Research Article

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The stability with general decay rate of hybrid stochastic fractional differential equations driven by Lévy noise with impulsive effects

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Abstract: In this paper, our aims are to study the stability with general decay rate of hybrid stochastic fractional differential equations driven by Lévy noise with impulsive effects. Using Lyapunov function, nonnegative semi-martingale convergence theorem, we obtain the almost sure stability with general decay rate, including the exponential stability and the polynomial stability. Moreover, we give an upper bound of each coefficient at any mode according to the theory of M-matrix. Finally, one example is given to show the effectiveness of the obtained theory.

Keywords: stability, hybrid stochastic fractional differential equations, Lévy noise, general decay, impulsive effects

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1 Introduction

With the wide application of stochastic differential equations driven by Lévy noise in biology, engineering, finance and economy, more and more experts and scholars pay attention to stochastic differential equations [1–3]. The stability has become one of the important topics, such as stochastic stability, stochastic asymptotic stability, moment exponential stability, almost everywhere stability and mean square polynomial stability (see [4–9]). Li and Deng [10] studied the almost sure stability with general decay rate of neutral delay stochastic differential equations with Lévy noise, while Shen et al. [11] studied the stability of solutions of neutral stochastic functional hybrid differential equations with Lévy noise. Shen's conclusion is more specific and universal than Deng's.

As is known to all, the integer order differential equations determine the local characteristics of the function, while the fractional order differential equations describe the overall information of the function in the form of weighting, so it is more flexible and widely used in the model. Abouagwa et al. [12] studied the existence and uniqueness by using Carathéodory approximation under non-Lipschitz conditions. Shen et al. [13] obtained an averaging principle and stability of hybrid stochastic fractional differential equations driven by Lévy noise. Recently, the classical mathematical modelling approach coupled with the stochastic methods were used to develop stochastic dynamic models for financial data (stock price). In order to extend

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this approach to more complex dynamic processes in science and engineering operating under internal structural and external environmental perturbations, Pedjeu and Ladde [14] modified the existing mathematical models by incorporating certain significant attributable parameters or variables with state variables, explicitly. Meanwhile, they obtained the existence and uniqueness of the solution by using the Picard-Lindel successive approximations. This motivates us to initiate to partially characterize intrastructural and external environmental perturbations by a set of linearly independent time-scales. For example, t, $B(t)$, t^α, $\widetilde{N}$, where $B(t)$ is the standard Wiener process, and $\widetilde{N}$ is the Lévy process, $\alpha \in (0, 1]$.

In addition, impulsive stochastic differential systems [15–17] with Markovian switching have been investigated. Zhu [18] has determined the pth moment exponential stability of impulsive stochastic functional differential equations with Markovian switching. Then Kao et al. [19] proved the pth moment exponential stability, almost exponential stability and instability on the basis of predecessors. Tan et al. [20] discussed the stability of hybrid impulsive and switching stochastic systems with time delay.

However, it is worth mentioning that to the best of our knowledge, the stability with general decay rate of hybrid stochastic fractional differential equations driven by Lévy noise with impulsive effects has not been investigated yet and this arouses our interest in research. In order to fill this gap, in this paper, combined with previous work, we consider the following stochastic differential equations driven by Lévy noise with impulsive effects

$$\begin{cases} \mathrm{d}x(t) = u(x(t-), t, r(t))\mathrm{d}t + b(x(t-), t, r(t))\mathrm{d}B(t) + \sigma(x(t-), t, r(t))(\mathrm{d}t)^\alpha \\ \qquad\qquad + \displaystyle\int_{|y|<c} h(x(t-), y, t, r(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}y), \quad t \neq t_k, t \geq 0, \\ \Delta x(t_k) = I_k(x(t_k^-), t_k), \quad k \in \mathbb{N}, \end{cases} \tag{1}$$

where $0 < \alpha \leq 1$, $x(0) = x_0 \in \mathbb{R}^n$ is the initial value satisfying $\mathbb{E}|x_0|^2 < \infty$, the constant c is the maximum allowable jump size and the mappings $u, \sigma : \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{S} \to \mathbb{R}^n$, $b : \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{S} \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S} \to \mathbb{R}^n$ are continuous functions. The fixed impulse time sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$ (as $k \to \infty$), $\Delta x(t_k) = I_k(x(t_k^-), t_k)$ denotes the state jumping at impulsive time instant t_k. Denote by $x(t; 0, x_0)$ the solution to the system which is assumed to be right continuous, i.e., $x(t_k^+) = x(t_k)$.

In this paper, we utilize the local Lipschitz condition and a weaker condition to replace the linear growth condition to obtain a unique global solution for system (1). According to the method of Lyapunov function, we can prove there is a unique global solution. Then we present a kind of λ-type function which will be introduced in Section 2. By means of nonnegative semi-martingale convergence theorem and Lyapunov function, we derive a kind of almost sure λ-type stability, including almost sure exponential stability, almost sure polynomial stability and almost sure logarithmic stability. The main features of this paper are as follows:

(i) The presented hybrid stochastic fractional differential equations driven by Lévy noise with impulsive effects have not been considered before.
(ii) A more general almost sure stability (including almost sure exponential stability, almost sure polynomial stability and almost sure logarithmic stability) problem has been investigated under much weaker conditions.
(iii) The upper bound of each coefficient at any mode is obtained.

The article is organized as follows. In Section 2, we present several definitions and preliminaries. In Section 3, the conditions for the existence and uniqueness of the global solution and the sufficient conditions for λ-type stability are established, respectively. In Section 4, we prove the λ-type stability about the upper bound of each coefficient at any mode by using the theory of the M-matrix. Finally, an example is given to illustrate the obtained theory.

2 Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $|x|$ denotes the Euclidean norm of a vector x. $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$. Diag$(\zeta_1, \dots, \zeta_N)$ denotes a diagonal matrix with diagonal entries $\zeta_1, \dots, \zeta_N$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). $B(t) = (B_1(t), B_2(t), \dots, B_m(t))^T$ be an m-dimensional $\mathcal{F}_t$-adapted Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$, and $N(t, z)$ be a $\mathcal{F}_t$-adapted Poisson random measure on $[0, +\infty) \times \mathbb{R}$ with a σ-finite intensity measure $\nu(\mathrm{d}z)$, the compensator martingale measure $\widetilde{N}(t, z)$ satisfies $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$. Let $r(t)$, $t \geq 0$ be a right-continuous Markov chain defined on the probability space taking values in a finite state $\mathbb{S} = \{1, 2, \dots N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ij}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$, and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. And we assume that the Markov chain $r(t)$ is $\mathcal{F}_t$-adapted but independent of the Brownian motion $B(t)$.

Next, we give some definitions about fractional calculus and λ-type function, which will be used in this paper.

Definition 2.1. [21] (Riemann-Liouville fractional integrals, Samko et al., 1993): For any $\alpha \in (0, 1)$ and function $f \in L^1[[a, b]; \mathbb{R}^n]$, the left-sided and right-sided Riemann-Liouville fractional integrals of order α are defined for almost all $a < t < b$ by

$$(I^\alpha_{a+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad t > a$$

and

$$(I^\alpha_{b-} f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad t < b,$$

where $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1}e^{-s}\mathrm{d}s$ is the Gamma function and $L^1[a, b]$ is the space of integrable functions in a finite interval $[a, b]$ of $\mathbb{R}$.

Definition 2.2. [21] (Riemann-Liouville fractional derivatives, Samko et al., 1993): For any $\alpha \in (0, 1)$ and well-defined absolutely continuous function f on an interval $[a, b]$, the left-sided and right-sided Riemann-Liouville fractional derivatives are defined, respectively, by

$$(D^\alpha_{a+} f)(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(a)}{(t-a)^\alpha} + \int_a^t (t-s)^{-\alpha} f'(s)\mathrm{d}s \right]$$

and

$$(D^\alpha_{b-} f)(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(b)}{(b-t)^\alpha} - \int_t^b (s-t)^{-\alpha} f'(s)\mathrm{d}s \right].$$

Definition 2.3. [22] (Jumarie, 2005): Let $\sigma(t)$ be a continuous function, then its integration with respect to $(\mathrm{d}t)^\alpha$, $0 < \alpha \leq 1$, is defined by

$$\int_0^t \sigma(s)(\mathrm{d}s)^\alpha = \alpha \int_0^t (t-s)^{\alpha-1}\sigma(s)\mathrm{d}s.$$

Definition 2.4. The function $\lambda : \mathbb{R} \to (0, \infty)$ is said to be λ-type function if the function satisfies the following three conditions:
(1) It is continuous and nondecreasing in $\mathbb{R}$ and differentiable in $\mathbb{R}^+$,
(2) $\lambda(0) = 1, \lambda(\infty) = \infty$ and $r = \sup_{t\geq 0}\left[\frac{\lambda'(t)}{\lambda(t)}\right] < \infty$,
(3) For any $s, t \geq 0, \lambda(t) \leq \lambda(s)\lambda(t-s)$.

Remark 2.5. It is obvious that the functions $\lambda(t) = e^t, \lambda(t) = (1+t^+)$ and $\log(1+t^+)$ are λ-type functions since they satisfy the aforementioned three conditions. Next, we give the definition of the almost sure stability with general decay rate based on Definition 2.4.

Definition 2.6. Let the function $\lambda(t) \in C(\mathbb{R}^+; \mathbb{R}^+)$ be a λ-type function. Then for any initial $x_0 \in \mathbb{R}^n$, the trivial solution is said to be almost surely stable with decay $\lambda(t)$ of order γ if

$$\limsup_{t\to\infty}\frac{\log|x(t, x_0)|}{\log\lambda(t)} \leq -\gamma, \quad \text{a.s.}$$

Remark 2.7. It is obvious that this almost sure λ-type stability implies the almost sure exponential stability, almost sure polynomial stability and almost sure logarithmic stability when $\lambda(t)$ is replaced by e^t, $1+t^+$, $\log(1+t^+)$, respectively. Because we have a wide choice for λ-type functions, thus our results will be more general than some classical results.

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{S} \to \mathbb{R}^+)$ denote the family of all functions $V(x, t, i)$ on $\mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{S}$, which are continuously twice differentiable in x and once in t. Define three functions $L_1V, L_2V, L_3V : \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{S} \to \mathbb{R}$ by

$$\begin{aligned}
L_1V(x,t,i) &= V_t(x,t,i) + V_x(x,t,i)u(x,t,i) + \sum_{j=1}^{N}\gamma_{ij}V(x,t,j) + \frac{1}{2}\text{trace}[b^T(x,t,i)V_{xx}(x,t,i)b(x,t,i)],\\
L_2V(x,t,i) &= \int_{|y|<c}[V(x+h(x,y,t,i),t,i) - V(x,t,i) - V_x(x,t,i)h(x,y,t,i)]\nu(\mathrm{d}y),\\
L_3V(x,t,i) &= V_x(x,t,i)\sigma(x,t,i),
\end{aligned}$$

where $V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t}$, $V_x(x,t,i) = \left(\frac{\partial V(x,t,i)}{\partial x_1}, \ldots, \frac{\partial V(x,t,i)}{\partial x_n}\right)$, $V_{xx}(x,t,i) = \left(\frac{\partial V^2(x,t,i)}{\partial x_k \partial x_l}\right)_{n\times n}$.
By the generalized Itô formula, for $t \in [t_{j-1}, t_j)$

$$\begin{aligned}
V(x(t),t,r(t)) = {} & V(x(t_{j-1}),t_{j-1},r(t_{j-1})) + \int_{t_{j-1}}^{t}L_1V(x(s),s,r(s))\mathrm{d}s + \int_{t_{j-1}}^{t}L_2V(x(s),s,r(s))\mathrm{d}s\\
& + \int_{t_{j-1}}^{t}\alpha(t-s)^{\alpha-1}L_3V(x(s),s,r(s))\mathrm{d}s + G(t),
\end{aligned}$$

where

$$\begin{aligned}
G(t) = {} & \int_{t_{j-1}}^{t}V_x(x(s),s,r(s))b(x(s-),s,r(s))\mathrm{d}B(s) + \int_{t_{j-1}}^{t}\int_{|y|<c}[V(x(s)+h(x(s-),y,s,r(s)),s,r(s))\\
& - V(x(s),s,r(s))]\tilde{N}(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

In fact, if a stochastic process is a martingale, then it is a local martingale. Hence, we can easily know that $\{G(t)\}_{t\geq 0}$ is a local martingale.

3 Main result

Before we state our main results in this section, the following hypotheses are imposed.

Assumption 3.1. (Local Lipschitz condition). For arbitrary $x_1, x_2 \in \mathbb{R}^n$, and $|x_1| \vee |x_2| \le n$, there is a positive constant L_n such that

$$|u(x_1,t,i) - u(x_2,t,i)| \vee |b(x_1,t,i) - b(x_2,t,i)| \vee |\sigma(x_1,t,i) - \sigma(x_2,t,i)| \vee \int\limits_{|y|<c} |h(x_1,y,t,i) - h(x_2,y,t,i)|\nu(\mathrm{d}y) \le L_n(|x_1 - x_2|^2).$$

Assumption 3.2. For any $(t,i) \in \mathbb{R} \times \mathbb{S}$, $u(0,t,i) = b(0,t,i) = \sigma(0,t,i) = h(0,y,t,i) = 0$.

For the stability analysis, Assumption 3.2 implies that $x(t) = 0$ is the trivial solution.

Assumption 3.3. There are several nonnegative number C and p_u, real number K_u and β_u $(1 \le u \le U$, for positive integer $U)$, a Lyapunov function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{S}; \mathbb{R}^+)$, such that

(i) $\lim_{|x|\to\infty} \inf_{0\le t<\infty} V(x(t),t,i) = \infty$.

(ii) $L_1V(x,s,i) + L_2V(x,s,i) + \alpha(t-s)^{\alpha-1}L_3V(x,s,i) \le C + \sum_{u=1}^{U} K_u|x|^{p_u}$.

(iii) $V(x(t_k^-) + I_k(x(t_k^-),t_k),t_k,r(t_k)) \le d(x(t_k^-),t_k) + V(x(t_k^-),t_k,r(t_k))$.

(iv) $\sum_{j=n}^{k} d(x(t_{j-1}^-),t_{j-1}) \le \sum_{u=1}^{U}\beta_u\int_{t_{j-1}}^{t_k}|x|^{p_u}\mathrm{d}s$.

For any $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $0 \le s < t$, $i \in \mathbb{S}$.

According to the above assumptions, let us state the following existence and uniqueness theorem:

Theorem 3.4. *Assume that Assumptions* 3.1–3.3 *hold, then for any initial data* $x_0 \in \mathbb{R}^n$, *there is a unique global solution* $x(t;0,x(0))$ *on* $t > 0$ *to system* (1).

Proof. Applying the standing truncation technique, Assumptions 3.1 and 3.2 admit a unique maximal local solution to system (1)). Let $x(t)(t \in [0,\varrho_\infty))$ be the maximal local solution to system (1) and ϱ_∞ be the explosion time. And let $a_0 \in \mathbb{R}^+$ be sufficiently large for $|x_0| \le a_0$. For any integer $a \ge a_0$, define the stopping time

$$\tau_a = \inf\{t \in [0,\varrho_\infty);\ |x(t)| \ge a\},$$

where $\inf\phi = \infty$. Obviously, the sequence τ_a is increasing. So we have a limit $\tau_\infty = \lim_{a\to\infty}\tau_a$, whence $\tau_\infty \le \varrho_\infty$. If we can show that $\tau_\infty = \infty$ a.s., then we have $\varrho_\infty = \infty$ a.s. Therefore, we only need to devote to prove $\tau_\infty = \infty$ a.s., which is equivalent to proving that $P(\tau_a \le t) \to 0$ as $a \to \infty$ for any $t > 0$. In fact, by the generalized Itô formula, for $t \in [t_{j-1}, t_j)$, we have

$$\begin{aligned}
&\mathbb{E}\Big(I_{\tau_a\le t}V(x(\tau_a),\tau_a,r(\tau_a))\Big)\\
&\le \mathbb{E}V(x(t\wedge\tau_a),t\wedge\tau_a,r(t\wedge\tau_a))\\
&= \mathbb{E}V(x(t_{j-1}),t_{j-1},r(t_{j-1})) + \mathbb{E}\int\limits_{t_{j-1}}^{t\wedge\tau_a} L_1V(x(s),s,r(s))\mathrm{d}s + \mathbb{E}\int\limits_{t_{j-1}}^{t\wedge\tau_a} L_2V(x(s),s,r(s))\mathrm{d}s\\
&\quad + \mathbb{E}\int\limits_{t_{j-1}}^{t\wedge\tau_a}\alpha(t-s)^{\alpha-1}L_3V(x(s),s,r(s))\mathrm{d}s.
\end{aligned} \tag{2}$$

By condition (iii), we have

$$\begin{aligned}
V(x(t_{j-1}),t_{j-1},r(t_{j-1})) &= V(x(t_{j-1}^-) + I_{j-1}(x(t_{j-1}^-),t_k),t_{j-1}^-,r(t_{j-1}))\\
&\le d(x(t_{j-1}^-),t_{j-1}) + V(x(t_{j-1}^-),t_{j-1}^-,r(t_{j-1})).
\end{aligned}$$

Hence, for all $t \geq 0$,

$$\begin{aligned}\mathbb{E}\left(I_{\tau_a \leq t} V(x(\tau_a), \tau_a, r(\tau_a))\right) &\leq \mathbb{E} V(x(0), 0, r(0)) + \mathbb{E} \int_0^{t \wedge \tau_a} L_1 V(x(s), s, r(s)) \mathrm{d}s + \mathbb{E} \int_0^{t \wedge \tau_a} L_2 V(x(s), s, r(s)) \mathrm{d}s \\ &\quad + \mathbb{E} \int_0^{t \wedge \tau_a} \alpha (t-s)^{\alpha-1} L_3 V(x(s), s, r(s)) \mathrm{d}s + \sum_{j: 0 < t_j \leq t} \mathbb{E} d(x(t_{j-1}^-), t_{j-1}) \\ &\leq \mathbb{E} V(x(0), 0, r(0)) + Ct + \sum_{u=1}^{U} K_u \mathbb{E} \int_0^{t \wedge \tau_a} |x(s)|^{p_u} \mathrm{d}s + \sum_{j: 0 < t_j \leq t} \mathbb{E} d(x(t_{j-1}^-), t_{j-1}).\end{aligned} \tag{3}$$

For almost all $\omega \in \Omega$, there is an integer $m_0 = m_0(\omega)$, for any $m \geq m_0$ and $0 \leq t \wedge \tau_a < m$, define

$$t_{k_m} = \max\{t_k : t_k \leq t \wedge \tau_a\}.$$

Combining with condition (iv),

$$\sum_{j: 0 < t_j \leq t} d(x(t_{j-1}^-), t_{j-1}) = \sum_{j=1}^{t_{k_m}} d(x(t_{j-1}^-), t_{j-1}) = \sum_{u=1}^{U} \beta_u \int_0^{t_{k_m}} |x(s)|^{p_u} \mathrm{d}s. \tag{4}$$

Substituting (4) to (3), for $0 < t \wedge \tau_a < m$,

$$\begin{aligned}\mathbb{E}\left(I_{\tau_a \leq t} V(x(\tau_a), \tau_a, r(\tau_a))\right) &\leq \mathbb{E} V(x(0), 0, r(0)) + Ct + \sum_{u=1}^{U} K_u \mathbb{E} \int_0^{t \wedge \tau_a} |x(s)|^{p_u} \mathrm{d}s + \sum_{u=1}^{U} \beta_u \mathbb{E} \int_0^{t_{k_m}} |x(s)|^{p_u} \mathrm{d}s \\ &\leq \mathbb{E} V(x(0), 0, r(0)) + Ct + \sum_{u=1}^{U} (K_u + \beta_u) \mathbb{E} \int_0^{t} |x(s)|^{p_u} \mathrm{d}s.\end{aligned} \tag{5}$$

By $\lim_{|x| \to \infty} \inf_{0 \leq t < \infty} V(x(t), t, r(t)) = \infty$, let $\mu_a = \inf_{|x| \geq a, 0 \leq t < \infty} V(x(t), t, r(t))$, for $a \geq a_0$. Therefore, we have

$$P(\tau_a \leq t) \mu_a \leq \mathbb{E}\left(I_{\tau_a \leq t} V(x(\tau_a), \tau_a, r(\tau_a))\right).$$

Using the idea of Theorem 3.1 in [8], letting $a \to \infty$, by Fatou's lemma, we can derive

$$\begin{aligned}0 \leq P(\tau_\infty \leq t) &\leq \lim_{a \to \infty} P(\tau_a \leq t) \\ &= \lim_{a \to \infty} \frac{\mathbb{E} V(x(0), 0, r(0)) + Ct + \sum_{u=1}^{U} (K_u + \beta_u) \mathbb{E} \int_0^t |x(s)|^{p_u} \mathrm{d}s}{\mu_a} \\ &= 0.\end{aligned} \tag{6}$$

This implies that there exists a unique global solution $x(t; 0, x(0))$ for system (1). □

Next, in order to obtain sufficient conditions of the almost sure stability with general decay rate, we need the following lemmas.

Lemma 3.5. [23] *Let $\{M_t\}_{t \geq 0}$ be a local martingale and $\{N_t\}_{t \geq 0}$ be a locally bounded predictable process, then the stochastic integral $\int_0^t N_s \mathrm{d}M_s$ is also a local martingale.*

Lemma 3.6. (Nonnegative semi-martingale convergence theorem) [10] *Assume $\{A_t\}$ and $\{U_t\}$ are two continuous predictable increasing processes vanishing at $t = 0$ a.s. and $\{M_t\}$ is a real-valued continuous local martingale with $M_0 = 0$ a.s. Let $X(t)$ be a nonnegative adapted process and ξ be a nonnegative $\mathcal{F}_0$-measurable random variable satisfying*

$$X_t \leq \xi + A_t - U_t + M_t, \quad t \geq 0.$$

If $A_\infty := \lim_{t\to\infty} A_t < \infty$, *a.s., then, we have*

$$\limsup_{t\to\infty} X_t < \infty, \quad a.s.$$

With these above assumptions and lemmas, we can now state our results about the almost sure stability with general decay rate.

Theorem 3.7. *Let Assumptions* 3.1 *and* 3.2 *hold. If there are positive numbers* $c_1, c_2, \alpha_1, \alpha_2, q > 2$, *such that the functions* $V(x,t,i)$ *and* $L_jV(x,t,i)(j = 1,2,3)$ *satisfy*

(*i*) $c_1|x|^2 \le V(x,t,i) \le c_2|x|^2$.

(*ii*) $L_1V(x,s,i) + L_2V(x,s,i) + \alpha(t-s)^{\alpha-1}L_3V(x,s,i) \le -\alpha_1|x|^2 - \alpha_2|x|^q$.

(*iii*) $V(x(t_k^-) + I_k(x(t_k^-), t_k), t_k, r(t_k)) \le d(x(t_k^-), t_k) + V(x(t_k^-), t_k, r(t_k))$.

(*iv*) $\Sigma_{j=n}^k \lambda^\varepsilon(t_{j-1}) d(x(t_{j-1}^-), t_{j-1}) \le \alpha_3 \int_{t_{j-1}}^{t_k} \lambda^\varepsilon(s)|x|^2 ds + \alpha_4 \int_{t_{j-1}}^{t_k} \lambda^\varepsilon(s)|x|^q ds$.

If there exists a small enough $\varepsilon > 0$ *such that* $\alpha_1 - c_2\varepsilon\alpha - \alpha_3 > 0$, $\alpha_2 - \alpha_4 > 0$.

Therefore, for any initial data x_0, *the inequality*

$$\limsup_{t\to\infty} \frac{\log|x(t,x_0)|}{\log\lambda(t)} < -\frac{\varepsilon}{2}$$

holds, that is, the trivial solution of system (1) *is almost surely stable with decay* $\lambda(t)$ *of order* $\frac{\varepsilon}{2}$.

Proof. Note that conditions (i)–(iv) are stronger than Assumption 3.3, so there is a unique global solution for system (1). Let $\lambda(t)$ be a λ-type function, and applying the generalized Itô formula to $\lambda^\varepsilon(t)V(x(t),t,r(t))$, for $t \in [t_{j-1}, t_j)$

$$\begin{aligned}\lambda^\varepsilon(t)V(x(t),t,r(t)) = \lambda^\varepsilon(t_{j-1})V(x(t_{j-1}),t_{j-1},r(t_{j-1})) &+ \int_{t_{j-1}}^t \varepsilon\frac{\lambda'(s)}{\lambda(s)}\lambda^\varepsilon(s)V(x(s),s,r(s))ds \\ &+ \int_{t_{j-1}}^t \lambda^\varepsilon(s)L_1V(x(s),s,r(s))ds + \int_{t_{j-1}}^t \lambda^\varepsilon(s)L_2V(x(s),s,r(s))ds \\ &+ \int_{t_{j-1}}^t \lambda^\varepsilon(s)\alpha(t-s)^{\alpha-1}L_3V(x(s),s,r(s))ds + M_{t_{j-1}}^t,\end{aligned} \tag{7}$$

where $M_{t_{j-1}}^t = \int_{t_{j-1}}^t \lambda^\varepsilon(s)dG(s)$ is a real-valued continuous local martingale with $M_0 = 0$ by Lemma 3.5. By condition (iii), we have

$$\begin{aligned}\lambda^\varepsilon(t_{j-1})V(x(t_{j-1}),t_{j-1},r(t_{j-1})) &= \lambda^\varepsilon(t_{j-1})V(x(t_{j-1}^-) + I_{j-1}(x(t_{j-1}^-),t_k),t_{j-1}^-,r(t_{j-1})) \\ &\le \lambda^\varepsilon(t_{j-1})[d(x(t_{j-1}^-),t_{j-1}) + V(x(t_{j-1}^-),t_{j-1}^-,r(t_{j-1}))] \\ &= \lambda^\varepsilon(t_{j-1})V(x(t_{j-1}^-),t_{j-1}^-,r(t_{j-1})) + \lambda^\varepsilon(t_{j-1})d(x(t_{j-1}^-),t_{j-1}).\end{aligned} \tag{8}$$

Hence, for all $t \ge 0$, we have

$$\begin{aligned}\lambda^\varepsilon(t)V(x(t),t,r(t)) = V(x(0),0,r(0)) &+ \int_0^t \varepsilon\frac{\lambda'(s)}{\lambda(s)}\lambda^\varepsilon(s)V(x(s),s,r(s))ds + \int_0^t \lambda^\varepsilon(s)L_1V(x(s),s,r(s))ds \\ &+ \int_0^t \lambda^\varepsilon(s)L_2V(x(s),s,r(s))ds \int_0^t \lambda^\varepsilon(s)\alpha(t-s)^{\alpha-1}L_3V(x(s),s,r(s))ds \\ &+ \sum_{j:0<t_j\le t} \lambda^\varepsilon(t_{j-1})d(x(t_{j-1}^-),t_{j-1}) + M_0^t.\end{aligned} \tag{9}$$

For almost all $\omega \in \Omega$, there is an integer $m_0 = m_0(\omega)$, for any $m \geq m_0$ and $0 \leq t < m$, define

$$t_{k_m} = \max\{t_k : t_k \leq t\},$$

while $0 < t < m$,

$$\begin{aligned}\sum_{j:0<t_j\leq t} \lambda^{\varepsilon}(t_j) d(x(t_{j-1}^-), t_{j-1}) &= \sum_{j=1}^{t_{k_m}} \lambda^{\varepsilon}(t_j) d(x(t_{j-1}^-), t_{j-1}) \\ &\leq \alpha_3 \int_0^{t_{k_m}} \lambda^{\varepsilon}(s)|x|^2 \mathrm{d}s + \alpha_4 \int_0^{t_{k_m}} \lambda^{\varepsilon}(s)|x|^q \mathrm{d}s. \\ &\leq \alpha_3 \int_0^{t} \lambda^{\varepsilon}(s)|x|^2 \mathrm{d}s + \alpha_4 \int_0^{t} \lambda^{\varepsilon}(s)|x|^q \mathrm{d}s.\end{aligned} \tag{10}$$

By condition (ii) and the definition of λ-type function, we have

$$\begin{aligned}\lambda^{\varepsilon}(t)V(x(t), t, r(t)) \leq V(x(0), 0, r(0)) &+ \varepsilon\alpha \int_0^t \lambda^{\varepsilon}(s)V(x(s), s, r(s))\mathrm{d}s - \alpha_1 \int_0^t \lambda^{\varepsilon}(s)|x(s)|^2 \mathrm{d}s \\ &- \alpha_2 \int_0^t \lambda^{\varepsilon}(s)|x(s)|^q \mathrm{d}s + \alpha_3 \int_0^t \lambda^{\varepsilon}(s)|x(s)|^2 \mathrm{d}s + \alpha_4 \int_0^t \lambda^{\varepsilon}(s)|x(s)|^q \mathrm{d}s + M_0^t.\end{aligned} \tag{11}$$

Moreover,

$$\begin{aligned}\lambda^{\varepsilon}(t)V(x(t), t, r(t)) \leq V(x(0), 0, r(0)) &+ c_2\varepsilon\alpha \int_0^t \lambda^{\varepsilon}(s)|x(s)|^2 \mathrm{d}s - \alpha_1 \int_0^t \lambda^{\varepsilon}(s)|x(s)|^2 \mathrm{d}s \\ &- \alpha_2 \int_0^t \lambda^{\varepsilon}(s)|x(s)|^q \mathrm{d}s + \alpha_3 \int_0^t \lambda^{\varepsilon}(s)|x(s)|^2 \mathrm{d}s + \alpha_4 \int_0^t \lambda^{\varepsilon}(s)|x(s)|^q \mathrm{d}s + M_0^t \\ &\leq V(x(0), 0, r(0)) - (\alpha_1 - c_2\varepsilon\alpha - \alpha_3) \int_0^t \lambda^{\varepsilon}(s)|x(s)|^2 \mathrm{d}s - (\alpha_2 - \alpha_4) \int_0^t \lambda^{\varepsilon}(s)|x(s)|^q \mathrm{d}s + M_0^t.\end{aligned} \tag{12}$$

Consequently, by inequality $\alpha_1 - c_2\varepsilon\alpha - \alpha_3 > 0$, $\alpha_2 - \alpha_4 > 0$ and (12) imply

$$\lambda^{\varepsilon}(t)V(x(t), t, r(t)) \leq V(x(0), 0, r(0)) + M_0^t.$$

Applying Lemma 3.6, we have

$$\limsup_{t\to\infty} \lambda^{\varepsilon}(t)V(x(t), t, r(t)) < \infty, \quad \text{a.s.}$$

Thus, there exists a positive constant H such that for any $t > 0$,

$$c_1\lambda^{\varepsilon}(t)|x(t)|^2 \leq \lambda^{\varepsilon}(t)V(x(t), t, r(t)) \leq H.$$

Moreover,

$$\lambda^{\varepsilon}(t)|x(t)|^2 \leq \frac{H}{c_1} < \infty.$$

Hence,

$$\limsup_{t\to\infty} \frac{\log|x(t, x_0)|}{\log\lambda(t)} < -\frac{\varepsilon}{2}. \qquad \square$$

4 Almost surely stable with decay $\lambda(t)$

We find that conditions (i) and (ii) of Theorem 3.7 are somewhat inconvenient in applications since they are not related to the coefficient u, b, σ, h explicitly. In this section, we will give the visualized conditions for the coefficient u, b, σ, h to study the λ-type stability.

We first recall some definitions and preliminaries.

Definition 4.1. A square matrix $A = \{a_{ij}\}_{N\times N}$ is called a Z-matrix if its off-diagonal entries are less than or equal to zero, namely $a_{ij} \le 0$ for $i \ne j$.

Definition 4.2. ((M-matrix) [24]) Let A be a $N \times N$ real Z-matrix. And the matrix A is also a nonsingular M-matrix if it can be expressed in the form $A = sI - B$ while all the elements of $B = (b_{ij})$ are nonnegative and $s \ge \rho(B)$, where I is an identity matrix and $\rho(B)$ the spectral radius of B.

Noting that if $A = \{a_{ij}\}_{N\times N}$ is an M-matrix, then it has positive diagonal entries and nonpositive off-diagonal entries, that is, $a_{ii} \ge 0$ while $a_{ij} \ge 0$, $i \ne j$.

Lemma 4.3. ([24]) *If $A = \{a_{ij}\}_{N\times N}$ is a Z-matrix, then the following statements are equivalent*:
(1) *A is a nonsingular M-matrix.*
(2) *Every real eigenvalue of A is positive.*
(3) *All of the principle minors of A are positive.*
(4) *A^{-1} exist and its elements are all nonnegative.*

Assumption 4.4. For each $i \in \mathbb{S}$, $x \in \mathbb{R}^n$ and $0 \le s < t$, there exists a positive integer $q > 2$, suppose the following conditions hold.
(1) There exist constants α_{i1}, α_{i2} such that

$$x^T(u(x,s,i) + \int_{|y|<c} h(x,y,s,i)\nu(\mathrm{d}y)) \le -\alpha_{i1}|x|^2 + \alpha_{i2}|x|^q.$$

(2) There exist constants β_{i1}, β_{i2} such that

$$x^T(\sigma(x,s,i)\alpha(t-s)^{\alpha-1} - \int_{|y|<c} h(x,y,s,i)\nu(\mathrm{d}y)) \le \beta_{i1}|x|^2 - \beta_{i2}|x|^q.$$

(3) There exist constants η_{i1}, η_{i2} such that

$$|b(x,s,i)|^2 \le \eta_{i1}|x|^2 + \eta_{i2}|x|^q.$$

(4) There exist constants ρ_{i1}, ρ_{i2} such that

$$\int_{|y|<c} |h(x,y,s,i)|^2\nu(\mathrm{d}y) \le \rho_{i1}|x|^2 + \rho_{i2}|x|^q.$$

Remark 4.5. Note that $q > 2$ in Assumption 4.4 which means that we can allow the coefficients u, b, σ, h of system (1) to be high-order nonlinear.

Assumption 4.6. According to the definition of M-matrix, let

$$\mathcal{A} := -\mathrm{diag}(-2\alpha_{11} + 2\beta_{11} + \eta_{11} + \rho_{11}, \dots, -2\alpha_{N1} + 2\beta_{N1} + \eta_{N1} + \rho_{N1}) - \Gamma$$

be a nonsingular M-matrix, where $\Gamma = (\gamma_{ij})_{N\times N}$ is given in Section 2. Moreover, from the fourth equivalent condition in Lemma 4.3, we obtain

$$(\theta_1, \theta_2, \dots, \theta_N)^T := \mathcal{A}^{-1}\vec{1} > 0,$$

where $\vec{1} = (1, 1, \dots, 1)^T$. That is, $\theta_i > 0$ for all $i \in \mathbb{S}$.

Theorem 4.7. *Let Assumptions* 3.1–3.2, 4.4 *and* 4.6 *hold. Those constants satisfy the following inequalities*

$$0 < \underline{a} < \bar{a},$$

where

$$\underline{a} = \min_{i\in\mathbb{S}}[\theta_i(-2\alpha_{i2} + 2\beta_{i2} - \eta_{i2} - \rho_{i2})],$$
$$\bar{a} = \max_{i\in\mathbb{S}}[\theta_i(-2\alpha_{i2} + 2\beta_{i2} - \eta_{i2} - \rho_{i2})].$$

Then for any given initial data x_0, there is a unique global solution $x(t; x_0)$ and the trivial solution is almost surely stable with decay $\lambda(t)$.

Proof. Let function $V : \mathbb{R}^n \times \mathbb{R}^+ \times S \to \mathbb{R}^+$ by $V(x, s, i) = \theta_i|x|^2$, choose two positive numbers $c_1 = \min_{i\in\mathbb{S}}\{\theta_i\}$, $c_2 = \max_{i\in\mathbb{S}}\{\theta_i\}$, such that

$$c_1|x|^2 \leq V(x, s, i) \leq c_2|x|^2.$$

By the definition of operator L_jV, $j = 1, 2, 3$ in Section 2, we obtain

$$\begin{aligned}
&\sum_{j=1}^{2} L_iV(x(s), s, i) + \alpha(t - s)^{\alpha-1}L_3V(x, s, i)\\
&\quad= 2\theta_i x^T u(x, s, i) + \theta_i|b(x, s, i)|^2 + \sum_{j=1}^{N}\gamma_{ij}\theta_j|x|^2 + 2\theta_i x^T\sigma(x, s, i)\alpha(t - s)^{\alpha-1}\\
&\qquad + \theta_i \int\limits_{|y|<c} [|x + h(x, y, s, i)|^2 - |x|^2]\nu(\mathrm{d}y) - \int\limits_{|y|<c} 2\theta_i x^T h(x, y, s, i)\nu(\mathrm{d}y)\\
&\quad\leq 2\theta_i x^T[u(x, s, i) + \int\limits_{|y|<c} h(x, y, s, i)\nu(\mathrm{d}y)] + \theta_i|b(x, s, i)|^2 + \sum_{j=1}^{N}\gamma_{ij}\theta_j|x|^2\\
&\qquad + \theta_i \int\limits_{|y|<c} |h(x, y, s, i)|^2\nu(\mathrm{d}y) + 2\theta_i x^T[\sigma(x, s, i)\alpha(t - s)^{\alpha-1} - \int\limits_{|y|<c} h(x, y, s, i)\nu(\mathrm{d}y)].
\end{aligned}$$

By Assumption 4.4, we have

$$\begin{aligned}
&\sum_{j=1}^{2} L_jV(x(s), s, i) + \alpha(t - s)^{\alpha-1}L_3V(x, s, i)\\
&\quad\leq 2\theta_i(-\alpha_{i1}|x|^2 + \alpha_{i2}|x|^q) + 2\theta_i(\beta_{i1}|x|^2 - \beta_{i2}|x|^q) + \theta_i(\eta_{i1}|x|^2 + \eta_{i2}|x|^q) + \theta_i(\rho_{i1}|x|^2 + \rho_{i2}|x|^q) + \sum_{j=1}^{N}\gamma_{ij}\theta_j|x|^2\\
&\quad= (-2\theta_i\alpha_{i1} + 2\theta_i\beta_{i1} + \theta_i\eta_{i1} + \theta_i\rho_{i1})|x|^2 + \sum_{j=1}^{N}\gamma_{ij}\theta_j|x|^2 - (-2\theta_i\alpha_{i2} + 2\theta_i\beta_{i2} - \theta_i\eta_{i2} - \theta_i\rho_{i2})|x|^q\\
&\quad= [\theta_i(-2\alpha_{i1} + 2\beta_{i1} + \eta_{i1} + \rho_{i1}) + \sum_{j=1}^{N}\gamma_{ij}\theta_j]|x|^2 - \theta_i(-2\alpha_{i2} + 2\beta_{i2} - \eta_{i2} - \rho_{i2})|x|^q.
\end{aligned} \tag{13}$$

By Assumption 4.6, we have

$$\sum_{j=1}^{2} L_jV(x(s), s, i) + \alpha(t - s)^{\alpha-1}L_3V(x, s, i) \leq -|x|^2 - \theta_i(-2\alpha_{i2} + 2\beta_{i2} - \eta_{i2} - \rho_{i2})|x|^q \leq -|x|^2 - \underline{a}|x|^q. \tag{14}$$

By Assumptions 3.1–3.2, conditions (iii)–(iv) of Assumption 3.3, inequality (14) and Theorem 3.4, there is a unique global solution for any initial data. By inequality (14) and Theorem 3.7, the trivial solution is almost surely λ-type stable. □

5 Example

We consider two scalar stochastic fractional hybrid differential equations driven by Lévy noise with impulsive effects, and 2-state Markovian switching such that our results have been illustrated simply.

Example 5.1.

$$\begin{cases} \mathrm{d}x(t) = u(x(t-), t, r(t))\mathrm{d}t + b(x(t-), t, r(t))\mathrm{d}B(t) + \sigma(x(t-), t, r(t))(\mathrm{d}t)^{\alpha} \\ \qquad + \int\limits_{|y|<c} h(x(t-), y, t, r(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}y), \quad t \neq t_k, t \geq 0, \\ \Delta x(t_k) = I_k(x(t_k^-), t_k), \quad k \in \mathbb{N}, \end{cases} \tag{15}$$

where $\alpha = 0.5$, $c = 1$, and Lévy measure ν satisfies $\nu(\mathrm{d}y) = |y|\mathrm{d}y$, $r(t)$ is a Markov chain in the state space $\mathbb{S} = 1, 2$ with the generator Γ

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Let

$$\begin{aligned} &u(x, t, 1) = -6x - 3x^2, && u(x, t, 2) = -5x - 1.5x^2, \\ &b(x, t, 1) = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}x^{\frac{3}{2}}, && b(x, t, 2) = 0.5x + 0.5x^{\frac{3}{2}}, \\ &\sigma(x, t, 1) = -2x, && \sigma(x, t, 2) = -2x^3, \\ &h(x, y, t, 1) = \frac{\sqrt{3}}{2}x^{\frac{3}{2}}y - 2x, && h(x, y, t, 2) = -x. \end{aligned}$$

By computing, we have

$$\begin{aligned} &\alpha_{11} = 8, && \alpha_{12} = -3, && \alpha_{21} = 6, && \alpha_{22} = -1.5, \\ &\beta_{11} = 2, && \beta_{12} = 0, && \beta_{21} = 1, && \beta_{22} = 0, \\ &\eta_{11} = 1, && \eta_{12} = 1, && \eta_{21} = 0.5, && \eta_{22} = 0.5, \\ &\rho_{11} = 4, && \rho_{12} = 0.375, && \rho_{21} = 1, && \rho_{22} = 0 \end{aligned}$$

and

$$\mathcal{A} = \begin{pmatrix} 8 & -1 \\ -2 & 10.5 \end{pmatrix}, \quad \mathcal{A}^{-1} = \frac{1}{82}\begin{pmatrix} 10.5 & 1 \\ 2 & 8 \end{pmatrix}$$

$$\theta_1 = 0.1402, \qquad \theta_2 = 0.1220.$$

Then we take Lyapunov functions

$$V(x, t, 1) = 0.1402|x(t)|^2, \quad V(x, t, 2) = 0.1220|x(t)|^2,$$

by Theorem 4.7, we derive that

$$\underline{a} = 0.305, \quad \bar{a} = 0.6484.$$

Let

$$I_k(x(t_k^-), t_k) = \left(\frac{\sqrt{2}}{2} - 1\right)x + \frac{\sqrt{2}}{2}x(t_k - t_{k-1})^{\frac{1}{2}}, \quad \lambda^{\varepsilon}(t) = e^{0.01t}.$$

By a simple calculation, we have

$$d(x(t_{k-1}^-), t_{k-1}) = 0.7x^2(t_k - t_{k-1}), \quad \sum_{j=n}^{k} e^{0.01t_{j-1}} 0.7x^2(t_j - t_{j-1}) \leq 0.7 \int\limits_{t_{n-1}}^{t_k} e^{0.01s}|x|^2\mathrm{d}s,$$

which verifies the condition of Theorem 3.7 with

$$\alpha_1 = 1, \quad \alpha_2 = 0.305, \quad c_2 = 0.1402, \quad \varepsilon = 0.01, \quad \alpha = 1, \quad \alpha_3 = 0.7, \quad \alpha_4 = 0.$$

Moreover,

$$\alpha_1 - c_2\varepsilon\alpha - \alpha_3 = 0.2986 > 0, \quad \alpha_2 - \alpha_4 = 0.305 > 0.$$

Therefore, system (5.1) is almost surely stable with decay $\lambda(t)$ of order $\frac{\varepsilon}{2}$.

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