Research Article

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The intersection graph of graded submodules of a graded module

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Abstract: In this article, we introduce and study the intersection graph of graded submodules of a graded module. Let M be a left G-graded R-module. We define the intersection graph of G-graded R-submodules of M, denoted by $\Gamma(G, R, M)$, to be a simple undirected graph whose set of vertices consists of all nontrivial G-graded R-submodules of M, where two vertices are adjacent if their intersection is nonzero. We study properties of these graphs, such as connectivity, diameter, and girth. We also investigate the intersection graph of graded submodules for certain types of gradings such as faithful and strong gradings.

Keywords: graded ring, graded module, intersection graph, submodule, homogeneous

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1 Introduction

Studies of graphs associated with algebraic structures developed remarkably in recent years. Usually, the purpose of associating a graph with an algebraic structure is to investigate the algebraic properties using concepts in graph theory. Zero-divisors graph, total graphs, annihilating-ideal graph, and unit graphs are very interesting examples of graphs associated with rings, see [1–4]. For studies on graphs associated with graded rings and graded modules, in particular, see [5,6].

Among the types of graphs associated with rings are intersection graphs. In 2009, Chakrabarty et al. [7] introduced and studied the intersection graph of ideals of a ring R, which is an undirected simple graph, denoted by $G(R)$, whose vertices are the nontrivial left ideals of R and two vertices I and J are adjacent if their intersection is nonzero. Inspired by their work, Akbari et al. [8] introduced the intersection graph of submodules of a module. For a ring R with unity and a unitary left R-module M, the set of all R-submodules of M is denoted by $S^*(M)$. The intersection graph of submodules of M, denoted by $G(M)$, is an undirected simple graph defined on $S^*(M)$, where two non-trivial submodules are adjacent if they have a nonzero intersection. Since they were introduced, intersection graphs of ideal and submodules have attracted many researchers to study their graph-theoretic properties and investigate their structures (see [9–17]). Alraqad et al. [18] introduced and studied the intersection graph of graded ideals of a graded ring.

Motivated by all previous works, we introduce the intersection graph of graded submodules of a graded module. Let G be a group. A ring R is said to be G-graded if there exist additive subgroups $\{R_\sigma | \sigma \in G\}$ such that $R = \oplus_{\sigma \in G} R_\sigma$ and $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. A left R-module M is said to be G-graded if there exist additive subgroups M_σ of M indexed by the elements $\sigma \in G$ such that $M = \oplus_{\sigma \in G} M_\sigma$ and $R_\tau M_\sigma \subseteq M_{\tau\sigma}$ for all $\tau, \sigma \in G$. The elements of M_σ are called homogeneous of degree σ. If $x \in M$, then x can be written uniquely as $\sum_{\sigma \in G} x_\sigma$, where x_σ is the component of x in M_σ. An R-submodule N of M is called G-graded provided that $N = \oplus_{\sigma \in G}(N \cap M_\sigma)$. We denote by $hS^*(M)$ the set of all nontrivial G-graded R-submodules of M.

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Definition 1.1. Let R be a G-graded ring and M be a G-graded left R-module. The intersection graph of G-graded submodules of M, denoted by $\Gamma(G, R, M)$, is defined to be an undirected simple graph whose set of vertices is $hS^*(M)$ and two vertices N and K are adjacent if $N \cap K \neq \{0\}$.

We aim to study the properties of these graphs analogous to the nongraded case. In addition, we investigate connections and relationships among $G(M_\sigma)$, $\Gamma(G, R, M)$, and $G(M)$ under certain types of gradings.

The organization of the paper is as follows: Section 2 is devoted to the study of graph-theoretic properties of $\Gamma(G, R, M)$. We discuss their connectivity, diameter, regularity, completeness, domination numbers, and girth. In Section 3, we investigate the relationships between $\Gamma(G, R, M)$ and $G(M_\sigma)$ (where M_σ is considered as a left R_e-module) under some types of gradings such as faithful grading and strong grading. This section also presents some results regarding the relationship between $\Gamma(G, R, M)$ and $G(M)$ when the grading group is a linearly ordered group.

For standard terminology and notion in the graph theory, we refer the reader to the textbook [19]. For a simple graph, Γ, the set of vertices and set of edges are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The cardinality $|V(\Gamma)|$ is referred to as the order of Γ. If $x, y \in V(\Gamma)$ are adjacent, we denote that as $x \leftrightarrow y$. The neighborhood of a vertex x is $\mathcal{N}(x) = \{y \in V(\Gamma) | y \leftrightarrow x\}$, and the degree of x is $\deg(x) = |\mathcal{N}(x)|$. The graph Γ is said to be regular if all of its vertices have the same degree. A graph is called complete (resp. null) if any pair of its vertices are adjacent (resp. not adjacent). A complete (resp. null) graph with n vertices is denoted by K_n (resp. N_n). A graph is said to be connected if any pair of its vertices is connected by a path.

Throughout this article, all rings are associated with unity $1 \neq 0$, and all modules are left modules. When a ring R is G-graded, we denote that by (R, G). The support of (R, G) is defined as $\mathrm{supp}(R, G) = \{\sigma \in G : R_\sigma \neq 0\}$. If $r \in R$, then r can be written uniquely as $\sum_{\sigma \in G} r_\sigma$, where r_σ is the component of r in R_σ. It is well known that R_e is a subring of R with $1 \in R_e$. An ideal I of R is said to be G-graded if $I = \oplus_{\sigma \in G}(I \cap R_\sigma)$. Let M be a G-graded R-module. It is known that M_σ is R_e-submodule of M for all $\sigma \in G$. Also, we write $h(M) = \cup_{\sigma \in G} M_\sigma$ and $\mathrm{supp}(M, G) = \{\sigma \in G : M_\sigma \neq 0\}$. A G-graded R-submodule of M is said to be G-graded maximal (resp. simple or minimal) if it is maximal (resp. minimal) among all proper (resp. nonzero) G-graded R-submodules of M. We denote by $\mathrm{GMax}(M)$ (resp. $\mathrm{GMin}(M)$) the set of all nontrivial G-graded maximal (resp. simple) R-submodules. A G-graded R-module M is called G-graded local (resp. G-graded simple) if $|\mathrm{GMax}(M)| = 1$ (resp. $\mathrm{GMax}(M) = \{(0)\}$). We say that M is G-graded left Noetherian (resp. Artinian) if M satisfies the ascending (resp. descending) chain condition for the G-graded R-submodules of M.

2 Graph theoretic properties of $\Gamma(G, R, M)$

We present the following well-known technical lemma in this section.

Lemma 2.1. [20, Lemma 2.1] *Let R be a G-graded ring and M be a G-graded R-module.*
(i) *If I and J are G-graded ideals of R, then $I + J$ and $I \bigcap J$ are G-graded ideals of R.*
(ii) *If N and K are G-graded R-submodules of M, then $N + K$ and $N \bigcap K$ are G-graded R-submodules of M.*
(iii) *If N is a G-graded R-submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a G-graded ideal of R, then Rx, IN, and rN are G-graded R-submodules of M. Moreover, $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a G-graded ideal of R.*

The following two results from [8] classify disconnected intersection graphs of submodules.

Theorem 2.2. [8, Theorem 2.1] *Let M be an R-module. Then, the graph $G(M)$ is disconnected if and only if M is a direct sum of two simple R-modules.*

Corollary 2.3. [8, Corollary 2.3] *Let M be an R-module. Then, the graph $G(M)$ is disconnected if and only if it is null graph with at least two vertices.*

Analogues to the nongraded case, next we characterize disconnected intersection graphs of graded submodules.

Theorem 2.4. *Let R be a G-graded ring and M be a G-graded R-module such that $|\Gamma(G,R,M)| \geq 2$. Then, the followings are equivalent*:
(1) *$\Gamma(G,R,M)$ is disconnected.*
(2) *$\Gamma(G,R,M)$ is a null graph.*
(3) *Every nontrivial G-graded R-submodule of M is G-graded maximal as well as G-graded simple.*
(4) *M is a direct sum of two G-graded simple (or maximal) R-modules.*

Proof.
(1) ⇒ (2) Suppose that $\Gamma(G,R,M)$ is disconnected. For a contradiction, assume N and K are two adjacent vertices. So N, K, and $N \cap K$ belong to the same component of $\Gamma(G,R,M)$. Since $\Gamma(G,R,M)$ is disconnected, there is a vertex L that is not connected to any of the vertices N, K, and $N \cap K$. If $(N \cap K) + L \neq M$, then $(N \cap K) \leftrightarrow ((N \cap K) + L) \leftrightarrow L$ is a path connecting $N \cap K$ and L, a contradiction. So $(N \cap K) + L = M$. Now let $a \in N$. Then, $a = t + c$ for some $t \in N \cap K$ and $c \in L$. So $a - t = c \in N \cap L = \{0\}$; consequently, $a = t \in N \cap K$. This implies that $N = N \cap K$. Similarly, we obtain $K = N \cap K$. Hence, we have $N = K$, a contradiction. Therefore, $\Gamma(G,R,M)$ contains no edges, and hence, it is a null graph.
(2) ⇔ (3) Straightforward.
(3) ⇒ (4) Let N and K be G-graded maximal as well as G-graded simple R-submodules of M. Then, $N + K = M$ and $N \cap K = \{0\}$. Hence, N and K are G-graded simple R-modules and $M = N \oplus K$.
(4) ⇒ (1) Suppose $M = N \oplus K$, where N and K are G-graded simple R-modules. Then, N and K are G-graded simple R-submodules. Also, they are G-graded maximal because $N \cong \frac{M}{K}$ and $K \cong \frac{M}{N}$. Thus, N and K are isolated vertices. Therefore, $\Gamma(G,R,M)$ is disconnected. □

Roshan-Shekalgourabi and Hassanzadeh-Lelekaami [6] associated a graph G_M with a G-graded R-module M, where $V(G_M) = hS^*(M)$ and two nontrivial G-graded R-submodules N and K are adjacent if $N + K = M$. Clearly, the two concepts G_M and $\Gamma(G,R,M)$ are distinct. The next theorem presents an obvious relation between these two graphs.

Corollary 2.5. *Let M be a G-graded R-module. Then, $\Gamma(G,R,M)$ is disconnected if and only if G_M is a complete graph with at least two vertices.*

Proof. The result follows by Theorem 2.4 and [6, Theorem 2.2]. □

The proof of the next corollary is straightforward.

Corollary 2.6. *Let M be a G-graded R-module. If $\Gamma(G,R,M)$ is connected, then every pair of G-graded maximal R-submodules intersect nontrivially.*

The distance $d(x,y)$ between any two vertices x, y in a graph Γ is the length of the shortest path between them, and $\operatorname{diam}(\Gamma)$ is the supremum of $\{d(x,y) | x, y \in V(\Gamma)\}$.

Theorem 2.7. *Let M be a G-graded R-module. If $\Gamma(G,R,M)$ is connected, then* $\operatorname{diam}(\Gamma(G,R,M)) \leq 2$.

Proof. Suppose N and K are distinct vertices in $\Gamma(G,R,M)$. If N and K are adjacent, then $d(N,K) = 1$. If N and K are nonadjacent, then $d(N,K) \geq 2$. If $N \oplus K \neq M$, then we have the path $N \leftrightarrow N \oplus K \leftrightarrow K$, and hence, $d(N,K) = 2$. If $N \oplus K = M$, then either N or K is not G-graded simple, say N. Let $(0) \neq L \subsetneq N$. Thus, we have the path $N \leftrightarrow L \oplus K \leftrightarrow K$, and hence, $d(N,K) = 2$. As a result, $d(N,K) \leq 2$. □

Theorem 2.8. *Let M be a G-graded Artinian R-module such that $\Gamma(G,R,M)$ is not a null graph. Then, the followings are equivalent*:
(*i*) $\Gamma(G,R,M)$ *is regular*.
(*ii*) $|\mathrm{GMin}(M)| = 1$.
(*iii*) $\Gamma(G,R,M)$ *is complete*.

Proof.
(*i*) ⇒ (*ii*) Suppose $\Gamma(G,R,M)$ is regular. Assume that M contains two distinct G-graded simple R-submodules N and K. Clearly, N and K are nonadjacent. By Theorem 2.7, there is a G-graded R-submodule Y that is adjacent to both N and K. Hence, by minimality of N, we obtain $N \subsetneq Y$. This implies that $\mathcal{N}(N) \subsetneq \mathcal{N}(Y)$; consequently, $\deg(Y) > \deg(N)$, a contradiction. Hence, M contains a unique G-graded simple R-submodule.
(*ii*) ⇒ (*iii*) Suppose M contains a unique G-graded simple R-submodule, say N. Since M is G-graded Artinian, $N \subseteq K$ for all $K \in hS^*(M)$. Thus, $\Gamma(G,R,M)$ is complete.
(*iii*) ⇒ (*i*) Straightforward. □

Remark 2.9. In a G-graded R-module M, a G-graded submodule N of M is called G-graded essential if $N \cap K \neq (0)$ for all $K \in hS^*(M)$. The graded socle, $\mathrm{Gsoc}(M)$, of M is the sum of all G-graded simple R-submodules of M. Equivalently $\mathrm{Gsoc}(M)$ equals the intersection of all G-graded essential R-submodules of M, see [21, page 48]. So, if M is G-graded Artinian and $\Gamma(G,R,M)$ is complete, then every G-graded R-submodule is G-essential, and thus, by Theorem 2.8, $\mathrm{GMin}(M) = \mathrm{Gsoc}(M)$.

Recall that the girth of a graph Γ, denoted by $g(\Gamma)$, is the length of its shortest cycle. If Γ has no cycles, then $g(\Gamma) = \infty$.

Theorem 2.10. *If M is a G-graded R-module, then $gr(\Gamma(G,R,M)) \in \{3,\infty\}$.*

Proof. Assume $g(\Gamma(G,R,M)) < \infty$ and $g(\Gamma(G,R,M)) \geq 4$. This implies that every pair of distinct nontrivial G-graded submodules of M with nonzero intersection should be comparable, otherwise $\Gamma(G,R,M)$ will have a cycle of length 3, a contradiction. Since $g(\Gamma(G,R,M)) \geq 4$, $\Gamma(G,R,M)$ contains a path of length 3, say $N \leftrightarrow L \leftrightarrow K \leftrightarrow P$. Since any two submodules in this path are comparable and any chain of nontrivial G-graded submodules of length 2 induces a cycle of length 3 in $\Gamma(G,R,M)$, the only possible two cases are $N \subseteq L$, $K \subseteq L$, $K \subseteq P$ or $L \subseteq N$, $L \subseteq K$, $P \subseteq K$. The first case yields $K \subseteq L \cap P$, and hence, $L \cap P \neq (0)$. Thus, $L \leftrightarrow K \leftrightarrow P \leftrightarrow L$ is a cycle of length 3 in $\Gamma(G,R,M)$, a contradiction. In the second case, we have $(0) \neq L \subseteq N \cap K$, and therefore, $N \leftrightarrow L \leftrightarrow K \leftrightarrow N$ is a cycle of length 3 in $\Gamma(G,R,M)$, which again yields a contradiction. Therefore, $g(\Gamma(G,R,M)) = 3$. □

The next theorem gives a characterization of G-graded R-modules such that $g(\Gamma(G,R,M)) = \infty$. Recall that a graph is called star if it has no cycles and has one vertex (the center) that is adjacent to all other vertices.

Theorem 2.11. *Let M be a G-graded Noetherian R-module such that $\Gamma(G,R,M)$ is not a null graph with $|\Gamma(G,R,M)| \geq 2$. If $g(\Gamma(G,R,M)) = \infty$, then M is a G-graded local module and $\Gamma(G,R,M)$ is a star graph whose center is the unique G-graded maximal R-submodule of M.*

Proof. By Theorem 2.4, $\Gamma(G,R,M)$ is connected. Suppose that N_1 and N_2 are two distinct G-graded maximal R-submodules of M. By Theorem 2.7, $d(N_1,N_2) \leq 2$. If $N_1 \cap N_2 \neq (0)$, then $N_1 \leftrightarrow (N_1 \cap N_2) \leftrightarrow N_2 \leftrightarrow N_1$ is a 3-cycle, a contradiction. So $N_1 \cap N_2 = (0)$. Since N_1 and N_2 are G-graded maximal R-submodules, we obtain $M = N_1 \oplus N_2$. Thus, $\Gamma(G,R,M)$ is null, which contradicts the assumption that $\Gamma(G,R,M)$ is not null. Therefore, M is G-graded local module. Let N be the G-graded maximal submodule of M. It is easy to see that every proper graded submodule of a G-graded Noetherian module is contained in a G-graded maximal submodule. So $N \cap K = K \neq (0)$ for all $K \in hS^*(M)$. However, since $\Gamma(G,R,M)$ has no cycles, we conclude that $\Gamma(G,R,M)$ is a star graph. □

A subgraph Υ of a graph Γ is called an induced subgraph if any edge in Γ that joins two vertices in Υ is in Υ. A complete induced subgraph of a graph Γ is called a clique, and the order of the largest clique in Γ, denoted by $\omega(\Gamma)$, is the clique number of Γ.

Lemma 2.12. *Let M be a G-graded R-module. If $\omega(\Gamma(G,R,M)) < \infty$, then M is G-graded Artinian and G-graded Noetherian.*

Proof. Members of any ascending or descending chain of G-graded R-submodules form a clique in $\Gamma(G,R,M)$, and hence, the chain is finite. □

Corollary 2.13. *Let M be a G-graded R-module. If $\Gamma(G,R,M)$ is connected, then $|\mathrm{GMax}(M)| \le \omega(\Gamma(G,R,M))$.*

Proof. If $\Gamma(G,R,M)$ is connected, then by Corollary 2.6, $\mathrm{GMax}(M)$ is a clique, and thus, $|\mathrm{GMax}(M)| \le \omega(\Gamma(G,R,M))$. □

A subset D of the set of vertices of a graph Γ is called a dominating set in Γ if every vertex of Γ is in D or adjacent to a vertex in D. The domination number of Γ, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set in Γ. In the next theorem, we determine the domination number of $\Gamma(G,R,M)$. In this result, we use the notion of graded decomposable modules. A G-graded R-module M is called G-graded decomposable, if it is a direct sum of two nontrivial G-graded R-submodules. If M is not G-graded decomposable, then it is called G-graded indecomposable.

Theorem 2.14. *Let M be a G-graded R-module that contains a G-graded maximal submodule. Then, $\gamma(\Gamma(G,R,M)) \le 2$. Furthermore, if M is G-graded indecomposable, then $\gamma(\Gamma(G,R,M)) = 1$.*

Proof. Let N be a G-graded maximal R-submodule of M. If there exists $K \in hS^*(M)$ such that $N \cap K = (0)$, then $N + K = M$, and hence, $M = N \oplus K$. So the set $\{N, K\}$ is a dominating set, and thus, $\gamma(\Gamma(G,R,M)) \le 2$. This proves the first part.

If M is G-graded indecomposable, then $N \cap K \ne (0)$ for all $K \in hS^*(M)$. Consequently, $\{N\}$ is a dominating set, and hence, $\gamma(\Gamma(G,R,M)) = 1$. □

3 Intersection graph of types of gradings

In this section, we study some relationships between $\Gamma(G,R,M)$ and $G(M_\sigma)$ and between $\Gamma(G,R,M)$ and $G(M)$. It is well known that if M is a G-graded R-module, then M_σ is an R_e-module for each $\sigma \in G$. So $G(M_\sigma)$ here represents the intersection graph of R_e-submodules of M_σ. We also note that if N_σ is an R_e-submodule of M_σ, then RN_σ is a G-graded R-submodule of M and $RN_\sigma \cap M_\sigma = N_\sigma$.

Theorem 3.1. *Let M be a G-graded R-module. If for some $\sigma \in G$, $G(M_\sigma)$ is connected with at least two vertices, then $\Gamma(G,R,M)$ is connected, and hence, $G(M)$ is connected.*

Proof. Since $G(M_\sigma)$ is connected, it must contain an edge. Let N_σ, K_σ be two adjacent vertices in $G(M_\sigma)$. Then, RN_σ and RK_σ are vertices in $\Gamma(G,R,M)$. Moreover, $RN_\sigma \cap M_\sigma = N_\sigma$ and $RK_\sigma \cap M_\sigma = K_\sigma$, and so, $RN_\sigma \ne RK_\sigma$. In addition, we have $\{0\} \ne N_\sigma \cap K_\sigma \subseteq RN_\sigma \cap RK_\sigma$. Therefore, $\Gamma(G,R,M)$ is not null, and hence, it is connected. The last part follows from Corollary 2.3 because $\Gamma(G,R,M)$ is a subgraph of $G(M)$. □

Remark 3.2. The converse of Theorem 3.1 needs not to be true in general. Let $R = \mathbb{Z}_6$ with trivial $\mathbb{Z}$-grading; that is, $R_0 = \mathbb{Z}_6$, and $R_k = 0$, for all $k \ne 0$, and choose $M = \mathbb{Z}_6[x]$ as Z_6-module with grading $M_k = \mathbb{Z}_6 x^k$, $k \ge 0$, and $M_k = 0$, $k < 0$. The $\mathbb{Z}$-graded $\mathbb{Z}_6$-submodules $\mathbb{Z}_6$ and $\mathbb{Z}_6 + \mathbb{Z}_6 x$ are adjacent in $\Gamma(\mathbb{Z}, \mathbb{Z}_6, \mathbb{Z}_6[x])$, and by Theorem 2.4, we have $\Gamma(\mathbb{Z}, \mathbb{Z}_6, \mathbb{Z}_6[x])$ is connected. On the other hand, for each $k \geq 0$, $\langle 2x^k \rangle$ and $\langle 3x^k \rangle$ are the only $\mathbb{Z}_6$-submodules of $\mathbb{Z}_6 x^k$, and their intersection is (0). So $G(\mathbb{Z}_6 x^k)$ is disconnected for all $k \geq 0$.

A G-graded R-module M is said to be left σ-faithful for some $\sigma \in G$, if $R_{\sigma\tau^{-1}} x_\tau \neq \{0\}$ for every $\tau \in G$, and every nonzero $x_\tau \in M_\tau$. If M is left σ-faithful for all $\sigma \in G$, then it is called left faithful.

Lemma 3.3. [21, Proposition 2.6.3] *A G-graded R-module M is σ-faithful for some $\sigma \in G$ if and only if $N \cap M_\sigma \neq \{0\}$ for all $N \in hS^*(M)$.*

Let M be a G-graded R-module. Define the simple graph $\Gamma_\sigma(G, R, M)$ on $hS^*(M)$, where N and K are adjacent only if $N \cap K \cap M_\sigma \neq \{0\}$. We will call this graph the σ-intersection graph of G-graded R-modules of M. It is clear that if $N \cap K \cap M_\sigma \neq \{0\}$, then $N \cap K \neq \{0\}$. So $\Gamma_\sigma(G, R, M)$ is a subgraph of $\Gamma(G, R, M)$.

Theorem 3.4. *Let M be a G-graded R-module such that $\Gamma(G, R, M)$ is not null graph. Then, M is σ-faithful for some $\sigma \in G$ if and only if the map $\phi_\sigma : \Gamma_\sigma(G, R, M) \longrightarrow \Gamma(G, R, M)$ defined by $\phi(N) = N$ is a graph isomorphism.*

Proof. Suppose M is σ-faithful for some $\sigma \in G$. Clearly, ϕ is a set bijection. Let $N, K \in hS^*(M)$ such that $N \cap K \neq \{0\}$. Since M is σ-faithful, by Lemma 3.3, $N \cap K \cap M_\sigma \neq \{0\}$, which implies that N and K are adjacent in $\Gamma_\sigma(G, R, M)$. Therefore, ϕ_σ is a graph isomorphism. For the converse, suppose that there exists $N \in hS^*(M)$ such that $N \cap M_\sigma = \{0\}$. Then, N is an isolated vertex in $\Gamma_\sigma(G, R, M)$, which implies that N is an isolated vertex in $\Gamma(G, R, M)$ because ϕ_σ is an isomorphism. So $\Gamma(G, R, M)$ is null, a contradiction. Then, $N \cap M_\sigma \neq \{0\}$ for all $N \in hS^*(M)$. Hence, by Lemma 3.3, M is σ-faithful. □

Theorem 3.5. *Let M be a σ-faithful G-graded R-module. If $RM_\sigma = M$, then the following assertions hold:*
(i) *$\Gamma(G, R, M)$ is connected if and only if $G(M_\sigma)$ is connected.*
(ii) *$\gamma(\Gamma(G, R, M)) = \gamma(G(M_\sigma))$.*
(iii) *$\omega(\Gamma(G, R, M)) < \infty$ if and only if $\omega(G(M_\sigma)) < \infty$, and for each $N_\sigma \in S^*(M_\sigma)$, the set $\beta_{N_\sigma} = \{N \in hS^*(M) \mid N \cap M_\sigma = N_\sigma\}$ is finite.*

Proof.

(i) The "if" part is Theorem 3.1. For the "only if" part, assume $\Gamma(G, R, M)$ is connected and let $N_\sigma$and K_σ be two distinct vertices in $G(M_\sigma)$. If $RN_\sigma \cap RK_\sigma \neq \{0\}$, then by Theorem 3.4, $RN_\sigma \cap RK_\sigma \cap M_\sigma \neq \{0\}$. So we have $N_\sigma \cap K_\sigma = RN_\sigma \cap M_\sigma \cap RK_\sigma \cap M_\sigma = RN_\sigma \cap RK_\sigma \cap M_\sigma \neq \{0\}$, and hence, $N_\sigma \leftrightarrow K_\sigma$ is a path. Assume $RN_\sigma \cap RK_\sigma = \{0\}$. By Theorem 2.7, there is $Y \in hS^*(M)$ such that $RN_\sigma \cap Y \neq \{0\}$ and $RK_\sigma \cap Y \neq \{0\}$. Then, $RN_\sigma \cap Y \cap M_\sigma \neq \{0\}$ and $RK_\sigma \cap Y \cap M_\sigma \neq \{0\}$. Since ϕ_σ is a graph isomorphism, $N_\sigma \cap (Y \cap M_\sigma)$ and $K_\sigma \cap (Y \cap M_\sigma)$ are nontrivial. Moreover, $Y \cap M_\sigma \neq M_\sigma$ because $RM_\sigma = M$. Hence, we obtain a path connecting N_σ and K_σ in $G(M)$. Therefore, $G(M)$ is connected.

(ii) Let $S \subseteq S^*(M_\sigma)$ be a minimal dominating set in $G(M_\sigma)$, and let $\mathscr{S} = \{RN_\sigma \mid N_\sigma \in S\}$. Clearly, $|\mathscr{S}| = |S|$. Let $K \in hS^*(M)$ such that $K \notin \mathscr{S}$. By Lemma 3.3, we have $K \cap M_\sigma \neq \{0\}$, and hence, $K \cap M_\sigma \cap N_\sigma \neq \{0\}$ for some $N_\sigma \in S$. So we have $RN_\sigma \in \mathscr{S}$ and K is adjacent to RN_σ in $\Gamma(G, R, M)$. Hence, $\mathscr{S}$ is a dominating set in $\Gamma(G, R, M)$. Therefore, $\gamma(\Gamma(G, R, M)) \leq \gamma(G(M_\sigma))$. Now assume $\mathscr{S}$ is a minimal dominating set in $\Gamma(G, R, M)$, and let $S = \{N \cap M_\sigma \mid N \in \mathscr{S}\}$. Let $K_\sigma \in S^*(M_\sigma)$ such that $K_\sigma \notin S$. If $RK_\sigma \in \mathscr{S}$, then $RK_\sigma \cap M_\sigma \in \mathscr{S}$. We have $K_\sigma \cap (RK_\sigma \cap M_\sigma) = K_\sigma \neq (0)$. So K_σ is adjacent to $RK_\sigma \cap M_\sigma \in \mathscr{S}$. Now assume $RK_\sigma \notin \mathscr{S}$. So there exists $N \in \mathscr{S}$ such that $RK_\sigma \cap N \neq (0)$. Hence, by Theorem 3.4, we obtain $(0) \neq RK_\sigma \cap N \cap M_\sigma \subseteq K_\sigma \cap (N \cap M_\sigma)$. Thus, $K_\sigma \cap (N \cap M_\sigma) \neq (0)$, and so S is a dominating set in $G(M_\sigma)$. So $\gamma(G(M_\sigma)) \leq |S| \leq |\mathscr{S}| = \gamma(\Gamma(G, R, M))$.

(iii) Suppose $\omega(\Gamma(G, R, M)) < \infty$. Let $N_\sigma \in S^*(M_\sigma)$. Since all elements of β_{N_σ} contain N_σ, β_{N_σ} is a clique in $\Gamma(G, R, M)$. Hence, $|\beta_{N_\sigma}| \leq \omega(\Gamma(G, R, M)) < \infty$. Let C be a clique in $G(M_\sigma)$. Then, $\cup_{N_\sigma \in C} \beta_{N_\sigma}$ is a clique in $\Gamma(G, R, M)$. Thus, $\cup_{N_\sigma \in C} \beta_{N_\sigma}$ is finite, which yields C itself is finite. Therefore, $\omega(G(M_\sigma)) < \infty$.

For the converse, suppose that $\omega(G(M_\sigma)) < \infty$, and for each $N_\sigma \in S^*(M_\sigma)$, the set β_{N_σ} is finite. Let D be a clique in $\Gamma(G, R, M)$. Let Λ be the set of all $N_\sigma \in S^*(M_\sigma)$ such that $D \cap \beta_{N_\sigma}$ is nonempty. Clearly, the collection

$\{D \cap \beta_{N_\sigma} | N_\sigma \in \Lambda\}$ is a partition of D. So $D = \cup_{N_\sigma \in \Lambda}(D \cap \beta_{N_\sigma})$. We note that, by the assumption for the converse, $D \cap \beta_{N_\sigma}$ is finite for all $N_\sigma \in \Lambda$. Let $N_\sigma, K_\sigma \in \Lambda$. Then, there are $N, K \in D$ such that $N_\sigma = N \cap M_\sigma$ and $K_\sigma = K \cap M_\sigma$. Since D be a clique, $N \cap K \neq (0)$. In addition, because the grading is σ-faithful, by Lemma 3.3, we obtain that $(0) \neq (N \cap K) \cap M_\sigma = N_\sigma \cap K_\sigma$. So Λ is a clique in $G(M_\sigma)$, and hence, it is finite. This implies that $D = \cup_{N_\sigma \in \Lambda}(D \cap \beta_{N_\sigma})$ is finite. We just proved that every clique in $\Gamma(G, R, M)$ is finite. Therefore, $\omega(\Gamma(G, R, M)) < \infty$. □

Corollary 3.6. *Let M be a σ-faithful G-graded R-module such that $RM_\sigma = M$. Then, M_σ is a direct sum of two simple R_e-modules if and only if M is a direct sum of two G-graded simple R-modules.*

Proof. The proof follows directly from Theorems 2.2, 2.4, and Part (i) of Theorem 3.5. □

A grading (R, G) is called strong (resp. first strong) if $1 \in R_\sigma R_{\sigma^{-1}}$ for all $\sigma \in G$ (resp. $\sigma \in \text{supp}(R, G)$) (see [22,23]). In what follows, the symbol $\leq$ means "a subgroup of," while the symbol $\cong$ means "isomorphic to."

Lemma 3.7. [23, Fact 2.5] *A grading (R, G) is first strong if and only if $H = \text{supp}(R, G) \leq G$ and (R, H) is strong.*

Lemma 3.8. *Let (R, G) be first strong grading and M be a G-graded R-module. If $\text{supp}(M, G) \subseteq \text{supp}(R, G)$, then $\Gamma(G, R, M) \cong G(M_\sigma)$ for all $\sigma \in \text{supp}(M, G)$.*

Proof. Fix $\sigma \in \text{supp}(M, G)$. We claim that if N is G-graded R-submodule of M, then $N = R(N \cap M_\sigma)$. Let $0 \neq x \in N \cap M_\tau$ for some $\tau \in G$. Now $\sigma, \tau \in \text{supp}(M, G) \subseteq \text{supp}(R, G)$. Thus, since $\text{supp}(R, G) \leq G$, $\tau\sigma^{-1} \in \text{supp}(R, G)$. So $R_{\tau\sigma^{-1}} R_{\sigma\tau^{-1}} = R_e$. This implies that $1 = \sum_{i=1}^{n} r_i s_i$ for some $r_i \in R_{\tau\sigma^{-1}}$ and $s_i \in R_{\sigma\tau^{-1}}$. Hence, $x = \sum_{i=1}^{n} r_i s_i x$. Since $x \in M_\tau$ and $s_i \in R_{\sigma\tau^{-1}}$, $s_i x \in R_{\sigma\tau^{-1}} M_\tau \subseteq M_\sigma$, for all i. Also $s_i x \in N$ because N is an R-submodule. So $x \in R(N \cap M_\sigma)$. Hence, $N \cap M_\tau \subseteq R(N \cap M_\sigma)$ for all $\tau \in \text{supp}(M, G)$. This implies that $R(N \cap M_\sigma) \subseteq N = \oplus_{\tau \in G}(N \cap M_\tau) \subseteq R(N \cap M_\sigma)$. Hence, $N = R(N \cap M_\sigma)$. From the claim, we conclude that $M = R(M \cap M_\sigma) = RM_\sigma$ and $N \cap M_\sigma \neq \{0\}$ for all $N \in hS^*(M)$. Therefore, the correspondence $N \to N \cap M_\sigma$ yields an isomorphism between $\Gamma(G, R, M)$ and $G(M_\sigma)$. □

Corollary 3.9. *Let (R, G) be strong grading and M be a G-graded R-module. Then, $\Gamma(G, R, M) \cong G(M_\sigma)$ for all $\sigma \in G$.*

Example 3.10. Let A be a ring, and consider the ring $R = M_3(A)$ and the left R-module $M = M_{3\times 1}(A)$ with $\mathbb{Z}_2$-gradings given by

$$R_0 = \begin{bmatrix} A & A & 0 \\ A & A & 0 \\ 0 & 0 & A \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 & 0 & A \\ 0 & 0 & A \\ A & A & 0 \end{bmatrix}.$$

$$M_0 = \begin{bmatrix} A \\ A \\ 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 \\ 0 \\ A \end{bmatrix}.$$

Clearly, $(R, \mathbb{Z}_2)$ is strong. So by Corollary 3.9, $G(\mathbb{Z}_2, M_3(A), M_{3\times 1}(A)) \cong G(M_1)$. The nontrivial R_0-submodules of M_1 are given as follows:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \middle| I \in I^*(A) \right\}.$$

Hence, $G(\mathbb{Z}_2, M_3(A), M_{3\times 1}(A)) \cong G(M_1) \cong G(A)$, where A is considered as left A-module.

For the remainder of this section, we focus on the relationships between the graph-theoretic properties of $\Gamma(G, R, M)$ and $G(M)$ when the grading group is a linearly ordered group. For details on rings and modules graded by linearly ordered group, see [21, Chapter 5].

A linearly ordered group is a group G equipped with a total ordered relation $\leq$ such that for all $\alpha, \beta, \delta \in G$, $\alpha \leq \beta$ implies $\alpha\delta \leq \beta\delta$ and $\delta\alpha \leq \delta\beta$.

Suppose that M is G-graded R-module where G is a linearly ordered group. Then, any $x \in M$ can be written uniquely as $x = x_{\sigma_1} + x_{\sigma_2} + \ldots + x_{\sigma_n}$, with $\sigma_1 < \sigma_2 < \cdots < \sigma_n$. We call x_{σ_n} the homogeneous components of x of highest degree. For each R-submodule N of M, the G-graded R-submodule generated by the homogeneous components of the highest degrees of all elements of N is denoted by $N^{\sim}$; that is, y is one of the generators of $N^{\sim}$ if and only if there exists $x = x_{\sigma_1} + x_{\sigma_2} + \ldots + x_{\sigma_n} \in N$, with $\sigma_1 < \sigma_2 < \cdots < \sigma_n$ and $x_{\sigma_n} = y$. We have the following result from [21, Lemma 5.3.1, Corollary 5.3.3]

Lemma 3.11. *Let M be a G-graded R-module, where G is linearly ordered group and N and K are submodules of M. Then,*
(*i*) *$N = N^{\sim}$ if and only if N is G-graded R-submodule.*
(*ii*) *$N^{\sim} = \{0\}$ if and only if $N = \{0\}$.*
(*iii*) *If $N \subseteq K$, then $N^{\sim} \subseteq K^{\sim}$.*
(*iv*) *If $\operatorname{supp}(M, G)$ is well ordered subset of G and $N \subseteq K$, then $N = K$ if and only if $N^{\sim} = K^{\sim}$*

Theorem 3.12. *Let M be a G-graded R-module, where G is linearly ordered group. If $\operatorname{supp}(M, G)$ is well ordered subset of G, then $\Gamma(G, R, M)$ is connected if and only if $G(M)$ is connected.*

Proof. If $\Gamma(G, R, M)$ is connected, then $G(M)$ is not null graph, and therefore, it is connected. For the converse, assume that $G(M)$ is connected and let N and K be adjacent vertices of $G(M)$. Hence, $N \cap K \neq \{0\}$. Let $J = N \cap K$. Since $N \neq K$, either $J \subsetneq N$ or $J \subsetneq K$. Without loss of generality, assume $J \subsetneq N$. Then, by parts (ii)–(iv) of Lemma 3.11, we have $\{0\} \neq J^{\sim} \subsetneq K^{\sim}$. So $\Gamma(G, R, M)$ is not null, and hence, it is connected. □

Theorem 3.13. *Let M be a G-graded R-module, where G is a linearly ordered group and $\operatorname{supp}(M, G)$ is well ordered subset of G. If M is local or not Noetherian, then $g(\Gamma(G, R, M)) = g(G(M))$.*

Proof. Clearly, if $g(G(M)) = \infty$, then $g(\Gamma(G, R, M)) = \infty$. Assume that $g(G(M)) < \infty$, it follows from [11, Theorem 2.5] that $g(G(M)) = 3$. If M is not Noetherian, then there are three nontrivial R-submodules N_1, N_2, and N_3 such that $N_1 \subsetneq N_2 \subsetneq N_3$. Then, by part (iv) of Lemma 3.11, we obtain that $N_1^{\sim} \subsetneq N_2^{\sim} \subsetneq N_3^{\sim}$. Hence, $N_1^{\sim} \leftrightarrow N_2^{\sim} \leftrightarrow N_3^{\sim} \leftrightarrow N_1^{\sim}$ is a 3-cycle in $\Gamma(G, R, M)$. Now suppose that M is Noetherian and local with maximal R-submodule N. This implies that $K \subseteq N$ for all $K \in S^*(M)$. Since $G(M)$ is not a star graph, there are two distinct R-submodules $K, L \in S^*(M)\backslash\{N\}$ such that $K \cap L \neq \{0\}$. Without the loss of generality, we may assume that $K \cap L \subsetneq K$. So we have $\{0\} \neq K \cap L \subsetneq K \subsetneq N$. Again by part (iv) of Lemma 3.11, we obtain the 3-cycle $(K \cap L)^{\sim} \leftrightarrow K^{\sim} \leftrightarrow N^{\sim} \leftrightarrow (K \cap L)^{\sim}$ in $\Gamma(G, R, M)$. Therefore, $g(\Gamma(G, R, M)) = 3$. □

The length of an R-module M over R, denoted by $\ell(M)$, is the supremum of the lengths of chains of R-submodules of M.

Theorem 3.14. *Let M be a G-graded R-module, where G is a linearly ordered group and $\operatorname{supp}(R, G)$ is well ordered subset of G. If $g(G(M)) \neq g(\Gamma(G, R, M))$, then the following assertions hold*
(*i*) *$g(G(M)) = 3$ and $g(\Gamma(G, R, M)) = \infty$.*
(*ii*) *M is G-graded local but not local.*
(*iii*) *The length of M over R is $\ell(M) = 3$.*
(*iv*) *For every maximal R-submodule K, $K^{\sim} = N$, where N is the unique G-graded maximal R-submodule of M.*
(*v*) *If $\operatorname{rad}(M) = \{0\}$, then $\omega(G(M)) = |\operatorname{Max}(M)|$. Otherwise, $\omega(G(M)) = |\operatorname{Max}(M)| + 1$.* ($\operatorname{rad}(M)$ *is the intersection of all maximal R-submodules of M.*)

Proof.

(i) Follows directly from the fact that $\Gamma(G, R, M)$ is a subgraph of $G(M)$.

(ii) Since $g(\Gamma(G, R, M)) = \infty$, it follows from Theorem 2.11 that M is G-graded local, and since $g(G(M)) \neq g(\Gamma(G, R, M))$, by Theorem 3.13, M is not local.

(iii) Since M is not local, $\ell(M) \geq 3$. Assume there are three R-submodules N_1, N_2, and N_3 such that $N_1 \subsetneq N_2 \subsetneq N_3$. Then, by part (iv) of Lemma 3.11, we obtain that $N_1^{\sim} \subsetneq N_2^{\sim} \subsetneq N_3^{\sim}$; consequently, $g(\Gamma(G, R, M)) = 3$, a contradiction. So $\ell(M) = 3$.

(iv) Let K be a maximal R-submodule of M. Since $G(M)$ is connected, K is not simple. So there is a nontrivial R-submodule $L \subsetneq K$. So $L^{\sim} \subsetneq K^{\sim}$. Also $\Gamma(G, R, M)$ is a star graph, and hence, $K^{\sim} = N$.

(v) Since $G(M)$ is connected, $\mathrm{Max}(M)$ is a clique in $G(M)$. Now from part (iii), we obtain that every R-submodule is either maximal or simple. Moreover, $\mathrm{rad}(M) \notin \mathrm{Max}(M)$ because M is not local. Therefore, if $\mathrm{rad}(M) = \{0\}$, then $\mathrm{Max}(M)$ is clique of maximum size, otherwise $\mathrm{Max}(M) \cup \{\mathrm{rad}(M)\}$ is clique of maximum size. □

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