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Research Article

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The family of random attractors for nonautonomous stochastic higher-order Kirchhoff equations with variable coefficients

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Abstract: In this paper, the stochastic asymptotic behavior of the nonautonomous stochastic higher-order Kirchhoff equation with variable coefficients is studied. By using the Galerkin method, the solution of this kind of equation is obtained, and stochastic dynamical system under this kind of equation is obtained; by using the uniform estimation, the existence of the family of $\mathcal{D}_k$-absorbing sets of the stochastic dynamical system Φ_k is obtained, and the asymptotic compactness of Φ_k is proved by the decomposition method. Finally, the $\mathcal{D}_k$-stochastic attractor family of the stochastic dynamical system Φ_k in $V_{m+k}(\Omega) \times V_k(\Omega)$ is obtained.

Keywords: nonautonomous higher-order Kirchhoff equation, partial differential equations, variable coefficient, the family of random attractors, additive noise

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1 Introduction

Let $\Omega \subset R^N$ be a bounded domain with smooth boundary (i.e., the derivative of the function at the boundary exists and is continuous). In this paper, we study the asymptotic behavior of nonautonomous stochastic higher-order Kirchhoff equations with variable coefficients on Ω:

$$\begin{cases} u_{tt} + a(x)(-\Delta)^m u_t + b(x)M(\|\nabla^m u\|)(-\Delta)^m u + g(x, u) = f(x, t) + h(x)\dfrac{\partial w}{\partial t}, \\ u = 0, \quad \dfrac{\partial^i u}{\partial v^i} = 0, \quad i = 1, 2, \ldots, m-1, \quad x \in \Gamma, \quad t \geq \tau, \\ u(x, \tau) = u_\tau(x), \quad u_t(x, \tau) = u_{1\tau}(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Γ is the smooth boundary of Ω, v is the outer normal vector on the boundary Γ, $m > 1$, $a(x)$ and $b(x)$ are variable coefficient functions, $f(x, t) \in L^2_{\mathrm{loc}}(R, V_k(\Omega))$ is a time-dependent external force term, w is a one-dimensional bilateral standard Wiener process, $h(x)\frac{\partial w}{\partial t}$ describes white noise, and $g(x, u)$ is a nonlinear function that satisfies certain growth conditions and dissipation conditions.

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The Kirchhoff model was proposed in 1883 to describe the motion of elastic cross-section. Compared with classical wave equations, the Kirchhoff model can describe the motion of elastic rod more accurately. There has been a lot of in-depth research on the Kirchhoff equations. [1–6] studied the long-term dynamics of the autonomous low-order Kirchhoff equation; [7–11] studied the existence of global solutions and the blow-up of solutions of the higher-order Kirchhoff equations.

Stochastic wave equations are a very important class of stochastic partial differential equations, which are widely used in many fields such as fluid mechanics, physics, electricity, etc. The random attractor is an important tool for studying the long-term asymptotic behavior of stochastic dynamical systems. Using it to characterize the long-term behavior of random dynamical systems has laid a solid foundation for the study of random dynamical systems. After more than 30 years of development, random dynamical systems have also been extensively studied. Many scholars have conducted in-depth studies on the dynamical behavior of random wave equations in unbounded domains [12–17] and bounded domains [18–20]. For new trends in functional analysis and random attractors, see also [21–24].

Regarding the variable coefficients in the equation, it represents the wave velocity at the space coordinate x, which will appear in the wave phenomena in mathematical physics, marine acoustics, and other fields. It is of great practical significance to study the mathematical and physical equations with variable coefficients. In [25], they studied the global well-posedness and asymptotic behavior of solutions of Kirchhoff-type equations with variable coefficients and weak damping in unbounded domains. More relevant results can also be found in [26–33].

In recent years, Lin and Chen [34], Lin and Jin [35] have performed a detailed study on the long-term dynamical behavior of higher-order wave equations and proposed the concept of the family of attractors. Combined with the current research results, there are no relevant research results on the long-time dynamics of the nonautonomous stochastic higher-order Kirchhoff equation, and the asymptotic behavior of the higher-order Kirchhoff equation with variable coefficients has not been studied. By studying the nonautonomous stochastic higher-order Kirchhoff model with variable coefficients, the relevant results of the Kirchhoff model can be generalized, and the theoretical achievements of the Kirchhoff model can be enriched, which lays a theoretical foundation for later application. Therefore, this article will specifically study the family of random attractors of nonautonomous random higher-order Kirchhoff equation with variable coefficients. In the research process, the reasonable assumption and Leibniz formula are used to overcome how to define the L^p-weighted space and the difficulty of estimating the absorption sets and asymptotic compactness caused by the variable coefficients.

Section 2 of this article introduces related theories, related definitions, and theories of stochastic dynamical systems; Section 3 presents the family of the continuous cocycle of the problem; In Section 4, the uniform estimation of the solution of problem (1.1) is obtained, and the asymptotic compactness of Φ_k is obtained through the decomposition method; in Section 5, we get the family of $\mathcal{D}_k$-random attractors of Φ_k in X_k.

2 Preparatory knowledge

In this section, we mainly give the related theories of nonautonomous stochastic dynamical systems and random attractor (the family of random attractors).

First, the relevant notation needed in this paper is introduced: Define the inner product and norm on $H = L^2(\Omega)$ as $(\cdot,\cdot)$ and $(\|\cdot\|)$, $L^p = L^p(\Omega)$, $\|\cdot\|_p = \|\cdot\|_{L^p}$, where $p \geq 1$. Set variable coefficient $a(x)$, $b(x) = b_0 a(x)$, b_0 as a positive constant, satisfying $a \in C_0^\infty(\Omega)$, $a(x) \geq a_{00} > 0$, $\frac{\partial^i a}{\partial^i v}|_\Gamma = 0$, $a_0 = \|a(x)\|_\infty$, $a(x)^{-1} = \mu(x)$, $x \in \Omega$, and $\mu \in L^{\frac{N}{2}}(\Omega) \cap C_0^\infty(\Omega)$.

By $D^{1,2}$, we define the closure of the $C_0^\infty(\Omega)$ functions with respect to the "energy norm" $\|u\|_{D^{1,2}} = \int_\Omega |\nabla u|^2 \mathrm{d}x$. It is well known that

$$D^{1,2} \equiv D^{1,2}(\Omega) = \{u \in L^{2N/(N-2)}(\Omega) | \nabla u \in (L^2(\Omega))^N\},$$

and for $D^{1,2} \hookrightarrow L^{2N/(N-2)}(\Omega)$, there exists $\beta > 0$ such that $\|u\|_{2N/(N-2)} \leq \beta \|u\|_{D^{1,2}}$.

Lemma 2.1. [26] *Suppose that* $\mu \in L^{\frac{N}{2}}(\Omega) \cap C_0^\infty(\Omega)$, *then for all* $u \in C_0^\infty(\Omega)$, *there exists* $\alpha > 0$ *such that*

$$\alpha \int_\Omega \mu u^2 \mathrm{d}x \le \int_\Omega |\nabla u|^2 \mathrm{d}x,$$

where $\alpha = \beta^{-2}\|\mu\|_{N/2}^{-1}$.

Let $\mu > 0$ *be the weight function, and the weighted space* $L_\mu^p = L_\mu^p(\Omega)$ *with the following norm*:

$$\|u\|_{L_\mu^p}^p = \int_\Omega \mu |u|^p \mathrm{d}x = \|\mu^{\frac{1}{p}} u\|_p^p,$$

for $1 \le p < +\infty$. *Clearly* $L_\mu^2 = L_\mu^2(\Omega)$ *is a separable Hilbert space the inner product and norm are respectively*:

$$(u, v)_\mu = \int_\Omega \mu u v \mathrm{d}x = \left(\mu^{\frac{1}{2}} u, \mu^{\frac{1}{2}} v\right), \quad \|u\|_{L_\mu^2} = \|\mu^{\frac{1}{2}} u\|.$$

For $p : 1 \le p < \infty$, *the Banach space* L_μ^p *is uniformly convex, reflexive space, and* $(L_\mu^p)' = L_\mu^{p'}$, *where* p' *is the conjugate number of* p.

Lemma 2.2. [26] *Suppose that* $\mu \in L^{\frac{N}{2}}(\Omega) \cap C_0^\infty(\Omega)$, *then* $D^{1,2}$ *is compactly embedded in* L_μ^2. *Let*

$$V_m = H_0^m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega), \quad V_{m+k} = H_0^{m+k}(\Omega) = H^{m+k}(\Omega) \cap H_0^1(\Omega), \quad k = 0, 1, \dots, m,$$

and the corresponding inner product and norm are, respectively,

$$(u, v)_{V_{m+k}} = (\nabla^{m+k} u, \nabla^{m+k} v), \quad \|u\|_{V_{m+k}} = \|\nabla^{m+k} u\|_H.$$

At the same time, a general form of Poincare inequality: $\lambda_1 \|\nabla^r u\|^2 \le \|\nabla^{r+1} u\|^2$, *where* λ_1 *is the first eigenvalue of* $-\Delta$. *In the text,* C_i *is a positive constant,* $C(\cdot)$ *represents a positive constant that depends on the parameters in parentheses, and* C_m^n *is the corresponding number of combinations.*

Assuming that $(X, \|\cdot\|_X)$ is a separable Hilbert space, and $B(X)$ is the Borel σ-algebra of $X(\Omega_1, \mathcal{F}, P)$ is the metric probability space.

Definition 2.3. [12] Let $\theta_t : R \times \Omega_1 \to \Omega_1$ be a family of $(B(X) \times \mathcal{F}, \mathcal{F})$-measurable mappings such that $\theta_0(\cdot)$ is the identity on Ω_1 $\forall t, s \in R$, $\theta_{t+s}(\cdot) = \theta_t(\cdot) \circ \theta_s(\cdot)$, $P\theta_t(\cdot) = P$. A mapping $\Phi : R^+ \times R \times \Omega_1 \times X \to X$ is called a continuous cocycle or continuous random dynamical system (RDS) on X over R and $(\Omega_1, \mathcal{F}, P, (\theta_t)_{t \in R})$ if for all $\tau \in R$, $w \in \Omega_1$, $t, s \in R^+$ the following conditions are satisfied:

i: $\Phi(\cdot, \tau, \cdot, \cdot) : R^+ \times \Omega_1 \times X \to X$ is a $(B(R^+) \times \mathcal{F} \times B(X), B(X))$-measurable mapping;

ii: $\Phi(0, \tau, w, \cdot)$ is the identity on X;

iii: $\Phi(t + s, \tau, w, \cdot) = \Phi(t, \tau + s, \theta_s w, \Phi(s, \tau, w, \cdot))$;

iv: $\Phi(t, \tau, w, \cdot) : X \to X$ is continuous.

Let $D = \{D(\tau, w) \subseteq X : \tau \in R, w \in \Omega_1\}$ be a family of subsets parameterized by $(\tau, w) \in R \times \Omega_1$ in X.

Definition 2.4. [13] The family $D = \{D(\tau, w) \subseteq X : \tau \in R, w \in \Omega_1\}$ satisfies:

(1) for all $(\tau, w) \in R \times \Omega_1$ $D(\tau, w)$ is a closed nonempty subset of X;

(2) for every fixed $x \in X$ and any $\tau \in R$, the mapping $w \in \Omega_1 \to \mathrm{dist}_X(x, B(\tau, w))$ is $(\mathcal{F}, B(R^+))$ measurable, then the family D is measurable with to $\mathcal{F}$ in Ω_1.

Definition 2.5. [15] For all $\sigma > 0$, $w \in \Omega_1$ $D = \{D(\tau, w) \subseteq X : \tau \in R, w \in \Omega_1\}$ satisfies:

$$\lim_{t \to -\infty} e^{\sigma t} \|D(\tau + t, \theta_t w)\|_X = 0,$$

then $D = \{D(\tau, w) \subseteq X : \tau \in R, w \in \Omega_1\}$ is called tempered.

Let $\mathcal{D} = \mathcal{D}(X)$ be the set of all random tempered sets in X.

Definition 2.6. [12] A family $K = \{K(\tau, w) \subseteq X : \tau \in R, w \in \Omega_1\} \in \mathcal{D}$ of nonempty subsets of X is called a measurable $\mathcal{D}$-pullback attracting(or absorbing) set for $\{\Phi(t, \tau, w)\}_{t\geq 0, \tau\in R, w\in\Omega_1}$ if
(1) K is measurable with respect to the P completion of $\mathcal{F}$ in Ω_1;
(2) for all $\tau \in R$, $w \in \Omega_1$ and for every $D \in \mathcal{D}$, there exists $T(D, \tau, w) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}w, D(\tau - t, \theta_{-t}w)) \subseteq K(\tau, w), \quad \forall t \geq T(D, \tau, w).$$

Definition 2.7. [15] Φ is said to be asymptotically compact in X if for $\tau \in R, w \in \Omega_1, D = \{D(\tau, w) \subseteq X : \tau \in R, w \in \Omega_1\} \in \mathcal{D}, x_n \in B(\tau - t_n, \theta_{-t_n}w)$ $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}w, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$.

Definition 2.8. [13] A family $A = \{A(\tau, w) \subseteq X : \tau \in R, w \in \Omega_1\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback random attractor for $\{\Phi(t, \tau, w)\}_{t\geq 0, \tau\in R, w\in\Omega_1}$ if
(1) $A(\tau, w)$ is measurable in Ω_1 with respect to $\mathcal{F}$ and compact in X for $\forall \tau \in R$, $w \in \Omega_1$,
(2) A is invariant, i.e., for $\forall \tau \in R$ and $w \in \Omega_1$, $\forall t \geq 0$,

$$\Phi(t, \tau, w, A(\tau, w)) = A(t + \tau, \theta_t w);$$

(3) A attracts every member of $\mathcal{D}$, i.e., for every $D \in \mathcal{D}$, $\tau \in R$ and for every $w \in \Omega_1$,

$$\lim_{t\to+\infty} \operatorname{dist}_X(\Phi(t, \tau - t, \theta_{-t}w, B(\tau - t, \theta_{-t}w)), A(\tau, w)) = 0,$$

where $\operatorname{dist}_X(P, Q)$ denotes the Hausdorff semi-distance between two subsets P and Q of X.

If we change $\mathcal{D} = \mathcal{D}(X)$ to $\mathcal{D}_k = \mathcal{D}_k(X_k)$, where $k = 0, 1, \ldots, m$, then A in Definition 2.8 can be a family of random attractors $\{A_k\}$.

Lemma 2.9. [12] *Let $\mathcal{D}$ be a neighborhood-closed collection of (τ, w)-parametrized families of nonempty subsets of X and Φ be a continuous cocycle on X over R and $(\Omega_1, \mathcal{F}, P, \{\theta_t\}_{t\in R})$, then Φ has a pullback $\mathcal{D}$-attract A if and only if Φ is pullback $\mathcal{D}$ asymptotically compact in X and Φ has a closed, $\mathcal{F}$-measurable pullback $\mathcal{D}$-absorbing set K in $\mathcal{D}$ and the unique pullback $\mathcal{D}$-attractor $A = \{A(\tau, w)\}$ is given by*

$$A(\tau, w) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi(t, \tau - t, \theta_{-t}w, K(\tau - t, \theta_{-t}w))}, \quad \tau \in R, \quad w \in \Omega_1.$$

Similarly, Lemma 2.9 *can be extended to* Lemma 2.10 *of the family of pullback attractors*.

Lemma 2.10. *Let $\mathcal{D}_k$ be neighborhood-closed collections of (τ, w)-parametrized families of nonempty subsets of X_k, $k = 1, 2, \ldots, m$, and Φ_k be the family of continuous cocycles on X_k, $k = 1, 2, \ldots, m$ over R and $(\Omega_1, \mathcal{F}, P, \{\theta_t\}_{t\in R})$, then Φ_k has the family of pullback $\mathcal{D}_k$-attracts $\{A_k\}$ if and only if Φ_k is pullback $\mathcal{D}_k$-asymptotically compact in X_k and $\mathcal{D}_k$ has closed, $\mathcal{F}$-measurable pullback $\mathcal{D}_k$-absorbing sets K_k in $\mathcal{D}_k$ and the unique pullback $\mathcal{D}_k$-attractor $A_k = \{A_k(\tau, w)\}$ is given by*

$$A_k(\tau, w) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi_k(t, \tau - t, \theta_{-t}w, K_k(\tau - t, \theta_{-t}w))}, \quad \tau \in R, \quad w \in \Omega_1.$$

3 The family of cocycles of nonautonomous stochastic higher-order Kirchhoff equations with variable coefficients

Let $(\Omega_1, \mathcal{F}, P)$ be a probability space, where

$$\Omega_1 = \{w \in C(R, R), w(0) = 0\}.$$

w is a two-sided real-valued Winner processes on the probability space $(\Omega_1, \mathcal{F}, P)$. Define $\theta_t w(\cdot) = w(\cdot+t) - w(t)$, $w \in \Omega_1$, $t \in R$, thus, $(\Omega_1, \mathcal{F}, P, (\theta_t)_{t\in R})$ is an ergodic metric dynamical system.

For a small positive number ε, let z be a new variable given by $z = u_t + \varepsilon u$ and then, system (1.1) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} + \varepsilon u = z; \\ \dfrac{\partial z}{\partial t} = \varepsilon z - a(x)(-\Delta)^m z + \varepsilon a(x)(-\Delta)^m u - \varepsilon^2 u - b(x)M(\|\nabla^m u\|^2)(-\Delta)^m u - g(x,u) + f(x,t) + h(x)\dfrac{\mathrm{d}w}{\mathrm{d}t}; \\ u = 0, \quad \dfrac{\partial^i u}{\partial v^i} = 0, \ i = 1, 2, \dots, m-1, \ x \in \Gamma, \ t \geq \tau; \\ u(x,\tau) = u_\tau(x), \quad z(x,\tau) = z_\tau(x) = u_{1\tau}(x) + u_\tau(x), \ x \in \Omega, \end{cases} \tag{3.1}$$

where $(M) M \in C^1(R^+)$, $M' \geq 0$, and $M(s) \leq M_0 \cdot (1 + s^q)$, $0 < q < 1/2$, $M_0 = M(0)$ is a positive constant $\forall s \in R^+$, $h(x) \in V_{m+k}(\Omega)$, $x \in \Omega$, $t \geq \tau$, $\tau \in R$, $k = 0, 1, \dots, m$, $f(x,t) \in L^2_{\mathrm{loc}}(R, V_k(\Omega))$. In order to get the conclusion of this article, suppose that the nonlinear term $g(x,u)$ satisfies the following conditions: for $\forall u \in R$, $x \in \Omega$, there are positive constants $c_1, c_2, c_3, c_4, c_5 > 0$, satisfying

$$|g(x,u)| \leq c_1|p|^p + \phi_1(x), \ \ \phi_1 \in L^2_\mu(\Omega), \tag{3.2}$$

$$ug(x,u) - c_2 G(x,u) \geq \phi_2(x), \ \ \phi_2 \in L^1_\mu(\Omega), \tag{3.3}$$

$$G(x,u) \geq c_3|u|^{p+1} - \phi_3(x), \ \ \phi_3 \in L^1_\mu(\Omega), \tag{3.4}$$

$$|g_u(x,u)| \leq c_4|u|^{p-1} + \phi_4(x), \ \ \phi_4 \in V_m(\Omega), \tag{3.5}$$

$$|\nabla^k_x g(x,u)| \leq c_5|u|^p + \phi_5(x), \ \ \phi_5 \in V_k(\Omega), \tag{3.6}$$

where $1 \leq p < +\infty$, for $N = 1, 2$; $1 \leq p < \frac{N}{N-2}$ $N = 3, 4$; and $G(x,u) = \int_0^u g(x,s)\mathrm{d}s$. From equations (3.2) and (3.3), we can get

$$G(x,u) \leq c_6(|u|^2 + |u|^{p+1} + \phi_1^2 + \phi_2). \tag{3.7}$$

To show that problem (3.1) generates a random dynamical system, we let $v(t,\tau,w) = z(t,\tau,w) - hw(t)$, and then, (3.1) can be rewritten as the equivalent system with random coefficients but without white noise:

$$\begin{cases} \dfrac{\partial u}{\partial t} - v + \varepsilon u = hw(t); \\ \dfrac{\partial v}{\partial t} = \varepsilon v - a(x)(-\Delta)^m v + \varepsilon a(x)(-\Delta)^m u - \varepsilon^2 u - b(x)M(\|\nabla^m u\|^2)(-\Delta)^m u \\ - g(x,u) + f(x,t) + \varepsilon h(x)w(t) - a(x)(-\Delta)^m h(x)w(t); \\ u = 0, \quad \dfrac{\partial^i u}{\partial v^i} = 0, \ i = 1, 2, \dots, m-1, \ x \in \Gamma, \ t \geq \tau; \\ u(x,\tau) = u_\tau(x), \quad v(x,\tau) = v_\tau(x) = z_\tau(x) - hw(\tau), \ x \in \Omega. \end{cases} \tag{3.8}$$

Let $X_k = V_{m+k} \times V_k$, $k = 0, 1, \dots, m$, when $k = 0$ $V_0 = L^2_\mu$, endowed with the usual norm $\|(u,v)\|^2_{X_k} = \|u\|^2_{V_{m+k}} + \|v\|^2_{V_k}$. By the standard Galerkin method: If the assumptions (M) $h(x) \in V_{m+k}(\Omega)$, $x \in \Omega$, $t \geq \tau$, $\tau \in R$, $f(x,t) \in L^2_{\mathrm{loc}}(R, V_k(\Omega))$ conditions (3.2)–(3.6) hold the problem (3.8) is well posed in $X_k = V_{m+k} \times V_k$, i.e., for all $\tau \in R$ and $P-a.e.$ $w \in \Omega_1$, $(u_\tau, v_\tau) \in X_k$, the problem (3.8) has a unique global solution $(u(t,\tau,w,u_\tau), v(t,\tau,w,v_\tau)) \in C([\tau,\infty), X_k)$ and $(u(\tau,\tau,w,u_\tau), v(\tau,\tau,w,v_\tau)) = (u_\tau, v_\tau)$. Moreover, for $t \geq \tau$, $(u(t,\tau,w,u_\tau), v(t,\tau,w,v_\tau))$ is $(\mathcal{F}, B(X_k))$ measurable in w and continuous in (u_τ, v_τ) with respect to the X_k norm. Thus, the solution mapping can be used to define a family of continuous cocycles for (3.8). Let $\Phi_k : R^+ \times R \times \Omega_1 \times X_k \to X_k$ be mappings given by

$$\Phi_k(t,\tau,w,(u_\tau,v_\tau)) = (u(t+\tau,\tau,\theta_{-\tau}w,u_\tau), v(t+\tau,\tau,\theta_{-\tau}w,v_\tau)), \tag{3.9}$$

where $(t, \tau, w, (u_\tau, v_\tau)) \in R^+ \times R \times \Omega_1 \times X_k$, then Φ_k is a family of continuous cocycles over $(R, \tau + t)$ and $(\Omega_1, \mathcal{F}, P, \{\theta_t\}_{t\in R})$ on X_k. For $P - a.e.\ w \in \Omega_1$ and $t, s \geq 0,\ \tau \in R$:

$$\Phi_k(t + s, \tau, w, (u_\tau, v_\tau)) = \Phi_k(t, s + \tau, w, \Phi_k(s, \tau, w, (u_\tau, v_\tau))). \tag{3.10}$$

For any bounded nonempty subset B_k of X_k denote by $\|B_k\| = \sup_{\Phi_k\in R}\|\Phi\|_{X_k}$. Let $D_k = \{D_k(\tau, w) : \tau \in R, w \in \Omega_1\}$ be a family of bounded nonempty subsets of X_k, and for all $\tau \in R$, $w \in \Omega_1$,

$$\lim_{s\to-\infty} e^{\sigma s}\|D_k(\tau + s, \theta_s w)\|^2_{X_k} = 0. \tag{3.11}$$

Remember that $\mathcal{D}_k$ is the set of the aforementioned subset family D_k, that is, $\mathcal{D}_k = \{D_k = \{D_k(\tau, w) : \tau \in R, w \in \Omega_1\} : D_k$ satisfies (3.11)$\}$.

4 Uniform estimates of solutions

To prove the existence of the family of random attractors, we conduct uniform estimates on the solutions of the problem (3.8) defined on Ω, for the purposes of showing the existence of a family of $\mathcal{D}_k$ pullback absorbing sets and the pullback $\mathcal{D}_k$ asymptotic compactness of the random dynamical system. Let $\varepsilon > 0$ be small enough and satisfy $\alpha\lambda_1^{m-1} - 3\varepsilon > 0$, $2a_{00}\lambda_1^m - (a_{00}\lambda_1^m + 12)\varepsilon > 0$, $M_0 - \frac{5}{2}\varepsilon > 0$, $\frac{b_0 M_0}{8} - \varepsilon > 0$,

$$\sigma = \frac{1}{2}\min\left\{\alpha\lambda_1^{m-1} - 3\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon c_2}{2}\right\}, \quad \sigma_1 = \frac{1}{2}\min\{2a_{00}\lambda_1^m - (a_{00}\lambda_1^m + 12)\varepsilon, \varepsilon\}. \tag{4.1}$$

To obtain uniform estimates of the solutions, $f(x, t)$ needs to satisfy $(F_1)\int_{-\infty}^t e^{\sigma s}\|f(\cdot, s)\|^2_{V_k} ds < \infty$.

Lemma 4.1. *Suppose M satisfies (M), $h(x) \in V_{m+k}(\Omega)$, $k = 0, 1, \ldots, m$, (3.2)–(3.6) hold, $f(x, t)$ satisfies (F_1), and $B_k = \{B_k(\tau, w) : \tau \in R, w \in \Omega_1\} \in \mathcal{D}_k$ for $P - a.\ e.w \in \Omega_1, \tau \in R$ initial value satisfies $(u_{\tau-t}, v_{\tau-t}) \in B_k(\tau - t, \theta_{-\tau}w)$, there exists $T_k = T_k(\tau, w, B_k) > 0$ such that for all $t \geq T_k$, the solution $(u(\tau, \tau, w, u_{\tau-t}), v(\tau, \tau, w, v_{\tau-t})) = (u_{\tau-t}, v_{\tau-t})$ of problem (3.8) satisfies*

$$\|v(\tau, \tau - t, \theta_{-\tau}w, v_{\tau-t})\|^2_{V_k} + \|u(\tau, \tau - t, \theta_{-\tau}w, v_{\tau-t})\|^2_{V_{m+k}} \leq r_{1k}(\tau, w),$$

where $r_{1k}(\tau, w)$ will be given in detail later.

Proof. Taking the inner product of (3.8) with v in $L^2_\mu(\Omega)$, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2_\mu} = {}& \varepsilon\|v\|^2_{L^2_\mu} - \|\nabla^m v\|^2 + \varepsilon((-\Delta)^m u, v) - \varepsilon^2(u, v)_{L^2_\mu} - b_0(M(\|\nabla^m u\|^2)(-\Delta)^m u, v)\\ & - (g(x, u), v)_{L^2_\mu} + (f(x, t), v)_{L^2_\mu} + \varepsilon w(t)(h, v)_{L^2_\mu} - w(t)((-\Delta)^m h, v),\end{aligned} \tag{4.2}$$

for each term on the right-hand side of (4.2):

$$\varepsilon((-\Delta)^m u, v) = \varepsilon((-\Delta)^m u, u_t + \varepsilon u + hw(t)) = \frac{\varepsilon}{2}\frac{d}{dt}\|\nabla^m u\|^2 + \varepsilon^2\|\nabla^m u\|^2 - \varepsilon w(t)((-\Delta)^m u, h), \tag{4.3}$$

$$\varepsilon^2(u, v)_{L^2_\mu} = \varepsilon^2(u, u_t + \varepsilon u - hw(t))_{L^2_\mu} = \frac{\varepsilon^2}{2}\frac{d}{dt}\|v\|^2_{L^2_\mu} + \varepsilon^3\|v\|^2_{L^2_\mu} - \varepsilon^2 w(t)(u, h)_{L^2_\mu}, \tag{4.4}$$

$$\begin{aligned}& b_0(M(\|\nabla^m u\|^2)(-\Delta)^m u, v) = b_0(M(\|\nabla^m u\|^2)(-\Delta)^m u, u_t + \varepsilon u - hw(t))\\ & = \frac{b_0}{2}\frac{d}{dt}\int_0^{\|\nabla^m u\|^2} M(s)\, ds + \varepsilon b_0 M(\|\nabla^m u\|^2)\|\nabla^m u\|^2 - b_0 M(\|\nabla^m u\|^2)w(t)((-\Delta)^m u, h),\end{aligned} \tag{4.5}$$

$$(g(x, u), v)_{L^2_\mu} = (g(x, u), u_t + \varepsilon u - hw(t))_{L^2_\mu} = \frac{d}{dt}\int_\Omega \mu G(x, u)dx + \varepsilon(g(x, u), u)_{L^2_\mu} - w(t)(g(x, u), h)_{L^2_\mu}. \tag{4.6}$$

Substitute (4.3)–(4.6) into (4.2) to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2_{L^2_\mu}+b_0\int\limits_0^{\|\nabla^m u\|^2}M(s)\mathrm{d}s-\varepsilon\|\nabla^m u\|^2+\varepsilon^2\|u\|^2_{L^2_\mu}+2\int\limits_\Omega\mu G(x,u)\mathrm{d}x\right]+2\|\nabla^m v\|^2\\&-2\varepsilon\|v\|^2_{L^2_\mu}+2\varepsilon b_0M(\|\nabla^m u\|^2)\|\nabla^m u\|^2-2\varepsilon^2\|\nabla^m u\|^2+2\varepsilon^3\|u\|^2_{L^2_\mu}+2\varepsilon(g(x,u),u)_{L^2_\mu}\\&=2(f(x,t),v)_{L^2_\mu}+(2b_0M(\|\nabla^m u\|^2)-2\varepsilon)w(t)((-\Delta)^m u,h)-2\varepsilon^2w(t)(u,h)_{L^2_\mu}\\&\quad+2w(t)(g(x,u),h)_{L^2_\mu}+2w(t)(h,v)_{L^2_\mu}-2w(t)((-\Delta)^m h,v).\end{aligned}\tag{4.7}$$

Using the Cauchy-Schwarz inequality, Young's inequality and Holder's inequality, we have

$$2\varepsilon^2w(t)(u,h)_{L^2_\mu}\le\varepsilon^3\|u\|^2_{L^2_\mu}+\varepsilon|w(t)|^2\|h\|^2_{L^2_\mu},\tag{4.8}$$

$$\begin{aligned}&2b_0M(\|\nabla^m u\|^2)w(t)((-\Delta)^m u,h)\\&\le 2b_0M_0(1+\|\nabla^m u\|^{2q})|w(t)|\|\nabla^m u\|\|\nabla^m h\|\\&\le 2b_0M_0|w(t)|\|\nabla^m u\|\|\nabla^m h\|+2b_0M_0|w(t)|\|\nabla^m u\|^{2q+1}\|\nabla^m h\|\\&\le\frac{\varepsilon b_0M_0}{4}\|\nabla^m u\|^2+4\varepsilon^{-1}b_0M_0|w(t)|^2\|\nabla^m h\|^2+\frac{\varepsilon b_0M_0}{4}\|\nabla^m u\|^2\\&\quad+\frac{1-2q}{2}\left(\frac{\varepsilon}{8q+4}\right)^{\frac{2q+1}{2q-1}}b_0M_0|w(t)|^{\frac{2}{1-2q}}\|\nabla^m h\|^{\frac{2}{1-2q}}.\end{aligned}\tag{4.9}$$

By (3.2) and (3.4), we get

$$\begin{aligned}2w(t)(g(x,u),h)_{L^2_\mu}&\le 2|w(t)|\|\phi_1\|_{L^2_\mu}\|h\|_{L^2_\mu}+2c_1|w(t)|\left(\int\limits_\Omega\mu|u|^{p+1}\mathrm{d}x\right)^{\frac{p}{p+1}}\|h\|_{L^{p+1}_\mu}\\&\le 2|w(t)|\|\phi_1\|_{L^2_\mu}\|h\|_{L^2_\mu}+2c_1|w(t)|\left(\int\limits_\Omega(\mu G(x,u)+\mu\phi_3)\mathrm{d}x\right)^{\frac{p}{p+1}}\|h\|_{L^{p+1}_\mu}\\&\le 2|w(t)|\|\phi_1\|_{L^2_\mu}\|h\|_{L^2_\mu}+\varepsilon c_2\int\limits_\Omega\mu G(x,u)\mathrm{d}x+\varepsilon c_2\int\limits_\Omega\mu\phi_3(x)\mathrm{d}x\\&\quad+(2c_1)^{p+1}\frac{(\varepsilon c_2)^{-p}}{p+1}\left(\frac{p+1}{p}\right)^{-p}|w(t)|^{p+1}\|h\|^{p+1}_{L^{p+1}_\mu},\end{aligned}\tag{4.10}$$

$$\begin{aligned}&2(f(x,t),v)_{L^2_\mu}+2w(t)(h,v)_{L^2_\mu}-2w(t)((-\Delta)^m h,v)\\&\le 2\|f\|_{L^2_\mu}\|v\|_{L^2_\mu}+2\varepsilon|w(t)|\|h\|_{L^2_\mu}\|v\|_{L^2_\mu}+2|w(t)|\|\nabla^m h\|\|\nabla^m v\|\\&\le 2\alpha^{-\frac12}\lambda_1^{-\frac{m-1}{2}}\|f\|_{L^2_\mu}\|\nabla^m v\|+\varepsilon\|v\|^2_{L^2_\mu}+\varepsilon|w(t)|^2\|h\|^2_{L^2_\mu}+\frac12\|\nabla^m v\|^2+2|w(t)|^2\|\nabla^m h\|^2\\&\le\|\nabla^m v\|^2+\varepsilon\|v\|^2_{L^2_\mu}+2\alpha^{-1}\lambda_1^{1-m}\|f\|^2_{L^2_\mu}+2|w(t)|^2\|\nabla^m h\|^2+\varepsilon|w(t)|^2\|h\|^2_{L^2_\mu},\end{aligned}\tag{4.11}$$

$$2\varepsilon w(t)((-\Delta)^m u,h)\le 2\varepsilon|w(t)|\|\nabla^m h\|\|\nabla^m u\|\le\varepsilon^2\|\nabla^m u\|^2+|w(t)|^2\|\nabla^m h\|^2.\tag{4.12}$$

Substitute (4.8)–(4.12) into (4.7) to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2_{L^2_\mu}+b_0\int\limits_0^{\|\nabla^m u\|^2}M(s)\mathrm{d}s-\varepsilon\|\nabla^m u\|^2+\varepsilon^2\|u\|^2_{L^2_\mu}+2\int\limits_\Omega\mu G(x,u)\mathrm{d}x\right]+\|\nabla^m v\|^2\\&-3\varepsilon\|v\|^2_{L^2_\mu}+\left(2\varepsilon M(\|\nabla^m u\|^2)-\frac{\varepsilon M_0}{2}\right)b_0\|\nabla^m u\|^2-3\varepsilon^2\|\nabla^m u\|^2+\varepsilon^3\|u\|^2_{L^2_\mu}+2\varepsilon(g(x,u),u)_{L^2_\mu}\end{aligned}\tag{4.13}$$

$$\leq 2\alpha^{-1}\lambda_1^{1-m}\|f\|^2_{L^2_\mu} + (3 + 4\varepsilon^{-1}M_0b_0)|w(t)|^2\|\nabla^m h\|^2 + 2\varepsilon|w(t)|^2\|h\|^2_{L^2_\mu}$$
$$+ \frac{1-2q}{2}\left(\frac{\varepsilon}{8q+4}\right)^{\frac{2q+1}{2q-1}} b_0M_0|w(t)|^{\frac{2}{1-2q}}\|\nabla^m h\|^{\frac{2}{1-2q}} + 2|w(t)|\|\phi_1\|_{L^2_\mu}\|h\|_{L^2_\mu} + \varepsilon c_2\int_\Omega \mu G(x,u)\mathrm{d}x$$
$$+ \varepsilon c_2\int_\Omega \mu\phi_3(x)\mathrm{d}x + (2c_1)^{p+1}\frac{(\varepsilon c_2)^{-p}}{p+1}\left(\frac{p+1}{p}\right)^{-p}|w(t)|^{p+1}\|h\|^{p+1}_{L^{p+1}_\mu}.$$

By condition (3.3), we have

$$2\varepsilon(g(x,u),u)_{L^2_\mu} \geq 2\varepsilon\left(c_2\int_\Omega \mu G(x,u)\mathrm{d}x + \int_\Omega \mu\phi_2(x)\mathrm{d}x\right). \tag{4.14}$$

Substitute (4.14) into (4.12) to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2_{L^2_\mu} + b_0\int_0^{\|\nabla^m u\|^2} M(s)\mathrm{d}s - \varepsilon\|\nabla^m u\|^2 + \varepsilon^2\|u\|^2_{L^2_\mu} + 2\int_\Omega \mu G(x,u)\mathrm{d}x\right] + \|\nabla^m v\|^2 - 3\varepsilon\|v\|^2_{L^2_\mu}\\
&+ \left(2\varepsilon M(\|\nabla^m u\|^2) - \frac{\varepsilon M_0}{2}\right)b_0\|\nabla^m u\|^2 - 3\varepsilon^2\|\nabla^m u\|^2 + \varepsilon^3\|u\|^2_{L^2_\mu} + \varepsilon c_2\int_\Omega \mu G(x,u)\mathrm{d}x + 2\varepsilon\int_\Omega \mu\phi_2(x)\mathrm{d}x\\
&\leq 2\alpha^{-1}\lambda_1^{1-m}\|f\|^2_{L^2_\mu} + (3 + 4\varepsilon^{-1}M_0b_0)|w(t)|^2\|\nabla^m h\|^2 + 2\varepsilon|w(t)|^2\|h\|^2_{L^2_\mu}\\
&+ \frac{1-2q}{2}\left(\frac{\varepsilon}{8q+4}\right)^{\frac{2q+1}{2q-1}} b_0M_0|w(t)|^{\frac{2}{1-2q}}\|\nabla^m h\|^{\frac{2}{1-2q}} + 2|w(t)|\|\phi_1\|_{L^2_\mu}\|h\|_{L^2_\mu}\\
&+ \varepsilon c_2\int_\Omega \mu\phi_3(x)\mathrm{d}x + (2c_1)^{p+1}\frac{(\varepsilon c_2)^{-p}}{p+1}\left(\frac{p+1}{p}\right)^{-p}|w(t)|^{p+1}\|h\|^{p+1}_{L^{p+1}_\mu}.
\end{aligned} \tag{4.15}$$

According to (4.1), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2_{L^2_\mu} + b_0\int_0^{\|\nabla^m u\|^2} M(s)\mathrm{d}s - \varepsilon\|\nabla^m u\|^2 + \varepsilon^2\|u\|^2_{L^2_\mu} + 2\int_\Omega \mu G(x,u)\mathrm{d}x\right]\\
&+ \sigma\left[\|v\|^2_{L^2_\mu} + b_0\int_0^{\|\nabla^m u\|^2} M(s)\,\mathrm{d}s - \varepsilon\|\nabla^m u\|^2 + \varepsilon^2\|u\|^2_{L^2_\mu} + 2\int_\Omega \mu G(x,u)\mathrm{d}x\right]\\
&\leq 2\alpha^{-1}\lambda_1^{1-m}\|f\|^2_{L^2_\mu} + C_{01}(1 + |w(t)|^2 + |w(t)|^{\frac{2}{1-2q}} + |w(t)|^{p+1}),
\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}
C_{01} = \max\Bigg\{&(3 + 4\varepsilon^{-1}M_0b_0)\|\nabla^m h\|^2 + 2\varepsilon\|h\|^2_{L^2_\mu} + \|\phi_1\|^2_{L^2_\mu}\|h\|^2_{L^2_\mu}, \varepsilon c_2\int_\Omega \mu\phi_3(x)\mathrm{d}x,\\
&\frac{1-2q}{2}\left(\frac{\varepsilon}{8q+4}\right)^{\frac{2q+1}{2q-1}} b_0M_0\|\nabla^m h\|^{\frac{2}{1-2q}}, (2c_1)^{p+1}\frac{(\varepsilon c_2)^{-p}}{p+1}\left(\frac{p+1}{p}\right)^{-p}\|h\|^{p+1}_{L^{p+1}_\mu}\Bigg\}.
\end{aligned}$$

Using the Gronwall inequality to integrate (4.16) over $[\tau - t, \tau]$ with $t \geq 0$ and replacing w by $\theta_{-\tau}w$, we obtain

$$\begin{aligned}
&e^{\sigma\tau}\left[\|v\|^2_{L^2_\mu}+b_0\int\limits_0^{\|\nabla^m u\|^2}M(s)\mathrm{d}s-\varepsilon\|\nabla^m u\|^2+\varepsilon^2\|u\|^2_{L^2_\mu}+2\int\limits_\Omega \mu G(x,u)\mathrm{d}x\right]\\
&\quad\le e^{\sigma(\tau-t)}\left[\|v(\tau-t)\|^2_{L^2_\mu}+b_0\int\limits_0^{\|\nabla^m u(\tau-t)\|^2}M(s)\mathrm{d}s-\varepsilon\|\nabla^m u(\tau-t)\|^2\right.\\
&\qquad\left.+\varepsilon^2\|u(\tau-t)\|^2_{L^2_\mu}+2\int\limits_\Omega \mu G(x,u(\tau-t))\mathrm{d}x\right]+2\alpha^{-1}\lambda_1^{1-m}\int\limits_{\tau-t}^{\tau}e^{\sigma\xi}\|f(\cdot,\xi)\|^2_{L^2_\mu}\mathrm{d}\xi\\
&\qquad+C_{01}\int\limits_{\tau-t}^{\tau}e^{\sigma\xi}(1+|w(\xi)|^2+|w(\xi)|^{\frac{2}{1-2q}}+|w(\xi)|^{p+1})\mathrm{d}\xi,
\end{aligned}\tag{4.17}$$

then

$$\begin{aligned}
&\|v(\tau,\tau-t,\theta_{-\tau}w,v_{\tau-t})\|^2_{L^2_\mu}+b_0\int\limits_0^{\|\nabla^m u(\tau,\tau-t,\theta_{-\tau}w,v_{\tau-t})\|^2}M(s)\mathrm{d}s-\varepsilon\|\nabla^m u(\tau,\tau-t,\theta_{-\tau}w,v_{\tau-t})\|^2\\
&\quad+\varepsilon^2\|u(\tau,\tau-t,\theta_{-\tau}w,v_{\tau-t})\|^2_{L^2_\mu}+2\int\limits_\Omega \mu G(x,u(\tau,\tau-t,\theta_{-\tau}w,v_{\tau-t}))\mathrm{d}x\\
&\quad\le e^{-\sigma t}\left[\|v(\tau-t)\|^2_{L^2_\mu}+b_0\int\limits_0^{\|\nabla^m u(\tau-t)\|^2}M(s)\mathrm{d}s-\varepsilon\|\nabla^m u(\tau-t)\|^2\right.\\
&\qquad\left.+\varepsilon^2\|u(\tau-t)\|^2_{L^2_\mu}+2\int\limits_\Omega \mu G(x,u(\tau-t))\mathrm{d}x\right]+2\alpha^{-1}\lambda_1^{1-m}e^{-\sigma\tau}\int\limits_{\tau-t}^{\tau}e^{\sigma\xi}\|f(\cdot,\xi)\|^2_{L^2_\mu}\mathrm{d}\xi\\
&\qquad+C_{01}e^{-\sigma\tau}\int\limits_{\tau-t}^{\tau}e^{\sigma\xi}(1+|\theta_{-\tau}w(\xi)|^2+|\theta_{-\tau}w(\xi)|^{\frac{2}{1-2q}}+|\theta_{-\tau}w(\xi)|^{p+1})\mathrm{d}\xi.
\end{aligned}\tag{4.18}$$

By (3.7), we have

$$\int\limits_\Omega \mu G(x,u(\tau-t))\mathrm{d}x\le c_6\Big(\|\phi_1\|^2_{L^2_\mu}+\|\phi_2\|_{L^1_\mu}+\|u\|^2_{L^2_\mu}+\|u\|^{p+1}_{L^{p+1}_\mu}\Big)\le C_{02}\Big(1+\|u\|^2_{L^2_\mu}+\|u\|^{p+1}_{V_m}\Big).\tag{4.19}$$

Since $(u(\tau-t),v(\tau-t))\in B_0(\tau-t,\theta_{-\tau}w)$ when $t\to+\infty$

$$\begin{aligned}
&e^{-\sigma t}\left[\|v(\tau-t)\|^2_{L^2_\mu}+b_0\int\limits_0^{\|\nabla^m u(\tau-t)\|^2}M(s)\mathrm{d}s-\varepsilon\|\nabla^m u(\tau-t)\|^2+\varepsilon^2\|u(\tau-t)\|^2_{L^2_\mu}+2\int\limits_\Omega \mu G(x,u(\tau-t))\mathrm{d}x\right]\\
&\quad\le\ C_{03}e^{-\sigma t}\Big(1+\|v(\tau-t)\|^2_{L^2_\mu}+\|\nabla^m u(\tau-t)\|^2+\|\nabla^m u(\tau-t)\|^{p+1}\Big)\to 0,
\end{aligned}\tag{4.20}$$

and there exists $T_0=T_0(\tau,w,B_0)$ such that for all $t\ge T_0$,

$$C_{03}e^{-\sigma t}\Big(1+\|v(\tau-t)\|^2_{L^2_\mu}+\|\nabla^m u(\tau-t)\|^2+\|\nabla^m u(\tau-t)\|^{p+1}\Big)\le 1.\tag{4.21}$$

By (3.4), it is easy to get to any $t\ge 0$,

$$-2\int\limits_\Omega G(x,u)\mathrm{d}x\le -2c_3\int\limits_\Omega |u\,|^{p+1}\mathrm{d}x+2\int\limits_\Omega \phi_3\mathrm{d}x\le 2\int\limits_\Omega \phi_3\mathrm{d}x.\tag{4.22}$$

When $|\xi|\to\infty$, $w(\xi)$ at most polynomial growth,

$$C_{01}e^{-\sigma\tau}\int\limits_{-\infty}^{\tau}e^{\sigma\xi}(1+|\theta_{-\tau}w(\xi)|^2+|\theta_{-\tau}w(\xi)|^{\frac{2}{1-2q}}+|\theta_{-\tau}w(\xi)|^{p+1})\mathrm{d}\xi\equiv r_{00}(\tau,w).$$

We get from (4.18) and (4.21) that

$$\begin{aligned}&\|v(\tau,\tau-t,\theta_{-\tau}w,v_{\tau-t})\|^2_{L^2_\mu}+\|\nabla^m u(\tau,\tau-t,\theta_{-\tau}w,v_{\tau-t})\|^2\\ &\leq C_{04}\left(1+2\alpha^{-1}\lambda_1^{1-m}e^{-\sigma\tau}\int\limits_{-\infty}^{\tau}e^{\sigma\xi}\|f(\cdot,\xi)\|^2_{L^2_\mu}\mathrm{d}\xi\right)+r_{00}(\tau,w)\equiv r_{10}(\tau,w),\end{aligned}\tag{4.23}$$

and $r_{10}(\tau, w)$ is bounded. □

Taking the inner product of (3.8) with $(-\Delta)^k v$, $k = 1, 2, \dots, m-1$ in $L^2(\Omega)$, we find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k v\|^2=&\ \varepsilon\|\nabla^k v\|^2-(a(x)(-\Delta)^m v,(-\Delta)^k v)+\varepsilon(a(x)(-\Delta)^m u,(-\Delta)^k v)\\ &-\varepsilon^2(u,(-\Delta)^k v)-(b(x)M(\|\nabla^m u\|^2)(-\Delta)^m u,(-\Delta)^k v)-(g(x,u),(-\Delta)^k v)\\ &+(f(x,t),(-\Delta)^k v)+\varepsilon w(t)(h,(-\Delta)^k v)-w(t)(a(x)(-\Delta)^m h,(-\Delta)^k v).\end{aligned}\tag{4.24}$$

For each term on the right-hand side of (4.24):

$$(a(x)(-\Delta)^m v,(-\Delta)^k v)=(a(x)\nabla^{m+k}v,\nabla^{m+k}v)+\left(\sum_{i=1}^{m-k}C^i_{m-k}\nabla^{m+k-i}v\nabla^i a(x),\nabla^{m+k}v\right),\tag{4.25}$$

$$\begin{aligned}\varepsilon(a(x)(-\Delta)^m u,(-\Delta)^k v)&=\varepsilon(a(x)\nabla^{m+k}u,\nabla^{m+k}v)+\varepsilon\left(\sum_{i=1}^{m-k}C^i_{m-k}\nabla^{m+k-i}v\nabla^i a(x),\nabla^{m+k}u\right)\\ &=\varepsilon\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(a(x)\nabla^{m+k}u,\nabla^{m+k}u)+\varepsilon^2(a(x)\nabla^{m+k}u,\nabla^{m+k}u)\\ &\quad-\varepsilon w(t)(a(x)\nabla^{m+k}u,\nabla^{m+k}h)+\varepsilon\left(\sum_{i=1}^{m-k}C^i_{m-k}\nabla^{m+k-i}v\nabla^i a(x),\nabla^{m+k}u\right),\end{aligned}\tag{4.26}$$

$$\varepsilon^2(u,(-\Delta)^k v)=\varepsilon^2(u,(-\Delta)^k(u_t+\varepsilon u-hw(t)))=\frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k u\|^2+\varepsilon^3\|\nabla^k u\|^2-\varepsilon^2 w(t)(\nabla^k u,\nabla^k h),\tag{4.27}$$

$$\begin{aligned}&(b(x)M(\|\nabla^m u\|^2)(-\Delta)^m u,(-\Delta)^k v)\\ &=b_0M(\|\nabla^m u\|^2)(a(x)\nabla^{m+k}u,\nabla^{m+k}v)+M(\|\nabla^m u\|^2)\left(\sum_{i=1}^{m-k}C^i_{m-k}\nabla^{m+k-i}v\nabla^i b(x),\nabla^{m+k}u\right)\\ &=\frac{b_0}{2}M(\|\nabla^m u\|^2)\frac{\mathrm{d}}{\mathrm{d}t}(a(x)\nabla^{m+k}u,\nabla^{m+k}u)+\varepsilon b_0M(\|\nabla^m u\|^2)(a(x)\nabla^{m+k}u,\nabla^{m+k}u)\\ &\quad-b_0M(\|\nabla^m u\|^2)w(t)(a(x)\nabla^{m+k}u,\nabla^{m+k}h)+M(\|\nabla^m u\|^2)\left(\sum_{i=1}^{m-k}C^i_{m-k}\nabla^{m+k-i}v\nabla^i b(x),\nabla^{m+k}u\right),\end{aligned}\tag{4.28}$$

$$\begin{aligned}(g(x,u),(-\Delta)^k v)=(\nabla_x^k g(x,u),\nabla^k v)&\leq\left|\int\limits_\Omega(c_5|u|^p+\phi_5(x))\nabla^k v\,\mathrm{d}x\right|\\ &\leq c_5\left|\int\limits_\Omega|u|^p\nabla^k v\,\mathrm{d}x\right|+\left|\int\limits_\Omega\phi_5(x)\nabla^k v\,\mathrm{d}x\right|\\ &\leq c_5\|u\|^p_{L^{2p}}\|\nabla^k v\|+\|\phi_5(x)\|\|\nabla^k v\|\\ &\leq\frac{a_{00}}{8}\|\nabla^{m+k}v\|^2+C_{k1}(r^p_{01}(\tau,w)+\|\phi_5(x)\|^2),\end{aligned}\tag{4.29}$$

$$(f(x,t),(-\Delta)^k v)=(\nabla^k f(x,t),\nabla^k v)\leq\|\nabla^k f(x,t)\|\|\nabla^k v\|\leq\frac{a_{00}}{8}\|\nabla^{m+k}v\|^2+\frac{2\lambda_1^{-m}}{a_{00}}\|\nabla^k f(x,t)\|^2,\tag{4.30}$$

moreover,

$$(\nabla^{m+k-i}v\nabla^i a(x), \nabla^{m+k}v) \le a_i\|\nabla^{m+k-i}v\|\|\nabla^{m+k}v\|, \quad i = 1, 2, \dots, m-k,\ a_i = \|\nabla^i a(x)\|_\infty. \tag{4.31}$$

According to the interpolation inequality, we have

$$\|\nabla^{m+k-i}v\| \le C_i\|\nabla^{m+k}v\|^{\alpha_i}\|v\|^{1-\alpha_i}, \quad \alpha_i = \frac{m+k-i}{m+k},$$

then

$$\begin{aligned}(C_{m-k}^i\nabla^{m+k-i}v\nabla^i a(x), \nabla^{m+k}v) &\le C_{m-k}^i C_i a_i\|v\|^{1-\alpha_i}\|\nabla^{m+k}v\|^{1+\alpha_i}\\ &\le \frac{a_{00}}{8(m-k)}\|\nabla^{m+k}v\|^2 + \frac{1-\alpha_i}{2}\left(\frac{a_{00}}{4(1-\alpha_i)(m-k)}\right)^{\frac{1+\alpha_i}{1-\alpha_i}}(C_{m-k}^i C_i a_i)^{\frac{2}{1-\alpha_i}}\|v\|^2,\end{aligned} \tag{4.32}$$

$$\begin{aligned}(C_{m-k}^i\nabla^{m+k-i}v\nabla^i a(x), \nabla^{m+k}u) &\le \frac{a_{00}b_0M_0}{8(m-k)}\|\nabla^{m+k}u\|^2 + \frac{2(m-k)}{a_{00}b_0M_0}(C_{m-k}^i a_i)^2\|\nabla^{m+k-i}v\|^2\\ &\le \frac{a_{00}b_0M_0}{8(m-k)}\|\nabla^{m+k}u\|^2 + \frac{a_{00}}{8(m-k)}\|\nabla^{m+k}v\|^2\\ &\quad + (1-\alpha_i)\left(\frac{a_{00}}{8\alpha_i(m-k)}\right)^{\frac{\alpha_i}{\alpha_i-1}}\left(\frac{2(m-k)}{a_{00}b_0M_0}(C_{m-k}^i a_i C_i)^2\right)^{\frac{1}{1-\alpha_i}}\|v\|^2,\end{aligned} \tag{4.33}$$

$$\begin{aligned}&b_0M(\|\nabla^m u\|^2)w(t)(a(x)\nabla^{m+k}u, \nabla^{m+k}h)\\ &\le b_0a_0M_0(1+\|\nabla^m u\|^{2q})|w(t)|\|\nabla^{m+k}u\|\|\nabla^{m+k}h\|\\ &\le \frac{\varepsilon b_0a_{00}M_0}{8}\|\nabla^{m+k}u\|^2 + 2\varepsilon^{-1}a_{00}^{-1}b_0a_0^2M_0(1+\|\nabla^m u\|^{2q})^2|w(t)|^2\|\nabla^{m+k}h\|^2,\end{aligned} \tag{4.34}$$

$$\begin{aligned}&M(\|\nabla^m u\|^2)\left(\sum_{i=1}^{m-k}C_{m-k}^i\nabla^{m+k-i}v\nabla^i b(x), \nabla^{m+k}u\right)\\ &\le M_0b_0(1+\|\nabla^m u\|^{2q})\sum_{i=1}^{m-k}C_{m-k}^i a_i\|\nabla^{m+k-i}v\|\|\nabla^{m+k}u\|\\ &\le \frac{\varepsilon b_0a_{00}M_0}{8}\|\nabla^{m+k}u\|^2 + \varepsilon^{-1}a_{00}^{-1}b_0M_0\sum_{i=1}^{m-k}(C_{m-k}^i(1+\|\nabla^m u\|^{2q})a_i)^2\|\nabla^{m+k-i}v\|^2\\ &\le \frac{\varepsilon b_0a_{00}M_0}{8}\|\nabla^{m+k}u\|^2 + \frac{a_{00}}{8}\|\nabla^{m+k}v\|^2\\ &\quad + (\varepsilon^{-1}a_{00}^{-1}b_0M_0)^{\frac{1}{1-\alpha_i}}\sum_{i=1}^{m-k}(1-\alpha_i)\left(\frac{a_{00}}{8\alpha_i(m-k)}\right)^{\frac{\alpha_i}{\alpha_i-1}}(C_{m-k}^i a_iC_i(1+\|\nabla^m u\|^{2q}))^{\frac{2}{1-\alpha_i}}\|v\|^2.\end{aligned} \tag{4.35}$$

By (4.25)–(4.30) and (4.32)–(4.35), we get

$$\begin{aligned}&\frac{d}{dt}[\|\nabla^k v\|^2 - \varepsilon(a(x)\nabla^{m+k}u, \nabla^{m+k}u) + \varepsilon^2\|\nabla^k u\|^2] + b_0M(\|\nabla^m u\|^2)\frac{d}{dt}(a(x)\nabla^{m+k}u, \nabla^{m+k}u)\\ &+ 2(a(x)\nabla^{m+k}v, \nabla^{m+k}v) - \frac{4+\varepsilon}{4}a_{00}\|\nabla^{m+k}v\|^2 - 2\varepsilon\|\nabla^k v\|^2\\ &+ 2\varepsilon b_0M(\|\nabla^m u\|^2)(a(x)\nabla^{m+k}u, \nabla^{m+k}u) - \frac{3\varepsilon a_{00}b_0M_0}{4}\|\nabla^{m+k}u\|^2\\ &- 2\varepsilon^2(a(x)\nabla^{m+k}u, \nabla^{m+k}u) + 2\varepsilon^3\|\nabla^k u\|^2\\ &\le \sum_{i=1}^{m-k}(1-\alpha_i)\left(\frac{a_{00}}{4(1-\alpha_i)(m-k)}\right)^{\frac{1+\alpha_i}{1-\alpha_i}}(C_{m-k}^i a_iC_i)^{\frac{2}{1-\alpha_i}}\|v\|^2\\ &\quad + 2\varepsilon\sum_{i=1}^{m-k}(1-\alpha_i)\left(\frac{a_{00}}{8\alpha_i(m-k)}\right)^{\frac{\alpha_i}{\alpha_i-1}}\left(\frac{2(m-k)}{a_{00}b_0M_0}(C_{m-k}^i a_iC_i)^2\right)^{\frac{1}{1-\alpha_i}}\|v\|^2\end{aligned} \tag{4.36}$$

$$
\begin{aligned}
&+ 2\sum_{i=1}^{m-k}(\varepsilon^{-1}a_{00}^{-1}b_0M_0)^{\frac{1}{1-\alpha_i}}(1-\alpha_i)\left(\frac{a_{00}}{8\alpha_i(m-k)}\right)^{\frac{\alpha_i}{\alpha_i-1}}(C_{m-k}^i a_iC_i(1+\|\nabla^m u\|^{2q}))^{\frac{2}{1-\alpha_i}}\|v\|^2\\
&+ 4\varepsilon^{-1}a_{00}^{-1}b_0a_0^2M_0(1+\|\nabla^m u\|^{2q})^2|w(t)|^2\|\nabla^{m+k}h\|^2 + \frac{2\lambda_1^{-m}}{a_{00}}\|\nabla^k f(x,t)\|^2\\
&+ 2C_{k1}(r_{01}^p(\tau,w)+\|\phi_5(x)\|^2) - 2\varepsilon w(t)(a(x)\nabla^{m+k}u,\nabla^{m+k}h) + 2\varepsilon^2 w(t)(\nabla^k u,\nabla^k h)\\
&+ 2\varepsilon w(t)((-\Delta)^k u, h) - 2w(t)(a(x)(-\Delta)^m h,(-\Delta)^k v).
\end{aligned}
$$

Using the Cauchy-Schwarz inequality,Young's inequality and Holder's inequality, etc. we have

$$2\varepsilon w(t)(a(x)\nabla^{m+k}u,\nabla^{m+k}h) \le \varepsilon^2 a_{00}\|\nabla^{m+k}u\|^2 + a_{00}^{-1}a_0^2|w(t)|^2\|\nabla^{m+k}h\|^2, \tag{4.37}$$

$$2\varepsilon^2 w(t)(\nabla^k u,\nabla^k h) \le \varepsilon^3\|\nabla^k u\|^2 + \varepsilon|w(t)|^2\|\nabla^k h\|^2, \tag{4.38}$$

$$2\varepsilon w(t)((-\Delta)^k u, h) \le \varepsilon\|\nabla^k v\|^2 + \varepsilon|w(t)|^2\|\nabla^k h\|^2, \tag{4.39}$$

$$
\begin{aligned}
&2w(t)(a(x)(-\Delta)^m h,(-\Delta)^k v)\\
&= 2w(t)(a(x)\nabla^{m+k}v,\nabla^{m+k}h) + 2w(t)\left(\sum_{i=1}^{m-k}C_{m-k}^i\nabla^{m+k-i}\nabla^i a(x),\nabla^{m+k}h\right)\\
&\le \frac{a_{00}}{4}\|\nabla^{m+k}v\|^2 + 4a_{00}^{-1}a_0^2|w(t)|^2\|\nabla^{m+k}\|^2 + \frac{a_{00}}{4}\|\nabla^{m+k}v\|^2 + 4a_{00}^{-1}|w(t)|^2\sum_{i=1}^{m-k}(C_{m-k}^i a_i)^2\lambda_1^{-i}\|\nabla^{m+k}h\|^2\\
&\le \frac{a_{00}}{2}\|\nabla^{m+k}v\|^2 + 4a_{00}^{-1}a_0^2|w(t)|^2\|\nabla^{m+k}\|^2 + 4a_{00}^{-1}|w(t)|^2\sum_{i=1}^{m-k}(C_{m-k}^i a_i)^2\lambda_1^{-i}\|\nabla^{m+k}h\|^2.
\end{aligned}
\tag{4.40}
$$

Substitute (4.37)–(4.40) into (4.36) to obtain

$$
\begin{aligned}
&\frac{d}{dt}[\|\nabla^k v\|^2 - \varepsilon(a(x)\nabla^{m+k}u,\nabla^{m+k}u) + \varepsilon^2\|\nabla^k u\|^2] + b_0M(\|\nabla^m u\|^2)\frac{d}{dt}(a(x)\nabla^{m+k}u,\nabla^{m+k}u)\\
&+ 2(a(x)\nabla^{m+k}v,\nabla^{m+k}v) - \frac{6+\varepsilon}{4}a_{00}\|\nabla^{m+k}v\|^2 - 3\varepsilon\|\nabla^k v\|^2 + 2\varepsilon b_0M(\|\nabla^m u\|^2)(a(x)\nabla^{m+k}u,\nabla^{m+k}u)\\
&- \frac{3\varepsilon a_{00}b_0M_0}{4}\|\nabla^{m+k}u\|^2 - 2\varepsilon^2(a(x)\nabla^{m+k}u,\nabla^{m+k}u) - \varepsilon^2 a_{00}\|\nabla^{m+k}u\|^2 + \varepsilon^3\|\nabla^k u\|^2\\
&\le \sum_{i=1}^{m-k}(1-\alpha_i)\left(\frac{a_{00}}{4(1-\alpha_i)(m-k)}\right)^{\frac{1+\alpha_i}{1-\alpha_i}}(C_{m-k}^i a_iC_i)^{\frac{2}{1-\alpha_i}}\|v\|^2\\
&\quad + 2\varepsilon\sum_{i=1}^{m-k}(1-\alpha_i)\left(\frac{a_{00}}{8\alpha_i(m-k)}\right)^{\frac{\alpha_i}{\alpha_i-1}}\left(\frac{2(m-k)}{a_{00}b_0M_0}(C_{m-k}^i a_iC_i)^2\right)^{\frac{1}{1-\alpha_i}}\|v\|^2\\
&\quad + 2\sum_{i=1}^{m-k}(1-\alpha_i)(\varepsilon^{-1}a_{00}^{-1}b_0M_0)^{\frac{1}{1-\alpha_i}}\left(\frac{a_{00}}{8\alpha_i(m-k)}\right)^{\frac{\alpha_i}{\alpha_i-1}}(C_{m-k}^i a_iC_i(1+\|\nabla^m u\|^{2q}))^{\frac{2}{1-\alpha_i}}\|v\|^2\\
&\quad + 4\varepsilon^{-1}a_{00}^{-1}b_0a_0^2M_0(1+\|\nabla^m u\|^{2q})^2|w(t)|^2\|\nabla^{m+k}h\|^2 + \frac{2\lambda_1^{-m}}{a_{00}}\|\nabla^k f(x,t)\|^2\\
&\quad + 2C_{k1}(r_{01}^p(\tau,w)+\|\phi_5(x)\|^2) + 5a_{00}^{-1}a_0^2|w(t)|^2\|\nabla^{m+k}\|^2 + 2\varepsilon|w(t)|^2\|\nabla^k h\|^2\\
&\quad + 4a_{00}^{-1}|w(t)|^2\sum_{i=1}^{m-k}(C_{m-k}^i a_i)^2\lambda_1^{-i}\|\nabla^{m+k}h\|^2.
\end{aligned}
\tag{4.41}
$$

When $\frac{d}{dt}(a(x)\nabla^{m+k}u,\nabla^{m+k}u) + 2\varepsilon(a(x)\nabla^{m+k}u,\nabla^{m+k}u) \ge 0$,

$$
\begin{aligned}
&b_0M(\|\nabla^m u\|^2)\left(\frac{d}{dt}(a(x)\nabla^{m+k}u,\nabla^{m+k}u) + 2\varepsilon(a(x)\nabla^{m+k}u,\nabla^{m+k}u)\right)\\
&\ge \frac{d}{dt}(b_0M_0(a(x)\nabla^{m+k}u,\nabla^{m+k}u)) + 2\varepsilon b_0M_0(a(x)\nabla^{m+k}u,\nabla^{m+k}u);
\end{aligned}
$$

else

$$
\begin{aligned}
&b_0M(\|\nabla^m u\|^2)\left(\frac{\mathrm{d}}{\mathrm{d}t}(a(x)\nabla^{m+k}u,\nabla^{m+k}u)+2\varepsilon(a(x)\nabla^{m+k}u,\nabla^{m+k}u)\right)\\
&\quad\geq\frac{\mathrm{d}}{\mathrm{d}t}(b_0C(\|\nabla^m u\|^2)(a(x)\nabla^{m+k}u,\nabla^{m+k}u))+2\varepsilon b_0C(\|\nabla^m u\|^2)(a(x)\nabla^{m+k}u,\nabla^{m+k}u),
\end{aligned}
$$

then (4.41) is transformed into

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\|\nabla^k v\|^2+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)(a(x)\nabla^{m+k}u,\nabla^{m+k}u)+\varepsilon^2\|\nabla^k u\|^2]\\
&\quad+2(a(x)\nabla^{m+k}v,\nabla^{m+k}v)-\frac{6+\varepsilon}{4}a_{00}\|\nabla^{m+k}v\|^2-3\varepsilon\|\nabla^k v\|^2\\
&\quad+2\varepsilon b_0M_0(C(\|\nabla^m u\|^2))(a(x)\nabla^{m+k}u,\nabla^{m+k}u)-\frac{3\varepsilon a_{00}b_0M_0}{4}\|\nabla^{m+k}u\|^2\\
&\quad-2\varepsilon^2(a(x)\nabla^{m+k}u,\nabla^{m+k}u)-\varepsilon^2a_{00}\|\nabla^{m+k}u\|^2+\varepsilon^3\|\nabla^k u\|^2\\
&\quad\leq\sum_{i=1}^{m-k}(1-\alpha_i)\left(\frac{a_{00}}{4(1-\alpha_i)(m-k)}\right)^{\frac{1+\alpha_i}{1-\alpha_i}}(C_{m-k}^i a_iC_i)^{\frac{2}{1-\alpha_i}}\|v\|^2\\
&\quad+2\varepsilon\sum_{i=1}^{m-k}(1-\alpha_i)\left(\frac{a_{00}}{8\alpha_i(m-k)}\right)^{\frac{\alpha_i}{\alpha_i-1}}\left(\frac{2(m-k)}{a_{00}b_0M_0}(C_{m-k}^i a_iC_i)^2\right)^{\frac{1}{1-\alpha_i}}\|v\|^2\\
&\quad+2\sum_{i=1}^{m-k}(1-\alpha_i)(\varepsilon^{-1}a_{00}^{-1}b_0M_0)^{\frac{1}{1-\alpha_i}}\left(\frac{a_{00}}{8\alpha_i(m-k)}\right)^{\frac{\alpha_i}{\alpha_i-1}}(C_{m-k}^i a_iC_i(1+\|\nabla^m u\|^{2q}))^{\frac{2}{1-\alpha_i}}\|v\|^2\\
&\quad+4\varepsilon^{-1}a_{00}^{-1}b_0a_0^2M_0(1+\|\nabla^m u\|^{2q})^2|w(t)|^2\|\nabla^{m+k}h\|^2+\frac{2\lambda_1^{-m}}{a_{00}}\|\nabla^k f(x,t)\|^2\\
&\quad+2C_{k1}(r_{01}^p(\tau,w)+\|\phi_5(x)\|^2)+5a_{00}^{-1}a_0^2|w(t)|^2\|\nabla^{m+k}\|^2+2\varepsilon|w(t)|^2\|\nabla^k h\|^2\\
&\quad+4a_{00}^{-1}|w(t)|^2\sum_{i=1}^{m-k}(C_{m-k}^i a_i)^2\lambda_1^{-i}\|\nabla^{m+k}h\|^2.
\end{aligned}\tag{4.42}
$$

By $2(a(x)\nabla^{m+k}v,\nabla^{m+k}v)\geq 2a_{00}\|\nabla^{m+k}v\|^2$, $(a(x)\nabla^{m+k}u,\nabla^{m+k}u)\geq a_{00}\|\nabla^{m+k}u\|^2$, (4.1), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\|\nabla^k v\|^2+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)(a(x)\nabla^{m+k}u,\nabla^{m+k}u)+\varepsilon^2\|\nabla^k u\|^2]\\
&\quad+\sigma_1[\|\nabla^k v\|^2+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)(a(x)\nabla^{m+k}u,\nabla^{m+k}u)+\varepsilon^2\|\nabla^k u\|^2]\\
&\quad\leq C_{k2}(1+|w(t)|^2)+\frac{2\lambda_1^{-m}}{a_{00}}\|\nabla^k f(x,t)\|^2.
\end{aligned}\tag{4.43}
$$

When $k=m$, (4.43) is also true, which will not be detailed here.

Using the Gronwall inequality to integrate (4.43) over $[\tau-t,\tau]$ and replacing w by $\theta_{-\tau}w$ we obtain

$$
\begin{aligned}
&e^{\sigma_1\tau}[\|\nabla^k v\|^2+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)(a(x)\nabla^{m+k}u,\nabla^{m+k}u)+\varepsilon^2\|\nabla^k u\|^2]\\
&\quad\leq e^{\sigma_1(\tau-t)}[\|\nabla^k v(\tau-t)\|^2+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)(a(x)\nabla^{m+k}u(\tau-t),\nabla^{m+k}u(\tau-t))\\
&\quad+\varepsilon^2\|\nabla^k u(\tau-t)\|^2]+\int_{\tau-t}^{\tau}e^{\sigma_1\xi}C_{k2}(1+|w(\xi)|^2)\mathrm{d}\xi+\frac{2\lambda_1^{-m}}{a_{00}}\int_{\tau-t}^{\tau}e^{\sigma_1\xi}\|\nabla^k f(x,\xi)\|^2\mathrm{d}\xi,
\end{aligned}\tag{4.44}
$$

moreover,

$$
\begin{aligned}
&\|\nabla^k v\|^2+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)(a(x)\nabla^{m+k}u,\nabla^{m+k}u)+\varepsilon^2\|\nabla^k u\|^2\\
&\quad\leq e^{-\sigma_1 t}[\|\nabla^k v(\tau-t)\|^2+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)(a(x)\nabla^{m+k}u(\tau-t),\nabla^{m+k}u(\tau-t))\\
&\quad+\varepsilon^2\|\nabla^k u(\tau-t)\|^2]+e^{-\sigma_1\tau}\int_{\tau-t}^{\tau}e^{\sigma_1\xi}C_{k2}(1+|w(\xi)|^2)\mathrm{d}\xi+\frac{2\lambda_1^{-m}}{a_{00}}e^{-\sigma_1\tau}\int_{\tau-t}^{\tau}e^{\sigma_1\xi}\|\nabla^k f(x,\xi)\|^2\mathrm{d}\xi.
\end{aligned}\tag{4.45}
$$

Since $(u(\tau-t), v(\tau-t)) \in B_k(\tau-t, \theta_{-\tau}w)$, when $t \to +\infty$,

$$\begin{aligned} &e^{-\sigma_1 t}[\|\nabla^k v(\tau-t)\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)(a(x)\nabla^{m+k}u(\tau-t), \nabla^{m+k}u(\tau-t)) + \varepsilon^2\|\nabla^k u(\tau-t)\|^2] \\ &\le e^{-\sigma_1 t}[\|\nabla^k v(\tau-t)\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)a_0\|\nabla^{m+k}u(\tau-t)\|^2 + \varepsilon^2\|\nabla^k u(\tau-t)\|^2] \to 0, \end{aligned} \tag{4.46}$$

then there exists $T_k = T_k(\tau, w, B_k)$ such that for all $t \ge T_k$

$$e^{-\sigma_1 t}[\|\nabla^k v(\tau-t)\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)a_0\|\nabla^{m+k}u(\tau-t)\|^2 + \varepsilon^2\|\nabla^k u(\tau-t)\|^2] \le 1. \tag{4.47}$$

When $|\xi| \to \infty$ $w(\xi)$ at most polynomial growth,

$$e^{-\sigma_1\tau}\int_{-\infty}^{\tau} e^{\sigma_1\xi}C_{k2}(1 + |w(\xi)|^2)\mathrm{d}\xi \equiv r_{0k}(\tau, w).$$

We get from (4.45)–(4.47), $(a(x)\nabla^{m+k}u, \nabla^{m+k}u) \ge a_{00}\|\nabla^{m+k}u\|^2$ that

$$\begin{aligned} &\|\nabla^k v(\tau, \tau-t, \theta_{-\tau}w, v_{\tau-t})\|^2 + \|\nabla^{m+k}u(\tau, \tau-t, \theta_{-\tau}w, v_{\tau-t})\|^2 \\ &\le C_{k3}\left(1 + \frac{2\lambda_1^{-m}}{a_{00}}e^{-\sigma_1\tau}\int_{-\infty}^{\tau} e^{\sigma_1\xi}\|\nabla^k f(x,\xi)\|^2\mathrm{d}\xi\right) + r_{0k}(\tau, w) \equiv r_{1k}(\tau, w), \end{aligned} \tag{4.48}$$

and $r_{1k}(\tau, w)$ are bounded.

Lemma 4.1 is derived from (4.23) and (4.48).

Lemma 4.1 is proved.

Considering the eigenvalue problem

$$(-\Delta)^{m+k}u = \lambda^{m+k}u,\ u|_\Gamma = 0,$$

the problem (3.8) has a family of eigenfunctions $\{e_j\}_{j=1}^{\infty}$ with the eigenvalues $\{\lambda_j\}_{j=1}^{\infty} : \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to \infty$ $(j \to \infty)$, such that $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of $L^2_\mu(\Omega)$. Given n let $Q_n = \operatorname{span}\{e_1, \cdot, e_n\}$ and $P_n : V_k(\Omega) \to Q_n$ be the projection operator.

Lemma 4.2. *Suppose M satisfies (M), $h(x) \in V_{m+k}(\Omega)$, $k = 0, 1, \ldots, m$ (3.2)–(3.6) hold $f(x,t)$ satisfies (F_1) and for $\forall\eta > 0$, $\tau \in R$, $w \in \Omega_1$, $B_k = \{B_k(\tau, w) : \tau \in R, w \in \Omega_1\} \in \mathcal{D}_k$, there exists $T_k = T_k(\tau, w, B_k, \eta_k) > 0$, $N_k = N_k(\tau, w, \eta_k) \ge 0$ such that the solution of (3.8) satisfies for $t \ge T_k$, $n \ge N_k$*

$$\|(I-P_n)v(\tau, \tau-t, \theta_{-\tau}w)\|^2_{V_k} + \|(I-P_n)u(\tau, \tau-t, \theta_{-\tau}w)\|^2_{V_{m+k}} \le \eta_k.$$

Proof. Let $u_{n,1} = P_n u$, $u_{n,2} = u - u_{n,1}$, $v_{n,1} = P_n v$, $v_{n,2} = v - v_{n,1}$. Applying $(I - P_n)$ to the second equation of (3.8), we obtain

$$\begin{aligned} \frac{\mathrm{d}v_{n,2}}{\mathrm{d}t} &= \varepsilon v_{n,2} - a(x)(-\Delta)^m v_{n,2} + \varepsilon a(x)(-\Delta)^m u_{n,2} - \varepsilon^2 u_{n,2} - b(x)(I-P_n)M(\|\nabla^m\|^2)(-\Delta)^m u \\ &\quad - (I-P_n)(g(x,u) + f(x,t)) + \varepsilon h(x)w(t) - a(x)(-\Delta)^m h(x)w(t). \end{aligned} \tag{4.49}$$

Taking the inner product of the resulting equation (4.49) with $v_{n,2}$ in $L^2_\mu(\Omega)$, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{n,2}\|^2_{L^2_\mu} &= \varepsilon\|v_{n,2}\|^2_{L^2_\mu} - \|\nabla^m v_{n,2}\|^2 + \varepsilon((-\Delta)^m u_{n,2}, v_{n,2}) - \varepsilon^2(u_{n,2}, v_{n,2})_{L^2_\mu} \\ &\quad - b_0((I-P_n)M(\|\nabla^m u\|^2)(-\Delta)^m u, v_{n,2}) - ((I-P_n)g(x,u), v_{n,2})_{L^2_\mu} \\ &\quad + ((I-P_n)f(x,t), v_{n,2})_{L^2_\mu} + \varepsilon w(t)(h(x), v_{n,2})_{L^2_\mu} - w(t)((-\Delta)^m h, v_{n,2}). \end{aligned} \tag{4.50}$$

Applying $I - P_n$ to the first equation of (3.8), we get

$$v_{n,2} = \frac{\mathrm{d}u_{n,2}}{\mathrm{d}t} + \varepsilon u_{n,2} - hw(t). \tag{4.51}$$

For the third and fourth terms on the right-hand side of (4.49), we obtain

$$\begin{aligned}
&\varepsilon((-\Delta)^m u_{n,2}, v_{n,2}) - \varepsilon^2(u_{n,2}, v_{n,2})_{L^2_\mu} \\
&= \varepsilon\Big((-\Delta)^m u_{n,2}, \frac{\mathrm{d}u_{n,2}}{\mathrm{d}t} + \varepsilon u_{n,2} - hw(t)\Big) - \varepsilon^2\Big(u_{n,2}, \frac{\mathrm{d}u_{n,2}}{\mathrm{d}t} + \varepsilon u_{n,2} - hw(t)\Big)_{L^2_\mu} \\
&= \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m u_{n,2}\|^2 + \varepsilon^2\|\nabla^m u_{n,2}\|^2 - \varepsilon w(t)((-\Delta)^m u_{n,2}, h) \\
&\quad -\frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{n,2}\|^2_{L^2_\mu} - \varepsilon^3\|u_{n,2}\|^2_{L^2_\mu} + \varepsilon^2 w(t)(u_{n,2}, h)_{L^2_\mu},
\end{aligned} \tag{4.52}$$

for the fifth term on the right-hand side of (4.49), we have

$$\begin{aligned}
&b_0((I-P_n)M(\|\nabla^m u\|^2)(-\Delta)^m u, v_{n,2}) \\
&= b_0\Big(M(\|\nabla^m u\|^2)(-\Delta)^m u_{n,2}, \frac{\mathrm{d}u_{n,2}}{\mathrm{d}t} + \varepsilon u_{n,2} - hw(t)\Big) \\
&= \frac{1}{2}b_0M(\|\nabla^m u\|^2)\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m u_{n,2}\|^2 + \varepsilon b_0 M(\|\nabla^m u\|^2)|\nabla^m u_{n,2}\|^2 - b_0M(\|\nabla^m u\|^2)w(t)((-\Delta)^m u_{n,2}, h),
\end{aligned} \tag{4.53}$$

for the sixth term on the right-hand side of (4.49), we have

$$\begin{aligned}
((I-P_n)g(x,u), v_{n,2})_{L^2_\mu} &= \Big((I-P_n)g(x,u), \frac{\mathrm{d}u_{n,2}}{\mathrm{d}t} + \varepsilon u_{n,2} - hw(t)\Big)_{L^2_\mu} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}((I-P_n)g(x,u), u_{n,2})_{L^2_\mu} - ((I-P_n)g_u(x,u)u_t, u_{n,2})_{L^2_\mu} \\
&\quad + \varepsilon((I-P_n)g(x,u), u_{n,2})_{L^2_\mu} - ((I-P_n)g(x,u), hw(t))_{L^2_\mu},
\end{aligned} \tag{4.54}$$

for the seventh, eighth, and ninth terms on the right-hand side of (4.49), we have

$$((I-P_n)f(x,t), v_{n,2})_{L^2_\mu} \le \frac{1}{4}\|\nabla^m v_{n,2}\|^2 + \frac{1}{\alpha_{n+1}\lambda_{n+1}^{m-1}}\|(I-P_n)f(x,t)\|^2_{L^2_\mu}, \tag{4.55}$$

$$\varepsilon w(t)(h(x), v_{n,2})_{L^2_\mu} \le \frac{1}{4}\|\nabla^m v_{n,2}\|^2 + \frac{\varepsilon|w(t)|^2}{\alpha_{n+1}\lambda_{n+1}^{m-1}}\|h(x)\|^2_{L^2_\mu}, \tag{4.56}$$

$$w(t)((-\Delta)^m h, v_{n,2}) \le \frac{1}{4}\|\nabla^m v_{n,2}\|^2 + \varepsilon|w(t)|^2\|\nabla^m h(x)\|^2. \tag{4.57}$$

When $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m u_{n,2}\|^2 + 2\varepsilon\|\nabla^m v_{n,2}\|^2 \ge 0$

$$b_0M(\|\nabla^m u\|^2)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m u_{n,2}\|^2 + 2\varepsilon\|\nabla^m v_{n,2}\|^2\Big) \ge \frac{\mathrm{d}}{\mathrm{d}t}(b_0M_0\|\nabla^m u_{n,2}\|^2) + 2\varepsilon b_0M_0\|\nabla^m v_{n,2}\|^2, \tag{4.58}$$

else

$$b_0M(\|\nabla^m u\|^2)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m u_{n,2}\|^2 + 2\varepsilon\|\nabla^m v_{n,2}\|^2\Big) \ge \frac{\mathrm{d}}{\mathrm{d}t}(b_0C(\|\nabla^m u\|^2)\|\nabla^m u_{n,2}\|^2) + 2\varepsilon b_0C(\|\nabla^m u\|^2)\|\nabla^m v_{n,2}\|^2. \tag{4.59}$$

By using Young’s inequality and Holder’s inequality, we can get

$$\begin{aligned}
2b_0M(\|\nabla^m u\|^2)w(t)((-\Delta)^m u_{n,2}, h) &\le 2b_0M(\|\nabla^m u\|^2)w(t)\|\nabla^m u_{n,2}\||w(t)|\|\nabla^m h\| \\
&\le \frac{\varepsilon b_0M_0}{2}\|\nabla^m u_{n,2}\|^2 + 2\frac{b_0C(\|\nabla^m u\|^2)}{\varepsilon M_0}|w(t)|^2\|\nabla^m h\|^2,
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
&2\varepsilon^2 w(t)(h(x), u_{n,2})_{L^2_\mu} - 2\varepsilon w(t)((-\Delta)^m u_{n,2}, h(x)) \\
&\le \varepsilon^3\|u_{n,2}\|^2_{L^2_\mu} + \varepsilon^2\|\nabla^m u_{n,2}\|^2 + \varepsilon|w(t)|^2\|h\|^2_{L^2_\mu} + |w(t)|^2\|\nabla^m h\|^2.
\end{aligned} \tag{4.61}$$

By (3.5), we have

$$\begin{aligned}2((I-P_n)g_u(x,u)u_t,u_{n,2})_{L^2_\mu} &\le 2\|\phi_4\|_{L^4_\mu}\|u_t\|_{L^2_\mu}\|u_{n,2}\|_{L^4_\mu}+2c_4\|u_t\|_{L^2_\mu}\|u\|^{p-1}_{L^{2p}_\mu}\|u_{n,2}\|_{L^{2p}_\mu}\\ &\le \frac{\varepsilon b_0M_0}{2}\|\nabla^m u_{n,2}\|^2+\frac{C_{05}}{\varepsilon b_0M_0}\lambda^{1-m}_{n+1}\|u_t\|^2_{L^2_\mu}+C_{06}\lambda^{1-m}_{n+1}\|u_t\|^2_{L^2_\mu}\|\nabla^m u\|^{2p-2},\end{aligned}\tag{4.62}$$

$$\begin{aligned}2w(t)((I-P_n)g(x,u),h)_{L^2_\mu} &\le 2|w(t)|\|\phi_1\|_{L^2_\mu}\|h\|_{L^2_\mu}+2c_1|w(t)|\|u\|^p_{L^{2p}_\mu}\|h\|_{L^2_\mu}\\ &\le 2|w(t)|\|\phi_1\|_{L^2_\mu}\|h\|_{L^2_\mu}+2C_{07}|w(t)|\|\nabla^m u\|^p\|h\|_{L^2_\mu}.\end{aligned}\tag{4.63}$$

By substituting (4.52)–(4.63) into (4.50), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|v_{n,2}\|^2_{L^2_\mu}+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)\|\nabla^m u_{n,2}\|^2+\varepsilon^2\|u_{n,2}\|^2_{L^2_\mu}+2((I-P_n)g(x,u),u_{n,2})_{L^2_\mu}\Big]\\ &+\frac{1}{2}\|\nabla^m v_{n,2}\|^2-2\varepsilon\|v_{n,2}\|^2_{L^2_\mu}+(2\varepsilon M(\|\nabla^m u\|^2)-\varepsilon M_0)b_0\|\nabla^m u\|^2\\ &-3\varepsilon^2\|\nabla^m u\|^2+\varepsilon^3\|u\|^2_{L^2_\mu}+2\varepsilon((I-P_n)g(x,u),u_{n,2})_{L^2_\mu}\\ &\le\left(1+2\frac{b_0C(\|\nabla^m u\|^2)}{\varepsilon M_0}+2\varepsilon\right)|w(t)|^2\|\nabla^m h\|^2+\left(\varepsilon+\frac{2\varepsilon}{\alpha_{n+1}\lambda^{m-1}_{n+1}}\right)|w(t)|^2\|h(x)\|^2_{L^2_\mu}\\ &+\frac{C_{05}}{\varepsilon b_0M_0}\lambda^{1-m}_{n+1}\|u_t\|^2_{L^2_\mu}+C_{06}\lambda^{1-m}_{n+1}\|u_t\|^2_{L^2_\mu}\|\nabla^m u\|^{2p-2}+\Big(2\|\phi_1\|_{L^2_\mu}\|h\|_{L^2_\mu}+2C_{08}\|\nabla^m u\|^p\Big)|w(t)|\|h\|_{L^2_\mu}\\ &+\frac{2}{\alpha_{n+1}\lambda^{m-1}_{n+1}}\|(I-p_n)f(x,t)\|^2_{L^2_\mu}.\end{aligned}\tag{4.64}$$

Because, when $N=1,2$, then $1\le p<+\infty$; when $N=3,4$, then $1\le p<\frac{N}{N-2}$. Moreover, when $n\to\infty$, $\lambda_n\to\infty$ so given $\eta_0>0$, there exists $N_{01}=N_{01}(\eta_0)\ge 1$ for $n\ge N_{01}$

$$\begin{aligned}&\left(1+2\frac{b_0C(\|\nabla^m u\|^2)}{\varepsilon M_0}+2\varepsilon\right)|w(t)|^2\|\nabla^m h\|^2+\left(\varepsilon+\frac{2\varepsilon}{\alpha_{n+1}\lambda^{m-1}_{n+1}}\right)|w(t)|^2\|h(x)\|^2_{L^2_\mu}\\ &+\frac{C_{05}}{\varepsilon b_0M_0}\lambda^{1-m}_{n+1}\|u_t\|^2_{L^2_\mu}+C_{06}\lambda^{1-m}_{n+1}\|u_t\|^2_{L^2_\mu}\|\nabla^m u\|^{2p-2}\\ &+\Big(2\|\phi_1\|_{L^2_\mu}\|h\|_{L^2_\mu}+2C_{08}\|\nabla^m u\|^p\Big)|w(t)|\|h\|_{L^2_\mu}+\frac{2}{\alpha_{n+1}\lambda^{m-1}_{n+1}}\|(I-p_n)f(x,t)\|^2_{L^2_\mu}\\ &\le C_{09}\eta_0\Big(1+|w(t)|^2+\|u_t\|^6_{L^2_\mu}+\|\nabla^m u\|^6\Big)+\frac{2}{\alpha_{n+1}\lambda^{m-1}_{n+1}}\|(I-p_n)f(x,t)\|^2_{L^2_\mu}.\end{aligned}\tag{4.65}$$

Then, there is an appropriate positive constant σ_2 so that (4.64) can be reduced to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|v_{n,2}\|^2_{L^2_\mu}+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)\|\nabla^m u_{n,2}\|^2+\varepsilon^2\|u_{n,2}\|^2_{L^2_\mu}+2((I-P_n)g(x,u),u_{n,2})_{L^2_\mu}\Big]\\ &+\sigma_2\Big[\|v_{n,2}\|^2_{L^2_\mu}+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)\|\nabla^m u_{n,2}\|^2+\varepsilon^2\|u_{n,2}\|^2_{L^2_\mu}+2((I-P_n)g(x,u),u_{n,2})_{L^2_\mu}\Big]\\ &\le C_{09}\eta_0\Big(1+|w(t)|^2+\|u_t\|^6_{L^2_\mu}+\|\nabla^m u\|^6\Big)+\frac{2}{\alpha_{n+1}\lambda^{m-1}_{n+1}}\|(I-p_n)f(x,t)\|^2_{L^2_\mu},\end{aligned}\tag{4.66}$$

integrating (4.66) over $(\tau-t,\tau)$ with $t\ge 0$ we get for all $n\ge N_{01}$

$$\begin{aligned}&\|v_{n,2}(\tau,\tau-t,w)\|^2_{L^2_\mu}+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)\|\nabla^m u_{n,2}(\tau,\tau-t,w)\|^2\\ &+\varepsilon^2\|u_{n,2}(\tau,\tau-t,w)\|^2_{L^2_\mu}+2((I-P_n)g(x,u),u_{n,2}(\tau,\tau-t,w))_{L^2_\mu}\end{aligned}\tag{4.67}$$

$$\begin{aligned}&\leq e^{-\sigma_2 t}\Big[\|v_{n,2}(\tau-t)\|^2_{L^2_\mu}+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)\|\nabla^m u_{n,2}(\tau-t)\|^2\\&\quad+\varepsilon^2\|u_{n,2}(\tau-t)\|^2_{L^2_\mu}+2((I-P_n)g(x,u(\tau-t)),u_{n,2}(\tau-t))_{L^2_\mu}\Big]\\&\quad+C_{09}\eta_0\int_{\tau-t}^{\tau}e^{\sigma(s-t)}\Big(1+|w(s)|^2+\|u_t(s,\tau-t,w,u_{1\tau})\|^6_{L^2_\mu}+\|\nabla^m u(s,\tau-t,w,u_{1\tau})\|^6\Big)ds\\&\quad+\frac{2}{\alpha_{n+1}\lambda_{n+1}^{m-1}}\int_{\tau-t}^{\tau}e^{\sigma(s-t)}\|(I-p_n)f(x,s)\|^2_{L^2_\mu}ds.\end{aligned}$$

Replacing w by $\theta_{-\tau}w$ in (4.67) for every $t\in R^+$, $\tau\in R$, $w\in\Omega_1$, $n\geq N_{01}$, we obtain

$$\begin{aligned}&\|v_{n,2}(\tau,\tau-t,\theta_{-\tau}w)\|^2_{L^2_\mu}+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)\|\nabla^m u_{n,2}(\tau,\tau-t,\theta_{-\tau}w)\|^2\\&\quad+\varepsilon^2\|u_{n,2}(\tau,\tau-t,\theta_{-\tau}w)\|^2_{L^2_\mu}+2((I-P_n)g(x,u),u_{n,2}(\tau,\tau-t,\theta_{-\tau}w))_{L^2_\mu}\\&\leq e^{-\sigma_2 t}[\|v_{n,2}(\tau-t)\|^2_{L^2_\mu}+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)\|\nabla^m u_{n,2}(\tau-t)\|^2\\&\quad+\varepsilon^2\|u_{n,2}(\tau-t)\|^2_{L^2_\mu}+2((I-P_n)g(x,u(\tau-t)),u_{n,2}(\tau-t))_{L^2_\mu}]\\&\quad+C_{09}\eta_0\int_{\tau-t}^{\tau}e^{\sigma(s-t)}(1+|w(s)|^2+\|u_t(s,\tau-t,\theta_{-\tau}w,u_{1\tau})\|^6_{L^2_\mu}+\|\nabla^m u(s,\tau-t,\theta_{-\tau}w,u_{1\tau})\|^6)ds\\&\quad+\frac{2}{\alpha_{n+1}\lambda_{n+1}^{m-1}}\int_{\tau-t}^{\tau}e^{\sigma(s-t)}\|(I-p_n)f(x,s)\|^2_{L^2_\mu}ds.\end{aligned}\tag{4.68}$$

From (3.8), $f(x,t)$ satisfies (F_1), $h\in V_m$ and Lemma 4.1, we have

$$\begin{aligned}&\|v_{n,2}(\tau,\tau-t,\theta_{-\tau}w)\|^2_{L^2_\mu}+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)\|\nabla^m u_{n,2}(\tau,\tau-t,\theta_{-\tau}w)\|^2+\varepsilon^2\|u_{n,2}(\tau,\tau-t,\theta_{-\tau}w)\|^2_{L^2_\mu}\\&\quad+2((I-P_n)g(x,u),u_{n,2}(\tau,\tau-t,\theta_{-\tau}w))_{L^2_\mu}\\&\quad\leq e^{-\sigma_2 t}\Big[\|v_{n,2}(\tau-t)\|^2_{L^2_\mu}+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)\|\nabla^m u_{n,2}(\tau-t)\|^2+\varepsilon^2\|u_{n,2}(\tau-t)\|^2_{L^2_\mu}\\&\qquad+2((I-P_n)g(x,u(\tau-t)),u_{n,2}(\tau-t))_{L^2_\mu}\Big]+C_{09}\eta_0r_{00}^3\int_{\tau-t}^{\tau}e^{\sigma(s-t)}(1+|w(s)|^2+|w(s)|^6)ds\\&\qquad+\frac{2}{\alpha_{n+1}\lambda_{n+1}^{m-1}}\int_{\tau-t}^{\tau}e^{\sigma(s-t)}\|(I-p_n)f(x,s)\|^2_{L^2_\mu}\,ds,\end{aligned}\tag{4.69}$$

by $(u_{\tau-t},v_{\tau-t})\in B_0(\tau-t,\theta_{-\tau}w)$, then

$$\begin{aligned}&e^{-\sigma_2 t}[\|v_{n,2}(\tau-t)\|^2_{L^2_\mu}+(b_0M_0(C(\|\nabla^m u\|^2))-\varepsilon)\|\nabla^m u_{n,2}(\tau-t)\|^2+\varepsilon^2\|u_{n,2}(\tau-t)\|^2_{L^2_\mu}\\&\quad+2((I-P_n)g(x,u(\tau-t)),u_{n,2}(\tau-t))_{L^2_\mu}]\to 0,\quad t\to\infty.\end{aligned}\tag{4.70}$$

Taking the inner product of (4.49) with $(-\Delta)^k v_{n,2}$, $k=1,2,\ldots,m-1$ in $L^2(\Omega)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla^k v_{n,2}\|^2&=\varepsilon\|\nabla^k v_{n,2}\|^2-(a(x)(-\Delta)^m v_{n,2},(-\Delta)^k v_{n,2})+\varepsilon(a(x)(-\Delta)^m u_{n,2},(-\Delta)^k v_{n,2})\\&\quad-\varepsilon^2(u_{n,2},(-\Delta)^k v_{n,2})-(b(x)M(\|\nabla^m u\|^2)(-\Delta)^m u_{n,2},(-\Delta)^k v_{n,2})\\&\quad-((I-P_n)g(x,u),(-\Delta)^k v_{n,2})+(f(x,t),(-\Delta)^k v_{n,2})+\varepsilon w(t)(h,(-\Delta)^k v_{n,2})\\&\quad-w(t)(a(x)(-\Delta)^m h,(-\Delta)^k v_{n,2}).\end{aligned}\tag{4.71}$$

Then applying $I-P_n$ to the first equation of (3.8), we obtain

$$v_{n,2}=\frac{du_{n,2}}{dt}+\varepsilon u_{n,2}-hw(t).\tag{4.72}$$

Combining the processing method of Lemma 4.1,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\|\nabla^k v_{n,2}\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)(a(x)\nabla^{m+k}u_{n,2}, \nabla^{m+k}u_{n,2}) + \varepsilon^2\|\nabla^k u_{n,2}\|^2]\\
&\quad + 2(a(x)\nabla^{m+k}v_{n,2}, \nabla^{m+k}v_{n,2}) - \frac{6+\varepsilon}{4}a_{00}\|\nabla^{m+k}v_{n,2}\|^2 - 3\varepsilon\|\nabla^k v_{n,2}\|^2\\
&\quad + 2\varepsilon b_0M_0(C(\|\nabla^m u\|^2))(a(x)\nabla^{m+k}u_{n,2}, \nabla^{m+k}u_{n,2}) - \frac{3\varepsilon a_{00}b_0M_0}{4}\|\nabla^{m+k}u_{n,2}\|^2\\
&\quad - 2\varepsilon^2(a(x)\nabla^{m+k}u_{n,2}, \nabla^{m+k}u_{n,2}) - \varepsilon^2 a_{00}\|\nabla^{m+k}u_{n,2}\|^2 + \varepsilon^3\|\nabla^k u_{n,2}\|^2\\
&\le \sum_{i=1}^{m-k}(1-\alpha_i)\left(\frac{a_{00}}{4(1-\alpha_i)(m-k)}\right)^{\frac{1+\alpha_i}{1-\alpha_i}}(C_{m-k}^i a_i C_i)^{\frac{2}{1-\alpha_i}}\|v_{n,2}\|^2\\
&\quad + 2\varepsilon\sum_{i=1}^{m-k}(1-\alpha_i)\left(\frac{a_{00}}{8\alpha_i(m-k)}\right)^{\frac{\alpha_i}{\alpha_i-1}}\left(\frac{2(m-k)}{a_{00}b_0M_0}(C_{m-k}^i a_i C_i)^2\right)^{\frac{1}{1-\alpha_i}}\|v_{n,2}\|^2\\
&\quad + 2(\varepsilon^{-1}a_{00}^{-1}b_0M_0)^{\frac{1}{1-\alpha_i}}\sum_{i=1}^{m-k}(1-\alpha_i)\left(\frac{a_{00}}{8\alpha_i(m-k)}\right)^{\frac{\alpha_i}{\alpha_i-1}}(C_{m-k}^i a_i C_i(1+\|\nabla^m u\|^{2q}))^{\frac{2}{1-\alpha_i}}\|v_{n,2}\|^2\\
&\quad + 4\varepsilon^{-1}a_{00}^{-1}b_0a_0^2M_0(1+\|\nabla^m u\|^{2q})^2|w(t)|^2\|\nabla^{m+k}h\|^2 + \frac{2\lambda_{n+1}^{-m}}{a_{00}}\|\nabla^k f(x,t)\|^2\\
&\quad + 2C_{k1}(r_{01}^p(\tau,w) + \|\phi_5(x)\|^2) + 5a_{00}^{-1}a_0^2|w(t)|^2\|\nabla^{m+k}h\|^2 + 2\varepsilon|w(t)|^2\|\nabla^k h\|^2\\
&\quad + 4a_{00}^{-1}|w(t)|^2\sum_{i=1}^{m-k}(C_{m-k}^i a_i)^2\lambda_1^{-i}\|\nabla^{m+k}h\|^2.
\end{aligned} \tag{4.73}$$

Then there is a positive constant σ_1 (4.73) can be reduced to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\|\nabla^k v_{n,2}\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)(a(x)\nabla^{m+k}u_{n,2}, \nabla^{m+k}u_{n,2}) + \varepsilon^2\|\nabla^k u_{n,2}\|^2]\\
&\quad + \sigma_1[\|\nabla^k v_{n,2}\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)(a(x)\nabla^{m+k}u_{n,2}, \nabla^{m+k}u_{n,2}) + \varepsilon^2\|\nabla^k u_{n,2}\|^2]\\
&\le C_{k4}(1+|w(t)|^2) + \frac{2\lambda_{n+1}^{-m}}{a_{00}}\|\nabla^k f(x,t)\|^2.
\end{aligned} \tag{4.74}$$

when $k = m$, (4.74) also holds. □

Integrating (4.74) over $(\tau - t, \tau)$ with $t \ge 0$, we get for all $n \ge N_{k1}$

$$\begin{aligned}
&\|\nabla^k v_{n,2}\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)(a(x)\nabla^{m+k}u_{n,2}, \nabla^{m+k}u_{n,2}) + \varepsilon^2\|\nabla^k u_{n,2}\|^2\\
&\le e^{-\sigma_1 t}[\|\nabla^k v_{n,2}(\tau-t)\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)(a(x)\nabla^{m+k}u_{n,2}(\tau-t), \nabla^{m+k}u_{n,2}(\tau-t))\\
&\quad + \varepsilon^2\|\nabla^k u_{n,2}(\tau-t)\|^2] + e^{-\sigma_1\tau}\int_{\tau-t}^{\tau}e^{\sigma_1\xi}C_{k4}(1+|w(\xi)|^2)\mathrm{d}\xi + \frac{2\lambda_1^{-m}}{a_{00}}e^{-\sigma_1\tau}\int_{\tau-t}^{\tau}e^{\sigma_1\xi}\|\nabla^k f(x,\xi)\|^2 d\xi.
\end{aligned} \tag{4.75}$$

Replacing w by $\theta_{-\tau}w$ in (4.75), we obtain for every $t \in R^+$, $\tau \in R$, $w \in \Omega_1$, $n \ge N_{k1}$

$$\begin{aligned}
&\|\nabla^k v_{n,2}(\tau, \tau-t, \theta_{-\tau}w)\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)\\
&\quad \times (a(x)\nabla^{m+k}u_{n,2}(\tau,\tau-t,\theta_{-\tau}w), \nabla^{m+k}u_{n,2}(\tau,\tau-t,\theta_{-\tau}w)) + \varepsilon^2\|\nabla^k u_{n,2}(\tau,\tau-t,\theta_{-\tau}w)\|^2\\
&\le e^{-\sigma_1 t}[\|\nabla^k v_{n,2}(\tau-t)\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)(a(x)\nabla^{m+k}u_{n,2}(\tau-t), \nabla^{m+k}u_{n,2}(\tau-t))\\
&\quad + \varepsilon^2\|\nabla^k u_{n,2}(\tau-t)\|^2] + e^{-\sigma_1\tau}\int_{\tau-t}^{\tau}e^{\sigma_1\xi}C_{k4}(1+|w(\xi)|^2)\mathrm{d}\xi + \frac{2\lambda_1^{-m}}{a_{00}}e^{-\sigma_1\tau}\int_{\tau-t}^{\tau}e^{\sigma_1\xi}\|\nabla^k f(x,\xi)\|^2\mathrm{d}\xi.
\end{aligned} \tag{4.76}$$

From (3.8), $h \in V_{m+k}$, $f(x,t)$ satisfies (F_1) Lemma 4.1 and $(u_{\tau-t}, v_{\tau-t}) \in B_k(\tau-t, \theta_{-\tau}w)$ for $t \to +\infty$

$$\begin{aligned}
&e^{-\sigma_1 t}[\|\nabla^k v_{n,2}(\tau-t)\|^2 + (b_0M_0(C(\|\nabla^m u\|^2)) - \varepsilon)(a(x)\nabla^{m+k}u_{n,2}(\tau-t), \nabla^{m+k}u_{n,2}(\tau-t))\\
&\quad + \varepsilon^2\|\nabla^k u_{n,2}(\tau-t)\|^2] \to 0.
\end{aligned} \tag{4.77}$$

Combining (4.69), (4.70), (4.76), (4.77), and Lemma 4.1, we can get the conclusion of Lemma 4.2. Lemma 4.2 is proved.

5 The existence of the family of random attractors

In this section, we shall prove the existence of the family of random pullback attractors for system (3.8). From Lemma 4.1, we know that for $P-a.e.$ $D_k = \{D_k(\tau, w) : \tau \in R, w \in \Omega_1\} \in \mathcal{D}_k$ and $w \in \Omega_1$, there exists $T_k = T_k(D_k, w)$ such that for all $t \geq T_k$

$$\|v(\tau, \tau-t, \theta_{-\tau}w, v_{\tau-t})\|_{V_k}^2 + \|\nabla^{m+k}u(\tau, \tau-t, \theta_{-\tau}w, u_{\tau-t})\|_{V_{m+k}}^2 \leq r_{1k}(\tau, w). \tag{5.1}$$

Let

$$B_k(\tau, w) = \left\{(u, v) \in V_{m+k} \times V_k : \|v\|_{V_k}^2 + \|u\|_{V_{m+k}}^2 \leq r_{1k}(\tau, w)\right\}. \tag{5.2}$$

Then, by (5.2), $B_k = \{B_k(\tau, w)\}_{w\in\Omega_1}$ are the closed absorption sets of Φ_k in X_k. We are now ready to prove the asymptotic compactness of Φ_k in X_k.

Lemma 5.1. *Suppose M satisfies (M), $h(x) \in V_{m+k}(\Omega)$, (3.2)–(3.6) hold $f(x,t)$ satisfies, (F_1), then Φ_k is asymptotically compact in X_k, that is, for every $\tau \in R$, $w \in \Omega_1$, the sequence $\{\Phi_k(t_i, \tau-t_i, \theta_{-t_i}w, (u_{\tau,i}, v_{\tau,i}))\}$ has a convergent subsequence in X_k provided $t_i \to \infty$ and*

$$(u_{\tau,i}, v_{\tau,i}) \in D_k\big(\tau - t_i, \theta_{-t_i}w\big);\ D_k = \{D_k(\tau, w) : \tau \in R, w \in \Omega_1\} \in \mathcal{D}_k.$$

Proof. We first let $t_i \to \infty$, it follows from Lemma 4.1 that there exist $i_1 = i_1(\tau, w, D_k) > 0$ such that for every $i \geq i_1$

$$\|v(\tau, \tau-t_i, \theta_{-\tau}w, v_\tau)\|_{V_k}^2 + \|u(\tau, \tau-t_i, \theta_{-\tau}w, u_\tau)\|_{V_{m+k}}^2 \leq r_{1k}(\tau, w), \tag{5.3}$$

next by using Lemma 4.2 for $\forall\eta_k > 0$, there are $i_{k2} = i_{k2}(\eta_k, w, B_k)$ and $N_k = N_k(\eta_k, w) > 0$ such that for every $i_k \geq i_{k2}$

$$\|(I-P_n)v(\tau, \tau-t_{ki}, \theta_{-\tau}w, v_\tau)\|_{V_k}^2 + \|(I-P_n)u(\tau, \tau-t_{ki}, \theta_{-\tau}w, u_\tau)\|_{V_{m+k}}^2 \leq \eta_k. \tag{5.4}$$

By using (5.3), we find that $\{P_{N_k}(u(\tau, \tau-t_i, w), v(\tau, \tau-t_i, w))\}$ is bounded in $P_{N_k}X_k$ and $P_{N_k}X_k$ is finite dimensional, which associates with (5.4) implies that $\{(u(\tau, \tau-t_i, w), v(\tau, \tau-t_i, w))\}$ is precompact in X_k. ☐

Theorem 5.2. *Suppose M satisfies (M), $h(x) \in V_{m+k}(\Omega)$, (3.2)–(3.6) hold $f(x,t)$ satisfies (F_1), then the family of cocycles Φ_k generated by (3.8) has a family of pullback $\mathcal{D}_k$ attractors $\{A_k\} = \{\{A_k(\tau, w)\} \in \mathcal{D}_k\ (k = 1, 2, \ldots, m)\}$ in X_k and can be expressed as follows:*

$$A_k(\tau, w) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}\Phi_k(t, \tau-t, \theta_{-t}w, B_k(\tau-t, \theta_{-t}w))}, \quad \tau \in R, \quad w \in \Omega_1.$$

Proof. From (5.2), Lemmas 5.1 and 2.10, the conclusion of Theorem 5.2 can be obtained.

Theorem 5.2 is proved. ☐

Note 5.3. Theorem 5.2 shows the family of cocycles Φ_k generated by (3.8) has a unique pullback attractor A_k, respectively, in the space $X_k (k = 0, 1, \ldots, m)$, which together form a family of pullback attractors $\{A_k\}$. At the same time, according to Lemma 4.1 and (5.2) and the tight embedding of $X_k \hookrightarrow X_0$, $k = 1, 2, \ldots, m$, get the corresponding a family of pullback attractors $\{A_k\}$, which is (X_k, X_0) the family of random weak attractors, which means that the family of cocycles Φ_k has uniformly asymptotically compact absorption sets $B_k(\tau, w) \subset X_0$, $k = 1, 2, \ldots, m$, where $B_k(\tau, w)$ are the bounded sets in X_k, i.e., Φ_k are asymptotically compact in X_0.

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