at the cusp ∞ belongs to $K_f[q^{-1}]$. Here K_f denotes the number field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of f.
(2) The principal part of $\mathfrak{F}$ at the cusp 0 is constant.
(3) We have $\xi_{2-k}\mathfrak{F} = \frac{f^c}{(f^c,f^c)}$.

Remark 1.1.

(i) The existence of $\mathfrak{F}$, which is good for a Hecke eigenform f^c is guaranteed by [10, Proposition 5.1].
(ii) Let $\mathfrak{F}_0$ be good for a Hecke eigenform f^c and denote $t=\dim S_k^+(p)$ and $t'=\dim S_k^-(p)$. Let $M_{2-k}^{\sharp}(p)$ be the space of weakly holomorphic modular forms of weight $2-k$ for $\Gamma_0(p)$ with poles allowed only at the cusp ∞. For $p\in\{1,2,3,5,7,13\}$, it follows from [12, Theorem 1.1, Theorem 1.7, and line 6 in p. 123] that

$$\max\{\mathrm{ord}_\infty f \mid f\neq 0\in M_{2-k}^{\sharp}(p)\} = -1-t-t',$$

and for each integer m with $-m \le -1 - t - t'$, there exists $\mathfrak{f}^{\sharp}_{2-k,m} = q^{-m} + O(q^{-t-t'}) \in M^{\sharp}_{2-k}(p)$ with integral Fourier coefficients. By subtracting a suitable linear combination of $\mathfrak{f}^{\sharp}_{2-k,m}$'s from $\mathfrak{F}_0$ we can take a unique $\mathfrak{F}$, which is good for f^c and $\mathfrak{F}^+ = O(q^{-t-t'})$.

(iii) In [11] a direct method for relating the coefficients of f^c and $\mathfrak{F}$ is provided by means of p-adic coupling and an algebraic regularized mock modular form F_α. More precisely, if we let α be the coefficient of q^1 in $\mathfrak{F}^+$, then F_α is given by

$$F_\alpha = D^{k-1}\mathfrak{F}^+ - \alpha f = \sum_{n \gg -\infty} c_\alpha(n)q^n.$$

Moreover, F_α has coefficients in K_f and

$$\lim_{w\to+\infty} \frac{\sum_{n\gg-\infty} c_\alpha(\ell^w n)q^n}{c_\alpha(\ell^w)} = f(z) - \beta^{-1}\ell^{k-1}f(\ell z), \tag{6}$$

where ℓ is a prime number and β, β' are the roots of the equation $X^2 - a_f(\ell)X + \chi(\ell)\ell^{k-1} = (X-\beta)(X-\beta')$ ordered so that $\mathrm{ord}_\ell(\beta) \le \mathrm{ord}_\ell(\beta')$, χ is a trivial character modulo p, and we assume that $\beta \neq 0$ in the case $\ell = p$. We take the limit in (6) in ℓ-adic topology.

(iv) In [13], the structure of half-integral weight weakly holomorphic Hecke eigenforms was developed, and in [14] half-integral weight p-adic coupling was investigated.

Let $f, g \in M^!_k(p)$. We define a *regularized inner product* as follows. For $T > 0$, we denote by $\mathcal{F}_T$ the truncated fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$

$$\mathcal{F}_T = \{z \in \mathbb{H} \,|\, |x| \le 1/2,\ |z| \ge 1,\ \text{ and }\ y \le T\}.$$

Moreover, we define the truncated fundamental domain for $\Gamma_0(p)$ by

$$\mathcal{F}_T(\Gamma_0(p)) = \bigcup_{\gamma\in\mathcal{V}} \gamma\mathcal{F}_T,$$

where $\mathcal{V}$ is a fixed set of representatives of $\Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z})$. Now we define the regularized inner product $(f,g)^{\mathrm{reg}}$ as the constant term in the Laurent expansion at $s = 0$ of the function

$$\frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)]} \lim_{T\to\infty} \int_{\mathcal{F}_T(\Gamma_0(p))} f(z)\overline{g(z)}y^{k-s}\frac{dxdy}{y^2}.$$

As explained in [10] and [15], $(f,g)^{\mathrm{reg}}$ exists if f or g is a holomorphic modular form. If both f and g are holomorphic modular forms such that fg is a cusp form, then $(f,g)^{\mathrm{reg}}$ reduces to the Petersson inner product (f,g).

In the next theorems, we give an explicit construction of $D^{k-1}\mathfrak{F}$ in terms of polar eigenforms and a canonical basis. Theorem 1.2 applies for $p = 1$ only, while Theorems 1.3 and 1.5 cover higher levels.

Theorem 1.2. *Let $2 < k \in 2\mathbb{Z}$, and $t = \dim S^+_k(1)$. Then for each $n \in \{1,\dots,t\}$,*

$$-h_n + \frac{(f^*_n, f_n)^{\mathrm{reg}}}{(f_n, f_n)} f_n$$

is equal to $D^{k-1}\mathfrak{F}_n$ for the unique $\mathfrak{F}_n \in H_{2-k}(\Gamma_0(1))$, which is good for f^c_n and $\mathfrak{F}^+_n = O(q^{-t})$.

Theorem 1.3. *Let p be a prime for which $\Gamma_0(p)$ is of genus zero, i.e., $p \in \{2,3,5,7,13\}$, $2 < k \in 2\mathbb{Z}$, $t = \dim S^+_k(p)$, $t' = \dim S^-_k(p)$, and $n \in \{1,\dots,t\}$. Then the following assertions are true.*

(i) *Let A be a $t \times t$ matrix whose ij-entry is given by $\mathrm{CT}(f_i \cdot \mathfrak{f}^-_{2-k,t'+j})$, where $\mathrm{CT}(f)$ denotes the constant term of the Fourier expansion of f. Then the matrix A is invertible.*

(*ii*) *Let* β_{ij} *be the ij-entry of the matrix* A^{-1}. *Take the unique weakly holomorphic modular form* $w_n \in M^{!,+}_{2-k}(p)$ *such that*

$$w_n - \sum_{j=1}^{t} \beta_{jn}\, \mathfrak{f}^-_{2-k,t'+j} = O(q^{-t}).$$

Then

$$-h_n + \frac{(f_n^*, f_n)^{\text{reg}}}{(f_n, f_n)} f_n + D^{k-1}\left(w_n + \sum_{j=1}^{t} \beta_{jn}\, \mathfrak{f}^-_{2-k,t'+j} \right)$$

is equal to $D^{k-1}\mathfrak{F}_n$ *for the unique* $\mathfrak{F}_n \in H_{2-k}(\Gamma_0(p))$, *which is good for* f_n^c *and* $\mathfrak{F}_n^+ = O(q^{-t-t'})$.

Remark 1.4. In Theorem 1.3, we observe from duality that $m^{\varepsilon}_{2-k} = -\dim S^{\varepsilon}_k(p) - 1$. This observation makes the existence and uniqueness of w_n much clearer.

Theorem 1.5. *Let* $p \in \{1, 2, 3, 5, 7, 13\}$, $2 < k \in 2\mathbb{Z}$, $t = \dim S^+_k(p)$, *and* $n \in \{1, \dots, t\}$. *Then the coefficients of* f_n^* *are in* K_{f_n}.

Let P_{k-2} denote the space of all polynomials of degree at most $k-2$. For any meromorphic function f on the complex upper half plane $\mathfrak{H}$, we define the action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ by

$$(f|_k\gamma)(z) = (\det\gamma)^{k/2}(cz+d)^{-k} f(\gamma z).$$

For $p \in \{1, 2, 3\}$, a subspace $W^+_{k-2}(p)$ of P_{k-2} is defined by

$$W^+_{k-2}(p) = \left\{ g \in P_{k-2} \,|\, g + g|_{2-k} W_p = 0 = g + g|_{2-k} U + g|_{2-k} U^2 + \cdots + g|_{2-k} U^{n_p - 1} \right\}$$

with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = TW_p$, and $n_p = \begin{cases} 3 & \text{if } p = 1 \\ 2p & \text{if } p = 2, 3 \end{cases}$. We call the elements of the space $W^+_{k-2}(p)$ *period polynomials*. Period polynomials have been investigated in relation to Eichler integrals, cusp forms via Eichler-Shimura isomorphisms, and to special values of modular L-functions. Indeed, Eichler [16] discovered relations between periods of cusp forms, and Shimura [17] extended them. Later, Manin [18] made more explicit the connection of these relations with the Fourier coefficients, by using the Hecke operators and continued fractions. For more discussion on the classical theory of period polynomials, one is referred to [18–22] and [23, Chapter 12]. The period polynomials are also related to weakly holomorphic modular forms as follows. For each $f = \sum_{n \gg -\infty} a_f(n) q^n \in M^{!,+}_k(p)$ we define the *period polynomial for f* by

$$r^+(f) = r^+(f, z) := c_k(\mathcal{E}_f - \mathcal{E}_f|_{2-k} W_p)(z), \tag{7}$$

where $c_k = -\frac{\Gamma(k-1)}{(2\pi i)^{k-1}}$ and $\mathcal{E}_f$ denotes the *Eichler integral*

$$\mathcal{E}_f(z) := \sum_{\substack{n \gg -\infty \\ n \neq 0}} a_f(n) n^{1-k} q^n. \tag{8}$$

Let $S^{!,+}_k(p)$ be the subspace of $M^{!,+}_k(p)$ consisting of those $f \in M^{!,+}_k(p)$ with $a_f(0) = 0$. As described in [24, Section 4], r^+ gives a map from $S^{!,+}_k(p)$ to $W^+_{k-2}(p)$. For $p \in \{1, 2, 3\}$ and even $k > 2$, in virtue of [5, Theorem 1.6], [8, Theorem 1.1], [24, Theorem 1.2], and [25, Theorem 2], we have the following exact sequence:

$$0 \longrightarrow D^{k-1}(M^{!,+}_{2-k}(p)) \longrightarrow S^{!,+}_k(p) \xrightarrow{r^+} W^+_{k-2}(p)/\langle(\sqrt{p}z)^{k-2} - 1\rangle \longrightarrow 0. \tag{9}$$

Let $p \in \{1, 2, 3\}$, n be a positive integer relatively prime to p, and k be an even integer greater than 2. Following Knopp [19,20] we define a Hecke operator $\widehat{T}_n$ on the space $W^+_{k-2}(p)$. Suppose that $q(z) \in W^+_{k-2}(p)$ and $F(z)$ is a meromorphic function on $\mathfrak{H}$ satisfying

$$F|_{2-k} T = F \;\text{ and }\; F|_{2-k} W_p = F + q(z). \tag{10}$$

We also assume that $F(z)$ is meromorphic in the local uniformizing variable at the cusp ∞ of a fundamental region for $\Gamma_0^+(p)$. In such a case, $F(z)$ is called a *modular integral for $q(z)$ of weight $2-k$*. We define $\widehat{T}_n$ by

$$\widehat{T}_n(q(z)) = F_n\,|_{2-k}(W_p - 1), \tag{11}$$

where

$$F_n = n^{-k/2} \sum_{\substack{ad=n\\ b(d)}} F|_{2-k}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}.$$

We then obtain that $\widehat{T}_n(q(z)) \in W_{k-2}^+(p)$ and it follows from [6, Theorem 1.3] the following diagram

$$\begin{array}{ccc} S_k^{!,+}(p) & \xrightarrow{r^+} & W_{k-2}^+(p)\\ \big\downarrow \ell^{1-k}T_\ell & & \big\downarrow \widehat{T_\ell}\\ S_k^{!,+}(p) & \xrightarrow{r^+} & W_{k-2}^+(p)\end{array}$$

is commutative, i.e., r^+ is a Hecke equivariant homomorphism. We recall the Eisenstein series

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \quad \mathbb{G}_k(z) = -\frac{B_k}{2k}E_k(z)$$

and

$$\mathbb{G}_k^+(z) = \mathbb{G}_{k,p}^+(z) = \begin{cases}\mathbb{G}_k(z), & \text{if } p = 1\\ \mathbb{G}_k(z) + p^{k/2}\mathbb{G}_k(pz), & \text{if } p = 2 \text{ or } 3,\end{cases}$$

where B_k is the kth Bernoulli number and σ_{k-1} denotes the usual divisor sum.

With the same notation as above, let $p \in \{1, 2, 3\}$ and l be a prime different from p. We set $p_0(z) := (\sqrt{p}z)^{k-2} - 1$. We then obtain from [6, Theorem 1.4] the following assertions.

(EP1) $\widehat{T}_\ell(p_0) = (1 + \ell^{1-k})p_0$ and $\widehat{T}_\ell(r^+(f_n)) = \ell^{1-k}\lambda(n, \ell)r^+(f_n)$ for each $n \in \{1, \dots, t\}$.

(EP2) For each n with $1 \le n \le t$ we define

$$c_0(n) = -\frac{c_k}{t(0)}\sum_{m=1}^{t}\frac{\mu(-m, n)}{m^{k-1}}t(m),$$

where c_k is the constant defined in (7) and $t(m)$ denotes the coefficient of q^m in $\mathbb{G}_k^+(z)$. Then we have

$$\widehat{T}_\ell(c_0(n)p_0 + r^+(h_n)) = \ell^{1-k}\lambda(n, \ell)(c_0(n)p_0 + r^+(h_n)).$$

(EP3) The set

$$\{p_0(z), r^+(f_1), \dots, r^+(f_t), c_0(1)p_0 + r^+(h_1), \dots, c_0(t)p_0 + r^+(h_t)\}$$

forms a basis for $W_{k-2}^+(p)$ consisting of Hecke eigenpolynomials. As we see, the basis in **(EP3)** is constructed by using weakly holomorphic modular forms that are not holomorphic ones. Every period polynomial decomposes into an even and an odd part. By the Eichler-Shimura isomorphism, the space of even period polynomials consists of p_0 and those coming from cusp forms. Also the space of odd period polynomials can be constructed by using cusp forms. So it is natural to find a basis for the space $W_{k-2}^+(p)$ consisting of odd and even period polynomials induced by only cusp forms that are Hecke eigenpolynomials. Although this result is well-known from the literature [18,21,22], we state it in the next theorem and reprove it by using the theory of harmonic weak forms because its proof will be used to prove our main results. For $f \in S_k^+(p)$, $r_-^+(f)$ denotes $\frac{r(f,z)-r(f,-z)}{2}$ and $r_+^+(f)$ stands for $\frac{r(f,z)+r(f,-z)}{2i}$.

Theorem 1.6. *With the same notations as above, let $p \in \{1, 2, 3\}$, $2 < k \in 2\mathbb{Z}$, $t = \dim S_k^+(p)$, and l be a prime different from p. Then for each $n \in \{1, \dots, t\}$, the following assertions are true.*

(*i*) *The polynomial* $r^+(f_n, -z)$ *belongs to the space* $W^+_{k-2}(p)$ *and is a Hecke eigenpolynomial with the same eigenvalues as those of* $r^+(f_n, z)$, *i.e.,*

$$\widehat{T_\ell}(r^+(f_n, -z)) = l^{1-k}\lambda(n, l)r^+(f_n, -z).$$

(*ii*) *Both polynomials* $r^+_+(f_n, z)$ *and* $r^+_-(f_n, z)$ *belong to the space* $W^+_{k-2}(p)$ *and are Hecke eigenpolynomials with the same eigenvalues as those of* $r^+(f_n, \pm z)$.

(*iii*) *The set*

$$\{p_0(z), r^+_+(f_1), \dots, r^+_+(f_t), r^+_-(f_1), \dots, r^+_-(f_t)\}$$

forms a basis for $W^+_{k-2}(p)$ *consisting of Hecke eigenpolynomials.*

Since $r^+(f_n, -z) \in W^+_{k-2}(p)$, it follows from (9) that there exists $g \in S^{!,+}_k(p)$ such that

$$r^+(g) \equiv r^+(f_n, -z) \pmod{p_0}.$$

When $p = 1$, by utilizing harmonic weak Maass forms, Bringmann et al. [5] constructed such g. So their construction is not explicit. In Theorem 1.8, we will provide an explicit construction of such g for all $p \in \mathfrak{S}$ without using harmonic weak Maass forms.

Remark 1.7. Theorem 1.6 is related to the results in [18,21]. In [22], more general level is considered by using a cohomological approach.

Theorem 1.8. *Let* $p \in \mathfrak{S}$, $2 < k \in 2\mathbb{Z}$, *and* $t = \dim S^+_k(p)$. *For each* $n \in \{1, \dots, t\}$, *choose* $G_n(z) \in H_{2-k}(\Gamma + 0(p))$ *such that* $\xi_{2-k}G_n(z) = f_n(z)$. *Let*

$$g_n(z) := D^{k-1}G^c_n \text{ and } \mathfrak{g}_n := -\frac{(f_n, f_n)}{b_k}h_n + \frac{(f^*_n, f_n)^{\mathrm{reg}}}{b_k}f_n \quad \text{with } b_k = \frac{\Gamma(k-1)}{(4\pi)^{k-1}}.$$

Then the following assertions are true.

(*i*) *The class* $[g_n]$ *is independent of the choice of* G_n *and uniquely determined by* f_n. *Moreover, we have* $[g_n] = [b_k\mathfrak{g}_n]$.

(*ii*) *Let* $p \in \{1, 2, 3\}$. *Then we have*

$$r^+(\mathfrak{g}_n, z) = r^+(f_n, -z) + \frac{(f_n, f_n)}{b_k}c_0(n)p_0.$$

Remark 1.9. When $p = 1$, the formula in Theorem 1.8(ii) is related to the results in [23, Theorem 12.10] and the following remark.

We define a map $\mathfrak{r} : S^+_k(p) \times S^+_k(p) \to W^+_{k-2}(p)$ by $\mathfrak{r}(f, g) = r^+_-(f) + ir^+_+(g)$ and $W_0 = \operatorname{im} \mathfrak{r}$. Then it is well known that $W^+_{k-2}(p) = W_0 \oplus \langle p_0\rangle$ and therefore we can consider a map $P : W^+_{k-2}(p) \to W_0$, which is the projection to the first component. When $p = 1$, Bringmann et al. [5] suggested the following diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & S^+_k(p) \times S^+_k(p) & \xrightarrow{\mathfrak{r}} & W_0 & & \\
 & & & & \uparrow{\scriptstyle \Phi} & & \uparrow{\scriptstyle P} & & \\
0 & \longrightarrow & D^{k-1}(S^{!,+}_{2-k}(p)) & \longrightarrow & S^{!,+}_k(p) & \xrightarrow{r^+} & W^+_{k-2}(p) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & D^{k-1}(S^{!,+}_{2-k}(p)) & \longrightarrow & D^{k-1}(M^{!,+}_{2-k}(p)) & \xrightarrow{r^+} & \langle p_0\rangle & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where a map $\Phi : S_k^{!,+}(p) \to S_k^+(p) \times S_k^+(p)$ is not given explicitly. So they did not show that the diagram is commutative in the part.

$$\begin{array}{ccc} S_k^+(p) \times S_k^+(p) & \xrightarrow{\mathfrak{r}} & W_0 \\ \uparrow \Phi & & \uparrow P \\ S_k^{!,+}(p) & \xrightarrow{r^+} & W_{k-2}^+(p) \end{array}$$

In the next two theorems, we will give an explicit construction of such a Hecke-equivariant map Φ for $p \in \{1, 2, 3\}$, which makes the diagram commutative. To this end, we are in need of the following proposition.

Proposition 1.10. *Let $p \in \mathfrak{S}$, $2 < k \in 2\mathbb{Z}$, and $t = \dim S_k^+(p)$. With the same notations as above, we have the following assertions.*

(*i*) *The set $\{[\mathfrak{g}_1], [\mathfrak{g}_2], \ldots, [\mathfrak{g}_t], [f_1], [f_2], \ldots, [f_t]\}$ forms a basis for the space $S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p))$.*

(*ii*) *The set $\{[-\mathfrak{g}_1 + f_1], [-\mathfrak{g}_2 + f_2], \ldots, [-\mathfrak{g}_t + f_t], [f_1], [f_2], \ldots, [f_t]\}$ forms a basis for the space $S_k^{!,+}(p)/D^{k-1}(M_{2-k}^{!,+}(p))$.*

We note from Proposition 1.10 that for any $F \in S_k^{!,+}(p)$, there exists unique constants $\alpha_n, \beta_n \in \mathbb{C}$ such that

$$F = \sum_n \alpha_n(-\mathfrak{g}_n + f_n) + \sum_n \beta_n f_n + D^{k-1} g_F \tag{12}$$

for some $g_F \in M_{2-k}^{!,+}(p)$. Now we define a map $\Phi : S_k^{!,+}(p) \to S_k^+(p) \times S_k^+(p)$ by

$$\Phi(F) = \left(2\sum_n \alpha_n f_n + \sum_n \beta_n f_n, \sum_n \beta_n f_n\right), \tag{13}$$

where α_n and β_n are the coefficients appearing in (12). We then easily check that the map Φ is a linear map.

Theorem 1.11. *Let $p \in \{1, 2, 3\}$ and $2 < k \in 2\mathbb{Z}$. With the same notations as above, we have the following assertions.*

(*i*) *The maps Φ and $\mathfrak{r}$ are Hecke equivariant homomorphisms, i.e., for $F \in S_k^{!,+}(p)$ and $f, g \in S_k^+(p)$,*

$$\Phi(T_\ell(F)) = (T_\ell \times T_\ell)(\Phi(F))$$

and

$$\ell^{1-k}\, \mathfrak{r}(T_\ell f, T_\ell g) = \widehat{T}_\ell \circ \mathfrak{r}(f, g).$$

(*ii*) $\ker\Phi = D^{k-1}(M_{2-k}^{!,+}(p))$.

(*iii*) $P \circ r^+ = \mathfrak{r} \circ \Phi$.

This paper is organized as follows. In Section 2, we give examples which illustrate Theorems 1.2, 1.3, and 1.5. In Sections 3 and 4, we prove Theorems 1.6 and 1.8, respectively. Next, proofs of Theorems 1.2, 1.3, and 1.5 are given in Section 5. Finally, in Section 6 we prove Proposition 1.10 and Theorem 1.11.

2 Examples

Example 2.1. (Cf. [6, Example 2.1] and [11, Examples in p. 6170]) Let $p = 1$ and $k = 12$. In this case, we have $t = \dim S_{12}^+(1) = 1$. Let $\eta(z)$ be the Dedekind eta function defined by $\eta(z) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1 - q^n)$.

Using $\Delta(z) = \eta(z)^{24} \in S_{12}^{+}(1)$ and the Hauptmodul $j_1(z) = E_4(z)^3/\Delta(z) - 744$ for $\Gamma_0^{+}(1)$ one can express $\mathfrak{f}_{12,m}$ $(-1 \le m \le 1)$ as follows:

$$\begin{aligned}
\mathfrak{f}_{12,-1}(z) &= \Delta(z)\\
&= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9 + O(q^{10})\\
\mathfrak{f}_{12,0}(z) &= \Delta(j_1 + 24)\\
&= 1 + 196560q^2 + 16773120q^3 + 398034000q^4 + 4629381120q^5 + 34417656000q^6 + O(q^7)\\
\mathfrak{f}_{12,1}(z) &= \Delta(j_1^2 + 24j_1 - 393444)\\
&= \frac{1}{q} + 47709536q^2 + 39862705122q^3 + 7552626810624q^4 + 609136463852480q^5 + O(q^6).
\end{aligned}$$

It is immediate from the definitions of f_1, f_1^*, h_1 that

$$f_1 = \mathfrak{f}_{12,-1} \text{ and } h_1 = f_1^* = \mathfrak{f}_{12,1}. \tag{14}$$

Then one has from Remark 1.1(iii) and Theorem 1.2,

$$F_\alpha = -h_1 = -\mathfrak{f}_{12,1} = -q^{-1} - \sum_{n=2}^{\infty} a_{12}(1,n)q^n,$$

and if we take $l = 3$ and $w = 1$ in (6), by using the Sturm bound one verifies that

$$\begin{aligned}
&\frac{-\sum_{n=1}^{\infty} a_{12}(1,3n)q^n}{-a_{12}(1,3)}\\
&= q + \frac{27947672851540608}{39862705122}q^2 + \frac{340389905850815087232}{39862705122}q^3 + \frac{6523525558635002468444416}{39862705122}q^4 + \cdots\\
&\equiv \Delta \pmod{3^{10}}.
\end{aligned}$$

Example 2.2. Let $p = 5$ and $k = 10$. In this case, one has $t = \dim S_{10}^{+}(5) = 1$ and $t' = \dim S_{10}^{-}(5) = 2$. Let $\Delta_5^{+}(z) = (\eta(z)\eta(5z))^4$ be the unique cusp form in $S_4^{+}(5)$ and $E_6(z)$ be the normalized Eisenstein series of weight six. We also let $j_5^{+}(z)$ be the Hauptmodul for $\Gamma_0^{+}(5)$, given by

$$j_5^{+}(z) = \left(\frac{\eta(z)}{\eta(5z)}\right)^6 + 6 + 5^3\left(\frac{\eta(5z)}{\eta(z)}\right)^6.$$

One then computes that

$$\begin{aligned}
\mathfrak{f}_{10,-1} &= \Delta_5^{+}E_6^{+} = q - 8q^2 - 114q^3 - 448q^4 - 625q^5 + 912q^6 + 4242q^7 + 7680q^8 - 6687q^9 + \cdots,\\
\mathfrak{f}_{10,1} &= \Delta_5^{+}E_6^{+}(j_5^{+2} + 8j_5^{+} - 90) = \frac{1}{q} - 192q^2 - 14511q^3 - 370176q^4 - 5152500q^5 - \cdots,\\
\mathfrak{f}_{-8,2} &= (\Delta_5^{+})^{-2} = \frac{1}{q^2} + \frac{8}{q} + 44 + 192q + 726q^2 + 2472q^3 + 7768q^4 + 22880q^5 + \cdots,\\
\mathfrak{f}_{-8,3} &= (\Delta_5^{+})^{-2}(j_5^{+} - 8) = \frac{1}{q^3} + \frac{114}{q} + 1672 + 14511q + 94848q^2 + 515774q^3 + 2454144q^4 + 10533315q^5 + \cdots,\\
\mathfrak{f}_{-8,3}^{-} &= (\Delta_5^{+})^{-3}E_4^{-} = \frac{1}{q^3} + \frac{2}{q^2} - \frac{120}{q} - 1740 - 14855q - 96200q^2 - 520532q^3 - 2469320q^4 - 10578425q^5 + \cdots,
\end{aligned}$$

where $E_6^{+} = \frac{1}{1+5^3}(E_6 + E_6\,|_6 W_5)$ and $E_4^{-} = \frac{1}{1-5^2}(E_4 - E_4\,|_4 W_5)$. We then obtain that

$$f_1 = \mathfrak{f}_{10,-1}, \quad h_1 = f_1^* = \mathfrak{f}_{10,1}, \quad \text{and} \quad w_1 = \beta_{11}\,\mathfrak{f}_{-8,3} + 2\beta_{11}\,\mathfrak{f}_{-8,2},$$

where $\beta_{11} = \frac{1}{\mathrm{CT}(f_1 \cdot \mathfrak{f}_{-8,3}^{-})} = -\frac{1}{250}$. Then one has from Remark 1.1(iii) and Theorem 1.3(ii),

$$F_\alpha = -h_1 + D^9(w_1 + \beta_{11}\,\mathfrak{f}_{-8,3}^{-}) + \frac{4}{25}f_1 = \sum_{n=-3}^{\infty} c_\alpha(n)q^n,$$

and if we take $l = 3$ and $w = 1$ in (6), by using the Sturm bound one verifies that

$$\begin{aligned}\frac{\sum_{n=-1}^{\infty} c_\alpha(3n)q^n}{c_\alpha(3)} &= -\frac{6561}{6308q} - \frac{18528264}{1577}q^2 + \frac{808269273}{1577}q^3 + \frac{68622811200}{1577}q^4 - \frac{533626633125}{1577}q^5 + \frac{2832551189208}{1577}q^6 \\ &\quad - \cdots \\ &\equiv f_1 \pmod{3^8}.\end{aligned}$$

Example 2.3. (Cf. [1, Example 2.10]) Let $p = 5$ and $k = 12$. In this case, $t = \dim S_{12}^{+}(5) = 3$ and the space $S_{12}^{+}(5)$ is spanned by

$$\begin{aligned}\mathfrak{f}_{12,-3}(z) &= \Delta_5^{+}(z)^3 = q^3 - 12q^4 + 54q^5 - 88q^6 + \cdots ,\\ \mathfrak{f}_{12,-2}(z) &= \Delta_5^{+}(z)^3(j_5^{+}(z) + 12) = q^2 + 44q^4 - 288q^5 + 306q^6 + \cdots ,\\ \mathfrak{f}_{12,-1}(z) &= \Delta_5^{+}(z)^3(j_5^{+}(z)^2 + 12j_5^{+}(z) - 178) = q + 2608q^4 - 65q^5 + 23472q^6 + \cdots .\end{aligned}$$

It then follows from [1, Example 2.10] that the Hecke eigenforms are given by

$$\begin{aligned}f_1 &= \mathfrak{f}_{12,-1} - 24\mathfrak{f}_{12,-2} + 252\mathfrak{f}_{12,-3},\\ f_2 &= \mathfrak{f}_{12,-1} + (-10 + 6\sqrt{151})\mathfrak{f}_{12,-2} + (-110 + 32\sqrt{151})\mathfrak{f}_{12,-3},\\ f_3 &= \mathfrak{f}_{12,-1} + (-10 - 6\sqrt{151})\mathfrak{f}_{12,-2} + (-110 - 32\sqrt{151})\mathfrak{f}_{12,-3},\end{aligned}$$

so that $K_{f_1} = \mathbb{Q}$ and $K_{f_2} = K_{f_3} = \mathbb{Q}(\sqrt{151})$. Now utilizing [6, Theorem 1.2(i)] one finds that

$$\begin{aligned}f_1^* &= \frac{17}{131}\mathfrak{f}_{12,1} - \frac{16384}{655}\mathfrak{f}_{12,2} + \frac{531441}{1310}\mathfrak{f}_{12,3},\\ f_2^* &= \frac{3(2869 + 43\sqrt{151})}{19781}\mathfrak{f}_{12,1} + \frac{512(2416 + 181\sqrt{151})}{98905}\mathfrak{f}_{12,2} + \frac{177147(-453 + 7\sqrt{151})}{395620}\mathfrak{f}_{12,3},\\ f_3^* &= \frac{3(2869 - 43\sqrt{151})}{19781}\mathfrak{f}_{12,1} + \frac{512(2416 - 181\sqrt{151})}{98905}\mathfrak{f}_{12,2} + \frac{177147(-453 - 7\sqrt{151})}{395620}\mathfrak{f}_{12,3},\end{aligned}$$

so that $K_{f_i^*} = K_{f_i}$ for each $i \in \{1, 2, 3\}$, as expected from Theorem 1.5.

3 Proof of Theorem 1.6

We recall that every $\mathcal{F} \in H_{2-k}(\Gamma_0(p))$ has a canonical decomposition [23, Section 4.2]

$$\mathcal{F}(z) = \mathcal{F}^{-}(z) + \mathcal{F}^{+}(z),$$

where $\mathcal{F}^{-}$ (respectively, $\mathcal{F}^{+}$) is nonholomorphic (respectively, holomorphic) on the complex upper-half plane $\mathbb{H}$. The holomorphic part $\mathcal{F}^{+}$ has a Fourier expansion

$$\mathcal{F}^{+}(z) = \sum_{n \gg -\infty} a_{\mathcal{F}}^{+}(n)q^n \quad (q^n := e^{2\pi i z}).$$

Then we call $\mathcal{F}^{+}$ a mock modular form if $\mathcal{F}^{-} \neq 0$. We define

$$H_{2-k}(\Gamma_0^{+}(p)) := \{\mathcal{F} \in H_{2-k}(\Gamma_0(p)) \mid \mathcal{F}|_{2-k}W_p = \mathcal{F}\}.$$

For each $\mathcal{F} \in H_{2-k}(\Gamma_0^{+}(p))$ we define the W_p-mock modular period function for $\mathcal{F}^{+}$ by

$$\mathbb{P}(\mathcal{F}^{+}, W_p; z) := b_k^{-1}(\mathcal{F}^{+} - \mathcal{F}^{+}|_{2-k}W_p)(z),$$

where $b_k = \frac{\Gamma(k-1)}{(4\pi)^{k-1}}$. Let L_k be the Maass lowering operator L_k defined by

$$L_k := -2iy^2\frac{\partial}{\partial\bar{\tau}} = -iy^2\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then for $k \geq 2$ the differential operator $\xi_{2-k} := y^{-k}\overline{L_{2-k}}$ defines antilinear maps

$$\xi_{2-k} : H_{2-k}(\Gamma_0^+(p)) \to S_k(\Gamma_0^+(p)).$$

Lemma 3.1. *Choose $G_n \in H_{2-k}(\Gamma + 0(p))$ such that $\xi_{2-k}(G_n) = f_n$. We then have*
(*i*) $\xi_{2-k}(G_n^c) = f_n^c = f_n$ *and*
(*ii*) $G_n^c \in H_{2-k}(\Gamma_0^+(p))$.

Proof.
(i) For simplicity, we let $G = G_n$. If we write the Fourier expansion of G as

$$G = G^+ + \sum_{\substack{n\ll\infty\\ \neq 0}} c_G^-(n)\Gamma(k-1, -4\pi ny)q^n,$$

then

$$G^c = (G^+)^c + \sum_{\substack{n\ll\infty\\ \neq 0}} \overline{c_G^-(n)}\Gamma(k-1, -4\pi ny)q^n,$$

so that

$$\xi_{2-k}(G^c) = \sum_{n>0}(4\pi n)^{k-1}c_G^-(n)q^n.$$

Since $\xi_{2-k}(G(z)) = f_n(z) = \sum_{n>0}(4\pi n)^{k-1}\overline{c_G^-(n)}q^n$, we must have $\xi_{2-k}(G^c) = f_n^c$. Moreover since $f_n \in \mathbb{R}[[q]]$ (see [26, p.263]), we readily have $f_n^c = f_n$.

(ii) To show that $G^c \in H_{2-k}(\Gamma_0^+(p))$, we first easily check that

$$G^c\,|_{2-k}\gamma = G^c \quad \text{for all } \gamma \in \Gamma_0^+(p).$$

Second, we obtain that

$$\Delta_{2-k}(G^c) = -\xi_k \circ \xi_{2-k}(G^c) = -\xi_k(f_n^c) = 0.$$

Third, the growth condition is immediate from that of G. Thus, the assertion is proved. □

Lemma 3.2. *With the same notations as above, we have*

$$r^+(f_n, -z) = c_k \cdot \mathbb{P}(\overline{G_n^+(-\bar{z})}, W_p; z),$$

where $c_k = -\frac{\Gamma(k-1)}{(2\pi i)^{k-1}}$.

Proof. By Theorem 1.5 in [8] we have $\overline{\mathbb{P}(G_n^+, W_p; z)} = c_k^{-1}r^+(f_n, \bar{z})$. Thus, we obtain that

$$r^+(f_n, -z) = c_k \cdot \overline{\mathbb{P}(G_n^+, W_p; -\bar{z})}. \tag{15}$$

Since $\mathbb{P}(G_n^+, W_p; z)$ is, by definition, $b_k^{-1}(G_n^+ - G_n^+\,|_{2-k}W_p)(z)$, we compute that

$$\overline{\mathbb{P}(G_n^+, W_p; -\bar{z})} = \mathbb{P}(\overline{G_n^+(-\bar{z})}, W_p; z). \tag{16}$$

Combining (15) and (16), we obtain the assertion. □

Lemma 3.3. *With the same notations as above, we write the Fourier expansion of G_n^+ as*

$$G_n^+(z) = \sum_{m\gg-\infty} c_{G_n}^+(m)q^m.$$

We then have

$$r^+(f_n, -z) = \frac{r^+(D^{k-1}(G_n^c), z) - c^+_{G_n}(0)c_k \cdot p_0}{b_k} \in W^+_{k-2}(p) \tag{17}$$

and

$$\widehat{T_\ell}(r^+(f_n, -z)) = \frac{l^{1-k} r^+(T_\ell(D^{k-1}(G_n^c))) - (1 + l^{1-k})c^+_{G_n}(0)c_k \cdot p_0}{b_k}. \tag{18}$$

Proof. For simplicity, we let $G = G_n$ and $a = c_G^+(0)$. Note that $D^{k-1}(G^c) \in S_k^{!,+}(p)$ has Fourier expansion

$$D^{k-1}(G^c) = \sum_{n \neq 0} n^{k-1} \overline{c_G^+(n)} q^n$$

and

$$\mathcal{E}_{D^{k-1}(G^c)} = \sum_{n \neq 0} \overline{c_G^+(n)} q^n = \overline{G^+(-\bar{z})} - a.$$

Thus, we come up with

$$\begin{aligned} b_k^{-1} \cdot \mathbb{P}(\overline{G^+(-\bar{z})}, W_p; z) &= \mathcal{E}_{D^{k-1}(G^c)} - \mathcal{E}_{D^{k-1}(G^c)}|_{2-k} W_p + a - a|_{2-k} W_p \\ &= c_k^{-1} \cdot r^+(D^{k-1}(G^c), z) - a \cdot p_0. \end{aligned}$$

Then (17) is immediate from the above equation and Lemma 3.2. Applying the Hecke operator $\widehat{T_\ell}$ to (17) and then employing the fact that $\widehat{T_\ell} \circ r^+ = r^+ \circ \ell^{1-k} T_\ell$ with (**EP1**) we obtain (18). □

In what follows, we will simply denote $G = G_n$ and $\lambda_\ell = \lambda(n, \ell)$. It then follows from [27, Theorem 7.10] that

$$\xi_{2-k}(T_\ell(G^c)) = \ell^{1-k}\lambda_\ell \xi_{2-k}(G^c) = \ell^{1-k}\lambda_\ell f_n.$$

Thus, we have

$$T_\ell(G^c) = \ell^{1-k}\lambda_\ell G^c + \mathfrak{h}_\ell \tag{19}$$

for some $\mathfrak{h}_\ell \in M_{2-k}^{!,+}(p)$. Comparing the constant terms of holomorphic parts in both sides of (19) we obtain that

$$c_G^+(0) + \ell^{1-k} c_G^+(0) = \ell^{1-k}\lambda_l c_G^+(0) + c_{\mathfrak{h}_\ell}(0), \tag{20}$$

where $c_{\mathfrak{h}_\ell}(0)$ denotes the constant term of $\mathfrak{h}_\ell$. Now, applying D^{k-1} to both sides of (19) we obtain that

$$D^{k-1}(T_\ell(G^c)) = \ell^{1-k}\lambda_\ell D^{k-1}(G^c) + D^{k-1}(\mathfrak{h}_\ell). \tag{21}$$

Meanwhile, we compute that

$$\begin{aligned} T_\ell(D^{k-1}(G^c)) &= T_\ell\left(\sum_n n^{k-1}\overline{c_G^+(n)} q^n\right) \\ &= \sum_n \left((n\ell)^{k-1}\overline{c_G^+(n\ell)} + \ell^{k-1}\overline{\left(\frac{n}{\ell}\right)^{k-1} c_G^+\left(\frac{n}{\ell}\right)}\right) q^n \\ &= \ell^{k-1}\left(D^{k-1}\left(\sum_n \left(\overline{c_G^+(n\ell)} + \ell^{1-k}\overline{c_G^+\left(\frac{n}{\ell}\right)}\right) q^n\right)\right) \\ &= \ell^{k-1} D^{k-1}(T_\ell(G^c)). \end{aligned} \tag{22}$$

Therefore, we find that

$$\ell^{1-k}\lambda_\ell D^{k-1}(G^c) + D^{k-1}(\mathfrak{h}_\ell) = \ell^{1-k} T_\ell(D^{k-1}(G^c)), \tag{23}$$

which gives rise to

$$\begin{aligned} r^+(T_\ell(D^{k-1}(G^c))) &= \ell^{k-1}r^+(D^{k-1}(\mathfrak{h}_\ell)) + \lambda_\ell r^+(D^{k-1}(G^c)) \\ &= \ell^{k-1}r^+(D^{k-1}(\mathfrak{h}_\ell)) + \lambda_\ell(b_k r^+(f_n, -z) + a \cdot c_k p_0) \quad \text{by (17)}, \end{aligned}$$

where $a = c_G^+(0)$. Thus, we obtain that

$$\begin{aligned} b_k \widehat{T}_\ell(r^+(f_n, -z)) &= \ell^{1-k}r^+(T_\ell(D^{k-1}(G^c))) - (1 + \ell^{1-k})a \cdot c_k \cdot p_0 \quad \text{by (18)} \\ &= \ell^{1-k}(\ell^{k-1}r^+(D^{k-1}(\mathfrak{h}_\ell)) + \lambda_\ell(b_k r^+(f_n, -z) + a \cdot c_k p_0)) - (1 + \ell^{1-k})a \cdot c_k \cdot p_0 \\ &= r^+(D^{k-1}(\mathfrak{h}_\ell)) + \ell^{1-k}\lambda_\ell b_k r^+(f_n, -z) + \ell^{1-k}\lambda_\ell a \cdot c_k p_0 - (1 + \ell^{1-k})a \cdot c_k \cdot p_0 \\ &= c_k c_{\mathfrak{h}_\ell}(0)p_0 + \ell^{1-k}\lambda_l b_k r^+(f_n, -z) + \ell^{1-k}\lambda_\ell a \cdot c_k p_0 - (1 + \ell^{1-k})a \cdot c_k \cdot p_0 \quad \text{by [8, p. 3373 line 9]} \\ &= \ell^{1-k}\lambda_\ell b_k r^+(f_n, -z) \ \text{by (20)}. \end{aligned}$$

This proves Theorem 1.6(i). Theorem 1.6(ii) is an immediate consequence of Theorem 1.6(i). Now it remains to prove Theorem 1.6(iii). Since we know that $\dim W_{k-2}^+(p) = 2t + 1$, it suffices to show that the set

$$\{p_0(z), r_+^+(f_1), \dots, r_+^+(f_t), r_-^+(f_1), \dots, r_-^+(f_t)\}$$

is linearly independent. Suppose that

$$\sum_{n=1}^{t} a_n r_+^+(f_n) + \sum_{n=1}^{t} b_n r_-^+(f_n) + cp_0 = 0 \tag{24}$$

for some $a_n, b_n, c \in \mathbb{C}$. Thus, we have

$$\sum_{n=1}^{t} b_n r_-^+(f_n) = -\sum_{n=1}^{t} a_n r_+^+(f_n) - cp_0.$$

Since the left side of the above equation is an odd polynomial while the right side is even, we must have

$$\sum_{n=1}^{t} b_n r_-^+(f_n) = 0 = -\sum_{n=1}^{t} a_n r_+^+(f_n) - cp_0.$$

It follows from [28, Theorem 1.1 and Remark 1.3] that the maps r_+^+ and r_-^+ are injective and the image of r_+^+ does not contain $\langle p_0\rangle$. Thus, we have $c = 0 = \sum_n a_n f_n = \sum_n b_n f_n$ and therefore $c = 0 = a_n = b_n = 0$ for all $n \in \{1, \dots, t\}$.

4 Proof of Theorem 1.8

(i) Suppose that $H_n \in H_{2-k}(\Gamma_0^+(p))$ such that

$$\xi_{2-k}H_n = f_n = \xi_{2-k}G_n.$$

Thus, we see that $H_n - G_n \in M_{2-k}^{!,+}$ and therefore $D^{k-1}(H_n) - D^{k-1}(G_n) \in D^{k-1}(M_{2-k}^{!,+})$. We then have $[D^{k-1}(H_n)] = [g_n]$, which implies that the class $[g_n]$ is independent of the choice of G_n and uniquely determined by f_n. Since g_n belongs to the space $S_k^{!,+}(p)$ we can write g_n as

$$g_n = \sum_{m=1}^{t} a_m^{(n)} f_m^* + \sum_{m=1}^{t} b_m^{(n)} f_m + D^{k-1}v_n$$

for some $a_m^{(n)}, b_m^{(n)} \in \mathbb{C}$, and $v_n \in M_{2-k}^{!,+}(p)$. We observe from [8, Theorem 1.1] that

$$\{D^{k-1}v_n, f\} = 0 \ \text{for every } f \in S_k^{!,+}(p). \tag{25}$$

For each $m \in \{1,\dots,t\}$, pairing with f_m yields

$$\begin{aligned}\{g_n, f_m\} &= \sum_{j=1}^{t} a_j^{(n)}\{f_j^*, f_m\} + \sum_{j=1}^{t} b_j^{(n)}\{f_j, f_m\} + \{D^{k-1}v_n, f_m\} \quad \text{since our pairing is bilinear}\\ &= a_m^{(n)} + \{D^{k-1}v_n, f_m\} \text{ since } \{f_j^*, f_m\} = \delta_{jm} \text{ and } \{f_j, f_m\} = 0\\ &= a_m^{(n)} \text{ by (25).}\end{aligned}$$

Similarly, pairing with f_m^* we obtain that

$$\{g_n, f_m^*\} = -b_m^{(n)}.$$

Moreover, one has

$$\begin{aligned}a_m^{(n)} &= \{g_n, f_m\} = -\{f_m, g_n\} = -\{f_m, D^{k-1}G_n^c\}\\ &= -(f_m, \xi_{2-k}G_n^c)^{\mathrm{reg}} \text{ by [5, (1.16)] or [8, Lemma 2.2] or [29, Theorem 3.1]}\\ &= -(f_m, f_n)^{\mathrm{reg}} \text{ by Lemma 3.1(i)}\\ &= -(f_m, f_n) = \begin{cases} 0, & \text{if } m \neq n,\\ -(f_n, f_n), & \text{if } m = n.\end{cases}\end{aligned}$$

Similarly, one finds that

$$b_m^{(n)} = \{f_m^*, g_n\} = (f_m^*, f_n)^{\mathrm{reg}}.$$

Thus, g_n can be rewritten as

$$\begin{aligned}g_n &= a_n^{(n)} f_n^* + \sum_{m=1}^{t} b_m^{(n)} f_m + D^{k-1}v_n\\ &= a_n^{(n)}\Big(h_n - \sum_{\substack{i=1\\ i\neq n}}^{t} x_i(n) f_i\Big) + \sum_{m=1}^{t} b_m^{(n)} f_m + D^{k-1}v_n\\ &= a_n^{(n)} h_n + b_n^{(n)} f_n + \sum_{\substack{i=1\\ i\neq n}}^{t} \big(b_i^{(n)} - a_n^{(n)} x_i(n)\big) f_i + D^{k-1}v_n.\end{aligned} \tag{26}$$

Now let

$$C_n := \sum_{\substack{i=1\\ i\neq n}}^{t} \big(b_i^{(n)} - a_n^{(n)} x_i(n)\big) f_i. \tag{27}$$

Then for each prime l with $\gcd(l, p) = 1$, we observe that

$$\begin{aligned}T_\ell(C_n) &= T_\ell\big(g_n - a_n^{(n)} h_n - b_n^{(n)} f_n - D^{k-1}v_n\big)\\ &= \lambda(n, \ell)\big(g_n - a_n^{(n)} h_n - b_n^{(n)} f_n\big) + D^{k-1}E \text{ for some } E \in M_{2-k}^{!,+}(p), \text{ by (23) and } \textbf{(EF2)}\\ &= \lambda(n, \ell) C_n + D^{k-1}\tilde{E} \text{ for some } \tilde{E} \in M_{2-k}^{!,+}(p).\end{aligned}$$

This implies that

$$D^{k-1}\tilde{E} = T_\ell C_n - \lambda(n, l) C_n \in S_k^+(p) \cap D^{k-1}M_{2-k}^{!,+}(p) = \{0\},$$

and therefore

$$T_\ell C_n = \lambda(n, l) C_n \text{ for each prime } l \text{ with } \gcd(l, p) = 1.$$

Thus, by multiplicity one theorem has $C_n \in \langle f_n\rangle$, which together with (27) yields that

$$C_n \in \langle f_n\rangle \cap \langle f_1, \dots, f_{n-1}, f_{n+1}, \dots, f_t\rangle = \{0\}.$$

It then follows from (26) that

$$g_n = a_n^{(n)} h_n + b_n^{(n)} f_n + D^{k-1} v_n.$$

Thus, we have

$$g_n = -(f_n, f_n) h_n + (f_n^*, f_n)^{\mathrm{reg}} f_n + D^{k-1} v_n = b_k \mathfrak{g}_n + D^{k-1} v_n, \tag{28}$$

which means that $[g_n] = [b_k \mathfrak{g}_n]$.

(ii) We obtain from (17) that

$$b_k \cdot r^+(f_n, -z) + c_{G_n}^+(0) c_k \cdot p_0 = r^+(g_n) = a_n^{(n)} r^+(h_n) + b_n^{(n)} r^+(f_n) + c_k \cdot c_{v_n}(0) \cdot p_0, \tag{29}$$

where $c_{v_n}(0)$ stands for the constant term in the Fourier expansion of v_n. Thus, we have an equality

$$A_n := b_k \cdot r^+(f_n, -z) - a_n^{(n)}(r^+(h_n) + c_0(n) p_0) - b_n^{(n)} r^+(f_n) = \big(c_k \cdot c_{v_n}(0) - a_n^{(n)} c_0(n) - c_{G_n}^+(0) c_k\big) p_0 =: B_n,$$

where $c_0(n)$ is the constant appeared in **(EP2)**. Applying $\widehat{T_\ell}$ and utilizing **(EP1)** and **(EP2)** we obtain that

$$\ell^{1-k} \lambda(n, \ell) A_n = (1 + \ell^{1-k}) B_n. \tag{30}$$

Now choose a prime l different from p such that $\ell^{1-k}\lambda(n, \ell) \neq 1 + \ell^{1-k}$. Indeed, if there is no such ℓ, then we have $\lambda(n, \ell) = \ell^{k-1} + 1$ for all ℓ, which contradicts a well-known estimate $|\lambda(n, \ell)| \leq 2\ell^{\frac{k-1}{2}}$ (see [26, (15)]). For such ℓ, since $A_n = B_n$, we obtain from (30) that $A_n = B_n = 0$, which renders

$$b_k \cdot r^+(f_n, -z) - a_n^{(n)}(r^+(h_n) + c_0(n) p_0) - b_n^{(n)} r^+(f_n) = 0$$

and

$$\big(c_k \cdot c_{v_n}(0) - a_n^{(n)} c_0(n) - c_{G_n}^+(0) c_k\big) p_0 = 0. \tag{31}$$

Combining (29) and (31) we obtain the assertion.

5 Proofs of Theorems 1.2, 1.3, and 1.5

Proof of Theorem 1.2. We recall that $\xi_{2-k}\Big(\frac{G_n^c}{(f_n, f_n)}\Big) = \frac{f_n^c}{(f_n, f_n)}$. It follows from (28) that

$$D^{k-1}(G_n^c) = -(f_n, f_n) h_n + (f_n^*, f_n)^{\mathrm{reg}} f_n + D^{k-1} v_n$$

for some $v_n \in M_{2-k}^{!,+}(1)$. Thus, we obtain that

$$D^{k-1}\left(\frac{G_n^c}{(f_n, f_n)} - \frac{v_n}{(f_n, f_n)}\right) = -h_n + \frac{(f_n^*, f_n)^{\mathrm{reg}}}{(f_n, f_n)} f_n,$$

whose principal part is equal to $\mathrm{prin}(-f_n^*) \in K_{f_n}[q^{-1}]$. Here $\mathrm{prin}(\cdot)$ means the principal part at the cusp ∞ except for the constant term. Let $\mathfrak{F}_n := \frac{G_n^c}{(f_n, f_n)} - \frac{v_n}{(f_n, f_n)}$. As mentioned in Remark 1.1(ii), let $Q \in H_{2-k}(\Gamma_0(1))$, which is good for f_n^c with $Q^+ = O(q^{-t})$. Since $\xi_{2-k}(Q - \mathfrak{F}_n) = 0$, we obtain $Q - \mathfrak{F}_n \in M_{2-k}^!(1)$. Moreover, we observe $Q - \mathfrak{F}_n = Q^+ - \mathfrak{F}_n^+ \in O(q^{-t})$ since the order at ∞ of the holomorphic part of a harmonic weak Maass form of weight $2 - k$ is the same as the order of the pole of its image under D^{k-1}. But it follows from [2, Remark 3.8] that

$$\max\{\mathrm{ord}_\infty f \mid f \neq 0 \in M_{2-k}^!(1)\} = -1 - t,$$

which forces $Q - \mathfrak{F}_n$ to be zero. Thus, $\mathrm{CT}(\mathfrak{F}_n^+) \in K_{f_n}$ and therefore $\mathfrak{F}_n$ is good for f_n^c and $D^{k-1}(\mathfrak{F}_n) = -h_n + \frac{(f_n^*, f_n)^{\mathrm{reg}}}{(f_n, f_n)} f_n$, as desired. □

Proof of Theorem 1.3. (i) Let $Q = Q_n$ be good for f_n^c. Write $Q = \left(\frac{Q+Q\mid W_p}{2}\right) + \left(\frac{Q-Q\mid W_p}{2}\right)$. Since $\xi_{2-k}(Q) = \xi_{2-k}\left(\frac{Q+Q\mid W_p}{2}\right) = \frac{f_n^c}{(f_n,f_n)}$, we obtain $\frac{Q-Q\mid W_p}{2} \in M_{2-k}^{!,-}(p)$. It then follows from Theorem 1.8 that

$$\left[D^{k-1}\left(\frac{Q+Q|W_p}{2}\right)\right] = \left[-h_n + \frac{(f_n^*,f_n)^{\mathrm{reg}}}{(f_n,f_n)}f_n\right],$$

so that

$$D^{k-1}\left(\frac{Q+Q|W_p}{2}\right) = -h_n + \frac{(f_n^*,f_n)^{\mathrm{reg}}}{(f_n,f_n)}f_n + D^{k-1}(v_n)$$

for some $v_n \in M_{2-k}^{!,+}(p)$. We then obtain that

$$\mathrm{prin}\left(D^{k-1}\left(\frac{Q-Q|W_p}{2}\right)\right) = \mathrm{prin}\left(D^{k-1}\left(\frac{Q+Q|W_p}{2}\right)\right) = -\mathrm{prin}(f_n^*) + \mathrm{prin}(D^{k-1}(v_n)),$$

which renders

$$\mathrm{prin}\left(D^{k-1}\left(\frac{Q-Q|W_p}{2} - v_n\right)\right) = -\mathrm{prin}(f_n^*). \tag{32}$$

Now we find

$$\mathrm{CT}\left(f_m\left(\frac{Q-Q|W_p}{2}\right)\right) = \mathrm{CT}\left(f_m\left(\frac{Q-Q|W_p}{2} - v_n\right)\right) = -\{f_m, f_n^*\}\ \text{(by (32))}\ = \delta_{mn},$$

where the first equality follows from the residue theorem because the residue of the meromorphic 1-form $d(f_m v_n)$ is given by $\mathrm{CT}(f_m v_n)$. Utilizing Remark 1.1(ii) we can take Q such that $Q^+ = O(q^{-t-t'})$. Thus, we can write

$$\frac{Q-Q|W_p}{2} = \sum_{j=1}^{t} b_{jn}\,\mathfrak{f}_{2-k,t'+j}^{-}\ \text{for some}\ b_{jn} \in \mathbb{C}. \tag{33}$$

Then we have

$$\sum_{j=1}^{t}\mathrm{CT}(f_m\,\mathfrak{f}_{2-k,t'+j}^{-})b_{jn} = \delta_{mn}, \tag{34}$$

which proves the assertion.

(ii) We write

$$D^{k-1}\left(w_n - \sum_{j=1}^{t}\beta_{jn}\,\mathfrak{f}_{2-k,t'+j}^{-}\right) = \sum_{i=1}^{t}\alpha_{in}q^{-i} + O(1)\ \text{for some}\ \alpha_{in} \in \mathbb{C}$$

so that

$$\mathrm{CT}\left(f_m\left(w_n - \sum_{j=1}^{t}\beta_{jn}\,\mathfrak{f}_{2-k,t'+j}^{-}\right)\right) = \sum_{i=1}^{t}\lambda(m,i)\frac{\alpha_{in}}{(-i)^{k-1}}.$$

Meanwhile, by the residue theorem the left-hand side of the above identity reduces to

$$-\mathrm{CT}\left(f_m\sum_{j=1}^{t}\beta_{jn}\,\mathfrak{f}_{2-k,t'+j}^{-}\right) = -\delta_{mn}.$$

Comparing this with the identities in [6, (20)–(22)] we obtain

$$\mathrm{prin}\left(D^{k-1}\left(w_n - \sum_{j=1}^{t}\beta_{jn}\,\mathfrak{f}_{2-k,t'+j}^{-}\right)\right) = \mathrm{prin}(f_n^*). \tag{35}$$

We recall from (28) that

$$D^{k-1}(G_n^c) = -(f_n, f_n)h_n + (f_n^*, f_n)^{\text{reg}} f_n + D^{k-1}v_n$$

for some $v_n \in M_{2-k}^{!,+}(p)$. Now we let

$$\mathfrak{F}_n := \frac{G_n^c}{(f_n, f_n)} - \frac{v_n}{(f_n, f_n)} + w_n + \sum_{j=1}^{t} \beta_{jn}\, \mathfrak{f}_{2-k,t'+j}^{-}.$$

We then have $\xi_{2-k}\mathfrak{F}_n = \frac{f_n^c}{(f_n, f_n)}$ and

$$D^{k-1}\mathfrak{F}_n = -h_n + \frac{(f_n^*, f_n)^{\text{reg}}}{(f_n, f_n)} f_n + D^{k-1}\left(w_n + \sum_{j=1}^{t} \beta_{jn}\, \mathfrak{f}_{2-k,t'+j}^{-} \right) \tag{36}$$

whose principal part is equal to

$$\text{prin}(-f_n^*) + \text{prin}\left(D^{k-1}\left(w_n + \sum_{j=1}^{t} \beta_{jn}\, \mathfrak{f}_{2-k,t'+j}^{-} \right) \right) = O(q^{-t-t'}).$$

Now we investigate the principal part of $\mathfrak{F}_n$ at the cusp 0. To this end, we consider

$$\begin{aligned}
&D^{k-1}\left(\mathfrak{F}_n \,|_{2-k} W_p \begin{pmatrix} 1/\sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix} \right) \\
&= D^{k-1}(\mathfrak{F}_n)|_k W_p \begin{pmatrix} 1/\sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix} \text{ by Bol's identity} \\
&= \left(-h_n + \frac{(f_n^*, f_n)^{\text{reg}}}{(f_n, f_n)} f_n \right)\Bigg|_k W_p \begin{pmatrix} 1/\sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix} + D^{k-1}(w_n)|_k W_p \begin{pmatrix} 1/\sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix} + D^{k-1}\left(\sum_{j=1}^{t} \beta_{jn}\, \mathfrak{f}_{2-k,t'+j}^{-} \right) |_k W_p \begin{pmatrix} 1/\sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix} \\
&= \sqrt{p}^{-k}\left(-h_n\left(\frac{z}{p}\right) + \frac{(f_n^*, f_n)^{\text{reg}}}{(f_n, f_n)} f_n\left(\frac{z}{p}\right) \right) + \sqrt{p}^{-k}\left(D^{k-1}(w_n)\left(\frac{z}{p}\right) - D^{k-1}\left(\sum_{j=1}^{t} \beta_{jn}\, \mathfrak{f}_{2-k,t'+j}^{-} \right)\left(\frac{z}{p}\right) \right)
\end{aligned}$$

whose principal part is equal to that of $\sqrt{p}^{-k}\left(-f_n^*\left(\frac{z}{p}\right)\right) + \sqrt{p}^{-k} f_n^*\left(\frac{z}{p}\right) = 0$ by (35). This means that the principal part of $\mathfrak{F}_n$ at the cusp 0 is constant. As mentioned in Remark 1.1(ii) and in the proof of the assertion (i), let $Q \in H_{2-k}(p)$, which is good for f_n^c with $Q^+ = O(q^{-t-t'})$. Since $\xi_{2-k}(Q - \mathfrak{F}_n) = 0$ and the principal part of $Q - \mathfrak{F}_n$ at the cusp 0 is constant, we obtain $Q - \mathfrak{F}_n \in M_{2-k}^{\sharp}(p)$. Moreover, we observe $Q - \mathfrak{F}_n = Q^+ - \mathfrak{F}_n^+ \in O(q^{-t-t'})$. But it follows from [12] that

$$\max\{\text{ord}_\infty f \mid f \neq 0 \in M_{2-k}^{\sharp}(p)\} = -1 - t - t',$$

which forces $Q - \mathfrak{F}_n$ to be zero. Thus, $\text{CT}(\mathfrak{F}_n^+) \in K_{f_n}$ and hence $\mathfrak{F}_n$ is good for f_n^c. Now the assertion follows from (36). □

Proof of Theorem 1.5. First we prove that $\text{prin}(h_n) \in K_{f_n}[q^{-1}]$. In the case of $p = 1$, this immediately follows from Theorem 1.2. Now let $p \in \{2, 3, 5, 7, 13\}$. We adopt the same notation as in Theorem 1.3. Since Q is good, $\text{prin}(Q) \in K_{f_n}[q^{-1}]$ and $Q|W_p$ has constant principal part. Meanwhile (33) implies that

$$\text{prin}\left(\frac{Q}{2}\right) = \text{prin}\left(\frac{Q - Q|W_p}{2}\right) = \sum_{j=1}^{t} b_{jn} q^{-t'+j} + O(q^{-t'}) \in K_{f_n}[q^{-1}] \text{ since } m_{2-k}^{-} = -1 - t' \text{ by [3]}.$$

We note from (34) that $\beta_{jn} = b_{jn} \in K_{f_n}$. Thus, by (36) we obtain that

$$-\text{prin}(h_n) + \text{prin}(D^{k-1}w_n) \in K_{f_n}[q^{-1}]. \tag{37}$$

Since $\beta_{jn} \in K_{f_n}$ and $f_{2-k,m}^{-}$ has rational Fourier coefficients, the definition of w_n implies that

$$\text{prin}(w_n) \in K_{f_n}[q^{-1}]. \tag{38}$$

By (37) and (38) one has $\mathrm{prin}(h_n) \in K_{f_n}[q^{-1}]$. Observing that $\mathrm{prin}(h_n) = \mathrm{prin}(f_n^*)$, we obtain that

$$\mathrm{prin}(f_n^*) \in K_{f_n}[q^{-1}]. \tag{39}$$

Since $f_n^* = \sum_{m=1}^{t} \mu(-m, n)\mathfrak{f}_{k,m}$, it then follows from (39) that $\mu(-m, n) \in K_{f_n}$ for each $n \in \{1, \dots t\}$ and hence every coefficient of f_n^* is contained in K_{f_n}. □

6 Proofs of Proposition 1.10 and Theorem 1.11

Proof of Proposition 1.10. The assertion follows immediately from the statement (**EF3**) and the relation between $\mathfrak{G}_n$ and h_n. □

Proof of Theorem 1.11. (i) It follows from (12) that

$$T_\ell(F) = \sum_n \alpha_n(-T_\ell(\mathfrak{g}_n) + \lambda_\ell f_n) + \sum_n \beta_n \lambda_\ell f_n + T_\ell(D^{k-1} g_F), \tag{40}$$

for some $g_F \in M_{2-k}^{!,+}(p)$. Meanwhile, we know from (21) and (22) that

$$l^{1-k} T_\ell(D^{k-1} G_n^c) = D^{k-1}(T_\ell(G_n^c)) = \ell^{1-k}\lambda_\ell D^{k-1}(G_n^c) + D^{k-1}(\mathfrak{h}_\ell)$$

for some $\mathfrak{h}_\ell \in M_{2-k}^{!,+}(p)$. Thus, we find that

$$T_\ell(g_n) = \lambda_\ell g_n + D^{k-1}(\ell^{k-1}\mathfrak{h}_\ell). \tag{41}$$

Combining (40) and (41) we obtain that

$$T_\ell(F) = \sum_n \alpha_n \lambda_\ell(-\mathfrak{g}_n + f_n) + \sum_n \beta_n \lambda_\ell f_n + D^{k-1} H$$

for some $H \in M_{2-k}^{!,+}(p)$. It then follows from the definition of the map Φ that

$$\Phi(T_\ell(F)) = \left(2\sum_n \alpha_n \lambda_\ell f_n + \sum_n \beta_n \lambda_\ell f_n, \sum_n \beta_n \lambda_\ell f_n\right) = (T_\ell \times T_\ell)(\Phi(F)).$$

Next for $f, g \in S_k^+(p)$ we write $f = \sum_n a_n f_n$ and $g = \sum_n b_n f_n$ for some $a_n, b_n \in \mathbb{C}$. We then have

$$\begin{aligned}
\widehat{T_\ell} \circ \mathfrak{r}(f, g) &= \widehat{T_\ell} \circ \mathfrak{r}\left(\sum_n a_n f_n, \sum_n b_n f_n\right) \\
&= \widehat{T_\ell}\left(r_-^+\left(\sum_n a_n f_n\right) + i r_+^+\left(\sum_n b_n f_n\right)\right) \\
&= \sum_n a_n \widehat{T_\ell}(r_-^+(f_n)) + i \sum_n b_n \widehat{T_\ell}(r_+^+(f_n)) \\
&= \sum_n a_n \ell^{1-k}\lambda(n, \ell) r_-^+(f_n) + i \sum_n b_n \ell^{1-k}\lambda(n, \ell) r_+^+(f_n) \text{ by Theorem 1.6(ii)} \\
&= \sum_n a_n \ell^{1-k} r_-^+(T_\ell f_n) + i \sum_n b_n \ell^{1-k} r_+^+(T_\ell f_n) \\
&= \ell^{1-k}\left(r_-^+\left(T_\ell\left(\sum_n a_n f_n\right)\right) + i r_+^+\left(T_\ell\left(\sum_n b_n f_n\right)\right)\right) \\
&= \ell^{1-k}(r_-^+(T_\ell f) + i r_+^+(T_\ell g)) \\
&= \ell^{1-k}\mathfrak{r}(T_\ell f, T_\ell g), \text{ as desired.}
\end{aligned}$$

(ii) Given $F \in S_k^{!,+}(p)$, by (12) we can write

$$F = \sum_n \alpha_n(-\mathfrak{g}_n + f_n) + \sum_n \beta_n f_n + D^{k-1}\mathfrak{g}_F, \quad \text{for some } \mathfrak{g}_F \in D^{k-1}M_{2-k}^{!,+}(p). \tag{42}$$

We then observe from (13) that

$$\begin{aligned} \Phi(F) = (0,0) \;&\Leftrightarrow\; 2\sum_n \alpha_n f_n + \sum_n \beta_n f_n = 0 \text{ and } \sum_n \beta_n f_n = 0 \\ &\Leftrightarrow\; \beta_n = \alpha_n = 0 \text{ for all } n \in \{1, 2, \ldots, t\} \\ &\Leftrightarrow\; F = D^{k-1}\mathfrak{g}_F, \end{aligned}$$

which proves the assertion.

(iii) First we note that

$$\begin{aligned} r^+(f_n, -z) + r^+(f_n, z) &= 2ir_+^+(f_n, z) \\ -r^+(f_n, -z) + r^+(f_n, z) &= 2r_-^+(f_n, z) \\ r^+(f_n, z) &= r_-^+(f_n, z) + ir_+^+(f_n, z). \end{aligned}$$

Now we compute that

$$\begin{aligned} P \circ r^+(F) &= P \circ r^+\left(\sum_n \alpha_n(-\mathfrak{g}_n + f_n) + \sum_n \beta_n f_n + D^{k-1}\mathfrak{g}_F\right) \\ &= P\left(\sum_n \alpha_n(-r^+(f_n, -z) + r^+(f_n, z)) + \sum_n \beta_n r^+(f_n, z) + ap_0\right) \text{ for some } a \in \mathbb{C} \text{ by Theorem 1.8(ii)} \\ &= P\left(\sum_n \alpha_n 2r_-^+(f_n, z) + \sum_n \beta_n(r_-^+(f_n, z) + ir_+^+(f_n, z)) + ap_0\right) \\ &= \sum_n \alpha_n 2r_-^+(f_n, z) + \sum_n \beta_n(r_-^+(f_n, z) + ir_+^+(f_n, z)). \end{aligned}$$

Meanwhile,

$$\mathfrak{r} \circ \Phi(F) = \mathfrak{r}\left(2\sum_n \alpha_n f_n + \sum_n \beta_n f_n, \sum_n \beta_n f_n\right) = 2\sum_n \alpha_n r_-^+(f_n, z) + \sum_n \beta_n r_-^+(f_n, z) + i\sum_n \beta_n r_+^+(f_n, z).$$

Thus, we have $P \circ r^+(F) = \mathfrak{r} \circ \Phi(F)$. □

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References

[1] S. Choi and C. H. Kim, *Congruences for Hecke eigenvalues in higher level cases*, J. Number Theory **131** (2011), 2023–2036.
[2] S. Choi and C. H. Kim, *Basis for the space of weakly holomorphic modular forms in higher level cases*, J. Number Theory **133** (2013), 1300–1311.

[3] S. Choi, C. H. Kim, and K. Lee, *Arithmetic properties for the minus space of weakly holomorphic modular forms*, J. Number Theory **196** (2019), 306–339.
[4] W. Duke and P. Jenkins, *On the zeros and coefficients of certain weakly holomorphic modular forms*, Pure Appl. Math. Q. **4** (2008), no. 4, 1327–1340.
[5] K. Bringmann, P. Guerzhoy, Z. Kent, and K. Ono, *Eichler-Shimura theory for mock modular forms*, Math. Ann. **355** (2013), 1085–1121.
[6] S. Choi and C. H. Kim, *Weakly holomorphic Hecke eigenforms and Hecke eigenpolynomials*, Adv. Math. **290** (2016), 144–162.
[7] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), 45–90.
[8] S. Choi and C. H. Kim, *Mock modular period functions and L-functions of cusp forms in higher level cases*, Proc. Amer. Math. Soc. **142** (2014), 3369–3386.
[9] P. Guerzhoy, *Hecke operators for weakly holomorphic modular forms and supersingular congruences*, Proc. Amer. Math. Soc. **136** (2008), no. 9, 3051–3059.
[10] J. Bruinier, K. Ono, and R. Rhoades, *Differential operators for harmonic Maass forms and the vanishing of Hecke eigenvalues*, Math. Ann. **342** (2008), 673–693.
[11] P. Guerzhoy, Z. Kent, and K. Ono, *p-adic coupling of mock modular forms and shadows*, Proc. Natl. Acad. Sci. USA **107** (2010), 6169–6174.
[12] J. Ahn and S. Choi, *Arithmetic of weakly holomorphic modular forms for Hecke groups*, JP Journal of Algebra, Number Theory and Applications **37** (2015), 105–124.
[13] K. Bringmann, P. Guerzhoy, and B. Kane, *Shintani lifts and fractional derivatives for harmonic weak Maass forms*, Adv. Math. **255** (2014), 641–671.
[14] K. Bringmann, P. Guerzhoy, and B. Kane, *Half-integral weight p-adic coupling of weakly holomorphic and holomorphic modular forms*, Res. Number Theory **1** (2015), no. 26, 1–13.
[15] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), 491–562.
[16] M. Eichler, *Eine Verallgemeinerung der Abelschen Integrale*, Math. Z. **67** (1957), 267–298.
[17] G. Shimura, *Sur les intégrales attachées aux formes automorphes*, J. Math. Soc. Japan **11** (1959), 291–311.
[18] Y. Manin *Periods of parabolic forms and p-adic Hecke series*, Mat. Sb. **21** (1973), 371–393.
[19] M. Knopp, *Rational period functions of the modular group*, Duke Math. J. **45** (1978), 47–62.
[20] M. Knopp, *Some new results on the Eichler cohomology of automorphic forms*, Bull. Amer. Math. Soc. **80** (1974), 607–632.
[21] W. Kohnen and D. Zagier, *Modular forms with rational periods*, Modular forms (Durham, 1983), Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., Horwood, Chichester, 1984, pp. 197–249.
[22] V. Pasol and A. Popa, *Modular forms and period polynomials*, Proc. Lond. Math. Soc. **107** (2013), no. 3, 713–743.
[23] K. Bringmann, A. Folsom, K. Ono, and L. Rolen, *Harmonic Maass Forms and Mock Modular Forms: Theory and Applications*, AMS Colloquium Publications, Providence, Rhode Island, 2017.
[24] S. Choi and C. H. Kim, *Rational period functions and cycle integrals in higher level cases*, J. Math. Anal. Appl. **427** (2015), 741–758.
[25] W. Duke, Ö. Imamoḡlu, and Á. Tóth, *Rational period functions and cycle integrals*, Abh. Math. Semin. Univ. Hambg. **80** (2010), no. 2, 255–264.
[26] D. Zagier, *Introduction to Modular Forms, From Number Theory to Physics* (*Les Houches, 1989*), Springer, Berlin, 1992, pp. 238–291.
[27] K. Ono, *Unearthing the visions of a master: Harmonic Maass forms and number theory*, Current Developments in Mathematics, 2008, Int. Press, Somerville, MA, 2009.
[28] S. Choi and C. H. Kim, *Eichler-Shimura isomorphism in higher level cases and its applications*, J. Number Theory **177** (2017), 353–380.
[29] B. Cho, S. Choi, and C. H. Kim, *Harmonic weak Maass-modular grids in higher level cases*, Acta Arith. **160** (2013), 129–141.