DE GRUYTER Open Mathematics 2022; 20: 366–390

Review Article

Octav Olteanu*

On Hahn-Banach theorem and some of its applications

https://doi.org/10.1515/math-2022-0001
received June 2, 2021; accepted December 5, 2021

Abstract: First, this work provides an overview of some of the Hahn-Banach type theorems. Of note, some of these extension results for linear operators found recent applications to isotonicity of convex operators on a convex cone. Next, the work investigates applications of the Krein-Milman theorem to representation theory and elements of Choquet theory. A sandwich theorem of intercalating an affine function h between f and g, where f and $-g$ are convex, $f \leq g$ on a finite-simplicial set, is recalled. Its recent topological version is also noted. Here, the novelty is that a finite-simplicial set may be unbounded in any locally convex topology on the domain space. Third, the paper summarizes and comments on recently published applications of a Hahn-Banach extension result for positive linear operators, combined with polynomial approximation on unbounded subsets, to the Markov moment problem. Some applications to concrete spaces are detailed as well. Finally, this work provides a characterization of a finite-dimensional convex bounded subset in terms of the property that any convex function defined on that subset is bounded below. This last property remains valid for a large class of convex operators.

Keywords: Hahn-Banach theorems, representations, finite simplicial sets, Markov moment problem, polynomial approximation

MSC 2020: 46A22, 47H07, 41A10, 46A55

1 Introduction

This work generally provides commentary on old and new aspects and applications of Hahn-Banach type results in representations related to the Choquet theory (such as sandwich results on finite-simplicial sets), the Markov moment problem and polynomial approximation on unbounded subsets, and characterization of finite-dimensional convex bounded subsets. All vector spaces appearing in the sequel are real vector spaces. One of the main consequences of the Hahn-Banach theorem is that on a Hausdorff locally convex space X, there are many enough linear continuous functionals, which separate the points of X. On the other hand, recent results require other consequences of a more general Hahn-Banach type theorem. Main such results are stated and discussed in the following sections and accompanied by applications. Versions in the framework of ordered topological vector spaces with normal cones are also emphasized, since concrete spaces have natural such structures. The assumption on order completeness of Y is motivated by applications of Hahn-Banach type results for extension of linear operators from a subspace of X to Y; usually, this extension preserves a sandwich condition defined by means of a dominating convex operator and a dominated concave operator. Generally, these operators are defined on arbitrary convex subsets of a real linear space X. When there is already a given linear order relation on X, the concave operator mentioned earlier is usually defined on X_+ and eventually might be null. When this is the case, the result is the

* **Corresponding author: Octav Olteanu,** Department of Mathematics-Informatics, University Politehnica of Bucharest, Bucharest 060042, Romania, e-mail: octav.olteanu50@gmail.com

positivity of the linear extension, while the convex dominating constraint controls the continuity and determines or evaluates the norm of the linear extension. Such constraints on the extension are motivated by concrete problems mentioned in the Abstract.

In Section 2, this work reviews almost all important versions and applications [1–10] of the Hahn-Banach theorem, starting with its geometric form [1] and going to the general forms stated and proved in [8–10]. Such results have found recent applications to the isotonicity of convex operators on convex cones [11]. If X, Y are ordered vector spaces, an operator $P: X_+ \to Y$ is called isotone if it is monotone increasing:

$$\mathbf{0} \le x_1 \le x_2 \text{ in } X \text{ implies } P(x_1) \le P(x_2).$$

A similar definition works for isotone operators defined on the entire space X (for example, if X is a vector lattice, then $P: X \to X,\ P(x) := x^+ = x \vee \mathbf{0},\ x \in X,$ is sublinear and isotone on X).

In Section 3, this work reviews elements of representations, Choquet's theory, and related results. This part is covered by references [2,3,12–14]. The first part of Section 3 is dedicated to the important notion of a barycenter of a probability measure on a compact convex subset K in a locally convex space X. Any point of such a subset is the barycenter of a probability measure μ on K, for which only the behavior on the extreme points of K matters. When K is metrizable, the representing measure μ is supported by the extreme points of K (Choquet's theorem). Conversely, any probability measure on an arbitrary compact convex subset K has a unique barycenter. This is unique because linear continuous functionals on X separate the points of X. All locally convex spaces involved in this work are assumed to be Hausdorff. To conclude this third part, the recent topological version [14] of one of the results of [13] on sandwich theorems over finite-simplicial sets is especially noted. Notably, a convex subset C of the real vector space X is called finite-simplicial if for any finite-dimensional compact subset $K \subseteq C$, there exists a finite-dimensional simplex S_y such that $K \subseteq S_y \subseteq C$. For example, in $\mathbb{R}^n$, $n \ge 2$, any convex cone C having a base that is a simplex is an unbounded finite-simplicial set. According to Theorem 3.12, on such a subset, the following result holds (see also [13], Corollary 3.5 for the proof).

Let X be an arbitrary vector space, C a finite-simplicial subset, $f: C \to \mathbb{R}$ a convex function, $g: C \to \mathbb{R}$ a concave function such that $f \le g$ on C. Then, there exists an affine function $h: C \to \mathbb{R}$ such that $f \le h \le g$.

Notably, in the aforementioned statement, the dominating function is concave, while the dominated one is convex. This differentiates this particular result from the usual Hahn-Banach type theorems based on the separation of convex subsets. Theorem 3.13 provides a topological version of this result (see [14], pp. 9 and 10 for the proof).

Section 4 is devoted to another field of applications of Hahn-Banach theorems and other results in Analysis and Functional Analysis, namely, to the classical moment problem. Being given a sequence $(y_j)_{j \in \mathbb{N}^n}$, $n \in \{1, 2, \ldots\}$ of real numbers and a closed subset $F \subseteq \mathbb{R}^n$, find a positive regular Borel measure (or a positive Radon measure) μ on F such that

$$\int_F t^j d\mu = y_j,\ j = (j_1, \ldots, j_n) \in \mathbb{N}^n,\ t^j = t_1^{j_1} \cdots t_n^{j_n}.$$

This is an inverse problem, because the measure μ is not known. Finding μ means characterizing its existence such that the aforementioned moment conditions are satisfied, studying its uniqueness (called determinacy) and eventually constructing it. All these should be done starting from the known moments y_j, $j \in \mathbb{N}^n$. This is a full moment problem since it involves all the moments $\int_F t^j d\mu$, $j \in \mathbb{N}^n$ of the measure μ. If we require only

$$\int_F t^j d\mu = y_j,\ j_k \in \{0, 1, \ldots, d\},\ k = 1, \ldots, n,$$

for some fixed natural number d, we have a truncated moment problem. For $n = 1$, the moment problem is called one dimensional, while for $n \ge 2$, we have a multidimensional moment problem. The references [15–24] concern various aspects of the moment problem. If an upper boundedness condition on μ is required, we have a Markov moment problem. Such a condition usually controls the norm of the linear positive continuous functional defined by the measure μ on a function space containing polynomials and compactly supported continuous real-valued function on F. Usually, such a space is a Banach lattice (for example, an L^p space, $1 \le p \le \infty$). The interested reader can find more information on Banach lattices in specialized monographs (see [25] and a part of [26]). For the present work, results from [1,11,26] on this topic are sufficient. For more details, see the introductory portion of Section 4. The papers [27–30] refer mainly to the Markov moment problem. Since the existence of a solution of the moment problem is an extension type result of a linear form defined on polynomials to a larger space, Section 4 is directly related to Hahn-Banach type results of Section 2. In the present work, existence and uniqueness of the solution for the full Markov moment problem are of special interest. The construction of a polynomial solution for the truncated moment problem is proposed in [28,29]. On the other hand, the notion of a moment determinate measure is basic because it leads to the existence and uniqueness of the linear operator solution, also controlling its norm (see [27,29]). An improved version of a result of [30] on operator-valued Markov moment problem is also stated.

Finally, in Section 5, this work seeks to revisit the result of [31] on convex operators $P : B \to Y$, where $B \subset \mathbb{R}^n$ is a convex bounded subset and Y is an order complete vector lattice. In [31], we proved that any such operator is bounded below on B. The proof was done by means of the existence of a subgradient of P at an arbitrary relative interior point of B. This way, the strong relationship between convex and linear operators is pointed out once more. Conversely, if B is a convex subset of an arbitrary (infinite-dimensional) real vector space X, such that any real convex function defined on B is bounded below, then B is finite-dimensional (and bounded). This is presented in Section 5.

The subjects reviewed later and the attached references relate to other fields of analysis and algebra (self-adjoint operators, symmetric matrixes, quadratic forms, fixed point theory, convex analysis, elements of Choquet theory, and polynomial approximation on Cartesian products of unbounded closed intervals).

Section 6 concludes the paper.

2 Various Hahn-Banach type results

The following lemma is the key result for the direct proof of the geometric version of Hahn-Banach theorem.

Lemma 2.1. (See [1], pp. 45–46). *Let X be a real topological vector space (t.v.s.) of dimension at least 2. If D is an open convex subset and* $\mathbf{0}$ *is not an element of D, then there exists a one-dimensional subspace of X not intersecting D.*

Lemma 2.1 and a standard application of Zorn's lemma yield:

Theorem 2.2. (See [1], p. 46). *Let X be a real t.v.s., let M be a linear manifold in X, and let D be a nonempty open convex subset of X, not intersecting M. Then, there exists a closed hyperplane H in X, containing M and not intersecting D.*

Corollary 2.3. *Let E be a t.v.s., C an open convex subset of E, E_1 a vector subspace of E such that $E_1 \cap C \neq \varnothing$, $T_1 \in L(E_1, \mathbb{R})$ a continuous linear functional, $P : C \to \mathbb{R}$ a convex upper semi-continuous functional such that $T_1(x) \le P(x)$ for all $x \in E_1 \cap C$. Then, there exists a continuous linear functional $T \in L(E, \mathbb{R})$, which extends T_1, such that $T(x) \le P(x)$ for all $x \in C$.*

To deduce Corollary 2.3 from Theorem 2.2, one applies Theorem 2.2, where X stands for $E \times \mathbb{R}$, M stands for the graph of T_1 ($M = \{(x, T_1(x)); x \in E_1\}$), D stands for $\{(x, t) \in C \times \mathbb{R}; P(x) < t\}$. According to Theorem 2.2, there exists a closed hyperplane H in $E \times \mathbb{R}$, which contains M, such that $H \cap D = \varnothing$. Due to condition $E_1 \cap C \neq \varnothing$, H cannot be vertical and hence is the graph of a linear functional $T \in L(E, \mathbb{R})$. From the details of this sketch of the proof, it is easy to observe that T extends T_1, $T(x) \le P(x)$, $x \in C$ and T is continuous (and linear) from E to $\mathbb{R}$ (see also [1, Exercise 6, p. 69]).

The next result holds in locally convex spaces. All such spaces are assumed to be Hausdorff.

Theorem 2.4. (See [1, Theorem 4.2, p. 49]). *Let X be a t.v.s., whose topology is locally convex. If T_1 is a linear form, defined and continuous on a subspace M of X, then T_1 has a continuous extension T to the entire space X.*

Corollary 2.5. *Given $n \in \{1, 2, \dots\}$ and n linearly independent elements x_ν of a l.c.s. X, there exist n continuous linear forms T_μ on X such that $T_\mu(x_\nu) = \delta_{\mu\nu}$, $(\mu, \nu = 1,\dots,n)$.*

The next result is basic in the finite-dimensional convex analysis due to its applications, including the maximum principle for convex functions.

Theorem 2.6. (Carathéodory; see [2], p. 7). *Let $K \subset \mathbb{R}^n$ ($n \in \{1, 2, \dots\}$) be a convex compact subset. Then, any $x \in K$ can be written as convex combination of at most $n + 1$ extreme points of K.*

A simple proof of Theorem 2.6 (by induction on the dimension n) is given in [2], pp. 7–8, essentially using Theorem 2.2 stated earlier. Here is a main application of Theorem 2.6 to convex optimization (in particular to linear optimization).

Corollary 2.7. (See [1, Exercise 26, p. 71]). *Let $K \subset \mathbb{R}^n$ be a nonempty compact subset. Then, its convex hull $co(K)$ is compact.*

Theorem 2.8. (See [3, p. 171). *If f is a continuous convex real function on a convex compact subset $K \subset \mathbb{R}^n$ ($n \in \{1, 2, \dots\}$), then f attains a global maximum at an extreme point of K.*

Theorem 2.9. (The maximum principle [3], p. 171). *Let C be a convex subset of $\mathbb{R}^n$. If a convex function $f : C \to \mathbb{R}$ attains its maximum on C at a point from the relative interior of C, then f is constant on C.*

Next, we recall the following basic results, derived from Theorem 2.2.

Theorem 2.10. (First separation theorem [1], p. 64). *Let A be a convex subset of a t.v.s. X, such that* $\mathrm{int}(A) \neq \emptyset$ *and let B be a nonempty convex subset of X, not intersecting the interior* $\mathrm{int}(A)$ *of A. There exists a closed hyperplane H separating A and B; if A and B are both open, H separates A and B strictly.*

Theorem 2.11. (Second separation theorem [1], p. 65). *Let A, B be nonempty, disjoint convex subsets of a locally convex Hausdorff space (l.c.s.) X, such that A is closed and B is compact. There exists a closed hyperplane in X strictly separating A and B.*

Corollary 2.12. *Let X be a l.c.s. and $x_1, x_2 \in X$, $x_1 \neq x_2$. Then, there exists a continuous linear functional $x^* \in X^*$ such that $x^*(x_1) \neq x^*(x_2)$.*

The preceding corollary states that the topological dual X^* of a l.c.s. X separates the points of X. On the other hand, by the definition of weak topology on a l.c.s. X, any weak closed subset of X is closed in the initial topology on X. For convex closed subsets, the reverse implication holds as well. Namely, we recall the following well-known consequence of Theorem 2.11:

Corollary 2.13. (See [1], p. 65) *Let X be a locally convex space and $C \subset X$ a convex closed subset. Then, C is the intersection of all closed half-spaces containing it. In particular, C is closed with respect to the weak topology $w(X, X^*)$ on X.*

The following result (Theorem 2.15) has a natural geometric meaning; it is based on Lemma 2.14 and Theorem 2.10. It is worth noticing that in the latter theorem, if we additionally assume that A is open, then A

is contained in the open half-space defined by H ([1]). Before stating Theorem 2.15, we have to review Lemma 2.14, which generalizes the formula for the distance from a point to a hyperplane in $\mathbb{R}^n$, $n \geq 2$, that is well known from analytical geometry.

Lemma 2.14. *Let X be a normed (real) linear space, $H = \{x \in X; T(x) = \alpha\}$ a closed hyperplane in X, $x_0 \in X$. Then, the distance $d(x_0, H) := \inf_{h \in H} \|x_0 - h\|$ is given by the following formula:*

$$d(x_0, H) = |T(x_0) - \alpha| / \|T\|$$

(here $T \in X^*$, $T \neq \mathbf{0}$, $\alpha \in \mathbb{R}$).

Theorem 2.15. (See [4]). *Let X be a normed linear space, A, B two convex subsets of X such that $d(A, B) := \inf_{(a,b) \in A \times B} \|a - b\| > 0$. Then, there exists two closed parallel hyperplanes H_1, H_2 in X, which separate the subsets A and B, such that $d(H_1, H_2) = d(A, B)$.*

The next key lemma is used in the proof of the main Theorem 2.17 (Krein-Milman).

Lemma 2.16. (See [1], p. 67). *If C is a compact, convex subset of a locally convex space, every closed hyperplane supporting C contains at least one extreme point of C.*

We recall that, by definition, a closed hyperplane H in the locally convex space X under attention is supporting C if $C \cap H \neq \emptyset$ and C is contained in one of the two half-spaces defined by H. A point $e \in C$ is called an extreme point of C if from $x_1, x_2 \in C$, $t \in (0, 1)$, the equality $e = (1 - t)x_1 + tx_2$ implies $x_1 = x_2 = e$. In other words, e cannot be an interior element of any line segment of ends elements of C.

Theorem 2.17. (Krein-Milman; see [1], p. 67). *Every compact convex subset of a locally convex space is the closed convex hull of its extreme points.*

Krein-Milman theorem says that in any compact convex subset C of a l.c.s., there are many extreme points, which generate C (any element of C is the limit of a net whose elements are convex combinations of extreme points of C).

Theorem 2.18. (See [1, Theorem 10.5, p. 68]). *If K is a compact subset of a locally convex space such that the closed convex hull C of K is compact, then each extreme point of C is an element of K.*

From Theorem 2.6 (Carathéodory), Corollary 2.7, and Theorem 2.18, the following consequence follows:

Corollary 2.19. *If $K \subset \mathbb{R}^n$ is a compact nonempty subset, then its convex hull $\mathrm{co}(K)$ is compact and $\mathrm{co}(K) = \mathrm{co}(\mathrm{Extr}(K))$. Moreover, each point of $\mathrm{co}(K)$ can be written as convex combination of at most $n + 1$ extreme points of K.*

The aforementioned results are more or less deduced from the geometric form of the Hahn-Banach theorem. In most of the cases motivated by further applications, analytic proofs of Hahn-Banach type theorems are more suitable. Here is the first main result, completely proved in [3, pp. 339–340].

Theorem 2.20. (The Hahn-Banach theorem). *Let X be a vector space, $P : X \to \mathbb{R}$ a sublinear functional, $M \subset X$ a vector subspace $L : M \to \mathbb{R}$ a linear functional, such that $L(x) \leq P(x)$ for all $x \in M$. Then, L has a linear extension $T : X \to \mathbb{R}$, such that T is dominated by P on the entire space X.*

Corollary 2.21. (See [3], p. 340) *If P is a sublinear functional on a real vector space X, then for every element $x_0 \in X$, there exists a linear functional T such that*

$$T(x_0) = P(x_0) \text{ and } T(x) \leq P(x) \text{ for all } x \in X.$$

Theorem 2.22. (The Hahn-Banach theorem on normed vector spaces; see [3], p. 341). *Let X_0 be a vector subspace of the real normed vector space X and $T_0 : X_0 \to \mathbb{R}$ a continuous linear functional. Then, T_0 has a continuous linear extension $T : X \to \mathbb{R}$, with $\|T\| = \|T_0\|$.*

Corollary 2.23. (See [3], p. 341). *If X is normed vector space, then for each $x_0 \in X,\ x_0 \neq 0$, there exists a linear functional T on X, such that $T(x_0) = \|x_0\|$, and $\|T\| = 1$.*

One of the reasons for using analytic proofs of Hahn-Banach type theorems is that they work not only for extending linear functional but also for operators. As in the case of functional, the proofs of such type results are quite simple, by means of Zorn's lemma and extension of linear operators from a subspace S of the involved domain space X, to a space $S \oplus \mathrm{Span}\{x_0\}$, where $x_0 \in X \backslash S$, preserving some constraints on the extension. The codomain of the operators for which Hahn-Banach type theorems hold must be order complete vector spaces, or even order complete vector lattices. We recall that an ordered vector space is a vector space Y endowed with an order relation, which is compatible with the algebraic structure of a vector space. Namely, the following two properties are satisfied:

$$y_1 \leq y_2,\ y \in Y \Rightarrow y_1 + y \leq y_2 + y,\ y_1 \leq y_2,\ \alpha \in \mathbb{R}_+ \Rightarrow \alpha y_1 \leq \alpha y_2.$$

We say that such an order relation is linear. If Y is an ordered vector space, then $Y_+ = \{y \in Y;\ y \geq 0\}$ is a convex cone, called the positive cone of Y. We always assume that the positive cone is generating ($Y = Y_+ - Y_+$). An ordered vector space Y is called order complete (Dedekind complete) if for any upper-bounded subset $B \subset Y$, there exists a least upper bound for B in Y, denoted by $\sup(B)$. A vector lattice is an ordered vector space Y with the property that for any $y_1, y_2 \in Y$, there exists $\sup\{y_1, y_2\} \in Y$. In a vector lattice Y, for any element $y \in Y$, one denotes $|y| = \sup\{y, -y\}$. An ordered Banach space is a Banach space Y, which is also an ordered vector space, such that the positive cone Y_+ is closed and the norm is monotone on Y_+:

$$0 \leq y_1 \leq y_2 \Rightarrow \|y_1\| \leq \|y_2\|.$$

A Banach lattice Y is a Banach space, which is also a vector lattice, such that

$$y_1, y_2 \in Y,\ |y_1| \leq |y_2| \Rightarrow \|y_1\| \leq \|y_2\|.$$

Obviously, any Banach lattice is an ordered Banach space. In an ordered Banach space, there exists also the compatibility of the topology defined by the norm with the order relation. There exist ordered Banach spaces that are not lattices. For example, the space Y of all $n \times n$ symmetric matrixes with real coefficients, endowed with the norm

$$\|V\| = \max_{\|x\| \leq 1} |\langle Vx, x\rangle|$$

and the order relation $V \preccurlyeq W \Leftrightarrow \langle Vx, x\rangle \leq \langle Wx, x\rangle$, for all $x \in \mathbb{R}^n,\ V, W \in Y$, is an ordered Banach space, which is not a lattice for $n \geq 2$. Here, the norm x is the Euclidean norm of the vector $\|x\| \in \mathbb{R}^n$. In the same way, if H is a real or complex Hilbert space, the real vector space $Y = \mathcal{A}(H)$ of all self-adjoint operators acting on H, with the norm and order relation defined similarly to the case of symmetric matrixes, is an ordered Banach space, which is not a lattice (here $\mathbb{R}^n$ is replaced by H). Almost all usual function spaces and sequence spaces have natural structures of Banach lattices. On a vector space $\mathcal{F}(S)$ of real-valued functions defined on a set S, the usual order relation is: $f \leq g \Leftrightarrow f(t) \leq g(t)$ for all $t \in S$. For example, if K is a compact Hausdorff topological space, the space $C(K)$ of all real-valued continuous functions over K is a Banach lattice with respect to the aforementioned order relation and usual norm. If we assume that K is compact, is connected, nonempty, and not reduced to a singleton, then $C(K)$ is not order complete. A particular such a Banach lattice is $C([0, 1])$. In other words, the only case when $C(K)$ is order complete is that of a totally disconnected space K. The Lebesgue spaces $L^p(F),\ 1 \leq p \leq \infty,\ F \subseteq \mathbb{R}^n$, and the sequence spaces l^p, $1 \leq p \leq \infty$, are order complete Banach lattices.

Here is one of the old results on this subject, with many applications to the vector-valued moment problem. Let X_1 be an ordered vector space whose positive cone $X_{1,+}$ is generating ($X_1 = X_{1,+} - X_{1,+}$). Recall that in such an ordered vector space X_1, a vector subspace S is called a majorizing subspace if for any $x \in X_1$

there exists $s \in S$ such that $x \leq s$. The following theorem holds. Here is a significant example of a majorizing subspace. Let $F \subseteq \mathbb{R}^n$ be a closed unbounded subset and $1 \leq \alpha < +\infty$. Let ν be a positive regular Borel measure on F, with finite moments of all orders. We denote $X := L_\nu^\alpha(F)$, X_1 the vector subspace of all functions $f \in X$ for which there exists a polynomial p such that $|f| \leq p$ on F. Then, the subspace $S := \mathcal{P}$ of all polynomial functions on F is a majorizing subspace of X_1. The space X_1 contains $C_0(F)$ (the subspace of all continuous compactly supported real functions on F), as well as the subspace $\mathcal{P}$ ($p \in \mathcal{P} \Rightarrow |p| = \sqrt{\mathbf{1}p^2} \leq (1 + p^2)/2 \in \mathcal{P}$). The subspace X_1 is dense in X, since it contains $C_0(F)$, which is dense in $L_\nu^\alpha(F) = X$.

Theorem 2.24. (See [5], Theorem 1.2.1). *Let X_1 be an ordered vector space whose positive cone is generating, $X_0 \subset X_1$ a majorizing vector subspace, Y an order complete vector space, $T_0 : X_0 \to Y$ a positive linear operator. Then, T_0 admits a positive linear extension $T : X_1 \to Y$.*

We continue with Hahn-Banach type theorems. Now a condition on the operator solution of being dominated by a convex operator defined on a convex subset of the domain space is required. In other words, a generalized Hahn-Banach theorem will be reviewed. The relationship between the next result and its corollary (existence of subgradients of convex operators) will appear clearly. A point x_0 of the subset A of a vector space X is called an (algebraic) interior point of A if for each $x \in X$ there is a positive λ_0 such that $\lambda x + (1 - \lambda)x_0 \in A$ for $|\lambda| \leq \lambda_0$. The point x_0 is said to be an (algebraic) relative interior point of A if for each x of the affine variety generated by A (affine hull of A) there is a positive λ_0 such that $\lambda x + (1 - \lambda)x_0 \in A$ for $|\lambda| \leq \lambda_0$. The set of all interior points of A is denoted by A^{int} and the set of all relative interior points by A^{ri}. For the next result, see [6, Theorem 2.1, pp. 284–286].

Theorem 2.25. (A generalized Hahn-Banach theorem; see [6], Theorem 2.1, p. 284). *Let X be a vector space, $M \subset X$ a vector subspace, Y an order complete vector space, $A \subseteq X$ a convex subset, $P : A \to Y$ a convex operator, $T_M : M \to Y$ a linear operator such that*

$$T_M(x) \leq P(x) \text{ for all } x \in M \cap A.$$

If $A^{\text{int}} \cap M \neq \emptyset$, then there exists a linear operator $T : X \to Y$ such that

$$T(x) = T_M(x) \text{ for all } x \in M \text{ and } T(x) \leq P(x) \text{ for all } x \in A.$$

Corollary 2.26. (See [6, Corollary 2.7, p. 286]). *Let X be a vector space, Y an order complete vector space, $A \subseteq X$ a convex subset, $P : A \to Y$ a convex operator. If $x_0 \in A^{ri}$, then there exists a linear operator $T : X \to Y$ such that*

$$T(x) - T(x_0) \leq P(x) - P(x_0) \text{ for all } x \in A. \tag{1}$$

A linear operator T satisfying (1) is called a subgradient of P at x_0. Corollary 2.26 says that a convex operator having as codomain an order complete vector space admits a subgradient at every relative interior point of its domain. This result (with a somewhat different proof) goes back to [7]. The set of all subgradients of P at x_0 is called the subdifferential of P at x_0 and is denoted by $\partial_{x_0}P$. This is a convex set, and, for convex operators P satisfying the hypothesis of Corollary 2.26, is nonempty.

In the results stated earlier, the order relation that naturally exists on concrete spaces does not appear on the domain space X in any way. The next theorems take into consideration linear order structures on X as well. This way, from now on, we have three conditions on the linear operator solution T. Namely, T must extend a given linear operator defined on a subspace of X, it is dominated by a given convex operator P and dominates a given concave operator Q. If $Q|_{X_+} \geq \mathbf{0}$, then the linear extension T is positive: $x \in X_+ \Rightarrow T(x) \in Y_+$. Recall that an ordered vector space X, which is also a topological vector space, is called an ordered topological vector space if the positive cone X_+ is topologically closed. The next result was published by H. Bauer, and independently by I. Namioka, with different proofs, in different journals, in 1957 (for citation of the original sources see [1, p. 227]).

Theorem 2.27. (See [1, Theorem 5.4, p. 227]). *Let X be an ordered t.v.s. with positive cone X_+ and M a vector subspace of X. For a linear form T_0 on M to have a linear continuous positive extension $T : X \to \mathbb{R}$ it is necessary and sufficient that T_0 be bounded above on $M \cap (U - X_+)$, where U is a suitable convex $\mathbf{0}$ – neighborhood in X.*

The next result is motivated by Theorem 2.27 and the discussion preceding it. Subsequently, all theorems are valid for operators. In particular, the corresponding cases of real-valued functionals follow as consequences. In the next theorem, X will be a real vector space, Y an order-complete vector lattice, $A, B \subseteq X$ convex subsets, $Q : A \to Y$ a concave operator, $P : B \to Y$ a convex operator, $M \subset X$ a vector subspace, and $T_0 : M \to Y$ a linear operator. All vector spaces and linear operators are considered over the real field.

Theorem 2.28. (See [8], Theorem 1). *Assume that $T_0(x) \geq Q(x)\ \forall x \in M \cap A$, $T_0(x) \leq P(x)\ \forall x \in M \cap B$. The following two statements are equivalent.*

(*a*) *There exists a linear extension $T : X \to Y$ of the operator T_0 such that*

$$T|_A \geq Q,\ \ T|_B \leq P;$$

(*b*) *There exists $P_1 : A \to Y$, convex, and $Q_1 : B \to Y$ concave operator such that for all*

$$(\rho, t, \lambda, a_1, a, b_1, b, v) \in [0,1]^2 \times (0, \infty) \times A^2 \times B^2 \times M,$$

the following implication holds:

$$(1-t)a_1 - tb_1 = v + \lambda((1-\rho)a - \rho b)$$
$$\Rightarrow (1-t)P_1(a_1) - tQ_1(b_1) \geq T_0(v) + \lambda((1-\rho)Q(a) - \rho P(b)).$$

It is worth noticing that the extension T of Theorem 2.28 satisfies the following conditions: is an extension of T_0, is dominated by P on B, and dominates Q on A. Here, the convex subsets A, B are arbitrary, with no restriction on the existence of relative interior points or on their position with respect to the subspace M.

The following theorems follow more or less directly as corollaries of Theorem 2.28. For details, see [8,9], while for applications to the abstract Markov moment problem, see all the results of [10]. For applications to characterizing the isotonicity of a convex operator over a convex cone, see [11] (for example, the proof of Theorem 5 of [11] uses Theorem 2.33 of this article, Theorem 6 of [11] uses Theorem 2.34 of this article, and Proposition 1 of [11] applies Theorem 2.30 of this article. The same article [11] contains a large class of examples of concrete spaces and operators for which the developed theory works. Also, the article [11] gives a new proof for a known result: any linear positive operator acting between two ordered Banach spaces is continuous. In particular, this theorem works for operators acting between Banach lattices.

Theorem 2.29. (See [8], Theorem 2). *Let E be an ordered vector space, F an order complete vector space, $M \subset E$ a vector subspace, $T_1 : M \to F$ a linear operator, and $P : E \to F$ a convex operator. The following two statements are equivalent.*

(*a*) *There exists a positive linear extension $T : E \to F$ of T_1 such that $T \leq P$ on E;*

(*b*) *We have $T_1(h) \leq P(x)$ for all $(h, x) \in M \times E$ such that $h \leq x$.*

One observes that in the very particular case $E_+ = \{\mathbf{0}\}$, when the order relation on E is the equality, from Theorem 2.29, one obtains Hahn-Banach extension theorem for linear operators dominated by convex operators. When the convex operator P is defined only on the positive cone of E, one obtains the following variant of Theorem 2.29 (see [9] and [14], Theorem 5):

Theorem 2.30. *Let E be an ordered vector space, F an order complete vector space, $M \subset E$ be a vector subspace, $T_1 : M \to F$ be a linear operator, and $P : E_+ \to F$ be a convex operator. The following two statements are equivalent.*

(*a*) *There exists a positive linear extension* $T : E \to F$ *of* T_1 *such that* $T|_{E_+} \le P$;
(*b*) *We have* $T_1(h) \le P(x)$ *for all* $(h, x) \in M \times E_+$ *such that* $h \le x$.

In Theorem 5 of [14], a direct sharp proof for Theorem 2.30 is pointed out. The next result provides a sufficient condition on the given linear operators for the existence of the linear extensions. When $X = \mathbb{R}^2$, $Y = \mathbb{R}$, it has an interesting geometric meaning.

Theorem 2.31. (See [9]). *Let* X *be a locally convex space,* Y *an order complete vector lattice with strong order unit* u_0 *and* $S \subset X$ *a vector subspace. Let* $A \subset X$ *be a convex subset with the following properties:*
(*a*) *There exists a neighborhood* V *of the origin such that* $(S + V) \cap A = \varnothing$ (*that is, by definition,* A *and* S *are distanced*);
(*b*) A *is bounded.*

Then for any equicontinuous family of linear operators $\{f_j\}_{j\in J} \subset \mathcal{L}(S, Y)$ *and for any* $\tilde{y} \in Y_+ \backslash \{\mathbf{0}\}$, *there exists an equicontinuous family* $\{T_j\}_{j\in J} \subset \mathcal{L}(X, Y)$ *such that*

$$T_j(s) = f_j(s),\ s \in S,\ T_j(\psi) \ge \tilde{y},\ \psi \in A,\ j \in J.$$

Moreover, if V *is a convex balanced neighborhood of the origin such that*

$$f_j(V \cap S) \subset [-u_0, u_0], \quad (S + V) \cap A = \varnothing,$$

and if $\alpha > 0$ *such that* $P_V(a) \le \alpha\ \forall a \in A$ *and* $\alpha_1 > 0$ *is large enough such that* $\tilde{y} \le \alpha_1 u_0$, *then the following relations hold:*

$$T_j(x) \le (1 + \alpha + \alpha_1) P_V(x)\, u_0,\ x \in X, j \in J.$$

We have denoted by P_V the gauge attached to V.

The following theorem is also a Hahn-Banach type result (see Theorem 2.29), but is formulated in terms similar to those of the abstract Markov moment problem [10]. However, the condition $T(x_j) = y_j,\ j \in J$ of the abstract moment problem is replaced by $T(x_j) \ge y_j,\ j \in J$.

Theorem 2.32. (Mazur-Orlicz: see [10], Theorem 5). *Let* X *be a preordered vector space,* Y *an order complete vector space,* $\{x_j\}_{j\in J}$, $\{y_j\}_{j\in J}$ *families of elements in* X, *respectively in* Y, *and* $P : X \to Y$ *a sublinear operator. The following two statements are equivalent:*
(*a*) *There exists a linear positive operator* $T : X \to Y$ *such that*

$$T(x_j) \ge y_j,\ j \in J,\ T(x) \le P(x),\ x \in X;$$

(*b*) *For any finite subset* $J_0 \subset J$ *and any* $\{\alpha_j\}_{j\in J_0} \subset \mathbb{R}_+ = [0, +\infty)$, *the following implication holds true*

$$\sum_{j\in J_0} \alpha_j x_j \le x \in X \Rightarrow \sum_{j\in J_0} \alpha_j y_j \le P(x).$$

If in addition we assume that P *is isotone, the assertions* (*a*) *and* (*b*) *are equivalent to* (*c*), *where*

(*c*) *for any finite subset* $J_0 \subset J$ *and any* $\{\alpha_j\}_{j\in J_0} \subset \mathbb{R}_+$, *the following inequality holds:*

$$\sum_{j\in J_0} \alpha_j y_j \le P\left(\sum_{j\in J_0} \alpha_j x_j\right).$$

The next two variants of the same controlled regularity property of some linear operators are also consequence of Theorem 2.28. Recall that a linear operator T is called regular if it can be written as a difference of two positive linear operators V, W: $T = V - W$. If V is dominated by a given convex operator Ψ, we say that we have a controlled regularity for T. This terminology is motivated by the fact that in the topological framework, Ψ is assumed to be continuous and $V \le \Psi$ on the entire domain space usually implies the continuity of V. Sometimes, the norm of V can be evaluated as well.

Theorem 2.33. (See [9]) *Suppose that X is an ordered vector space, Y is an order complete vector lattice, and $P : X_+ \to Y$ is a convex operator. Then for any linear operator $T : X \to Y$, the following two statements are equivalent.*
(*a*) *There exist two positive linear operators $V, W : X \to Y$ such that $T = V - W$, $V|_{X_+} \le P$;*
(*b*) *$T(x_1) \le P(x_2)$ for all x_1, x_2 in X such that $0 \le x_1 \le x_2$.*

Most of convex operators P appearing in applications are defined on the entire domain space. Therefore, we recall the similar statement to that of Theorem 2.33, but for convex operators $P : X \to Y$.

Theorem 2.34. (See [8], Theorem 3). *Assume that X is an ordered vector space, Y is an order complete vector lattice and $P : X \to Y$ is a convex operator. For any linear operator $T : X \to Y$, the following two statements are equivalent:*
(*a*) *There exist two positive linear operators $V, W : X \to Y$ such that $T = V - W$, $V \le P$;*
(*b*) *$T(x_1) \le P(x_2)$ for all x_1, x_2 in X such that $0 \le x_1 \le x_2$.*

In the end of this section, we state a general constrained extension result, which can be proved as a consequence of Theorem 2.28. Probably, Theorems 2.28 and 2.35 are equivalent.

Theorem 2.35. (See [9]). *Let X be a vector space, Y be an order complete vector lattice, $M \subset X$ be a vector subspace, $T_0 : M \to Y$ be a linear operator, $A \subseteq X$ be a convex subset, and $Q : A \to Y$ be a concave operator. Assume that $T_0(x) \ge Q(x)\ \forall x \in M \cap A$. The following two statements are equivalent.*
(*a*) *There exists a linear operator $T : X \to Y$ which extends T_0, such that $T|_A \ge Q$;*
(*b*) *There exists a convex operator $P : A \to Y$ such that for all $(x, r, a) \in M \times (0, \infty) \times A$, the following implication holds:*

$$x + ra \in A \Rightarrow T_0(x) + rQ(a) \le P(x + ra).$$

Moreover, if P satisfies the requirements of (*b*), *then the extension T of* (*a*) *verifies the relation $T|_A \le P$.*

Since all concrete spaces are endowed with a natural linear order relation, we restate Theorem 2.35 in the framework of ordered vector spaces.

Theorem 2.36. *Let X be an ordered vector space, Y be an order complete vector lattice, $M \subset X$ be a vector subspace, $T_0 : M \to Y$ be a linear operator, $Q : X_+ \to Y$ be a supralinear operator, and $P : X_+ \to Y$ be a convex operator. The following two statements are equivalent.*
(*a*) *There exists a linear operator $T : X \to Y$, which extends T_0, such that $Q \le T|_{X_+} \le P$;*
(*b*) *For all $(h, \varphi_1, \varphi_2) \in M \times X_+ \times X_+$, the following implication holds:*

$$h = \varphi_2 - \varphi_1 \Rightarrow T_0(h) \le P(\varphi_2) - Q(\varphi_1).$$

Corollary 2.37. *Let X, Y, P, Q be as in the statement of* Theorem 2.36. *Assume that $Q \le P$ on X_+. Then, there exists a linear operator $T : X \to Y$, such that $Q \le T|_{X_+} \le P$.*

The last result of this section has also been deduced from the general Theorem 2.28. Theorem 2.38 is applied in the proof of Theorem 3.12 of the next section.

Theorem 2.38. (See [10, Theorem 4]). *Let X, Y, $\{x_j\}_{j \in J}, \{y_j\}_{j \in J}$ be as in* Theorem 2.32, *$T_1, T_2 \in L(X, Y)$ two linear operators. Assume also that Y is a vector lattice. The following two statements are equivalent.*
(*a*) *There is a linear operator $T \in L(X, Y)$ such that*

$$T_1(x) \le T(x) \le T_2(x),\ x \in X_+,\ T(x_j) = y_j,\ j \in J;$$

(*b*) *For any finite subset* $J_0 \subset J$ *and any* $\{\alpha_j\}_{j\in J_0} \subset \mathbb{R}$, *the following implication holds true:*

$$\left(\sum_{j\in J_0}\alpha_j x_j = \psi_2 - \psi_1,\ \psi_1, \psi_2 \in X_+\right) \Rightarrow \sum_{j\in J_0}\alpha_j y_j \le T_2(\psi_2) - T_1(\psi_1).$$

If X *is a vector lattice, then assertions* (*a*) *and* (*b*) *are equivalent to* (*c*), *where*

(*c*) $T_1(w) \le T_2(w)$ *for all* $w \in X_+$, *and for any finite subset* $J_0 \subset J$ *and* $\forall\{\alpha_j; j \in J_0\} \subset \mathbb{R}$, *we have*

$$\sum_{j\in J_0}\alpha_j y_j \le T_2\left(\left(\sum_{j\in J_0}\alpha_j x_j\right)^+\right) - T_1\left(\left(\sum_{j\in J_0}\alpha_j x_j\right)^-\right).$$

3 Krein-Milman theorem and elements of representation theory

We start with an interpretation of Carathéodory's Theorem 2.6 as an integral representation theorem (by means of a discrete measure). Then, by using Krein-Milman Theorem 2.17 and a passing to the limit procedure (eventually involving convergent subnets), one obtains integral representations in terms of arbitrary probability measures. In what follows, K is a compact convex nonempty subset of a (Hausdorff) locally convex space E. For $y \in K$, one denotes by δ_y the "point mass" at y, that is, δ_y is the Borel measure, which equals 1 on any Borel subset of K, which contains y, and equals 0 otherwise. According to these comments, if $x \in K$ and K is contained in an n-dimensional subspace of E, there exist $e_1, \ldots, e_{n+1}$ extreme points of K and $\alpha_1, \ldots, \alpha_{n+1}$ in $\mathbb{R}_+$, $\sum_{j=1}^{n+1}\alpha_j = 1$, such that $x = \sum_{j=1}^{n+1}\alpha_j e_j$. Let us denote $\mu = \sum_{j=1}^{n+1}\alpha_j\delta_{e_j}$. Then, for any continuous linear form L on E, one obtains:

$$\delta_x(L) = L(x) = \sum_{j=1}^{n+1}\alpha_j L(e_j) = \sum_{j=1}^{n+1}\alpha_j\delta_{e_j}(L) = \mu(L) := \int_K L\,d\mu. \tag{2}$$

Here, we recall that the first equality in (2) is actually the definition of the Dirac measure associated with the point $x \in K$, applied to the restriction to K of the continuous linear functional L on E. The conclusion $\delta_x(L) = \int_K L\,d\mu$ for all linear continuous forms L on E one reads as μ represents x. In the last equality (2), there is an abuse of notation: we denote in two different ways (μ and $d\mu$) the same measure μ on K. In what follows, a probability measure on K is a nonnegative regular Borel measure μ on K, with $\mu(K) = 1$.

Definition 3.1. Suppose that K is a nonempty compact subset of a locally convex space X and μ is a probability measure on K. A point x in X is said to be represented by μ if

$$L(x) = \int_K L\,d\mu$$

for every continuous linear functional L on X (other terminology: "x is the barycenter of μ" and "x is the resultant of μ").

Note that any point $x \in K$ is trivially represented by δ_x; the interesting fact pointed out by (2) is that for a convex compact subset K of a finite-dimensional space, each x in K may be represented by a probability measure, which "is supported" by the extreme points of K. A similar result holds for arbitrary convex compact metrizable subsets K of X (see Theorem 3.3).

Definition 3.2. If μ is a nonnegative regular Borel measure on the compact Hausdorff space K and B is a Borel subset of K, we say that μ is supported by B if $\mu(K \backslash B) = 0$.

Theorem 3.3. (Choquet). *Suppose that K is a metrizable compact convex subset of the locally convex space X, and that x_0 is an element of K. Then, there is a probability measure μ on K, which represents x_0 and is supported by the extreme points of K.*

For the proof of the preceding theorem, see [2, pp. 14–15]. The next result is somehow similar to Choquet's theorem, without requiring metrizability condition on K (see [2, p. 17]).

Theorem 3.4. (Choquet-Bishop-de Leeuw). *Suppose that K is a compact convex subset of the locally convex space X, and that x_0 is in K. Then, there is a probability measure μ on K, which represents x_0 and which vanishes on every Baire subset of K, which is disjoint from the set of extreme points of K.*

Theorems 3.3 and 3.4 claim that any point in K is the barycenter of a probability measure essentially defined by its behavior on the set of extreme points of K. The following question arises naturally: does any probability measure on K have a barycenter? The answer is affirmative, and, moreover, for a given probability measure μ on K, there exists a unique corresponding barycenter denoted $\mathrm{bar}(\mu)$. Namely, the following result holds:

Theorem 3.5. (See [3, Lemma 7.2.3, p. 310]). *If K is a compact convex subset in the locally convex space X and μ is a probability measure on K, there exists a unique point* $\mathrm{bar}(\mu) \in K$ *such that*

$$L(\mathrm{bar}(\mu)) = \int_K L d\mu$$

for all continuous linear functionals L on X.

Since all the locally convex spaces are assumed to be Hausdorff, the uniqueness of $\mathrm{bar}(\mu)$ follows from the fact that the topological dual X^* of X separates the points of X. The next result follows from the more general Theorem 7.2.4 of [3] and represents the Jensen integral inequality for a barycenter and probability measures.

Theorem 3.6. (Jensen; see [3]). *Suppose that μ is a probability measure on the convex compact subset K of the locally convex space X. Then,*

$$f(\mathrm{bar}(\mu)) \leq \int_K f(x) \mathrm{d}\mu(x)$$

for all continuous convex functions $f : K \to \mathbb{R}$.

Next, we recall some results on the uniqueness of the representing measure. The uniqueness holds if and only if the compact convex subset K is a simplex. Before going to infinite-dimensional simplexes, we review the definition of a finite-dimensional simplex. The sets of the form $C = \mathrm{co}(\{x_0, \ldots, x_N\})$ are called polytopes. If $x_1 - x_0, \ldots, x_N - x_0$ are linearly independent, then C is called an N-simplex, with vertices $x_0, \ldots, x_N$. In this case, $\dim C = N$ and any point x of C has a unique representation as a convex combination of vertices:

$$x = \sum_{k=0}^{N} \alpha_k x_k, \ \alpha_k \in \mathbb{R}_+ = [0, \infty), \ \sum_{k=0}^{N} \alpha_k = 1.$$

The numbers $\alpha_0, \ldots, \alpha_N$ are called the barycentric coordinates of x. The standard N-simplex (or unit N-simplex) in $\mathbb{R}^{N+1}$ is defined by:

$$\Delta^N = \left\{ (\alpha_0, \ldots, \alpha_N) \in \mathbb{R}^{N+1}; \ \sum_{k=0}^{N} \alpha_k = 1, \ \alpha_k \geq 0, \ k = 0, \ldots, N \right\}.$$

We go on with infinite-dimensional simplexes. As is shown in [2, pp. 51–52], for studying a compact convex subset K of a locally convex space X and see when K is a simplex, it is easier to assume that K is the base of a convex cone C (with vertex at the origin), i.e., $K \subset C$ and $y \in C$ if and only if there exists a unique $\alpha \geq 0$ and x in K such that $y = \alpha x$. Moreover, as discussed in [2, p. 52], whenever a compact convex subset is a base for a cone C, we can always assume that it is of the form $H \cap C$ for some closed hyperplane H in X, which misses the origin.

Definition 3.7. If a convex set K (not necessarily compact) is a base of a cone $\tilde{K}$, we call K a simplex if the space $\tilde{K} - \tilde{K}$ is a vector lattice in the ordering induced by $\tilde{K}$.

Definition 3.8. Let $K \subset X$ be a compact convex subset; if ν and μ are nonnegative regular Borel measures on K, we write $\nu \succ \mu$ if $\nu(f) \geq \mu(f)$ for all continuous convex functions f on K, where $\nu(f) := \int_K f d\nu$.

Lemma 3.9. (See [2, p. 18]). *If ν is a nonnegative measure on K, then there exists a maximal measure μ such that $\mu \succ \nu$.*

Theorem 3.10. (Choquet-Meyer; see [2], pp. 56–57). *Suppose that K is a nonempty compact convex subset of the locally convex space X. Then, K is a simplex if and only if for each point x in K there is a unique maximal measure μ_x on K such that $\mu_x(h) = h(x)$ for all continuous affine functions $h : K \to \mathbb{R}$.*

Next, we recall the statement of D.A. Edwards' separation theorem (Theorem 16.7 of [2]).

Theorem 3.11. (Edwards). *If f and $-g$ are convex upper semicontinuous real-valued functions on a simplex K contained in a locally convex space, with $f \leq g$, then there exists a continuous affine function h on K such that $f \leq h \leq g$.*

Of note, sandwich-type theorems such as Theorem 3.11 can be proved when the simplex K is replaced by a finite-simplicial set, as discussed in [13]. Here, the novelty is that a finite-simplicial set can be unbounded in any locally convex topology on E. A convex subset F of a vector space X is called finite simplicial if for any finite-dimensional compact subset $K \subseteq F$, there exists a finite-dimensional simplex S_y such that $K \subseteq S_y \subseteq F$. Here are a few examples:

1) In $\mathbb{R}^n$, $n \geq 2$, any convex cone C having a base that is a simplex (the corresponding order relation is laticial) is an unbounded finite simplicial set.
2) In $\mathbb{R}^n$, $n \geq 2$, for each $\alpha \in (1, \infty)$, the convex cone C defined by

$$C = \left\{ (x_1, \ldots, x_n);\; x_n \geq \left(\sum_{j=1}^{n-1} |x_j|^\alpha \right)^{1/\alpha} \right\}$$

   has a compact base, but C is not finite-simplicial.
3) Let X be an arbitrary infinite or finite-dimensional vector space (of dimension ≥2), $T : X \to \mathbb{R}$ a non-null linear functional and $r \in \mathbb{R}$. Then, the sets $F_1 = \{x; T(x) \geq r\}$, $F_2 = \{x; T(x) \leq r\}$ are finite-simplicial.
4) Let X, T be as in Example 3), α, β two real numbers such that $\alpha < \beta$. The set

$$\{x \in X;\; \alpha \leq T(x) \leq \beta\}$$

is not finite-simplicial. From the last two examples, we easily infer that generally the intersection of two finite-simplicial sets is not finite-simplicial.

The following sandwich type result holds true:

Theorem 3.12. (See [13], Corollary 3.5). *Let X be an arbitrary vector space, F a finite-simplicial subset, $f : F \to \mathbb{R}$ a convex function, $g : F \to \mathbb{R}$ a concave function such that $f \leq g$ on F. Then, there exists an affine function $h : F \to \mathbb{R}$ such that $f \leq h \leq g$.*

The proof of Theorem 3.12 is using Theorem 2.38 of Section 2. Next, we state a topological version of Theorem 3.12.

Theorem 3.13. (See [14], Theorem 4, pp. 8–10). *Let X be an ordered Banach space. Assume that the positive cone X_+ is finite-simplicial and there exists $x_0 \in X_+$ such that $X_+ - x_0$ contains a balanced and absorbing convex subset. Let $f, -g : X_+ \to \mathbb{R}$ be convex continuous functions such that $f \le g$. Assume also that $f(\mathbf{0}) = g(\mathbf{0}) = 0$. Then, there exists a continuous linear functional $L : X \to \mathbb{R}$ such that $f \le L \le g$ on X_+.*

4 The moment problem and related results

We recall the classical formulation of the moment problem, under the terms of T. Stieltjes, given in 1894–1895: find the repartition of the positive mass on the nonnegative semi-axis, if the moments of arbitrary orders k $(k = 0, 1, 2,\dots)$ are given. Precisely, in the Stieltjes moment problem, a sequence of real numbers $(y_k)_{k\ge 0}$ is given and one looks for a nondecreasing real function $\sigma(t)$ $(t \ge 0)$, which verifies the moment conditions:

$$\int_0^\infty t^k \mathrm{d}\sigma = y_k, \ (k = 0, 1, 2, \dots).$$

This is a one-dimensional moment problem, on an unbounded interval. Namely, it is an interpolation problem with the constraint on the positivity of the measure $\mathrm{d}\sigma$. The numbers y_k, $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ are called the moments of the measure $\mathrm{d}\sigma$. Existence, uniqueness, and construction of the solution $\mathrm{d}\sigma$ are studied. The moment problem is an inverse problem: we are looking for an unknown measure, starting from its given moments. The direct problem might be: being given the measure $\mathrm{d}\sigma$ compute its moments $\int_0^\infty t^k \mathrm{d}\sigma$, $k = 0, 1, 2,\dots$. The connection with the positive polynomials and extensions of linear positive functional and operators is quite clear. Namely, if one denotes by φ_j, $\varphi_j(t) := t^j$, $j \in \mathbb{N}$, $t \in [0, +\infty)$, $\mathcal{P}$ the vector space of polynomials with real coefficients, and

$$T_0 : \mathcal{P} \to \mathbb{R}, \ T_0\left(\sum_{j\in J_0} \alpha_j \varphi_j\right) := \sum_{j\in J_0} \alpha_j y_j, \tag{3}$$

where $J_0 \subset \mathbb{N}$ is a finite subset, then the moment conditions $T_0(\varphi_j) = y_j$, $j \in \mathbb{N}$ are clearly satisfied. It remains to check whether the linear form T_0 defined by (3) has nonnegative value at each nonnegative polynomial. If this condition is also accomplished, then one looks for the existence of a linear positive extension T of T_0 to a larger ordered function space X, which contains both $\mathcal{P}$ and the space of continuous compactly supported functions, then representing T by means of a positive regular Borel measure μ on $[0, +\infty)$, via Riesz representation theorem or applying Haviland theorem. Usually, the positive linear extension is defined on a Banach lattice of functions. For example, if ν is a positive regular Borel measure on $[0, +\infty)$, with finite moments $\int_0^\infty t^k \mathrm{d}\nu$ of all orders $k \in \mathbb{N}$, and $X = L^\alpha_\nu([0, +\infty))$, $1 \le \alpha < \infty$, one denotes by X_1 the vector subspace of X defined by $X_1 := \{g \in X; \exists p \in \mathcal{P}, \ |g| \le p\}$, X_1 contains $\mathcal{P}$ and all continuous real-valued compactly supported functions on $[0, +\infty)$. If T_0 defined by (3) is a positive (linear) functional on $\mathcal{P}$, one extends T_0 to a linear positive functional T on X_1, by means of Theorem 2.24 ($\mathcal{P}$ is a majorizing subspace in X_1). Usually, this extension is also continuous on the subspace X_1 of X. In this case, T can be extended to a linear continuous functional $\tilde{T}$ defined on the entire space X, via density of X_1 in X (the subspace of all continuous compactly supported functions on $[0, +\infty)$ is contained in X_1 and is dense in X). If an interval (for example, $[a, b]$, $\mathbb{R}$, or $[0, +\infty)$) is replaced by a closed subset F of $\mathbb{R}^n$, $n \ge 2$, we have a multidimensional moment problem. Passing to an example of the multidimensional real classical moment problem, let us denote

$$\varphi_j(t) = t^j = t_1^{j_1} \cdots t_n^{j_n}, \ j = (j_1,\dots,j_n) \in \mathbb{N}^n, \ t = (t_1,\dots,t_n) \in \mathbb{R}^n_+, \ n \in \mathbb{N}, \ n \ge \ 2.$$

If a sequence $(y_j)_{j\in\mathbb{N}^n}$ is given, one studies the existence, uniqueness, and construction of a linear positive form T defined on a function space containing polynomials and continuous compactly supported real functions, such that the moment conditions

$$T(\varphi_j) = y_j,\ j \in \mathbb{N}^n \tag{4}$$

are satisfied. Usually, the positive linear form T (that is called a solution for the moment problem defined by (4)) can be represented by means of a positive regular Borel measure μ on $\mathbb{R}_+^n$. In this case, we say that μ is a representing measure for the sequence $y = (y_j)_{j\in\mathbb{N}^n}$, and this sequence is called a moment sequence. Similar definitions and terminology are valid when we replace $\mathbb{R}_+^n$ with an arbitrary closed subset F of $\mathbb{R}^n$. When an upper constraint on the solution T is required too, we have a Markov moment problem (see the last part of this section). From solutions linear functional, many authors considered linear operators solutions. Of course, in this case, the moments $y_j,\ j \in \mathbb{N}^n$ are elements of an ordered vector space Y (usually Y is an order complete Banach lattice). The order completeness is necessary to apply Hahn-Banach type results for operators defined on polynomials and having Y as codomain. The classical moment problem is clearly related to the form of positive polynomials on the involved closed subsets of $\mathbb{R}^n$. As it is known, there exist nonnegative polynomials on the entire space $\mathbb{R}^n$, $n \geq 2$, which are not sums of squares of polynomials, unlike the case $n = 1$ (see [17], Proposition 13.4, p. 318; see also the comments which precede and follows this result). The analytic form of positive polynomials on closed intervals is crucial in solving classical moment problems. Such results are useful in characterizing the existence of a positive solution by means of signatures of quadratic forms. In the case of the Markov moment problem, approximation of nonnegative compactly supported continuous functions (with their support contained in a closed unbounded subset F) by special nonnegative polynomials on that subset, having known analytic form, is very important. For the multidimensional Markov moment problem on Cartesian products of closed unbounded intervals, this method works, provided that each interval is endowed with a moment determinate positive regular Borel measure. Recall that a measure is called *M-determinate* (*moment determinate*, or simply *determinate*) if it is uniquely determined by its classical moments, or, equivalently, by its values on polynomials. A moment sequence is called determinate if it has only one representing measure. If a sequence $\boldsymbol{y}$ has a representing measure supported on a compact subset F, then $\boldsymbol{y}$ is determinate thanks to the Weierstrass approximation theorem. We start reviewing existence of a solution for the simplest classical one-dimensional moment problems: the Hamburger moment problem (when $F = \mathbb{R}$), Stieltjes moment problem (when $F = \mathbb{R}_+$), and Hausdorff moment problem (when $F = [0, 1]$). In the sequel, the following notations are used: $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$, $C_0(F)$ is the vector space of all real-valued compactly supported continuous functions defined on F, $(C_0(F))_+$ is the convex cone of all functions in $C_0(F)$, which take nonnegative values at each point of F. $\mathcal{P}_+ = \mathcal{P}_+(F)$ is the convex cone of all polynomial functions with real coefficients, which are nonnegative on F.

Theorem 4.1. (Hamburger's theorem: see [17], Theorem 3.8, p. 63). *For a real sequence $y = (y_n)_{n\in\mathbb{N}}$, the following statements are pairwise equivalent.*

(*i*) *The sequence $\boldsymbol{y}$ is a Hamburger moment sequence, that is, there is a nonnegative Radon measure μ on $\mathbb{R}$ such that $t^j \in L^1_\mu(\mathbb{R}),\ j \in \mathbb{N}$ and*

$$\int_{\mathbb{R}} t^j d\mu(t) = y_j,\ j \in \mathbb{N}.$$

(*ii*) *The sequence y is positive semidefinite, i.e., for all $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n \in \mathbb{R}$, we have*

$$\sum_{i,j=0}^{n} y_{i+j}x_i x_j \geq 0.$$

(*iii*) *All Hankel matrices $H_n(y) = (y_{i+j})_{i,j=0}^n,\ n \in \mathbb{N}$ are positive semidefinite.*

(*iv*) *T_0 defined by (3) is a positive linear functional on $\mathbb{R}[t]$, that is, $T_0(p^2) \geq 0$ for $p \in \mathbb{R}[t]$.*

(*v*) *$T_0(q) \geq 0$ for all $q \in \mathcal{P}_+(\mathbb{R})$.*

Theorem 4.2. (See [17], p. 65). *For a real sequence $y = (y_n)_{n\in\mathbb{N}}$, the following statements are pairwise equivalent.*

(*i*) ***y*** *is a Stieltjes moment sequence, that is, there is a nonnegative Radon measure μ on $[0,\infty)$ such that $t^j \in L^1_\mu(\mathbb{R}_+)$, $j \in \mathbb{N}$ and*

$$\int_0^\infty t^j \mathrm{d}\mu(t) = y_j,\ j \in \mathbb{N}.$$

(*ii*) *For all $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n \in \mathbb{R}$, we have*

$$\sum_{i,j=0}^n y_{i+j}x_ix_j \geq 0,\ \sum_{i,j=0}^n y_{i+j+1}x_ix_j \geq 0.$$

(*iii*) *All Hankel matrixes $(y_{i+j})_{i,j=0}^n$, $(y_{i+j+1})_{i,j=0}^n$, $n \in \mathbb{N}$, are positive semidefinite.*

(*iv*) *$T_0(p^2) \geq 0$ and $T_0(tq^2) \geq 0$ for $p, q \in \mathbb{R}[t]$.*

(*v*) *$T_0(q) \geq 0$ for all $q \in \mathcal{P}_+(\mathbb{R}_+)$.*

Theorem 4.1 (respectively 4.2) gives necessary and sufficient conditions for a sequence $(y_n)_{n\in\mathbb{N}}$ of real numbers to be an $\mathbb{R}$-moment sequence (respectively an $\mathbb{R}_+$-moment sequence). Next, we go on with the corresponding problem on $[0,1]$ (the Hausdorff moment problem).

Theorem 4.3. (See [17], p. 66). *For a real sequence **y**, the following statements are pairwise equivalent:*

(*i*) ***y*** *is a $[0,1]$-moment sequence.*

(*ii*) *$T_0((1-t)^n t^k) \geq 0$ for $n.\ k \in \mathbb{N}$.*

(*iii*) *$\sum_{j=0}^n (-1)^j \binom{n}{j} y_{j+k} \geq 0$, for $n.\ k \in \mathbb{N}$.*

Next, we go on with the problem of determinacy. A Hamburger moment sequence is determinate if it has a unique representing measure, while a Stieltjes moment sequence is called determinate if it has only one representing measure supported on $[0,\infty)$. The Carleman theorem contains a powerful sufficient condition for determinacy.

Theorem 4.4. (See [17], Theorem 4.3, pp. 80–81). *Suppose that $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$ is a positive semidefinite sequence. The following assertions hold.*

(*i*) *If **y** satisfies the Carleman condition*

$$\sum_{n=1}^\infty y_{2n}^{-\frac{1}{2n}} = +\infty,$$

*then **y** is a determinate Hamburger moment sequence.*

(*ii*) *If in addition $(y_{n+1})_{n\in\mathbb{N}}$ is positive definite and*

$$\sum_{n=1}^\infty y_n^{-\frac{1}{2n}} = +\infty,$$

*then **y** is a determinate Stieltjes moment sequence.*

The following theorem of Krein consists in a sufficient condition for indeterminacy (for measures given by densities).

Theorem 4.5. (Krein condition: see [17], Theorem 4.14, pp. 85–86). *Let f be a nonnegative Borel function on $\mathbb{R}$. Suppose that the measure μ defined by $\mathrm{d}\mu = f(t)\mathrm{d}t$ is a Radon measure on R and has finite moments $y_n := \int_{\mathbb{R}} t^n \mathrm{d}\mu$ for all $n \in \mathbb{N}$.*

If

$$\int_{\mathbb{R}} \frac{-\ln(f(x))}{1+x^2}\mathrm{d}x < +\infty,$$

then the moment sequence $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$. *is M-indeterminate.*

Next, we give new checkable sufficient conditions on distributions of random variables that imply Carleman condition, ensuring determinacy. Consider two random variables $V \sim \Psi$, V with values in $\mathbb{R}$, $W \sim \Lambda$, W with values in $\mathbb{R}_+$. Assume that both Ψ and Λ belong to the class C^1 and let $\psi = \Psi'$, and $\lambda = \Lambda'$ be the corresponding densities. All moments of V, W are assumed to be finite. The symbol $\nearrow$ used later has the usual meaning of "monotone increasing."

Theorem 4.6. (See [19], Theorem 1, p. 498: Hamburger case). *Assume that the distribution ψ of V is symmetric on $\mathbb{R}$ and continuous and strictly positive outside an interval $(-t_0, t_0)$, $t_0 > 1$, such that the following conditions hold:*

$$\int_{|t|\geq t_0} \frac{-\ln \psi(t)}{t^2 \ln(|t|)}\mathrm{d}t = \infty,$$

$$\frac{-\ln \psi(t)}{\ln t} \nearrow \infty \text{ as } t_0 \leq t \to \infty.$$

Under these conditions, $V \sim \Psi$ satisfies Carleman's condition, and hence, it is M-determinate.

Theorem 4.7. (See [19], Theorem 2, p. 498: Stieltjes case). *Assume that the density λ of W is continuous and strictly positive on $[a, \infty)$ for some $a > 1$ such that the following conditions hold:*

$$\int_a^\infty \frac{-\ln \lambda(t^2)}{t^2 \ln t}\mathrm{d}t = +\infty,$$

$$\frac{-\ln \lambda(t)}{\ln t} \nearrow \infty \text{ as } a \leq t \to \infty.$$

Under these conditions, $W \sim \Lambda$ satisfies Carleman's condition, and hence, it is M-determinate.

Example 4.8. The distribution function Λ having as density $\lambda(u) = \exp(-u)$, $u \in \mathbb{R}_+$, satisfies the conditions of Theorem 4.7; hence, it is M-determinate.

Going back to the existence problem for a solution, we consider the multidimensional case, which is much more complicated than the one-dimensional moment problem. The main reason is that the analytic form of nonnegative polynomials on closed subsets of $\mathbb{R}^n$, $n \geq 2$, is generally not known in terms of sums of squares of polynomials. A case when this difficulty can be solved is that of semi-algebraic compact subsets of $\mathbb{R}^n$. Here is one of the main results on this subject. If $\mathbf{y} = (y_j)_{j\in\mathbb{N}^n}$ $n \geq 2$, is a sequence of real numbers, one denotes by $T_{\mathbf{y}}$ the linear functional defined on $\mathbb{R}[t_1,\ldots, t_n]$ by

$$T_{\mathbf{y}}\left(\sum_{j\in J_0} \alpha_j t^j\right) = \sum_{j\in J_0} \alpha_j y_j,$$

where $J_0 \subset \mathbb{N}^n$ is a finite subset and α_j are arbitrary real coefficients. Let $\{f_1,\ldots,f_k\}$ be a finite subset of $\mathbb{R}[t_1,\ldots,t_n]$, where $\mathbb{R}[t_1,\ldots,t_n]$ is the real vector space of all polynomials with real coefficients, of n real variables $t_1,\ldots, t_n$. Then, the closed subset given by

$$K = \{t \in \mathbb{R}^n; f_1(t) \geq 0,\ldots, f_k(t) \geq 0\} \tag{5}$$

is called a semi-algebraic set. The following result was proved for compact semi-algebraic sets (see [21] Theorem 1.4, and [22] Theorem II.2.4 for related or more general results). On the other hand, important results on resolution of the moment problem on any compact (not necessarily semi-algebraic) subset with nonempty interior in $\mathbb{R}^n$ had been proved in [20] (see [20], Theorems 1, 2, and 4). The expression of positive polynomials on such a compact is also deduced in Theorem 4 of [20].

Theorem 4.9. (See [18]). *Let K be a compact semi-algebraic set as defined earlier. Then, there is a positive Borel measure μ* supported on K such that

$$\int_K t^j \mathrm{d}\mu = y_j, \ \forall j \in \mathbb{N}^n,$$

if and only if

$$T_y(f_1^{e_1} \cdots f_k^{e_k} p^2) \geq 0, \ \forall p \in \mathbb{R}[t_1,\ldots,t_n], \forall e_1,\ldots, e_k \in \{0, 1\}.$$

Corollary 4.10. (See [18]). *With the aforementioned notations, if $p \in \mathbb{R}[t_1,\ldots,t_n]$ is such that $p(t) > 0$ for all t in the semi-algebraic compact K defined by (5), then p is a finite sum of special polynomials of the form*

$$f_1^{e_1} \cdots f_k^{e_k} q^2 \geq 0,$$

for some $q \in \mathbb{R}[t_1,\ldots,t_n]$ and $e_1,\ldots, e_k \in \{0, 1\}$.

The next results of this section are based on polynomial approximation on unbounded subsets, also using Hahn-Banach type results. We start by recalling the following key approximation lemma.

Lemma 4.11. (See [27], Lemma 3). *Let $F \subseteq \mathbb{R}^n$ be an unbounded closed subset, and let ν be an M-determinate measure on F (with finite moments of all natural orders). Then, for any $x \in C_0(F), \ x(t) \geq 0, \ \forall t \in F$, there exists a sequence of polynomials $(p_m)_m, p_m \geq x, \ m \in \mathbb{N}, \ p_m \to x$ in $L^1_\nu(F)$. In particular, we have*

$$\lim_m \int_F p_m(t)\mathrm{d}\nu = \int_F x(t)\mathrm{d}\nu,$$

the cone $\mathcal{P}_+$ of nonnegative polynomials is dense in $(L^1_\nu(F))_+$, and $\mathcal{P}$ is dense in $L^1_\nu(F)$.

Proof. To prove the assertions of the statement, it is sufficient to show that for any $x \in (C_0(F))_+$, we have

$$Q_1(x) := \inf\left\{ \int_F p(t)\mathrm{d}\nu; p \geq x, \ p \in \mathcal{P} \right\} = \int_F x(t)\mathrm{d}\nu.$$

Obviously, one has

$$Q_1(x) \geq \int_F x(t)\mathrm{d}\nu. \tag{6}$$

To prove the converse, we define the linear form

$$T_0 : X_0 := \mathcal{P} \oplus Sp\{x\} \to \mathbb{R}, \ T_0(p + \alpha x) := \int_F p(t)\mathrm{d}\nu + \alpha Q_1(x), \ p \in \mathcal{P}, \ \alpha \in \mathbb{R}.$$ □

Next, we show that T_0 is positive on X_0. In fact, for $\alpha < 0$, one has (from the definition of Q_1, which is a sublinear functional on X_1):

$$p + \alpha x \geq 0 \Rightarrow p \geq -\alpha x \Rightarrow (-\alpha)Q_1(x) = Q_1(-\alpha x) \leq \int_F p(t)\mathrm{d}\nu \Rightarrow T_0(p + \alpha x) \geq 0.$$

If $a \geq 0$, we infer that:

$$0 = Q_1(0) = Q_1(\alpha x - \alpha x) \leq \alpha Q_1(x) + Q_1(-\alpha x) \Rightarrow$$

$$\int_F p(t)\mathrm{d}\nu \geq Q_1(-\alpha x) \geq -\alpha Q_1(x) \Rightarrow T_0(p + \alpha x) \geq 0$$

where, in both possible cases, we have $x_0 \in (X_0)_+ \Rightarrow T_0(x_0) \geq 0$. Since X_0 contains the space of the polynomial functions, which is a majorizing subspace of X_1, there exists a linear positive extension $T : X \to \mathbb{R}$ of T_0, which is continuous on $C_0(F)$ with respect to the sup-norm. Therefore, T has a representation by means of a positive Borel regular measure μ on F, such that

$$T(x) = \int_F x(t)\mathrm{d}\mu, \ x \in C_0(F).$$

Let $p \in \mathcal{P}_+$ be a nonnegative polynomial function. There is a nondecreasing sequence $(x_m)_m$ of continuous nonnegative function with compact support, such that $x_m \nearrow p$ pointwise on F. Positivity of T and Lebesgue's dominated convergence theorem for μ yield

$$\int_F p(t)\mathrm{d}\nu = T(p) \geq \sup T(x_m) = \sup \int_F x_m(t)\mathrm{d}\mu = \int_F p(t)\mathrm{d}\mu, \ p \in \mathcal{P}_+.$$

Thanks to Haviland's theorem, there exists a positive Borel regular measure λ on F, such that

$$\lambda(p) = \nu(p) - \mu(p) \Leftrightarrow \nu(p) = \lambda(p) + \mu(p), \ p \in \mathcal{P}.$$

Since ν is assumed to be M-determinate, it follows that

$$\nu(B) = \lambda(B) + \mu(B)$$

for any Borel subset B of F. From this last assertion, approximating each $x \in (L^1_\nu(F))_+$, by a nondecreasing sequence of nonnegative simple functions, and also using Lebesgue's convergence theorem, one obtains first for positive functions, then for arbitrary ν-integrable functions, φ:

$$\int_F \varphi \mathrm{d}\nu = \int_F \varphi \mathrm{d}\lambda + \int_F \varphi \mathrm{d}\mu, \ \varphi \in L^1_\nu(F).$$

In particular, we must have

$$\int_F x\mathrm{d}\nu \geq \int_F x\mathrm{d}\mu = T(x) = T_0(x) = Q_1(x). \tag{7}$$

Then, equations (6) and (7) conclude the proof. □

Lemma 4.12. *Let $\nu = \nu_1 \times \cdots \times \nu_n$ be a product of n M- determinate measures on $\mathbb{R}$; we can approximate any nonnegative continuous compactly supported function in $X = L^1_\nu(\mathbb{R}^n)$ with dominating sums of products:*

$$p_1 \otimes \cdots \otimes p_n, \ (p_1 \otimes \cdots \otimes p_n)(t_1, \ldots, t_n) := p_1(t_1) \cdots p_n(t_n),$$

where p_j are nonnegative polynomial on the entire real line, $j = 1, \ldots, n$.

To prove Lemma 4.12, one uses approximating Bernstein polynomial of n variables. Then, Lemma 4.11 is applied in each separate variable, for $n = 1$, $F = \mathbb{R}$, and the Fubini theorem.

Theorem 4.13. (See [27], Theorem 7). *Let ν be as in* Lemma 4.12, *$X = L^1_\nu(\mathbb{R}^n)$, and Y be a Banach lattice. Assume that T is a linear bounded operator from X to Y. The following statements are equivalent:*

(*a*) *T is positive on the positive cone of X;*

(*b*) *For any finite subsets* $J_k \subset \mathbb{N}$, $k = 1,\dots, n$, *and any* $\{\lambda_{j_k}\}_{j_k \in J_k} \subset \mathbb{R}$, $k = 1,\dots, n$, *the following inequalities hold true:*

$$\mathbf{0} \le \sum_{i_1,j_1 \in J_1}\left(\cdots \left(\sum_{i_n,j_n \in J_n} \lambda_{i_1}\lambda_{j_1} \cdots \lambda_{i_n}\lambda_{j_n} T(x_{i_1+j_1,\dots,i_n+j_n}) \right) \cdots \right).$$

Proof. Note that (b) says that T is positive on the convex cone generated by special positive polynomials $p_1 \otimes \cdots \otimes p_n$, each factor of any term in the sum being nonnegative on the whole real axis. Consequently, (a) ⇒ (b) is clear. To prove the converse, observe that any nonnegative element of X can be approximated by nonnegative continuous compactly supported functions. Such functions can be approximated by the sums of tensor products of positive polynomials in each separate variable, the latter being sums of squares. The conclusion is that any nonnegative function from X can be approximated in $X = L^1_\nu(\mathbb{R}^n)$ by the sums of tensor products of squares of polynomials in each separate variable. We know that on such special polynomials, T admits values in Y_+, according to the condition (b). Now, the desired conclusion is a consequence of the continuity of T, also using the fact that the positive cone of Y is closed. This concludes the proof. □

Let H be a Hilbert space and U a self-adjoint operator acting on H. Let $Y = Y(U)$ the order complete Banach lattice (which is also a commutative real algebra) of self-adjoint operators discussed in [26, pp. 303–305]. Namely, if $\mathcal{A} = \mathcal{A}(H)$ is the space of all self-adjoint operators acting on H, the natural order relation on $\mathcal{A}$ is, by definition,

$$V \le W \Leftrightarrow \langle Vh, h\rangle \le \langle Wh, h\rangle \ \forall h \in H.$$

With respect to this order relation, $\mathcal{A}(H)$ is an ordered Banach space, which is not a lattice. Moreover, the multiplication operation on $\mathcal{A}(H)$ is not commutative. Therefore, we use the following notations, to define a suitable subspace $Y(U)$ of $\mathcal{A}(H)$:

$$Y_1(U) = \{V \in \mathcal{A}; UV = VU\}, \ Y(U) = \{W \in Y_1(U); WV = VW, \ \forall V \in Y_1(U)\}. \tag{8}$$

Then, $Y(U)$ is the codomain space we are interested in: it is an order complete Banach lattice (see [26], pp. 303–305) and is a commutative real algebra (this last assertion is obvious).

In the sequel, the condition $0 \le T \le T_2$ on X_+ is replaced by a more general requirements on the solution T, namely, $T_1 \le T \le T_2$ on X_+, where T_1, T_2 are given linear operators satisfying natural assumptions. We start with a one-dimensional Markov moment problem on an arbitrary compact subset $K \subset \mathbb{R}_+$. We denote by $X = C(K)$ the Banach lattice of all real-valued continuous functions on K and let Y be an arbitrary order complete Banach lattice. We use the following notation:

$$\varphi_j(t) = t^j, \ t \in \mathbb{R}_+, \ j \in \mathbb{N}.$$

Theorem 4.14. (See [29], Theorem 3). *Let T_1, T_2 be two linear operators from X to Y, such that* $0 \le T_1 \le T_2$ *on the positive cone of X, and $(y_n)_{n\ge 0}$ a given sequence of elements in Y. The following statements are equivalent:*

(*a*) *There exists a unique* (*bounded*) *linear operator* $T : X \to Y$ *such that* $T(\varphi_j) = y_j$, $j \in \mathbb{N}$, $T_1 \le T \le T_2$ *on the positive cone of X*, $\|T_1\| \le \|T\| \le \|T_2\|$.

(*b*) *For any polynomial* $\sum_{j=0}^m \alpha_j\varphi_j \ge 0$ *on K, we have* $\sum_{j=0}^m \alpha_j T_1(\varphi_j) \le \sum_{j=0}^m \alpha_j y_j$; *if* $J_0 \subset \mathbb{N}$ *is a finite subset and* $\{\alpha_j; j \in J_0\} \subset \mathbb{R}$, *then the following conditions are satisfied:*

$$\sum_{i,j \in J_0} \alpha_i\alpha_j y_{i+j+l} \le \sum_{i,j\in J_0} \alpha_i\alpha_j T_2(\varphi_{i+j+l}), \ l \in \{0, 1\}.$$

The next result holds also for the multidimensional Markov moment problem.

Theorem 4.15. (See [29], Theorem 4). *Let* $F \subseteq \mathbb{R}^n$ *be a closed unbounded subset, ν a positive Borel moment determinate measure on F, having finite moments of all orders,* $X = L^1_\nu(F)$, $\varphi_j(t) = t^j$, $t \in F$, $j \in \mathbb{N}^n$. *Let Y be* *an order complete Banach lattice, $(y_j)_{j\in\mathbb{N}^n}$ be a given sequence of elements in Y, T_1, and T_2 be two bounded linear operators from X to Y. Assume that there exists a sub − cone $\mathcal{P}_{++} \subseteq \mathcal{P}_+$ such that each $f \in (C_0(F))_+$ can be approximated in X by a sequence $(p_l)_l$, $p_l \in \mathcal{P}_{++}$, $p_l \geq f$ for all l. The following statements are equivalent:*

(a) *There exists a unique (bounded) linear operator $T : X \to Y$, $T(\varphi_j) = y_j$, $j \in \mathbb{N}^n$, $0 \leq T_1 \leq T \leq T_2$ on X_+,*

$$\|T_1\| \leq \|T\| \leq \|T_2\|;$$

(b) *For any finite subset $J_0 \subset \mathbb{N}^n$and any $\{\alpha_j;\ j \in J_0\} \subset \mathbb{R}$, the following implications hold true:*

$$\sum_{j\in J_0}\alpha_j\varphi_j \in \mathcal{P}_+ \Rightarrow \sum_{j\in J_0}\alpha_j T_1(\varphi_j) \leq \sum_{j\in J_0}\alpha_j y_j;\ \sum_{j\in J_0}\alpha_j\varphi_j \in \mathcal{P}_{++} \Rightarrow \sum_{j\in J_0}\alpha_j T_1(\varphi_j) \geq \mathbf{0},\ \sum_{j\in J_0}\alpha_j y_j \leq \sum_{j\in J_0}\alpha_j T_2(\varphi_j).$$

The proof of Theorem 4.15 is essential based on previous approximation results published in [27]. Next, we use approximation by special nonnegative polynomials [27] to express conditions of Theorem 4.15 in terms of quadratic forms. The following consequences follow.

Corollary 4.16. (See [29], Corollary 2). *Let $X = L^1_\nu(\mathbb{R})$, where ν is a moment determinate positive Borel measure on $\mathbb{R}$, with finite moments of all orders. Assume that Y is an arbitrary order complete Banach lattice, and $(y_n)_{n\geq 0}$ is a given sequence having its terms in Y. Let T_1, T_2 be two linear operators from X to Y such that $\mathbf{0} \leq T_1 \leq T_2$ on X_+. The following statements are equivalent:*
(a) *There exists a unique bounded linear operator T from X to Y, $T_1 \leq T \leq T_2$ on X_+, $\|T_1\| \leq \|T\| \leq \|T_2\|$, such that $T(\varphi_n) = y_n$ for all $n \in \mathbb{N}$.*
(b) *If $J_0 \subset \mathbb{N}$ is a finite subset and $\{\alpha_j; j \in J_0\} \subset \mathbb{R}$, then the following inequalities hold:*

$$\sum_{i,j\in J_0}\alpha_i\alpha_j T_1(\varphi_{i+j}) \leq \sum_{i,j\in J_0}\alpha_i\alpha_j y_{i+j} \leq \sum_{i,j\in J_0}\alpha_i\alpha_j T_2(\varphi_{i+j}).$$

Using the notation of Theorem 4.15, in Corollary 4.16, we have $\mathcal{P}_{++} = \mathcal{P}_+$. since any nonnegative polynomial on the entire real axis is a sum of squares.

We next write the solution for a multidimensional Markov moment problem, which follows via Theorem 4.15 as well. In the case of $S = \mathbb{R}^n_+$ (respectively $S = \mathbb{R}^n$), $n \geq 2$, the cone $\mathcal{P}_{++}$ consists in all polynomials that are sums of products of the form:

$$p_1 \otimes \cdots \otimes p_n,$$
$$(p_1 \otimes \cdots \otimes p_n)(t_1, \ldots, t_n) = p_1(t_1)\cdots p_n(t_n),$$

where each p_j, $j = 1, \ldots, n$, is a nonnegative polynomial on $\mathbb{R}_+$ (respectively on $\mathbb{R}$) and hence is expressible by means of sums of squares of polynomials of one variable. Proceeding this way, some of the conditions of Theorem 4.15 can be written in terms of quadratic forms.

Corollary 4.17. (See [29], Corollary 3). *Let $\nu = \nu_1 \times \cdots \times \nu_n$, where ν_j is an M-determinate (moment determinate) positive regular Borel measure on the real line, $j = 1, \ldots, n$, $X = L^1_\nu(\mathbb{R}^n)$, $\varphi_j(t) = t^j$, $t \in \mathbb{R}^n$, $j \in \mathbb{N}^n$. In addition, assume that ν_j has finite moments of all orders, $j = 1, \ldots, n$. Let Y be an order complete Banach lattice, $(y_j)_{j\in\mathbb{N}^n}$ be a given sequence of elements in Y, and T_1 T_2 two bounded linear operators from X to Y. The following two statements are equivalent.*
(a) *There exists a unique (bounded) linear operator $T : X \to Y$, $T(\varphi_j) = y_j$, $j \in \mathbb{N}^n$, $0 \leq T_1 \leq T \leq T_2$ on X_+,*

$$\|T_1\| \leq \|T\| \leq \|T_2\|.$$

(b) *For any finite subset $J_0 \subset \mathbb{N}^n$and any $\{\alpha_j; j \in J_0\} \subset \mathbb{R}$, the following implication holds true:*

$$\sum_{j\in J_0}\alpha_j\varphi_j \in \mathcal{P}_+ \Rightarrow \sum_{j\in J_0}\alpha_j T_1(\varphi_j) \leq \sum_{j\in J_0}\alpha_j y_j;$$

for any finite subsets $J_k \subset \mathbb{N}$, $k = 1, \ldots, n$, and any $\{\alpha_{j_k}\}_{j_k\in J_k} \subset \mathbb{R}$, the following inequalities hold true:

$$\mathbf{0} \le \sum_{i_1,j_1 \in J_1}\left(\cdots\left(\sum_{i_n,j_n \in J_n} \alpha_{i_1}\alpha_{j_1} \cdots \alpha_{i_n}\alpha_{j_n} T_1(\varphi_{i_1+j_1,\dots,i_n+j_n})\right)\cdots\right);$$

$$\sum_{i_1,j_1 \in J_1}\left(\cdots\left(\sum_{i_n,j_n \in J_n} \alpha_{i_1}\alpha_{j_1} \cdots \alpha_{i_n}\alpha_{j_n} y_{i_1+j_1,\dots,i_n+j_n}\right)\cdots\right)$$

$$\le \sum_{i_1,j_1 \in J_1}\left(\cdots\left(\sum_{i_n,j_n \in J_n} \alpha_{i_1}\alpha_{j_1} \cdots \alpha_{i_n}\alpha_{j_n} T_2(\varphi_{i_1+j_1,\dots,i_n+j_n})\right)\cdots\right).$$

We recall that a moment problem is called truncated if the interpolation moment conditions (4) are satisfied only for $j = (j_1,\dots,j_n) \in \mathbb{N}^n$, $j_k \le d$ for all $k \in \{1,\dots,n\}$, for some fixed natural number d. Therefore, related problems involve only a finite number of unknowns (or equations, or inequalities). For some results on truncated moment problem, see [28] Theorems 2.4 and 2.5, and perturbation of the solution in terms of the perturbations of the moments ([28], p. 35). See also [29] p, 11, Theorem 5 for a polynomial solution, as well as the references of [29] for other kinds of solutions. On the other hand, moment problems can be reduced, but not truncated in the sense mentioned earlier. Here is an illustrating example. Let $Y = Y(U)$ be defined by (8) and $\tilde{B} \in Y_+ \setminus \{0\}$. Let X be the vector space of all continuous complex functions defined on the unit closed polydisk

$$\bar{D}_1 = \{z = (z_1,\dots,z_n) : \ |z_i| \le 1, \ i \in \{1,\dots,n\}\}.$$

The norm on X is defined by $\|g\|_\infty = \sup\{|g(z)| : z \in \bar{D}_1\}$, $g \in X$. We denote $h_k(z) = z_1^{k_1} \cdots z_n^{k_n}$, $k = (k_1,\dots,k_n) \in \mathbb{N}^n$, $z \in \bar{D}_1$, $|k| = k_1 + \cdots + k_n$.

The next result is using this space $Y(U)$ as the codomain of the involved operators. Let $(B_k)_{k \in \mathbb{N}^n}$ be a multi-indexed sequence of operators in $Y = Y(U)$.

Theorem 4.18. (See [30], Theorem 3.1). *Assume that $A_1,\dots, A_n$ are elements of $Y(U)$ such that there exists a real number $M > 0$, with the property*

$$|B_k| \le M\frac{A_1^{2k_1}}{k_1!} \cdots \frac{A_n^{2k_n}}{k_n!}, \ \forall k \in \mathbb{N}^n, \ \sum_{p=1}^{n} A_p^2 \le I,$$

where $I : H \to H$ is the identity operator. Let $\{g_k\}_{k \in \mathbb{N}^n} \subset X$ be such that $1 = \|g_k\|_\infty = g_k(\mathbf{0})$, $\forall k \in \mathbb{N}^n$. Then, there exists a linear bounded operator $T \in B(X, Y)$ such that

$T(h_k) = B_k$, $|k| \ge 1$, $F(g_k) \ge \tilde{B}$, $\forall k \in \mathbb{N}^n$,

$$T(h) \le (2 + \tilde{B}M^{-1}e^{-1})\|h\|_\infty u_0, \ \forall h \in X, \ u_0 = MeI.$$

In particular, the following evaluation holds: $T \le 2Me + \tilde{B}$.

For this space X the proof from [30], Theorem 3.1, works.

5 On convex operators defined on convex-bounded finite-dimensional subsets

In [31], we emphasized a property of convex operators defined on convex bounded finite-dimensional subsets. The codomain was an order complete vector lattice having a strong order unit. This last condition was removed, without affecting the result of the next theorem.

Theorem 5.1. (See [31]). *Let X be an arbitrary real vector space, $B \subset X$ be a finite-dimensional convex bounded subset, Y be an order complete vector lattice, and $P : B \to Y$ be a convex operator. Then, there exists $y_0 \in Y$ such that $P(x) \geq y_0$ for all $x \in B$.*

Proof. Since B is finite-dimensional and convex, its relative interior $ri(B)$ is nonempty. Recall that by $ri(B)$ one denotes the interior of B with respect to the topology induced on B by that on the (finite-dimensional) linear variety generated by B. As is well known, P is subdifferentiable at any point of $ri(B)$ (this follows from Corollary 2.26 stated earlier). Let $b_0 \in ri(B)$ and T a translation of a subgradient of P at b_0, that is an affine operator $T : X \to Y$ such that $T(x_0) = P(x_0)$ and $T(x) \leq P(x)$ for all $x \in B$. On the other hand, let $x_1, \ldots, x_{p+1}$ (at most) $p + 1$ affine independent points in the linear variety generated by B, such that □

$$B \subseteq \mathrm{co}\{x_1, \ldots, x_{p+1}\}$$

(here, p is the linear dimension of the linear variety generated by B). Such a system of points does exist thanks to the fact that B is finite dimensional and bounded. Now the following relations hold

$$\begin{aligned} P(x) \geq T(x) = T\left(\sum_{j=1}^{p+1} \alpha_j x_j\right) = \sum_{j=1}^{p+1} \alpha_j T(x_j) \geq \left(\sum_{j=1}^{p+1} \alpha_j\right) \inf\{T(x_j); 1 \leq j \leq p+1\} \\ = \inf\{T(x_j); 1 \leq j \leq p+1\} := y_0, \end{aligned}$$

where $x = \sum_{j=1}^{p+1} \alpha_j x_j$, $\alpha_j \geq 0$, $j = 1, \ldots, p+1$, $\sum_{j=1}^{p+1} \alpha_j = 1$. This concludes the proof. □

The following question appeared naturally: given a convex subset B with the property stated in the conclusion of Theorem 5.1, is B necessarily finite dimensional? The answer is affirmative. Namely, using some results of [1], we prove the following theorem.

Theorem 5.2. (See [31]). *Let X be an arbitrary real infinite-dimensional vector space and $B \subset X$ a convex subset, such that any convex real function defined on B is bounded below. Then, B is contained in a finite-dimensional subspace of X and is bounded there.*

Proof. Let x^* be an arbitrary linear functional in the algebraic dual X^* of X. Then, x^* and $-x^*$ are convex, and, by hypothesis, both of them are bounded from below on B. Thus, $x^*(B)$ is bounded in $\mathbb{R}$. Hence, B is weakly bounded in X, endowed with the weak topology corresponding to the dual pair (X, X^*). Let us endow X with the finest locally convex topology, which is compatible with this dual pair (the Mackey topology), which is actually the finest locally convex topology on X. By [1], Chapter IV, Corollary 2, p. 132, we derive that B is bounded in this topology. Application of [1], exercise 7, Chapter II, p. 69, leads to the fact that B is contained in a finite-dimensional subspace and bounded there. This concludes the proof. □

Thus, according to the two results of this section mentioned earlier, the property stated in Theorem 5.1 characterizes the finite-dimensional bounded convex subsets of an arbitrary vector space.

6 Conclusion

One of the aims of the present work was to emphasize the relationships between the properties of convex operators and related properties of linear operators. In Section 1, we state the main purposes of this work and the corresponding references used or related to it. Section 2 overviewed main Hahn-Banach type theorems and some of their direct applications. This was done in the general ordered vector setting, as well as in concrete spaces framework. For example, some of the Hahn-Banach results of Section 2 have been recently applied to characterizing isotonicity of a convex operator on a convex cone. Also, Theorem 2.38 was applied to prove an unusual sandwich theorem for real functions defined on finite-simplicial sets (see

Theorem 3.12 of Section 3). Here, the new aspect is that such subsets can be unbounded in any locally convex topology on the entire space X. A topological version of this kind of sandwich results is Theorem 3.13. Other results of Section 3 concern applications of Krein-Milman theorem to representation theorems (the notion of the barycenter of a probability measure being the main point). In Section 4, both old and very recent results on the moment problem are analyzed. One of the main results from this section are Lemmas 4.11 and 4.12 and Theorem 4.13. From the point of view of this last mentioned theorem, we solve the difficulty created arising from the fact that on $\mathbb{R}^n$, $n \geq 2$, there exist nonnegative polynomials that are not sums of squares. Instead of looking for the expression of any nonnegative polynomial, we approximate an arbitrary nonnegative function from $L^1_\nu(\mathbb{R}^n)$ (where ν is as in Lemma 4.12), by sums of squares of polynomials $q_1 \otimes \cdots \otimes q_n$, $q_i \in \mathbb{R}[t]$, $i = 1, \ldots, n$. This result leads to Theorem 4.13, whose statement has nothing in common with the moment problem. However, Lemma 4.12 was initially proved and applied to solve a multidimensional Markov moment problem. Moreover, in Theorem 4.15 and its consequences, the solution T verifies the sandwich condition $T_1 \leq T \leq T_2$ on the positive cone of the domain space, where T_1 is not necessarily the null operator. In particular, this allows the control of the norm of the solution. Finally, Section 5 detailed a main property of convex operators defined on convex bounded finite-dimensional sets. This property characterizes such subsets. Going back to Section 2, take note that the general Theorems 2.28 and 2.35 are equivalent. This could be the first time that Theorem 2.35 has been submitted for publication in English.

Acknowledgements: The Author would like to thank the Anonymous reviewers for their comments and suggestions, leading to the improvement of the presentation of the paper. Special thanks are addressed to the Editors and Production team, for their kind support.

Funding information: The author states no funding involved.

Conflict of interest: The author states no conflict of interest.

Data availability statement: Data sharing is not applicable to this paper as no datasets were generated or analysed during the current study.

References

[1] H. H. Schaefer and M. P. Wolff, *Topological Vector Spaces*. Second Edition, Springer, New York, 1999.
[2] R. R. Phelps, *Lectures on Choquet's Theorem*. Second Edition, Springer-Verlag, Berlin Heidelberg, 2001.
[3] C. P. Niculescu and L.-E. Persson, *Convex functions and their applications. A contemporary approach*, 2nd Ed., CMS Books in Mathematics, Vol. 23, Springer-Verlag, New York, 2018.
[4] O. Olteanu, *A strong separation theorem in normed linear spaces*, Mathematica (Cluj), **35** (58) (1993), no. 1, 59–63.
[5] S. S. Kutateladze, *Convex operators*, Russian Math. Surveys **34** (1979), no. 1, 181–214, DOI: https://doi.org/10.1070/RM1979v034n01ABEH002874.
[6] J. Zowe, *Sandwich theorems for convex operators with values in an ordered vector space*, J. Math. Anal. Appl. **66** (1978), 282–296, DOI: https://doi.org/10.1016/0022-247X(78)90232-9.
[7] M. Valadier, *Sous-Différentiabilité de fonctions convexes à valeurs dans un espace vectoriel ordonné*, Math. Scand. **30** (1972), 65–74, DOI: https://doi.org/10.7146/math.scand.a-11064.
[8] O. Olteanu, *Convexité et prolongement d'opérateurs linéaires,* (*Convexity and extension of linear operators*), C. R. Math. Acad. Sci. Paris, Série A **286** (1978), 511–514.
[9] O. Olteanu, *Théorèmes de prolongement d'opérateurs linéaires* (*Theorems on extension of linear operators*), Rev. Roumaine Math. Pures Appl. **28** (1983), no. 10, 953–983.
[10] O. Olteanu, *Application de théorèmes de prolongement d'opérateurs linéaires au problème des moments e à une generalization d'un théorème de Mazur-Orlicz,* (*Applications of theorems on extension of linear operators to the moment problem and to a generalization of a Mazur-Orlicz theorem*), C. R. Math. Acad. Sci. Paris, t. **313**, Série I, (1991), 739–742.

[11] C. P. Niculescu and O. Olteanu, *From the Hahn-Banach extension theorem to the isotonicity of convex functions and the majorization theory*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114** (2020), no. 171, 1–19, DOI: https://doi.org/10.1007/s13398-020-00905-4.

[12] C. Berg, J. P. Reus Christensen, and P. Ressel, *Harmonic Analysis on Semigroups. Theory of Positive Definite and Related Functions*, Springer, New York Berlin Heidelberg Tokyo, 1984.

[13] C. Ambrozie and O. Olteanu, *A sandwich theorem, the moment problem, finite-simplicial sets and some inequalities*, Rev. Roumaine Math. Pures Appl. **49** (2004), no. 3, 189–210.

[14] O. Olteanu, *From Hahn-Banach type theorems to the Markov moment problem, sandwich theorems and further application*, Mathematics **8** (2020), no. 1328, 1–18, DOI: https://doi.org/10.3390/math8081328.

[15] E. K. Haviland, *On the momentum problem for distributions in more than one dimension*, Amer. J. Math. **58** (1936), 164–168, DOI: https://doi.org/10.2307/2371063.

[16] C. Berg, J. P. R. Christensen, and C. U. Jensen, *A remark on the multidimensional moment problem*, Math. Ann. **243** (1979), 163–169, DOI: https://doi.org/10.1007/BF01420423.

[17] K. Schmüdgen, *The moment problem*, In: Graduate Texts in Mathematics, Springer International Publishing AG, Cham, 2017.

[18] K. Schmüdgen, *The K – moment problem for compact semi-algebraic sets*, Math. Ann. **289** (1991), 203–206, DOI: https://doi.org/10.1007/BF01446568

[19] J. M. Stoyanov, G. D. Lin, and P. Kopanov, *New checkable conditions for moment determinacy of probability distributions*, Theory Probab. Appl. **65** (2020), no. 3, 497–509, DOI: https://doi.org/10.1137/S0040585X97T990083.

[20] G. Cassier, *Problèmes des moments sur un compact de* $\mathbb{R}^n$ *et décomposition des polynômes à plusieurs variables*, J. Funct. Anal. **58** (1984), 254–266, DOI: https://doi.org/10.1016/0022-1236(84)90042-9.

[21] M. Putinar, *Positive polynomials on compact semi-algebraic sets*, Indiana Univ. Math. J. **42** (1993), no. 3, 969–984, DOI: https://doi.org/10.1512/iumj.1993.42.42045.

[22] F. H. Vasilescu, *Spectral measures and moment problems*, In: A. Gheondea, M. Şabac (Eds.) Spectral Analysis and its Applications, Ion Colojoară Anniversary Volume, Theta, Bucharest, 2003, pp. 173–215.

[23] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis,* Oliver and Boyd, Edinburgh-London, 1965.

[24] M. G. Krein and A. A. Nudelman, *Markov Moment Problem and Extremal Problems*, Amer. Math. Soc. Providence, RI, 1977.

[25] P. Meyer-Nieberg, *Banach Lattices*, Springer, Berlin, 1991.

[26] R. Cristescu, *Ordered Vector Spaces and Linear Operators*, Academiei: Bucharest, and Abacus Press, Tunbridge Wells, Kent, 1976.

[27] O. Olteanu, *Polynomial approximation on unbounded subsets, Markov moment problem and other applications*, Mathematics **8** (2020), no. 1654, 1–12, DOI: https://doi.org/10.3390/math8101654.

[28] O. Olteanu, *On truncated and full classical Markov moment problems*, Contrib. Math. **3** (2021), 29–36, DOI: https://doi.org/10.47443/cm.2021.0011.

[29] O. Olteanu, *On Markov moment problem and related results*, Symmetry **13** (2021), no. 986, DOI: https://doi.org/10.3390/sym13060986.

[30] O. Olteanu and J. M. Mihăilă, *Operator – valued Mazur – Orlicz and moment problems in spaces of analytic functions*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **79** (2017), no. 1, 175–184.

[31] O. Olteanu, *Sur les fonctions convexes définies sur les ensembles convexes bornés de* $\mathbb{R}^n$ (*On convex functions defined on bounded convex subsets of* $\mathbb{R}^n$), C.R. Math. Acad. Sci. Paris, Série A, **290** (1980), 837–838.