Research Article

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Ground state solutions of nonlinear Schrödinger equations involving the fractional p-Laplacian and potential wells

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Abstract: The purpose of this paper is to investigate the ground state solutions for the following nonlinear Schrödinger equations involving the fractional p-Laplacian

$$(-\Delta)_p^s u(x) + \lambda V(x)u(x)^{p-1} = u(x)^{q-1}, \quad u(x) \geq 0, \ x \in \mathbb{R}^N,$$

where $\lambda > 0$ is a parameter, $1 < p < q < \frac{Np}{N-sp}$, $N \geq 2$, and $V(x)$ is a real continuous function on $\mathbb{R}^N$. For λ large enough, the existence of ground state solutions are obtained, and they localize near the potential well $\operatorname{int}(V^{-1}(0))$.

Keywords: nonlinear Schrödinger equation, ground state solution, fractional p-Laplacian, variational methods

MSC 2020: 35J60, 35B33

1 Introduction and main results

In this paper, we consider the following nonlinear Schrödinger equations involving the fractional p-Laplacian

$$\begin{cases} (-\Delta)_p^s u(x) + \lambda V(x)u(x)^{p-1} = u(x)^{q-1}, & x \in \mathbb{R}^N, \\ u(x) \geq 0, \qquad u(x) \in W^{s,p}(\mathbb{R}^N), & \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $1 < p < q < \frac{Np}{N-sp}$, $N \geq 2$, and $V(x)$ is a real continuous function on $\mathbb{R}^N$.

We are interested in the existence of ground state solutions for λ big enough, and their asymptotical behavior as $\lambda \to \infty$. As far as we know, these kinds of problems were first put forward in [1] by Bartsch and Wang, where they studied the Schrödinger equations. Under suitable conditions imposed on the potential, the loss of compactness caused by the whole space $\mathbb{R}^N$ can be recovered when parameter λ is big enough. Then, many authors began studying the problems with potential well. A lot of results have been obtained.

Bartsch and Parnet [2] also considered the nonlinear Schrödinger equation:

$$\begin{cases} -\Delta u + (a_0(x) + \lambda a(x))u = f(x,u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & |x| \to \infty, \end{cases}$$

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where $a_0(x) + \lambda a(x)$ is indefinite. By using a local linking theorem and the critical groups theory, they obtained the existence of solutions and their asymptotical behavior as $\lambda \to \infty$.

Xu and Chen [3] studied the following Kirchhoff problem:

$$\begin{cases} -\left(a + b \int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right) \Delta u + \lambda V(x) u = f(x, u), & x \in \mathbb{R}^N, \\ u(x) \in H^1(\mathbb{R}^N), \end{cases}$$

where $f(x, u)$ can be sublinear or superlinear. By using the genus theory, they obtained infinitely many negative solutions.

Aleves et al. [4] dealt with the following Choquard equation:

$$\begin{cases} -\Delta u + (\lambda a(x) + 1) u = \left(\frac{1}{|x|^\mu} * |u|^p\right) |u|^{p-2} u, & x \in \mathbb{R}^N, \\ u(x) \in H^1(\mathbb{R}^N), \end{cases}$$

where $\mu \in (0, 3)$, $p \in (2, 6 - \mu)$, and the potential well $\Omega = \bigcup_{j=1}^k \Omega_j$. They proved the existence of a solution, which is nonzero on any subset Ω_j. Furthermore, its asymptotical behavior was investigated.

Zhao et al. [5] studied the Schrödinger-Poisson system allowing the potential $V(x)$ changes sign

$$\begin{cases} -\Delta u + \lambda V(x) u + K(x) \phi u = |u|^{p-2} u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x) u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $p \in (3, 6)$ and $V \in C(\mathbb{R}^3, \mathbb{R})$ are bounded from below. Using the variational method, they obtained the existence and asymptotic behavior of nontrivial solutions.

For the critical problems, Clapp and Ding [6] have studied the nonlinear Schrödinger equation:

$$-\Delta u + \lambda V(x) u = \mu u + u^{2^*-1}, \quad x \in \mathbb{R}^N$$

for $N \geq 4$, $\lambda, \mu > 0$. By using variational methods, the authors established existence and multiplicity of positive solutions, which localize near the potential well for λ large and μ small.

Later, the corresponding results obtained in [6] were generalized to the fractional Schrödinger equations by Niu and Tang [7], where they have studied

$$\begin{cases} (-\Delta)^s u + (\lambda V(x) - \mu) u = |u|^{2_s^*-2} u, & x \in \mathbb{R}^N, \\ u \geq 0, \quad u \in H^s(\mathbb{R}^N). \end{cases}$$

Under the linear perturbation, [6] and [7] obtained the existence of solutions and their asymptotic behavior. For the nonlinear perturbation, Alves and Barros [8] considered

$$-\Delta u + \lambda V(x) u = \mu u^{p-1} + u^{2^*-1}, \quad x \in \mathbb{R}^N.$$

By employing the Ljusternik-Schnirelmann category, for λ big enough and μ small enough, the aforementioned problem has at last $\mathrm{cat}(\Omega)$ positive solutions.

For more results about these kinds of problems and fractional Schrödinger equations, see, for example, [9–24] and references therein. Motivated by the aforementioned results, we consider equation (1.1). The potential function $V(x)$ satisfies

(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ such that $V(x) \geq 0$, $\Omega := \mathrm{int} V^{-1}(0)$ is a nonempty open set of class $C^{0.1}$ with bounded boundary and $V^{-1}(0) = \bar{\Omega}$;

(V_2) There exists $M_0 > 0$ such that

$$\mu(\{x \in \mathbb{R}^N : V(x) \leq M_0\}) < \infty,$$

where μ denotes the Lebesgue measure on $\mathbb{R}^N$.

We first introdcue some notations. For $s \in (0,1)$, $p \in [1,+\infty)$, define

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dx dy \right)^{\frac{1}{p}},$$

where the term

$$[u]_{W^{s,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx dy \right)^{\frac{1}{p}}$$

is the so-called Gagliardo (semi)norm of u. Moreover, we define

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_{\Omega} |u|^p dx + \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

Let

$$E_\lambda := \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x)|u(x)|^p dx < \infty \right\},$$

with the norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx dy + \int_{\mathbb{R}^N} \lambda V(x)|u(x)|^p dx \right)^{\frac{1}{p}}.$$

The energy functional associated with (1.1) is

$$J_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx dy + \frac{\lambda}{p}\int_{\mathbb{R}^N} V(x)|u(x)|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} u^+(x)^q dx \quad \text{for } u \in E_\lambda, \tag{1.2}$$

where $u^+ = \max\{u, 0\}$. Then, we can define the Nehari manifold

$$\mathcal{M}_\lambda := \left\{ u \in E_\lambda \backslash \{0\} : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx dy + \lambda \int_{\mathbb{R}^N} V(x)|u(x)|^p dx = \int_{\mathbb{R}^N} u^+(x)^q dx \right\}$$

and

$$c_\lambda := \inf\{J_\lambda(u) : u \in \mathcal{M}_\lambda\}.$$

Consider the following "limit" problem of (1.1)

$$\begin{cases} (-\Delta)_p^s u(x) = u(x)^{q-1}, & x \in \Omega, \\ u(x) \geq 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

Define a subspace E_0 of $W^{s,p}(\mathbb{R}^N)$ as follows:

$$E_0 := \{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ in } \mathbb{R}^N \setminus \Omega\} \tag{1.4}$$
$$\mathrm{tr}_\Omega E_0 = \{u|_\Omega : u \in E_0\}.$$

The energy functional associated with (1.3) can be defined by

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y - \frac{1}{q}\int_\Omega u^+(x)^q \mathrm{d}x \quad \text{for } u \in E_0.$$

Then, the associated Nehari manifold is

$$\mathcal{N} := \left\{u \in E_0\setminus\{0\} : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y = \int_\Omega u^+(x)^q \mathrm{d}x\right\}$$

and

$$c(\Omega) := \inf\{\Phi(u) : u \in \mathcal{N}\}.$$

Definition 1.1. A function $u_\lambda(x)$ is a ground state solution of (1.1) if c_λ is achieved by $u_\lambda \in \mathcal{M}_\lambda$, which is a critical point of J_λ. Similarly, a function $u(x)$ is a ground state solution of (1.3) if $c(\Omega)$ is achieved by $u \in \mathcal{N}$, which is a critical point of Φ.

Definition 1.2. Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$. The function φ satisfies the $(PS)_c$ condition if any sequence $\{u_n\} \subseteq X$, such that

$$\varphi(u_n) \to c, \varphi'(u_n) \to 0 \tag{1.5}$$

has a convergent subsequence. The sequence $\{u_n\}$ that satisfies (1.5) is called to be a $(PS)_c$ sequence of φ.

Our main results read as follows:

Theorem 1.3. *Suppose* (V_1) *and* (V_2) *hold, then for* λ *large,* (1.1) *has a ground state solution* $u_\lambda(x)$. *Furthermore, any sequence* $\lambda_n \to \infty$, $\{u_{\lambda_n}(x)\}$ *has a subsequence such that* u_{λ_n} *converges in* $W^{s,p}(\mathbb{R}^N)$ *along the subsequence to a ground state solution* u *of* (1.3).

Theorem 1.4. *Suppose* (V_1) *and* (V_2) *hold. Let* $u_n, n \in \mathbb{N}$ *be a sequence of solutions of* (1.1) *with* λ *being replaced by* λ_n ($\lambda_n \to \infty$ *as* $n \to \infty$) *such that* $\limsup_{n\to\infty} J_\lambda(u_n) < \infty$. *Then,* $u_n(x)$ *converges strongly along a subsequence in* $W^{s,p}(\mathbb{R}^N)$ *to a solution* u *of* (1.3).

The following paper is organized as follows: In Section 2, we will give some preliminary results. Section 3 is devoted to the "limit" problem, and Section 4 contains the proofs of the main results. C denotes various generic positive constants, and $o(1)$ will be used to represent quantities that tend to 0 as λ(or n) $\to \infty$.

2 Preliminary results

Lemma 2.1. *Let* $\lambda_0 > 0$ *be a fixed constant. Then, for* $\lambda \geq \lambda_0 > 0$, $V(x)$ *satisfying* (V_1) *and* (V_2), E_λ *is continuously embedded in* $W^{s,p}(\mathbb{R}^N)$ *uniformly in* λ.

Proof. By the definition of $W^{s,p}(\mathbb{R}^N)$ and E_λ, we only need to prove the following inequality:

$$\int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x \leq C\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \lambda V(x)|u(x)|^p \mathrm{d}x\right). \tag{2.1}$$

Define

$$D := \{x \in \mathbb{R}^N : V(x) \le M_0\}$$

and

$$D^{\delta_0} := \{x \in \mathbb{R}^N : \mathrm{dist}(x, D) \le \delta_0\}.$$

Take $\zeta \in C^\infty(\mathbb{R}^N, \mathbb{R})$, $0 \le \zeta \le 1$, satisfying

$$\zeta(x) = \begin{cases} 1, & x \in D, \\ 0, & x \notin D^{\delta_0}, \end{cases} \quad |\nabla \zeta| \le C/\delta_0. \tag{2.2}$$

Then, for any function $u \in E_\lambda$, we can obtain

$$\int_{\mathbb{R}^N} (1 - \zeta^p)|u(x)|^p \mathrm{d}x = \int_{\mathbb{R}^N \setminus D} (1 - \zeta^p)|u(x)|^p \mathrm{d}x + \int_D (1 - \zeta^p)|u(x)|^p \mathrm{d}x \le \frac{1}{\lambda_0 M_0} \lambda \int_{\mathbb{R}^N} V(x)|u(x)|^p \mathrm{d}x \tag{2.3}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} \zeta^p |u(x)|^p \mathrm{d}x &= \int_{D^{\delta_0}} \zeta^p |u(x)|^p \mathrm{d}x \\ &\le \mu\left(D^{\delta_0}\right)^{1-\frac{p}{p_s^*}} \left(\int_{D^{\delta_0}} |u(x)|^{p_s^*} \mathrm{d}x \right)^{\frac{p}{p_s^*}} \\ &\le C\mu\left(D^{\delta_0}\right)^{1-\frac{p}{p_s^*}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{2.4}$$

where we have used (V_2) and the Sobolev trace inequality

$$\left(\int_{\mathbb{R}^N} |u(x)|^{p_s^*} \mathrm{d}x \right)^{1/p_s^*} \le C \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{1/p},$$

for $u \in W^{s,p}(\mathbb{R}^N)$ and $C = C(N, p, s) > 0$. Thus, (2.1) follows from (2.3) and (2.4). □

Lemma 2.2. *There exists $\sigma > 0$ independent of λ, such that $\|u\|_\lambda \ge \sigma$ for all $u \in \mathcal{M}_\lambda$.*

Proof. From Lemma 2.1, for any $u \in \mathcal{M}_\lambda$,

$$\begin{aligned} 0 = \langle J_\lambda'(u), u \rangle &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \lambda \int_{\mathbb{R}^N} V(x)|u(x)|^p \mathrm{d}x - \int_{\mathbb{R}^N} u^+(x)^q \mathrm{d}x \\ &\ge \|u\|_\lambda^p - C\|u\|_{W^{s,p}(\mathbb{R}^N)}^q \\ &\ge \|u\|_\lambda^p - C\|u\|_\lambda^q, \end{aligned}$$

where $C > 0$ is independent of $\lambda \ge 0$. The aforementioned inequality implies that $\|u\|_\lambda^{q-p} \ge \frac{1}{C}$. Choosing $\sigma = \left(\frac{1}{C}\right)^{\frac{1}{q-p}}$, we obtain $\|u\|_\lambda \ge \sigma$. □

Lemma 2.3. *Let λ_0 be a fixed positive constant, there exists $c_0 > 0$ independent of $\lambda \ge \lambda_0 > 0$, such that if $\{u_n\}$ is a $(PS)_c$ sequence of J_λ, then either $c \ge c_0$ or $c = 0$. Moreover,*

$$\limsup_{n \to \infty} \|u_n\|_\lambda^p \le \frac{pq}{q - p} c. \tag{2.5}$$

Proof. From the definition of $(PS)_c$ sequence,

$$\begin{aligned} c + \|u_n\|_\lambda \cdot o(1) &= J_\lambda(u_n) - \frac{1}{q}\langle J'_\lambda(u_n), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \lambda \int_{\mathbb{R}^N} V(x)|u_n(x)|^p \mathrm{d}x\right) \\ &= \frac{q-p}{pq}\|u_n\|_\lambda^p. \end{aligned}$$

Then, (2.5) holds. On the other side, there is a constant $C > 0$ independent of $\lambda \geq \lambda_0 > 0$, such that

$$\langle J'_\lambda(u), u\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \lambda \int_{\mathbb{R}^N} V(x)|u(x)|^p \mathrm{d}x - \int_{\mathbb{R}^N} u^+(x)^q \mathrm{d}x \geq \|u\|_\lambda^p - C\|u\|_\lambda^q.$$

Thus, there exists $\sigma_1 > 0$ independent of λ, such that

$$\frac{1}{4}\|u\|_\lambda^p \leq \langle J'_\lambda(u), u\rangle \ \text{ for } \ \|u\|_\lambda < \sigma_1. \tag{2.6}$$

If $c < \frac{\sigma_1^p(q-p)}{pq}$, then

$$\limsup_{n\to\infty}\|u_n\|_\lambda^p \leq \frac{cpq}{q-p} < \sigma_1^p.$$

Hence, $\|u_n\|_\lambda < \sigma_1$ for n large. It follows from (2.6) that

$$\frac{1}{4}\|u_n\|_\lambda^p \leq \langle J'_\lambda(u_n), u_n\rangle = o(1)\|u_n\|_\lambda,$$

which implies $\|u_n\|_\lambda \to 0$ as $n \to \infty$. Therefore, $J_\lambda(u_n) \to 0$, that is, $c = 0$. Thus, $c_0 = \frac{\sigma_1^p(q-p)}{qp}$ is as required. □

Lemma 2.4. *There exists $\delta_0 > 0$, such that any $(PS)_c$ sequence $\{u_n\}$ of J_λ with $\lambda \geq 0$ and $c > 0$ satisfies*

$$\liminf_{n\to\infty} \|u_n^+\|_{L^q(\mathbb{R}^N)}^q \geq \delta_0 c. \tag{2.7}$$

Proof. From the definition of $(PS)_c$ sequence,

$$c = \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{p}\langle J'_\lambda(u_n), u_n\rangle\right) = \left(\frac{1}{p} - \frac{1}{q}\right)\lim_{n\to\infty}\int_{\mathbb{R}^N} u_n^+(x)^q \mathrm{d}x = \frac{(q-p)}{qp}\lim_{n\to\infty}\|u_n^+(x)\|_{L^q(\mathbb{R}^N)}^q,$$

which implies (2.7) with $\delta_0 \leq \frac{qp}{q-p}$. □

Lemma 2.5. *Let C_1 be any fixed constant. Then, for any $\varepsilon > 0$, there exists $\Lambda_\varepsilon > 0$ and $R_\varepsilon > 0$, such that if $\{u_n\}$ is a $(PS)_c$ sequence of J_λ with $\lambda \geq \Lambda_\varepsilon$, $c \leq C_1$, then*

$$\limsup_{n\to\infty}\int_{B_{R_\varepsilon}^c} u_n^+(x)^q \mathrm{d}x \leq \varepsilon, \tag{2.8}$$

where $B_{R_\varepsilon}^c = \{x \in \mathbb{R}^N : |x| \geq R_\varepsilon\}$.

Proof. For $R > 0$, let

$$A(R) := \{x \in \mathbb{R}^N : |x| > R, V(x) \geq M_0\}$$

and

$$B(R) := \{x \in \mathbb{R}^N : |x| > R, V(x) < M_0\}.$$

It follows from Lemma 2.3 that

$$\begin{aligned}\int_{A(R)} |u_n(x)|^p \mathrm{d}x &\le \frac{1}{\lambda M_0} \int_{\mathbb{R}^N} \lambda V(x)|u_n(x)|^p \mathrm{d}x \\ &\le \frac{1}{\lambda M_0}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \lambda V(x)|u_n(x)|^p \mathrm{d}x \right) \\ &\le \frac{1}{\lambda M_0}\left(\frac{pq}{q-p} C_1 + o(1)\right).\end{aligned} \tag{2.9}$$

From Hölder inequality and (2.5), we can see that, for $1 < r < N/(N-ps)$,

$$\int_{B(R)} |u_n(x)|^p \mathrm{d}x \le \left(\int_{\mathbb{R}^N} |u_n(x)|^{pr} \mathrm{d}x \right)^{1/r} \mu(B(R))^{1/r'} \le C\|u_n\|_\lambda^p \cdot \mu(B(R))^{1/r'} \le C\frac{pq}{q-p} C_0 \cdot \mu(B(R))^{1/r'}, \tag{2.10}$$

where $C = C(N, r) > 0$ and $1/r + 1/r' = 1$. By interpolation inequality and Sobolev embedding inequality, we can obtain

$$\begin{aligned}\int_{B_R^c} u_n^+(x)^q \mathrm{d}x &\le \left(\int_{B_R^c} |u_n(x)|^p \mathrm{d}x \right)^{\frac{q(1-\theta)}{p}} \cdot \left(\int_{B_R^c} |u_n(x)|^{p_s^*} \mathrm{d}x \right)^{\frac{q\theta}{p_s^*}} \\ &\le \left(\int_{B_R^c} |u_n(x)|^p \mathrm{d}x \right)^{\frac{q(1-\theta)}{p}} \left(\int_{\mathbb{R}^N} |u_n(x)|^{p_s^*} \mathrm{d}x \right)^{\frac{q\theta}{p_s^*}} \\ &\le C\left(\int_{B_R^c} |u_n(x)|^p \mathrm{d}x \right)^{\frac{q(1-\theta)}{p}} \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{\frac{q\theta}{p}} \\ &\le C\left(\int_{A(R)} |u_n(x)|^p \mathrm{d}x + \int_{B(R)} |u_n(x)|^p \mathrm{d}x \right)^{\frac{q(1-\theta)}{p}} \|u_n\|_\lambda^{q\theta},\end{aligned}$$

where $\theta = \frac{N}{s}\frac{q-p}{pq}$. Then, the result follows from (2.9), (2.10) and (V_2). □

Lemma 2.6. (Brézis-Lieb lemma, 1983) *Let $\{u_n\} \subset L^p(\mathbb{R}^N)$, $1 \le p < \infty$. If*
(a) $\{u_n\}$ is bounded in $L^p(\mathbb{R}^N)$,
(b) $u_n \to u$ almost everywhere on $\mathbb{R}^N$, then

$$\lim_{n\to\infty} (|u_n|_p^p - |u_n - u|_p^p) = |u|_p^p. \tag{2.11}$$

Lemma 2.7. *Let $\lambda \ge \lambda_0 > 0$ be fixed and let $\{u_n\}$ be a $(PS)_c$ sequence of J_λ. Then, up to a subsequence, $u_n \rightharpoonup u$ in E_λ with u being a weak solution of* (1.1). *Moreover, $u_n^1 = u_n - u$ is $(PS)_{c'}$ sequence with $c' = c - J_\lambda(u)$.*

Proof. By Lemma 2.3, $\{u_n\}$ is bounded in E_λ. Then, up to a subsequence $u_n \rightharpoonup u$ in E_λ as $n \to \infty$, and

$$u_n \rightharpoonup u \quad \text{in} \quad W^{s,p}(\mathbb{R}^N), \tag{2.12}$$

$$u_n \rightharpoonup u \quad \text{in} \quad L^q(\mathbb{R}^N), \quad p \le q < p_s^*, \tag{2.13}$$

$$u_n \to u \quad \text{in} \quad L^q_{\mathrm{loc}}(\mathbb{R}^N), \quad p \le q < p_s^*, \tag{2.14}$$

$$u_n \to u \quad \text{a.e. in} \quad \mathbb{R}^N, \tag{2.15}$$

where $p_s^* = \frac{Np}{N-ps}$ is the fractional critical Sobolev exponent. Hence, for any $\varphi \in E_\lambda$, we have

$$\begin{aligned}
\langle J_\lambda'(u_n), \varphi\rangle = & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\
& + \lambda \int_{\mathbb{R}^N} V(x)|u_n(x)|^{p-2}u_n(x)\varphi(x)\mathrm{d}x - \int_{\mathbb{R}^N} u_n^+(x)^{q-1}\varphi(x)\mathrm{d}x \\
\to & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\
& + \lambda \int_{\mathbb{R}^N} V(x)u(x)^{p-1}\varphi(x)\mathrm{d}x - \int_{\mathbb{R}^N} u^+(x)^{q-1}\varphi(x)\mathrm{d}x = \langle J_\lambda'(u), \varphi\rangle.
\end{aligned}$$

Therefore,

$$\langle J_\lambda'(u), \varphi\rangle = \lim_{n\to\infty} \langle J_\lambda'(u_n), \varphi\rangle = 0, \tag{2.16}$$

which implies that u is a critical point of J_λ.

Let $u_n^1 = u_n - u$, we will show that as $n \to \infty$,

$$J_\lambda(u_n^1) \to c - J_\lambda(u) \tag{2.17}$$

and

$$J_\lambda'(u_n^1) \to 0. \tag{2.18}$$

To show (2.17), we observe that

$$\begin{aligned}
J_\lambda(u_n^1) = & \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n^1(x)-u_n^1(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \frac{\lambda}{p}\int_{\mathbb{R}^N} V(x)|u_n^1(x)|^p\mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N} u_n^{1+}(x)^q\mathrm{d}x \\
= & J_\lambda(u_n) - J_\lambda(u) + \frac{\lambda}{p}\int_{\mathbb{R}^N} V(x)(|u_n^1(x)|^p - |u_n(x)|^p + |u(x)|^p)\mathrm{d}x \\
& + \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n^1(x)-u_n^1(y)|^p - |u_n(x)-u_n(y)|^p + |u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\
& + \frac{1}{q}\int_{\mathbb{R}^N} u_n^+(x)^q\mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N} u_n^{1+}(x)^q\mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N} u^+(x)^q\mathrm{d}x.
\end{aligned} \tag{2.19}$$

From Lemma 2.6, $\int_{\mathbb{R}^N} u_n^+(x)^q\mathrm{d}x - \int_{\mathbb{R}^N} u^+(x)^q\mathrm{d}x - \int_{\mathbb{R}^N} u_n^{1+}(x)^q\mathrm{d}x \to 0$ as $n \to \infty$. Conversely, we know that $\|u_n\|_\lambda^p - \|u\|_\lambda^p - \|u_n^1\|_\lambda^p \to 0$, as $n \to \infty$. Thus, from (2.19), we indeed have obtained (2.17). Now we come to show (2.18). From (2.16), we have for any $\varphi \in E_\lambda$

$$\langle J_\lambda'(u_n^1), \varphi\rangle = \langle J_\lambda'(u_n), \varphi\rangle - \int_{\mathbb{R}^N} (u_n^{1+})^{q-1}\varphi(x)\mathrm{d}x + \int_{\mathbb{R}^N} (u_n^+)^{q-1}\varphi(x)\mathrm{d}x - \int_{\mathbb{R}^N} (u^+)^{q-1}\varphi(x)\mathrm{d}x + o(1).$$

Since $J_\lambda'(u_n) \to 0$ and $u_n \rightharpoonup u$ in $L^q(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty} \sup_{\|\varphi\|_\lambda\le 1} \int_{\mathbb{R}^N} ((u_n^{1+})^{q-1}(x)\varphi(x) - (u_n^+)^{q-1}\varphi(x) + (u^+)^{q-1}\varphi(x))\mathrm{d}x = 0.$$

Thus, we have

$$\lim_{n\to\infty} \langle J_\lambda'(u_n^1), \varphi\rangle = 0 \ \text{ for any } \varphi \in E_\lambda,$$

which implies (2.18), and this completes the proof. □

Proposition 2.8. *Suppose* (V_1) *and* (V_2) *hold. Then, for any* $C_0 > 0$, *there exists* $\Lambda_0 > 0$ *such that* J_λ *satisfies the* $(PS)_c$ *condition for all* $\lambda \geq \Lambda_0$ *and* $c \leq C_0$.

Proof. Choose $0 < \varepsilon < \delta_0 c_0/2$, where c_0 and δ_0 are the constants in Lemmas 2.3 and 2.4, respectively. Let $\Lambda_0 := \Lambda_\varepsilon$, where $\Lambda_\varepsilon > 0$ is from Lemma 2.5.

Assume $\{u_n\}$ is a $(PS)_c$ sequence of J_λ with $\lambda \geq \Lambda_0$ and $c \leq C_0$. By Lemma 2.7, $u_n^1 = u_n - u$ is a $(PS)_{c'}$ sequence of J_λ with $c' = c - J_\lambda(u)$. If $c' > 0$, it follows from Lemma 2.3 that $c' \geq c_0$. From Lemma 2.4, we can obtain

$$\liminf_{n\to\infty} \|u_n^{1+}(\cdot)\|^q_{L^q(\mathbb{R}^N)} \geq \delta_0 c' \geq \delta_0 c_0.$$

Conversely, Lemma 2.5 implies

$$\limsup_{n\to\infty} \int_{B^c_{R_\varepsilon}} u_n^{1+}(x)^q \leq \varepsilon < \frac{\delta_0 c_0}{2}.$$

Noting $u_n^1 \to 0$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$, $p \leq q < p_s^*$, a contradiction follows from the aforementioned two inequalities. Therefore, $c' = 0$. Thus, $u_n^1 \to 0$ in E_λ by Lemma 2.3. □

Corollary 2.9. *For any* $q \in (p, p_s^*)$, *there exists* $\Lambda_0 > 0$, *such that* c_λ *is achieved for all* $\lambda \geq \Lambda_0$ *at some* $u_\lambda \in E_\lambda$, *which is a ground state solution of* (1.1).

Proof. By Ekeland variational principle, there is a PS sequence $u_n \in E_\lambda$, such that

$$J_\lambda(u_n) \to c_\lambda \quad \text{and} \quad J'_\lambda(u_n) \to 0.$$

By Proposition 2.8, there exists some $u_\lambda \in E_\lambda$, such that, up to subsequence, $u_n \to u_\lambda$ in E_λ as $n \to \infty$ and λ is sufficiently large. It is not difficult to show that

$$J_\lambda(u_n) \to J_\lambda(u_\lambda) \quad \text{and} \quad J'_\lambda(u_n) \to J'_\lambda(u_\lambda).$$

Therefore, we have $J_\lambda(u_\lambda) = c_\lambda$ and $J'_\lambda(u_\lambda) = 0$. This means that u_λ is a ground state solution of (1.1). □

3 Limit problem

Lemma 3.1. *Let* $1 < p < q < p_s^* := \frac{pN}{N-ps}$, $N \geq 2$. *Then,* $tr_\Omega E_0$ *is compactly embedded in* $L^q(\Omega)$.

Proof. Since $tr_\Omega E_0 \subset W^{s,p}(\Omega)$ and $W^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$ are compact for $p < q < p_s^*$, $N \geq 2$, the result follows. □

Lemma 3.2. *The infimum* $c(\Omega)$ *is achieved by a function* $u \in \mathcal{N}$, *which is a ground state solution of* (1.3).

Proof. By Ekeland variational principle, there is a PS sequence $u_n \in E_0$, such that

$$\Phi(u_n) \to c(\Omega) \quad \text{and} \quad \Phi'(u_n) \to 0.$$

Thus, by Lemma 3.1, we can easily obtain a subsequence of $\{u_n\}$ (still denote it itself), such that $u_n \to u$ in E_0. Therefore, u is a ground state solution of (1.3). □

Remark 3.3. Assume set $\Omega = \mathrm{int} V^{-1}(0)$ has more than one isolated component, for example, $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \varnothing$. Suppose that $u \in \mathcal{N}$ is a nonnegative solution of (1.3) with $u(x) = 0$ in Ω_1 and $u(x) \gneqq 0$ in Ω_2. Then, we have $(-\Delta)_p^s u(x) = \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy < 0$ in Ω_1. Conversely, $(-\Delta)_p^s u(x) = u(x)^{q-1} = 0$ for

$x \in \Omega_1$. This contradiction shows that the nonnegative solution $u(x)$ of (1.3) must be $u(x) \gneqq 0$ in both Ω_1 and Ω_2. However, the Laplacian case can have a nonnegative solution $u(x)$ satisfying $u(x) = 0$ in Ω_1 and $u(x) \gneqq 0$ in Ω_2. The difference between the two phenomena is attributed to the nonlocality of fractional operators and the locality of Laplacian operators.

4 The proof of the main results

Lemma 4.1. *$c_\lambda \to c(\Omega)$ as $\lambda \to \infty$.*

Proof. From the definition of c_λ and $c(\Omega)$, we know that $c_\lambda \le c(\Omega)$, $\lambda > 0$. Furthermore, c_λ is monotone increasing about the parameter $\lambda > 0$. Then, there exists a constant k, such that

$$\lim_{n\to\infty} c_{\lambda_n} = k,$$

where $\lambda_n \to \infty$. It follows from Lemma 2.3 that $k > 0$. By Corollary 2.9, for n large enough, there exists a sequence $u_n \in \mathcal{M}_{\lambda_n}$, such that $J'_{\lambda_n}(u_n) = 0$ and $J_{\lambda_n}(u_n) = c_{\lambda_n}$. If $k < c(\Omega)$, it is easy to see that $\{u_n\}$ is bounded in $W^{s,p}(\mathbb{R}^N)$; thus, we can assume that $u_n \rightharpoonup u$ in E and

$$u_n(x) \to u(x) \quad \text{in} \quad L^{\theta}_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{for } p \le \theta < p_s^*. \tag{4.1}$$

Claim 1: $u\,|_{\Omega^c} = 0$. In fact, if $u|_{\Omega^c} \neq 0$, then there exists a compact subset $F \subset \Omega^c$ with $\mathrm{dist}(F, \Omega) > 0$, such that $u\,|_F \neq 0$. It follows from (4.1) that

$$\int_F |u_n(x)|^p \mathrm{d}x \to \int_F |u(x)|^p \mathrm{d}x > 0.$$

However, there exists $\varepsilon_0 > 0$, such that $V(x) \ge \varepsilon_0 > 0$, $x \in F$. Thus,

$$J_{\lambda_n}(u_n) \ge \frac{q-p}{pq}\lambda_n \int_F V(x)|u_n(x)|^p \mathrm{d}x \ge \frac{q-p}{pq}\lambda_n\varepsilon_0 \int_F |u_n(x)|^p \mathrm{d}x \to \infty \quad \text{as } n \to \infty,$$

which is a contradiction. Therefore, $u \in E_0$.

Claim 2: $u_n \to u$ in $L^q(\mathbb{R}^N)$ for $p < q < p_s^*$. Indeed, if not, then by the concentration-compactness lemma from the study by Loins [25], there exist $\delta > 0$, $\rho > 0$ and $x_n \in \mathbb{R}^N$ with $|x_n| \to \infty$, such that

$$\liminf_{n\to\infty} \int_{B_\rho(x_n)} |u_n(x) - u(x)|^p \mathrm{d}x \ge \delta > 0.$$

Then, we have

$$\begin{aligned}
J_{\lambda_n}(u_n) &= \frac{q-p}{pq} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \frac{q-p}{pq} \int_{\mathbb{R}^N} \lambda_n V(x)|u_n(x)|^p \mathrm{d}x \\
&\ge \frac{q-p}{pq}\lambda_n \int_{B_\rho(x_n)\cap\{x:V(x)\ge M_0\}} V(x)|u_n(x)|^p \mathrm{d}x \\
&= \frac{q-p}{pq}\lambda_n \int_{B_\rho(x_n)\cap\{x:V(x)\ge M_0\}} V(x)|u_n(x) - u(x)|^p \mathrm{d}x \\
&\ge \frac{q-p}{pq}\lambda_n \left(M_0 \int_{B_\rho(x_n)} |u_n(x) - u(x)|^p \mathrm{d}x - M_0 \int_{B_\rho(x_n)\cap\{x:V(x)\le M_0\}} |u_n(x)|^p \mathrm{d}x \right) \\
&\ge \frac{q-p}{pq}\lambda_n \left(M_0 \int_{B_\rho(x_n)} |u_n(x) - u(x)|^p \mathrm{d}x - o(1) \right) \to \infty \text{ as } n \to \infty,
\end{aligned}$$

as a contradiction. So $u_n \to u$ in $L^p(\mathbb{R}^N)$. Therefore, it is easy to see that $u \geq 0$ is a solution for problem (1.3). Furthermore,

$$k = \lim_{n\to\infty} c_{\lambda_n} = \lim_{n\to\infty} J_{\lambda_n}(u_n) = \lim_{n\to\infty}\left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^N} u_n^+(x)^q \mathrm{d}x = \left(\frac{1}{p}-\frac{1}{q}\right)\int_{\Omega} u^+(x)^q \mathrm{d}x,$$

which means $u \in \mathcal{N}$, and then, $k \geq c(\Omega)$, a contradiction. Hence, $\lim_{\lambda\to\infty} c_\lambda = c(\Omega)$. □

Proof of Theorem 1.3. By Corollary 2.9, there exists $u_n \in \mathcal{M}_{\lambda_n}$, such that $J_{\lambda_n}(u_n) = c_{\lambda_n}$ ($\lambda_n \to \infty$ as $n \to \infty$). It is easy to see that $\{u_n\}$ is bounded in $W^{s,p}(\mathbb{R}^N)$. Then, without loss of generality, $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$ and $u_n \to u$ in $L^\theta_{\mathrm{loc}}(\mathbb{R}^N)$ for $p < \theta < p_s^*$.

Now we prove that $u_n \to u$ strongly in $W^{s,p}(\mathbb{R}^N)$ and u is a ground state solution of (1.3). First, as the proof of Lemma 4.1, $u \geq 0$ is a solution of problem (1.3) and $u_n^+ \to u^+$ strongly in $L^q(\mathbb{R}^N)$.

Now we claim that

$$\lambda_n \int_{\mathbb{R}^N} V(x)|u_n(x)|^p \mathrm{d}x \to 0$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.$$

Indeed, if either

$$\liminf_{n\to\infty} \lambda_n \int_{\mathbb{R}^N} V(x)|u_n(x)|^p \mathrm{d}x > 0$$

or

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y > \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.$$

Thus, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y < \int_{\Omega} u^+(x)^q \mathrm{d}x.$$

Therefore, there is $\alpha \in (0,1)$, such that $\alpha u \in \mathcal{N}$ and

$$\begin{aligned}
c(\Omega) \leq \Phi(\alpha u) &= \frac{q-p}{pq}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\alpha u(x)-\alpha u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\
&< \frac{q-p}{pq}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\mathrm{d}x\mathrm{d}y \\
&\leq \lim_{n\to\infty} \frac{q-p}{pq}\left(\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \lambda_n V(x)|u_n(x)|^p \mathrm{d}x\right) \\
&= \lim_{n\to\infty} J_{\lambda_n}(u_n) = c(\Omega),
\end{aligned}$$

which is a contradiction. By now we complete the proof of Theorem 1.3. □

Proof of Theorem 1.4. Suppose $\{u_n\} \subset W^{s,p}(\mathbb{R}^N)$ is a solution of (1.1) with λ being replaced by λ_n ($\lambda_n \to \infty$ as $n \to \infty$). It follows from $\limsup_{n\to\infty} J_\lambda(u_n) < \infty$ that such a sequence $\{u_n\}$ is bounded in $W^{s,p}(\mathbb{R}^N)$.

Suppose that $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$ and $u_n \to u$ in $L^{\theta}_{\mathrm{loc}}(\mathbb{R}^N)$ for $p < \theta < p_s^*$. Similar to the proof of Lemma 4.1, $u\,|_{\Omega^c} = 0$ and $u \in E_0$ is solution of (1.3). Moreover, $u_n \to u$ in $L^{\theta}(\mathbb{R}^N)$ for $p < \theta < p_s^*$. Noting $u_n \in \mathcal{M}_{\lambda_n}$ and $u \in \mathcal{N}$, we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y) - u(x) + u(y)|^p}{|x-y|^{n+ps}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \lambda_n V(x)|u_n(x) - u(x)|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{n+ps}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \lambda_n V(x)|u_n(x)|^p \mathrm{d}x - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N} \lambda_n V(x)|u(x)|^p \mathrm{d}x \\
&\quad + o(1) \\
&= \int_{\mathbb{R}^N} u_n^+(x)^q \mathrm{d}x - \int_{\Omega} u^+(x)^q \mathrm{d}x + o(1) = o(1).
\end{aligned}$$

Thus, $u_n \to u$ in $W^{s,p}(\mathbb{R}^N)$. This completes the proof of Theorem 1.4. □

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