Research Article

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Bernstein-Walsh type inequalities for derivatives of algebraic polynomials in quasidisks

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Abstract: In this paper, we study Bernstein-Walsh type estimates for the higher-order derivatives of an arbitrary algebraic polynomial on quasidisks.

Keywords: algebraic polynomial, quasiconformal mapping, quasicircle

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1 Introduction

Let $\mathbb{C}$ be a complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region with boundary $L := \partial G$ such that $0 \in G$; $\Omega := \overline{\mathbb{C}} \backslash \overline{G} = \text{ext}L$; $\Delta := \Delta(0, 1) := \{w : |w| > 1\}$. Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto Δ such that $\Phi(\infty) = \infty$ and $\lim_{z\to\infty}\frac{\Phi(z)}{z} > 0$; $\Psi := \Phi^{-1}$. For $R > 1$, we take $L_R := \{z : |\Phi(z)| = R\}$, $G_R := \text{int}L_R$, and $\Omega_R := \text{ext}L_R$. Let $\wp_n$ denote the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

In this work, we consider the following weight function $h(z)$. Let $\{z_j\}_{j=1}^{l}$ be the fixed system of distinct points on curve L. For some fixed R_0, $1 < R_0 < \infty$, and $z \in \overline{G}_{R_0}$, consider generalized Jacobi weight function $h(z)$, which is defined as follows:

$$h(z) := h_0(z)\prod_{j=1}^{l}|z - z_j|^{\gamma_j}, \tag{1}$$

where $\gamma_j > -2$, for all $j = 1, 2, \ldots, l$, and h_0 is uniformly separated from zero in L_{R_0}, i.e., there exists a constant $c_1(G) > 0$ such that $h_0(z) \geq c_1(G) > 0$ for all $z \in G_{R_0}$.

Let $0 < p \leq \infty$ and σ be the two-dimensional Lebesgue measure. For the Jordan region G, we introduce:

$$\begin{aligned} \|P_n\|_p &:= \|P_n\|_{A_p(h,G)} := \left(\iint_G h(z)|P_n(z)|^p d\sigma_z\right)^{1/p}, \quad 0 < p < \infty, \\ \|P_n\|_\infty &:= \|P_n\|_{A_\infty(1,G)} := \max_{z\in\overline{G}}|P_n(z)|, \quad p = \infty, \end{aligned} \tag{2}$$

and $A_p(1, G) =: A_p(G)$.

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When L is rectifiable, for any $p > 0$, let

$$\|P_n\|_{\mathcal{L}_p(h,G)} := \left(\int_L h(z)|P_n(z)|^p |dz| \right)^{1/p} < \infty, \quad 0 < p < \infty,$$
$$\|P_n\|_{\mathcal{L}_\infty(1,G)} := \max_{z \in L} |P_n(z)|, \quad p = \infty, \tag{3}$$

and $\mathcal{L}_p(1, L) =: \mathcal{L}_p(L)$.

According to well-known Bernstein-Walsh lemma [1], we have:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n \|P_n\|_{C(\overline{G})}. \tag{4}$$

Hence, setting $R = 1 + \frac{1}{n}$, we see that the C norm of polynomials $P_n(z)$ in $\overline{G}_R$ and $\overline{G}$ have the same order of growth, that is, the norm $\|P_n\|_{C(\overline{G})}$ increases up to multiplication by a constant in $\overline{G}_R$.

In [1] also was given some similar estimates for various norms on the right-hand side of (3). Analogous estimation with respect to the quasinorm (3) for $p > 0$ was obtained in [2] for $h(z) \equiv 1$ (i.e., $\gamma_j = 0$ for all $j = 1, 2, \ldots, l$) as follows:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq R^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \quad p > 0.$$

Moreover, in [3, Lemma 2.4], this estimate has been generalized for $h(z) \neq 1$, defined as in (1) for the $\gamma_j > -1$, for all $j = 1, 2, \ldots, l$, and the following was proved:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \quad \gamma^* = \max\{0; \gamma_j : 1 \leq j \leq l\}. \tag{5}$$

To give a similar estimation to (5) for the $A_p(h, G)$ norm, first, we will give the following definition.

Let the function φ map G conformally and univalently onto $B := B(0, 1) := \{w : |w| < 1\}$, which is normalized by $\varphi(0) = 0$ and $\varphi'(0) > 0$; let $\psi := \varphi^{-1}$.

Definition 1. A bounded Jordan region G is called a κ-quasidisk, $0 \leq \kappa < 1$, if any conformal mapping ψ can be extended to a K-quasiconformal, $K = \frac{1+\kappa}{1-\kappa}$, homeomorphism of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$. In that case, the curve $L := \partial G$ is called a κ-quasicircle. The region G (curve L) is called a quasidisk (quasicircle), if it is κ-quasidisk (κ-quasicircle) with some $0 \leq \kappa < 1$.

A simple example of a κ-quasidisk may be a region bounded by two arcs of circle, symmetric with respect to the OX-axis and OY-axis, each of the arcs crosses the OX-axis at $\pm\varepsilon_0$, where $\varepsilon_0 > 0$ and the angle between the arcs is $\pi(1 - \kappa)$, where $0 \leq \kappa < 1$.

A Jordan curve L is called a quasicircle or quasiconformal curve, if it is the image of the unit circle under a quasiconformal mapping of $\mathbb{C}$ to $\mathbb{C}$ (see [4, p. 105], [5, p. 286]). On the other hand, it was given some geometric criteria of quasiconformality of the curves (see also [6, p. 81], [7, p. 107]). It is well-known that quasicircles can be nonrectifiable (see, e.g., [8], [4, p. 104]).

In [9] (see also [10]), the Bernstein-Walsh type estimates for the norm (2), the regions with quasiconformal boundary, weight function $h(z)$, defined in (1) with $\gamma_j > -2$, and for all $p > 0$, are as follows:

$$\|P_n\|_{A_p(h,G_R)} \leq c_1 R^{*^{n+\frac{1}{p}}} \|P_n\|_{A_p(h,G)},$$

where $R^* := 1 + c_2(R - 1)$, $c_2 > 0$ and, $c_1 := c_1(G, p, c_2) > 0$ constants, independent from n and R.

In [11, Theorem 1.1], analogous estimate was studied for $A_p(G)$ norm, $p > 0$, for arbitrary Jordan region and was obtained: for any $P_n \in \wp_n$, $R_1 = 1 + \frac{1}{n}$ and arbitrary R, $R > R_1$, the following estimate

$$\|P_n\|_{A_p(G_R)} \leq c R^{n+\frac{2}{p}} \|P_n\|_{A_p\left(G_{R_1}\right)}$$

is true, where $c = \left(\frac{2}{e^p - 1}\right)^{\frac{1}{p}} \left[1 + O\left(\frac{1}{n}\right)\right]$, $n \to \infty$.

Stylianopoulos in [12] replaced the norm $\|P_n\|_{C(\overline{G})}$ with norm $\|P_n\|_{A_2(G)}$ on the right-hand side of (4) and found a new version of the Bernstein-Walsh lemma: *Assume that L is quasiconformal and rectifiable. Then, there exists a constant $c = c(L) > 0$ depending only on L such that*

$$|P_n(z)| \le c\frac{\sqrt{n}}{d(z, L)}\, \|P_n\|_{A_2(G)}|\Phi(z)|^{n+1}, \quad z \in \Omega,$$

where $d(z, L) := \inf\{|\zeta - z| : \zeta \in L\}$ holds for every $P_n \in \wp_n$.

In this work, we study for κ-quasidisks G, $0 \le \kappa < 1$, pointwise estimation in unbounded region $\Omega_{1+\varepsilon_0 n^{-1}} = \overline{\mathbb{C}} \backslash \overline{G}_{1+\varepsilon_0 n^{-1}}$ for sufficiently small $\varepsilon_0 > 0$, for the derivative $|P_n^{(m)}(z)|$, $m = 0, 1, 2, \ldots$, in the following type:

$$|P_n^{(m)}(z)| \le \eta_n(G, h, p, m, d(z, L), |\Phi(z)|^{n+1})\|P_n\|_p, z \in \Omega_{1+\varepsilon_0 n^{-1}}, \tag{6}$$

where $\eta_n(\cdot) \to \infty$, as $n \to \infty$, depending on the properties of the G and h.

Analogous results of (6)-type for $m = 0$, different weight function h, unbounded region Ω, and some norms were obtained in [13, pp. 418–428] [14–22] and others.

To obtain estimates for $|P_n^{(m)}(z)|$ on the whole complex plane, it is necessary to use Bernstein-Markov-Nikolsky type estimate for $|P_n^{(m)}(z)|$, $z \in \overline{G}$, of the following type:

$$\|P_n^{(m)}\|_\infty \le \lambda_n(G, h, p)\|P_n\|_p, \quad m = 0, 1, 2, \ldots, \tag{7}$$

where $\lambda_n := \lambda_n(G, h, p, m) > 0$, $\lambda_n \to \infty$, $n \to \infty$, is a constant, depending on the geometrical properties of the region G and the weight function h in general.

Many mathematicians have studied inequalities of type (7) since the beginning of the twentieth century [23–25]. In recent years, such inequalities for various spaces have been studied (see, i.e., [10], [13, pp. 418–428], [14], [15, Secttion 5.3], [16], [21, pp. 122–133], [26–29] (see also the references cited therein). In recent years, analogous estimates to the (7)-type for $m = 0$ were continued to be studied in [9,10, 19,20,22–34] and others for various regions in the complex plane.

Therefore, combining estimates (6) and (7), we obtain in whole complex plane estimate for $|P_n^{(m)}(z)|$ for any $m = 1, 2, \ldots,$:

$$|P_n^{(m)}(z)| \le c_4\, \|P_n\|_p \begin{cases} \lambda_n(G, h, p) & z \in \overline{G}_{1+\varepsilon_0 n^{-1}}, \\ \eta_n(G, h, p, d(z, L)|\Phi(z)|^{n+1}) & z \in \Omega_{1+\varepsilon_0 n^{-1}}, \end{cases} \tag{8}$$

where $c_4 = c_4(G, p) > 0$ is a constant independent of n, h, P_n, and $\lambda_n(G, h, p) \to \infty$, $\eta_n(G, h, p, d(z, L)) \to \infty$, as $n \to \infty$, depending on the properties of the G and h.

2 Definitions and main results

Throughout this paper, $c, c_0, c_1, c_2, \ldots$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive constants (generally, different in different relations), which depends on G in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. For any $k \ge 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k+1, \ldots, m$.

First, we give estimate for $|P_n^{(m)}(z)|$, $z \in \overline{G}$, for $m \ge 0$.

Theorem A. [35, Theorem 1] *Let $0 < p \le \infty$, G be a κ-quasidisk for some $0 \le \kappa < 1$ and $h(z)$ be defined by* (1). *Then, for any $P_n \in \wp_n$, $n \in N$, and every $m = 0, 1, 2, \ldots,$ we have*:

$$\|P_n^{(m)}\|_\infty \le cn^{\left(\frac{\gamma^*+2}{p}+m\right)(1+\kappa)}\, \|P_n\|_p, \tag{9}$$

where here and throughout the text,

$$\gamma^* := \max\{0; \gamma_j,\ j = \overline{1, l}\}. \tag{10}$$

Recall, the estimate (9) is sharp and for $m = 0$ was given in [32, Theorem 2.1].

For $0 < \delta_j < \delta_0 := \frac{1}{4}\min\{|z_i - z_j| : i, j = 1, 2, \ldots, l, i \neq j\}$, let $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$; $\delta := \min_{1\leq j\leq l}\delta_j$; $U_\infty(L, \delta) := \bigcup_{\zeta\in L} U(\zeta, \delta)$-infinite open cover of the curve L; $U_N(L, \delta) := \bigcup_{j=1}^{N} U_j(L, \delta) \subset U_\infty(L, \delta)$-finite open cover of the curve L; $\Omega(\delta) := \Omega(L, \delta) := \Omega \cap U_N(L, \delta)$, $\widehat{\Omega} := \Omega\backslash\Omega(\delta)$; $\Omega_R(\delta) := \Omega(L_R, \delta) := \Omega_R \cap U_N(L_R, \delta)$, $\widehat{\Omega}_R := \Omega_R\backslash\Omega_R(\delta)$.

Now, we start to formulate the new results.

2.1 The general estimate (recurrence formula)

First, we present a general estimate for the $|P_n^{(m)}(z)|$, for which it will be possible to obtain estimates for the derivative for each order $m = 1, 2, \ldots$

Theorem 2. *Let $p \geq 2$, G be a κ-quasidisk for some $0 \leq \kappa < 1$, and $h(z)$ be defined by* (1). *Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 1, 2, \ldots$, we have*:

$$|P_n^{(m)}(z)| \leq c_1 \quad |\Phi^{n+1}(z)| \cdot \left\{ \frac{\|P_n\|_p}{d(z, L)} A_{n,p}^1(z, m) + \sum_{j=1}^{m} C_m^j B_{n,j}^1(z) |P_n^{(m-j)}(z)| \right\}, \tag{11}$$

where $c_1 = c_1(G, \gamma, m, p) > 0$ constant independent from n and z;

$$A_{n,p}^1(z, m) := \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)(1+\kappa)}, & p \geq 2, & m \geq 2, \ \gamma > -2, \\ n^{\left(\frac{\gamma+2}{p}+m-1\right)(1+\kappa)}, & 2 \leq p < 1 + (2+\gamma)(1+\kappa), & m = 1, \ \gamma \geq 0, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1 + (2+\gamma)(1+\kappa), & m = 1, \ \gamma \geq 0, \\ n^{1-\frac{1}{p}}, & p > 1 + (2+\gamma)(1+\kappa), & m = 1, \ \gamma \geq 0, \\ n^{\left(\frac{2}{p}+m-1\right)(1+\kappa)}, & p \geq 2, & m \geq 1, \ -2 < \gamma < 0, \end{cases}$$

$$B_{n,j}^1(z) := n^{j(1+\kappa)}, \quad j = 1, 2, \ldots, m,$$

if $z \in \Omega(\delta)$, and

$$A_{n,p}^1(z, m) := \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)(1+\kappa)}, & 2 \leq p < \frac{1}{2+\kappa} + (\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma \geq \frac{1}{1+\kappa}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{1}{2+\kappa} + (\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma \geq \frac{1}{1+\kappa}, \\ n^{1-\frac{1}{p}}, & p > \frac{1}{2+\kappa} + (\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma \geq \frac{1}{1+\kappa}, \\ n^{1-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(1-\frac{2}{p}\right)+\frac{1}{p}}, & p \geq 2, & -2 < \gamma < 0, \end{cases}$$

$$B_{n,j}^1(z) := n^{\kappa}, \quad j = 1, 2, \ldots, m,$$

if $z \in \widehat{\Omega}(\delta)$.

Theorem 3. *Let $1 < p < 2$, G be a κ-quasidisk for some $0 \leq \kappa < 1$ and $h(z)$ be defined by* (1). *Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and every $m = 1, 2, \ldots$, we have*:

$$|P_n^{(m)}(z)| \leq c_2 |\Phi^{n+1}(z)| \left\{ \frac{\|P_n\|_p}{d(z, L)} A_{n,p}^2(z, m) + \sum_{j=1}^{m} C_m^j B_{n,j}^1(z) |P_n^{(m-j)}(z)| \right\}, \tag{12}$$

where $c_2 = c_2(G, \gamma, m, p) > 0$ constant independent from n and z;

$$A_{n,p}^2(z, m) := n^{\left(\frac{\gamma^*+2}{p}+m-1\right)(1+\kappa)},$$

if $z \in \Omega(\delta)$,

$$A^2_{n,p}(z,m) := \begin{cases} n^{(\frac{\gamma+2}{p}+m-1)(1+\kappa)}, & 1<p<2, \quad \gamma \geq \frac{1}{1+\kappa},\\ n^{(\frac{\gamma+2}{p}+m-1)(1+\kappa)}, & p<1+\gamma(1+\kappa), \quad 0<\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\gamma(1+\kappa), \quad 0<\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & p>1+\gamma(1+\kappa), \quad 0<\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & 1<p<2, \quad -2<\gamma\leq 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$, $\gamma^* := \max\{0;\gamma\}$, *and* $B^1_{n,j}(z)$ *defined as in* Theorem 2.

2.2 Estimate for $|P_n(z)|$

As can be seen from (12), we also need an estimate for $|P_n(z)|$ as follows.

Theorem 4. *Let* $p>1$, G *be a* κ*-quasidisk for some* $0 \leq \kappa < 1$ *and* $h(z)$ *be defined by* (1). *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and* $z \in \Omega_R$, *we have*:

$$|P_n(z)| \leq c_3 \frac{|\Phi^{n+1}(z)|}{d(z,L)} A^3_{n,p} \, \|P_n\|_p, \tag{13}$$

where $c_3 = c_3(G,\gamma,p) > 0$ *constant independent from n and* z;

$$A^3_{n,p} := \begin{cases} n^{(\frac{\gamma+2}{p}-1)(1+\kappa)}, & 2 \leq p < \frac{1}{2+\kappa}+(2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa},\\ (n\ln n)^{1-\frac{1}{p}}, & p=\frac{1}{2+\kappa}+(2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa},\\ n^{1-\frac{1}{p}}, & p>\frac{1}{2+\kappa}+(2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa},\\ n^{1-\frac{1}{p}}, & p \geq 2, \quad 0 \leq \gamma < \frac{1}{1+\kappa},\\ n^{\kappa\left(1-\frac{2}{p}\right)+\frac{1}{p}}, & p \geq 2, \quad -2<\gamma<0, \end{cases}$$

for any $p \geq 2$ *and*

$$A^3_{n,p} := \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)(1+\kappa)}, & 1<p<2, \quad \gamma \geq \frac{1}{1+\kappa},\\ n^{\left(\frac{\gamma+2}{p}-1\right)(1+\kappa)}, & 1<p<1+\gamma(1+\kappa), \quad 0<\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\gamma(1+\kappa), \quad 0<\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & 1+\gamma(1+\kappa)<p<2, \quad 0 \leq \gamma < \frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & 1<p<2, \quad -2<\gamma<0, \end{cases}$$

for $1<p<2$.

We note that the estimate for $|P_n(z)|$ was previously received by us in [32, Theorem 3] for $p>0$. But this result is better for $p \geq 2$ and coincides with that [32, Theorem 3] for $1<p<2$.

2.3 Estimate for $|P_n'(z)|$

Now, using Theorems 2 and 3, we can give an estimate for the $|P_n'(z)|$ for $z \in \Omega_R$.

Theorem 5. *Let $p \geq 2$, G be a κ-quasidisk for some $0 \leq \kappa < 1$ and $h(z)$ be defined by* (1). *Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have*:

$$|P_n'(z)| \leq c_4 \frac{|\Phi^{2(n+1)}(z)|}{d(z,L)} \|P_n\|_p A^4_{n,p}(z), \tag{14}$$

where $c_4 = c_4(G, \gamma, p) > 0$ constant independent from n and z;

$$A^4_{n,p}(z) := \begin{cases} n^{\frac{\gamma+2}{p}(1+\kappa)}, & 2 \leq p < \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{2-\frac{1}{p}+\kappa}(\ln n)^{1-\frac{1}{p}}, & p = \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{2-\frac{1}{p}+\kappa}, & p > \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{2-\frac{1}{p}+\kappa}, & p \geq 2, \quad 0 \leq \gamma < \frac{1}{1+\kappa}, \\ n^{1+\frac{1}{p}+\left(2-\frac{2}{p}\right)\kappa}, & p \geq 2, \quad -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$, and

$$A^4_{n,p}(z) := \begin{cases} n^{\frac{\gamma+2}{p}(1+\kappa)}, & 2 \leq p < \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{1+\kappa-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p = \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{1+\kappa-\frac{1}{p}}, & p > \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{1+\kappa-\frac{1}{p}}, & p \geq 2, \quad 0 \leq \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(2-\frac{2}{p}\right)+\frac{1}{p}}, & p \geq 2, \quad -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

Theorem 6. *Let $1 < p < 2$, G be a κ-quasidisk for some $0 \leq \kappa < 1$ and $h(z)$ be defined by* (1). *Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have*:

$$|P_n'(z)| \leq c_5 \frac{|\Phi^{2(n+1)}(z)|}{d(z,L)} \|P_n\|_p A^5_{n,p}(z), \tag{15}$$

where $c_5 = c_5(G, \gamma, p) > 0$ constant independent from n and z;

$$A^5_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma+2}{p}\right)(1+\kappa)}, & 1 < p < 2, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{\left(\frac{\gamma+2}{p}\right)(1+\kappa)}, & 1 < p < 1+\gamma(1+\kappa), \quad 0 < \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+1+\kappa}, & 1+\gamma(1+\kappa) \leq p < 2, \quad 0 \leq \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+1+\kappa}, & 1 < p < 2, \quad -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$, *and*

$$A^5_{n,p}(z) = \begin{cases} n^{\frac{\gamma+2}{p}(1+\kappa)-1}, & 1 < p < 2, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{\frac{\gamma+2}{p}(1+\kappa)-1}, & 1 < p < 1+\gamma(1+\kappa), \quad 0 < \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+\kappa}(\ln n)^{1-\frac{1}{p}}, & p = 1+\gamma(1+\kappa), \quad 0 < \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+\kappa}, & 1+\gamma(1+\kappa) < p < 2, \quad 0 \leq \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+\kappa}, & 1 < p < 2, \quad -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

2.4 Estimate for $|P_n''(z)|$

Considering the estimates obtained in Theorems 5 and 6 for $||P_n'(z)|$ and Theorem 4 for $|P_n(z)|$ in Theorems 2 and 3, respectively, we can obtain the following.

Theorem 7. *Let* $p \geq 2$, *G be a* κ*-quasidisk for some* $0 \leq \kappa < 1$ *and* $h(z)$ *be defined by* (1). *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have*:

$$|P_n''(z)| \leq c_6 \frac{|\Phi^{3(n+1)}(z)|}{d(z,L)} \, \|P_n\|_p A^6_{n,p}(z), \tag{16}$$

where $c_6 = c_6(G,\gamma,p) > 0$ *constant independent from n and* z;

$$A^6_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & 2 \leq p < \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{1-\frac{1}{p}+2(1+\kappa)}(\ln n)^{1-\frac{1}{p}}, & p = \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{1-\frac{1}{p}+2(1+\kappa)}, & p > \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{1-\frac{1}{p}+2(1+\kappa)}, & 2 \leq p < 1+(2+\gamma)(1+\kappa), \quad 0 \leq \gamma < \frac{1}{1+\kappa}, \\ n^{2+\frac{1}{p}+\left(3-\frac{2}{p}\right)\kappa}, & p \geq 2, \quad -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$, *and*

$$A^6_{n,p}(z) := \begin{cases} n^{\frac{\gamma+2}{p}(1+\kappa)+\kappa}, & 2 \leq p < \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{1+2\kappa-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p = \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{1+2\kappa-\frac{1}{p}}, & p > \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa}, \\ n^{1+2\kappa-\frac{1}{p}}, & p \geq 2, \quad 0 \leq \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(2-\frac{2}{p}\right)+\frac{1}{p}+\kappa}, & p \geq 2, \quad -2 < \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

Theorem 8. *Let* $1 < p < 2$, G *be a* κ*-quasidisk for some* $0 \le \kappa < 1$ *and* $h(z)$ *be defined by* (1). *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have*:

$$|P_n''(z)| \le c_7 \frac{|\Phi^{3(n+1)}(z)|}{d(z,L)} \, \|P_n\|_p A^7_{n,p}(z), \tag{17}$$

where $c_7 = c_7(G,\gamma,p) > 0$ *constant independent from* n *and* z;

$$A^7_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & 1 < p < 2, \quad \gamma \ge \frac{1}{1+\kappa}, \\ n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & 1 < p < 1+\gamma(1+\kappa), \quad 0 < \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}+1\right)+\frac{1}{p}+2}(\ln n)^{1-\frac{1}{p}}, & p = 1+\gamma(1+\kappa), \quad 0 < \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}+1\right)+\frac{1}{p}+2}, & 1+\gamma(1+\kappa) < p < 2, \quad 0 \le \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}+1\right)+\frac{1}{p}+2}, & 1 < p < 2, \quad -2 < \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$, *and*

$$A^7_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & 1 < p < 2, \quad \gamma \ge \frac{1}{1+\kappa}, \\ n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & p < 1+\gamma(1+\kappa), \quad 0 < \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+2\kappa}(\ln n)^{1-\frac{1}{p}}, & p = 1+\gamma(1+\kappa), \quad 0 < \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+2\kappa}, & p > 1+\gamma(1+\kappa), \quad 0 < \gamma < \frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+2\kappa}, & 1 < p < 2, \quad -2 < \gamma \le 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

2.5 Estimates for $|P_n'(z)|$ and $|P_n''(z)|$ in whole plane

According to (4) (applied to the polynomial $Q_{n-1}(z) := P_n'(z)$), the estimation (9) is true also for the points $z \in \overline{G}_R$, $R = 1 + \varepsilon_0 n^{-1}$, with a different constant. Therefore, combining estimation (9) (for the $z \in \overline{G}_R$) with (14), (15), (16), and (17), we will obtain estimation on the growth of $|P_n'(z)|$ and $|P_n''(z)|$, respectively, in the whole complex plane.

Theorem 9. *Let* $p \ge 2$, G *be a* κ*-quasidisk for some* $0 \le \kappa < 1$ *and* $h(z)$ *be defined by* (1). *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have*:

$$|P_n'(z)| \le c_8 \, \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & z \in \overline{G}_R, \\ \frac{|\Phi^{2(n+1)}(z)|}{d(z,L)} A^4_{n,p}(z), & z \in \Omega_R, \end{cases}$$

where $c_8 = c_8(G,\gamma,p) > 0$ *constant independent from* n *and* z; $A^4_{n,p}(z)$ *defined as in* Theorem 5 *for all* $z \in \Omega_R$.

Theorem 10. *Let* $1 < p < 2$, G *be a* κ*-quasidisk for some* $0 \le \kappa < 1$ *and* $h(z)$ *be defined by* (1). *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have*:

$$|P_n'(z)| \le c_9 \, \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & z \in \overline{G}_R, \\ \dfrac{|\Phi^{2(n+1)}(z)|}{d(z,L)} A_{n,p}^5(z), & z \in \Omega_R, \end{cases}$$

where $c_9 = c_9(G, \gamma, p) > 0$ *constant independent from n and* z; $A_{n,p}^5(z)$ *defined as in* Theorem 6 *for all* $z \in \Omega_R$.

Theorem 11. *Let* $p \ge 2$, G *be a* κ*-quasidisk for some* $0 \le \kappa < 1$ *and* $h(z)$ *be defined by* (1). *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have*:

$$|P_n''(z)| \le c_{10} \, \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+2\right)(1+\kappa)}, & z \in \overline{G}_R, \\ \dfrac{|\Phi^{3(n+1)}(z)|}{d(z,L)} A_{n,p}^6(z), & z \in \Omega_R, \end{cases}$$

where $c_{10} = c_{10}(G, \gamma, p) > 0$ *constant independent from n and* z; $A_{n,p}^6(z)$ *defined as in* Theorem 7 *for all* $z \in \Omega_R$.

Theorem 12. *Let* $1 < p < 2$, G *be a* κ*-quasidisk for some* $0 \le \kappa < 1$ *and* $h(z)$ *be defined by* (1). *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *we have*:

$$|P_n''(z)| \le c_{11} \, \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+2\right)(1+\kappa)}, & z \in \overline{G}_R, \\ \dfrac{|\Phi^{3(n+1)}(z)|}{d(z,L)} A_{n,p}^7(z), & z \in \Omega_R, \end{cases}$$

where $c_{11} = c_{11}(G, \gamma, p) > 0$ *constant independent from n and* z; $A_{n,p}^7(z)$ *defined as in* Theorem 8 *for all* $z \in \Omega_R$.

Thus, using Theorems 2 and 3 and estimating the $|P_n^{(m)}(z)|$ sequentially for each $m \ge 3$ and combining the obtained estimates with Theorem A, we obtain estimates for the $|P_n^{(m)}(z)|$ on the whole complex plane.

3 Some auxiliary results

Throughout this paper, we denote "$a \preccurlyeq b$" and "$a \asymp b$" are equivalent to $a \le b$ and $c_1 a \le b \le c_2 a$ for some constants c, c_1, c_2, respectively.

Lemma 1. [36] *Let G be a quasidisk,* $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preccurlyeq d(z_1, L_{r_0})\}$, $w_j = \Phi(z_j)$, $j = 1, 2, 3$. *Then,*

(a) *The statements* $|z_1 - z_2| \preccurlyeq |z_1 - z_3|$ *and* $|w_1 - w_2| \preccurlyeq |w_1 - w_3|$ *are equivalent. So they are* $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$.

(b) *If* $|z_1 - z_2| \preccurlyeq |z_1 - z_3|$, *then,*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_1} \preccurlyeq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preccurlyeq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_2},$$

where $0 < r_0 < 1$ is *a constant, depending on G and* κ.

Lemma 2. *Let G be a* κ*-quasidisk for some* $0 \le \kappa < 1$. *Then,*

$$|w_1 - w_2|^{1-\kappa} \succcurlyeq |\Psi(w_1) - \Psi(w_2)| \succcurlyeq |w_1 - w_2|^{1+\kappa},$$

for all $w_1, w_2 \in \overline{\Omega}'$.

This fact follows from an appropriate result for the mapping $f \in \sum(\kappa)$ [5, p. 287] and estimation for the Ψ' [37, Theorem 2.8]:

$$|\Psi'(\tau)| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{18}$$

Lemma 3. [38] *Let L be a κ-quasidisk for some $0 \le \kappa < 1$; $R = 1 + \frac{c}{n}$. Then, for any fixed $\varepsilon \in (0,1)$, there exist a level curve $L_{1+\varepsilon(R-1)}$ such that the following holds for any polynomial $P_n(z) \in \wp_n$, $n \in \mathbb{N}$:*

$$\|P_n\|_{\mathcal{L}_p\left(\frac{h}{|\Phi'|}, L_{1+\varepsilon(R-1)}\right)} \preceq n^{\frac{1}{p}} \|P_n\|_p, \quad p > 0.$$

Let $\{z_j\}_{j=1}^{l}$ be a fixed system of the points on L and the weight function $h(z)$ defined as in (1). The following result is the integral analog of the familiar lemma of Bernstein-Walsh [1, p. 101] for the $A_p(h, G)$ norm.

Lemma 4. [19] *Let G be a quasidisk and $P_n(z)$, $\deg P_n \le n$, $n = 1, 2, \dots$, is arbitrary polynomial and weight function $h(z)$ satisfied the condition (1). Then, for any $R > 1$, $p > 0$ and $n = 1, 2, \dots$,*

$$\|P_n\|_{A_p(h, G_R)} \le c_3(1 + c(R-1))^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)},$$

where c, and c_3 are independent of n and G.

This fact shows that the order of norms $\|P_n\|_{A_p(h, G_{1+c/n})}$ and $\|P_n\|_{A_p(h,G)}$ for arbitrary polynomials $P_n(z)$ have the same growth order.

4 Proof of theorems

Proof of Theorems 2 and 3. The proofs of Theorems 2 and 3 are as follows. Let $G \in Q(\kappa)$ for some $0 \le \kappa < 1$ and let $R = 1 + \frac{1}{n}$, $R_1 := 1 + \frac{R-1}{2}$. For $z \in \Omega$, let us set:

$$H_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}.$$

Let us represent the mth derivative of $H_{n,p}(z)$ as follows:

$$\begin{aligned} H_n^{(m)}(z) &:= \left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)} \\ &= \sum_{j=0}^{m} C_m^j \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} P_n^{(m-j)}(z) \\ &= \frac{P_n^{(m)}(z)}{\Phi^{n+1}(z)} + \sum_{j=1}^{m} C_m^j \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} P_n^{(m-j)}(z), \end{aligned}$$

where $C_m^j = \binom{m}{j}$. Therefore,

$$P_n^{(m)}(z) = \Phi^{n+1}(z) \left\{ \left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)} - \sum_{j=1}^{m} C_m^j \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} P_n^{(m-j)}(z) \right\}, \quad z \in \Omega.$$

Then,

$$|P_n^{(m)}(z)| \le |\Phi^{n+1}(z)| \left\{ \left| \left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)} \right| + \sum_{j=1}^{m} C_m^j \left| \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} \right| |P_n^{(m-j)}(z)| \right\}. \tag{19}$$

As can be seen from (19), the following three statements on the right side need to be evaluated for $z \in \Omega$ to obtain the evaluation for $|P_n^{(m)}(z)|$:

$$\text{(A)} \left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right|; \quad \text{(B)} \left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right|; \quad \text{(C)} \ |P_n^{(m-j)}(z)|.$$

Now let us start the evaluations in order.

(A) Since the function $H_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}$, $H_n(\infty) = 0$, is analytic in Ω, continuous on $\overline{\Omega}$, then Cauchy integral representation for the region Ω_{R_1} gives

$$H_n^{(m)}(z) = -\frac{1}{2\pi i}\int\limits_{L_{R_1}} H_n(\zeta)\frac{d\zeta}{(\zeta - z)^{m+1}}, z \in \Omega_R, \quad m \geq 1.$$

Then,

$$\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right| \leq \frac{1}{2\pi}\int\limits_{L_{R_1}}\left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta - z|^{m+1}} \leq \frac{1}{2\pi d(z, L_{R_1})}\int\limits_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta - z|^m}. \tag{20}$$

We denote

$$A_n(z) := \int\limits_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta - z|^m}, \tag{21}$$

and estimate these integrals separately.

For this, we give some notations.

Let $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$. Without loss of generality, we will assume that $\varphi_l < 2\pi$. For $\eta_j = \min_{t\in\partial\Phi(\Omega(z_j,\delta_j))} |t - w_j| > 0$ and $\eta := \min\{\eta_j, j = \overline{1, l}\}$, let us set:

$$\Delta_j(\eta_j) := \{t : |t - w_j| \leq \eta_j\} \subset \Phi(\Omega(z_j, \delta_j)),$$

$$\Delta(\eta) := \bigcup_{j=1}^{l}\Delta_j(\eta), \widehat{\Delta}_j = \Delta\backslash\Delta(\eta_j); \ \widehat{\Delta}(\eta) := \Delta\backslash\Delta(\eta); \ \Delta_1' := \Delta_1'(1),$$

$$\Delta_1'(\rho) := \left\{t = R\cdot e^{i\theta} : R \geq \rho > 1, \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\},$$

$$\Delta_j' := \Delta_j'(1), \Delta_j'(\rho) := \left\{t = Re^{i\theta} : R \geq \rho > 1, \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_0}{2}\right\}, \quad j = 2, 3 \ldots, l,$$

where $\varphi_0 = 2\pi - \varphi_l$; $\Omega_j := \Psi(\Delta_j')$, $L_{R_1}^j := L_{R_1} \cap \Omega_j$; $\Omega = \bigcup_{j=1}^{l}\Omega_j$.

For simplicity, only one "critical" point z_1 on the boundary is taken for $h(z)$, i.e., $l = 1$. To estimate $A_n(z)$, first by replacing the variable $\tau = \Phi(\zeta)$ and multiplying the numerator and denominator of the integrant by $|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}|\Psi'(\tau)|^{\frac{2}{p}}$ and applying the Hölder inequality, we obtain:

$$A_n(z) = \int\limits_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta - z|^m} = \sum_{i=1}^{2}\int\limits_{F_{R_1}^i}\frac{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma_j}{p}}\left|P_n(\Psi(\tau))(\Psi'(\tau))^{\frac{2}{p}}\right||\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}|\Psi(\tau) - \Psi(w)|^m}|d\tau|$$

$$\leq \sum_{i=1}^{2}\left(\int\limits_{F_{R_1}^i}|\Psi(\tau) - \Psi(w_1)|^{\gamma}|P_n(\Psi(\tau))|^p|\Psi'(\tau)|^2|d\tau|\right)^{\frac{1}{p}}$$

$$\times\left(\int\limits_{F_{R_1}^i}\left(\frac{|\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}|\Psi(\tau) - \Psi(w)|^m}\right)^q|d\tau|\right)^{\frac{1}{q}} =: \sum_{i=1}^{2}A_n^i(z),$$

where $F_{R_1}^1 := \Phi(L_{R_1}^1) = \Delta_1' \cap \{\tau : |\tau| = R_1\}$, $F_{R_1}^2 := \Phi(L_{R_1}) \backslash F_{R_1}^1$ and

$$A_n^i(z) := \left(\int_{F_{R_1}^i} |f_{n,p}(\tau)|^p |d\tau| \right)^{\frac{1}{p}} \left(\int_{F_{R_1}^i} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}} |d\tau| \right)^{\frac{1}{q}} =: J_{n,1}^i \cdot J_{n,2}^i(z),$$

$$f_{n,p}(\tau) := (\Psi(\tau) - \Psi(w_1))^{\frac{\gamma}{p}} P_n(\Psi(\tau))(\Psi'(\tau))^{\frac{2}{p}}, \; |\tau| = R_1.$$

By applying Lemma 3, we obtain:

$$J_{n,1}^i \preccurlyeq n^{\frac{1}{p}} \, \|P_n\|_p, \quad i = 1, 2.$$

For the estimation of the integral

$$(J_{n,2}^i(z))^q = \int_{F_{R_1}^i} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}} |d\tau| \tag{22}$$

for $i = 1, 2$, we set:

$$E_{R_1}^{11} := \{\tau : \tau \in F_{R_1}^1, |\tau - w_1| < c_1(R_1 - 1)\}, \tag{23}$$

$$E_{R_1}^{12} := \{\tau : \tau \in F_{R_1}^1, \; c_1(R_1 - 1) \le |\tau - w_1| < \eta\}, \quad E_{R_1}^{13} := \{\tau : \tau \in \Phi\left(L_{R_1}\right), |\tau - w_1| \ge \eta\},$$

where $0 < c_1 < \eta$ is chosen so that $\{\tau : |\tau - w_1| < c_1(R_1 - 1)\} \cap \Delta \neq \varnothing$ and $\Phi(L_{R_1}) = \bigcup_{k=1}^3 E_{R_1}^{1k}$. Considering these notations, from (22), we have:

$$J_{n,2}^1(z) + J_{n,2}^2(z) =: J_2(z) = J_2\left(E_{R_1}^{11}\right) + J_2\left(E_{R_1}^{12}\right) + J_2\left(E_{R_1}^{13}\right) =: J_2^1(z) + J_2^2(z) + J_2^3(z)$$

and, consequently,

$$A_n(z) = A_n^1(z) + A_n^2(z) \preccurlyeq n^{\frac{1}{p}} \, \|P_n\|_p \cdot (J_2^1(z) + J_2^2(z) + J_2^3(z)) =: A_{n,1}(z) + A_{n,2}(z) + A_{n,3}(z), \tag{24}$$

where

$$A_{n,k}(z) := n^{\frac{1}{p}} \, \|P_n\|_p \cdot J_2^k(z), \quad k = 1, 2, 3,$$

$$(J_2^k(z))^q := \int_{E_{R_1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}}, \quad k = 1, 2, 3.$$

For any $k = 1, 2$, we denote

$$E_{R_1,1}^{1k} := \{\tau \in E_{R_1}^{1k} : |\Psi(\tau) - \Psi(w_1)| \ge |\Psi(\tau) - \Psi(w)|\}, \; E_{R_1,2}^{1k} := E_{R_1}^{1k} \backslash E_{R_1,1}^{1k},$$

$$(I\left(E_{R_1,1}^{1k}\right))^q := \begin{cases} \displaystyle\int_{E_{R_1,1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{\gamma(q-1)+qm}}, & \text{if } \gamma \ge 0, \\ \displaystyle\int_{E_{R_1,1}^{1k}} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma)(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}}, & \text{if } \gamma < 0, \end{cases} \tag{25}$$

$$(I\left(E_{R_1,2}^{1k}\right))^q := \int_{E_{R_1,2}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)+qm}}, \quad k = 1, 2,$$

and estimate the last integrals.

Given the possible values q ($q > 2$ and $q < 2$) and γ ($-2 < \gamma < 0$ and $\gamma \ge 0$), we will consider the cases separately.

Case 1. Let $1 < q \le 2 (p \ge 2)$. Then,

$$\left(I\left(E^{1k}_{R_1,1}\right)\right)^q = \int\limits_{E^{1k}_{R_1,1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}|\Psi(\tau)-\Psi(w)|^{qm}}.$$

1.1. Let $\gamma \ge 0$. If $z \in \Omega(\delta)$, applying Lemma 2 to (18), we get:

$$\begin{aligned}\left(I\left(E^{11}_{R_1,1}\right)\right)^q &\le \int\limits_{E^{11}_{R_1,1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \asymp \int\limits_{E^{11}_{R_1,1}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \\ &\preccurlyeq n^{(2-q)} \int\limits_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{[\gamma(q-1)+qm-(2-q)](1+\kappa)}} \preccurlyeq n^{(2-q)+[\gamma(q-1)+qm-(2-q)](1+\kappa)} \operatorname{mes} E^{11}_{R_1,1} \\ &\preccurlyeq n^{(2-q)+[\gamma(q-1)+qm-(2-q)](1+\kappa)-1},\end{aligned} \tag{26}$$

$$I\left(E^{11}_{R_1,1}\right) \preccurlyeq n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]},$$

$$\begin{aligned}\left(I\left(E^{11}_{R_1,2}\right)\right)^q &\preccurlyeq \int\limits_{E^{11}_{R_1,2}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+qm}} \\ &\preccurlyeq n^{(2-q)} \int\limits_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\tau-w_1|^{[\gamma(q-1)+qm-(2-q)](1+\kappa)}} \preccurlyeq n^{(2-q)+[\gamma(q-1)+qm-(2-q)](1+\kappa)} \operatorname{mes} E^{11}_{R_1,2} \\ &\preccurlyeq n^{(2-q)+[\gamma(q-1)+qm-(2-q)](1+\kappa)-1},\end{aligned}$$

$$I\left(E^{11}_{R_1,2}\right) \preccurlyeq n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]},$$

$$\begin{aligned}\left(I\left(E^{12}_{R_1,1}\right)\right)^q &\preccurlyeq \int\limits_{E^{12}_{R_1,1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \preccurlyeq \int\limits_{E^{12}_{R_1,1}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \\ &\preccurlyeq n^{(2-q)} \int\limits_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{[\gamma(q-1)+qm-(2-q)](1+\kappa)}} \\ &\preccurlyeq n^{(2-q)} \begin{cases} n^{[\gamma(q-1)+qm-(2-q)](1+\kappa)-1}, & [\gamma(q-1)+qm-(2-q)](1+\kappa) > 1, \\ \ln n, & [\gamma(q-1)+qm-(2-q)](1+\kappa) = 1, \\ 1, & [\gamma(q-1)+qm-(2-q)](1+\kappa) < 1, \end{cases}\end{aligned} \tag{27}$$

$$I\left(E^{12}_{R_1,1}\right) \preccurlyeq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & \gamma > -2, & m \ge 2, \\ n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & p < 1+(2+\gamma)(1+\kappa), & m = 1, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{\left(1-\frac{1}{p}\right)}, & p = 1+(2+\gamma)(1+\kappa), & m = 1, \\ n^{\left(1-\frac{2}{p}\right)}, & p > 1+(2+\gamma)(1+\kappa), & m = 1, \end{cases}$$

$$\begin{aligned}\left(I\left(E^{12}_{R_1,2}\right)\right)^q &\le \int\limits_{E^{12}_{R_1,2}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+qm}} \preccurlyeq n^{(2-q)} \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\tau-w_1|^{[\gamma(q-1)+qm-(2-q)](1+\kappa)}} \\ &\preccurlyeq n^{(2-q)} \begin{cases} n^{[\gamma(q-1)+qm-(2-q)](1+\kappa)-1}, & [\gamma(q-1)+qm-(2-q)](1+\kappa) > 1, \\ \ln n, & [\gamma(q-1)+qm-(2-q)](1+\kappa) = 1, \\ 1, & [\gamma(q-1)+qm-(2-q)](1+\kappa) < 1, \end{cases}\end{aligned} \tag{28}$$

$$I\left(E^{12}_{R_1,2}\right) \preccurlyeq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & \gamma > -2, \quad m \geq 2, \\ n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & p < 1+(2+\gamma)(1+\kappa), \quad m = 1, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{1-\frac{1}{p}}, & p = 1+(2+\gamma)(1+\kappa), \quad m = 1, \\ n^{\left(1-\frac{2}{p}\right)}, & p > 1+(2+\gamma)(1+\kappa), \quad m = 1, \end{cases}$$

$$I\left(E^{12}_{R_1,1}\right) + I\left(E^{12}_{R_1,2}\right) \preccurlyeq \begin{cases} n^{\left(\frac{\gamma+2}{p}+m\right)(1+\kappa)-\left[1+\kappa+\frac{1}{p}\right]}, & \gamma > -2, \quad m \geq 2, \\ n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & p < 1+(2+\gamma)(1+\kappa), \quad m = 1, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{1-\frac{1}{p}}, & p = 1+(2+\gamma)(1+\kappa), \quad m = 1, \\ n^{\left(1-\frac{2}{p}\right)}, & p > 1+(2+\gamma)(1+\kappa), \quad m = 1. \end{cases}$$

For $\tau \in E^{13}_{R_1}$, we see that $\eta < |\tau - w_1| < 2\pi\dot{R}_1$, $|\tau - w| \geq \eta - c_1$. Therefore, $|\Psi(\tau) - \Psi(w_1)| \succcurlyeq 1$, from Lemma 1 and for $|\tau - w_1| \geq \eta, |\Psi(\tau) - \Psi(w)| \succcurlyeq |\tau - w|^{1+\kappa}$, from Lemma 2. Then, for $w \in \Delta(w_1, \eta)$, applying (18), we obtain:

$$\begin{aligned} (J^3_2(z))^q &\preccurlyeq \int\limits_{E^{13}_{R_1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}} \preccurlyeq \int\limits_{E^{13}_{R_1}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}} \\ &\preccurlyeq n^{(2-q)} \int\limits_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau - w|^{[qm-(2-q)](1+\kappa)}} \\ &\preccurlyeq n^{(2-q)} \begin{cases} n^{[qm-(2-q)](1+\kappa)-1}, & [qm-(2-q)](1+\kappa) > 1, \quad m \geq 2, \\ n^{[qm-(2-q)](1+\kappa)-1}, & [qm-(2-q)](1+\kappa) > 1, \quad m = 1, \\ \ln n, & [qm-(2-q)](1+\kappa) = 1, \quad m = 1, \\ 1, & [qm-(2-q)](1+\kappa) < 1, \quad m = 1, \end{cases} \end{aligned}$$

$$J^3_2(z) \preccurlyeq \begin{cases} n^{\left(\frac{2}{p}+m\right)(1+\kappa)-\left[1+\kappa+\frac{1}{p}\right]}, & p \geq 2, \quad m \geq 2, \\ n^{\left(\frac{2}{p}+m\right)(1+\kappa)-\left[1+\kappa+\frac{1}{p}\right]}, & p < 1+2(1+\kappa), \quad m = 1, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{1-\frac{1}{p}}, & p = 1+2(1+\kappa), \quad m = 1, \\ n^{\left(1-\frac{2}{p}\right)}, & p > 1+2(1+\kappa), \quad m = 1, \end{cases} \quad z \in \Omega(\delta), \tag{29}$$

$$J^3_2(z) \preccurlyeq n^{\kappa\left(1-\frac{2}{p}\right)}, \ z \in \widehat{\Omega}(\delta).$$

Combining (26)–(29), for $p \geq 2$, $\gamma \geq 0$ and $z \in \Omega(\delta)$, we obtain:

$$\sum_{k=1}^{3} J^k_2(z) \preccurlyeq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & \gamma > -2, \quad m \geq 2, \\ n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & 2 \leq p < 1+(2+\gamma)(1+\kappa), \quad m = 1, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{\left(1-\frac{1}{p}\right)}, & p = 1+(2+\gamma)(1+\kappa), \quad m = 1, \\ n^{1-\frac{2}{p}}, & p > 1+(2+\gamma)(1+\kappa), \quad m = 1. \end{cases} \tag{30}$$

If $z \in \widehat{\Omega}(\delta)$, then

$$\begin{aligned} (J^1_2(z))^q &\preccurlyeq \int\limits_{E^{11}_{R_1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq \int\limits_{E^{11}_{R_1}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\tau - w_1|^{\gamma(q-1)(1+\kappa)}} \\ &\preccurlyeq n^{(2-q)} \int\limits_{E^{11}_{R_1}} \frac{|d\tau|}{|\tau - w_1|^{[\gamma(q-1)-(2-q)](1+\kappa)}} \preccurlyeq n^{(2-q)+[\gamma(q-1)-(2-q)](1+\kappa)} \mathrm{mes} E^{11}_{R_1} \\ &\preccurlyeq n^{(2-q)+[\gamma(q-1)-(2-q)](1+\kappa)-1}, \end{aligned} \tag{31}$$

$$J^1_2(z) \preccurlyeq n^{\frac{\gamma}{p}(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}.$$

$$(J_2^2(z))^q \preccurlyeq \int\limits_{E_{R_1}^{12}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq \int\limits_{E_{R_1}^{12}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\tau-w_1|^{\gamma(q-1)(1+\kappa)}}$$
$$\preccurlyeq n^{(2-q)} \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau-w_1|^{[\gamma(q-1)-(2-q)](1+\kappa)}}$$
$$\preccurlyeq n^{(2-q)} \begin{cases} n^{[\gamma(q-1)-(2-q)](1+\kappa)-1}, & [\gamma(q-1)-(2-q)](1+\kappa)>1,\\ \ln n, & [\gamma(q-1)-(2-q)](1+\kappa)=1,\\ 1, & [\gamma(q-1)-(2-q)](1+\kappa)<1,\end{cases}$$

$$J_2^2(z) \preccurlyeq \begin{cases} n^{\left(\frac{\gamma}{p}\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & 2\le p<\frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma>\frac{1}{1+\kappa},\\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{\left(1-\frac{1}{p}\right)}, & p=\frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma>\frac{1}{1+\kappa},\\ n^{1-\frac{2}{p}}, & p>\frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma>\frac{1}{1+\kappa},\\ n^{1-\frac{2}{p}}, & p\ge 2, & -2<\gamma\le\frac{1}{1+\kappa},\end{cases}$$

$$(J_2^3(z))^q \preccurlyeq \int\limits_{E_{R_1}^{13}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq \int\limits_{E_{R_1}^{13}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q}|d\tau| \preccurlyeq n^{\kappa(2-q)},$$
$$J_2^3(z) \preccurlyeq n^{\kappa\left(1-\frac{2}{p}\right)}.$$

From (24)–(31), we obtain:

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)(1+\kappa)}, & \gamma>-2, p\ge 2, & m\ge 2,\\ n^{\left(\frac{\gamma+2}{p}+m-1\right)(1+\kappa)}, & 2\le p<1+(2+\gamma)(1+\kappa), & m=1,\\ (n\ln n)^{\left(1-\frac{1}{p}\right)}, & p=1+(2+\gamma)(1+\kappa), & m=1,\\ n^{\left(1-\frac{1}{p}\right)}, & p>1+(2+\gamma)(1+\kappa), & m=1,\end{cases} \tag{32}$$

if $z\in\Omega(\delta)$ and

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)(1+\kappa)}, & 2\le p<\frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma>\frac{1}{1+\kappa},\\ (n\ln n)^{1-\frac{1}{p}}, & p=\frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma>\frac{1}{1+\kappa},\\ n^{1-\frac{1}{p}}, & p>\frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma>\frac{1}{1+\kappa},\\ n^{1-\frac{1}{p}}, & p\ge 2, & -2<\gamma\le\frac{1}{1+\kappa},\end{cases} \tag{33}$$

if $z\in\widehat{\Omega}(\delta)$.

1.2. If $\gamma<0$, for $w\in\Delta(w_1,\eta)\cap\Omega_R(\delta)$, such that $|\Psi(\tau)-\Psi(w_1)|\preccurlyeq|\Psi(\tau)-\Psi(w)|$, according to Lemma 1, analogously we have:

$$\left(I\left(E_{R_1,1}^{11}\right)\right)^q \preccurlyeq \int\limits_{E_{R_1,1}^{11}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}} \preccurlyeq n^{2-q} \int\limits_{E_{R_1,1}^{11}} \frac{|\tau-w_1|^{(-\gamma)(q-1)(1-\kappa)}}{|\tau-w|^{[qm-(2-q)](1+\kappa)}}|d\tau|$$
$$\preccurlyeq n^{2-q+[qm-(2-q)](1+\kappa)}\mathrm{mes}E_{R_1}^{11} \preccurlyeq n^{2-q+[qm-(2-q)](1+\kappa)-1}, \tag{34}$$
$$I\left(E_{R_1,1}^{11}\right) \preccurlyeq n^{m(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]},$$

$$\begin{aligned}(I(E^{11}_{R_1,2}))^q &\asymp \int\limits_{E^{11}_{R_1,2}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+qm}} \preccurlyeq n^{(2-q)} \int\limits_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\tau-w_1|^{[\gamma(q-1)+qm-(2-q)](1+\kappa)}}\\ &\preccurlyeq n^{(2-q)+[\gamma(q-1)+qm-(2-q)](1+\kappa)}\mathrm{mes}E^{11}_{R_1,2} \preccurlyeq n^{(2-q)+[\gamma(q-1)+qm-(2-q)](1+\kappa)-1},\\ I(E^{11}_{R_1,2}) &\preccurlyeq n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}.\end{aligned} \tag{35}$$

For $\tau \in E^{12}_{R_1}$, we see that $|\tau - w_1| < \eta$ and from Lemma 1, $|\Psi(\tau) - \Psi(w_1)| \preccurlyeq 1$. Then, for $w \in \Delta(w_1, \eta) \cap \Omega_R(\delta)$, such that $|\Psi(\tau) - \Psi(w_1)| \preccurlyeq |\Psi(\tau) - \Psi(w)|$, applying Lemma 2, we obtain:

$$\begin{aligned}(I(E^{12}_{R_1,1}))^q &= \int\limits_{E^{12}_{R_1,1}} \left|\frac{\Psi(\tau)-\Psi(w_1)}{\Psi(\tau)-\Psi(w)}\right|^{(-\gamma)(q-1)} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm+\gamma(q-1)}}\\ &\preccurlyeq n^{2-q} \int\limits_{E^{12}_{R_1,1}} \left|\frac{\tau-w_1}{\tau-w}\right|^{(-\gamma)(q-1)\frac{1}{c_1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm+\gamma(q-1)-(q-2)}}\\ &\preccurlyeq n^{2-q} \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{[qm+\gamma(q-1)-(2-q)](1+\kappa)}}\\ &\preccurlyeq n^{(2-q)}\begin{cases} n^{[qm+\gamma(q-1)-(2-q)](1+\kappa)-1}, & qm+\gamma(q-1)-(2-q)(1+\kappa)>1,\\ \ln n, & qm+\gamma(q-1)-(2-q)(1+\kappa)=1,\\ 1, & qm+\gamma(q-1)-(2-q)(1+\kappa)<1,\end{cases}\end{aligned} \tag{36}$$

$$I(E^{12}_{R_1,1}) \preccurlyeq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & \gamma>-2, & m\ge 2,\ \gamma>-2,\\ n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & 2\le p<1+(2+\gamma)(1+\kappa), & m=1,\ \gamma>-2+\dfrac{1}{1+\kappa},\\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{\left(1-\frac{1}{p}\right)}, & p=1+(2+\gamma)(1+\kappa), & m=1,\ \gamma>-2+\dfrac{1}{1+\kappa},\\ n^{\left(1-\frac{2}{p}\right)}, & p>1+(2+\gamma)(1+\kappa), & m=1,\ \gamma>-2+\dfrac{1}{1+\kappa},\\ n^{\left(1-\frac{2}{p}\right)}, & p\ge 2, & m=1,\ -2<\gamma\le -2+\dfrac{1}{1+\kappa},\end{cases}$$

$$\begin{aligned}(I(E^{12}_{R_1,2}))^q &\asymp \int\limits_{E^{12}_{R_1,2}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm+\gamma(q-1)}}\\ &\preccurlyeq n^{2-q} \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\tau-w|^{[qm+\gamma(q-1)-(2-q)](1+\kappa)}}\\ &\preccurlyeq n^{(2-q)}\begin{cases} n^{[qm+\gamma(q-1)-(2-q)](1+\kappa)-1}, & qm+\gamma(q-1)-(2-q)(1+\kappa)>1,\\ \ln n, & qm+\gamma(q-1)-(2-q)(1+\kappa)=1,\\ 1, & qm+\gamma(q-1)-(2-q)(1+\kappa)<1,\end{cases}\end{aligned} \tag{37}$$

$$I(E^{12}_{R_1,2}) \preccurlyeq \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & \gamma>-2, & m\ge 2,\ \gamma>-2,\\ n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & p<1+(2+\gamma)(1+\kappa), & m=1,\ \gamma>-2+\dfrac{1}{1+\kappa},\\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{1-\frac{1}{p}}, & p=1+(2+\gamma)(1+\kappa), & m=1,\ \gamma>-2+\dfrac{1}{1+\kappa},\\ n^{1-\frac{2}{p}}, & p>1+(2+\gamma)(1+\kappa), & m=1,\ \gamma>-2+\dfrac{1}{1+\kappa},\\ n^{1-\frac{2}{p}}, & p\ge 2, & m=1,\ -2<\gamma\le -2+\dfrac{1}{1+\kappa}.\end{cases}$$

For $\tau \in E^{13}_{R_1}$ and each $w \in \Delta(w_1, \eta) \cap \Omega_R(\delta)$, we see that $\eta < |\tau - w_1| < 2\pi \dot{R}_1$. Therefore, from Lemma 1 and applying (18), we obtain:

$$\begin{aligned}
\left(I\left(E^{13}_{R_1}\right)\right)^q &:= \int\limits_{E^{13}_{R_1,1}} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma)(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}} \asymp \int\limits_{E^{13}_{R_1}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm}} \\
&\preccurlyeq n^{2-q} \int\limits_{E^{13}_{R_1}} \frac{|d\tau|}{|\tau - w|^{[qm-(2-q)](1+\kappa)}} \preccurlyeq n^{1-q+[qm-(2-q)](1+\kappa)}, \qquad (38) \\
I\left(E^{13}_{R_1}\right) &\preccurlyeq n^{m(1+\kappa) - \left[1 + \left(1-\frac{2}{p}\right)\kappa - \frac{1}{p}\right]}.
\end{aligned}$$

Therefore, combining (34)–(38) in case of $\gamma < 0$ for $z \in \Omega(\delta)$, we have:

$$\begin{aligned}
\sum_{k=1}^{3} J_2^k(z) &\preccurlyeq n^{m(1+\kappa) - \left[1+\left(1-\frac{2}{p}\right)\kappa - \frac{1}{p}\right]} + \begin{cases}
n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa) - \left[1+\left(1-\frac{2}{p}\right)\kappa - \frac{1}{p}\right]}, & p \ge 2, \quad m \ge 2, \ \gamma > -2, \\
n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa) - \left[1+\left(1-\frac{2}{p}\right)\kappa - \frac{1}{p}\right]}, & p < 1 + (2+\gamma)(1+\kappa), \quad m = 1, \ \gamma > -2 + \frac{1}{1+\kappa}, \\
n^{\left(1-\frac{2}{p}\right)} (\ln n)^{1-\frac{1}{p}}, & p = 1 + (2+\gamma)(1+\kappa), \quad m = 1, \ \gamma > -2 + \frac{1}{1+\kappa}, \\
n^{1-\frac{2}{p}}, & p > 1 + (2+\gamma)(1+\kappa), \quad m = 1, \ \gamma > -2 + \frac{1}{1+\kappa}, \\
n^{1-\frac{2}{p}}, & p \ge 2, \quad m = 1, \ -2 < \gamma \le -2 + \frac{1}{1+\kappa},
\end{cases} \\
&\preccurlyeq n^{m(1+\kappa) - \left[1+\left(1-\frac{2}{p}\right)\kappa - \frac{1}{p}\right]}. \qquad (39)
\end{aligned}$$

If $z \in \widehat{\Omega}(\delta)$, then $|w - w_1| \ge \eta$, from Lemma 2 and (18), we obtain:

$$\begin{aligned}
(J_2^1(z))^q &= \int\limits_{E^{11}_{R_1}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{qm}} \preccurlyeq \int\limits_{E^{11}_{R_1}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1}\right)^{2-q} |\Psi(\tau) - \Psi(w_1)|^{(-\gamma)(q-1)} |d\tau| \\
&\preccurlyeq n^{\kappa(2-q)} \int\limits_{E^{11}_{R_1}} \left(\frac{1}{n}\right)^{(-\gamma)(q-1)(1-\kappa)} |d\tau| \preccurlyeq n^{\kappa(2-q)} \operatorname{mes} E^{11}_{R_1} \preccurlyeq n^{\kappa(2-q)-1} \preccurlyeq 1, \qquad (40) \\
J_2^1(z) &\preccurlyeq 1, \\
(J_2^2(z))^q &\preccurlyeq \int\limits_{E^{12}_{R_1}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1}\right)^{2-q} |\Psi(\tau) - \Psi(w_1)|^{(-\gamma)(q-1)} |d\tau| \preccurlyeq \int\limits_{E^{12}_{R_1}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1}\right)^{2-q} |d\tau| \preccurlyeq n^{\kappa(2-q)}, \\
J_2^2(z) &\preccurlyeq n^{\kappa\left(1-\frac{2}{p}\right)}, \\
(J_2^3(z))^q &\preccurlyeq \int\limits_{E^{13}_{R_1}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq \int\limits_{E^{13}_{R_1}} \left(\frac{d(\Psi(\tau), L)}{|\tau| - 1}\right)^{2-q} |d\tau| \preccurlyeq n^{\kappa(2-q)}, \\
J_2^3(z) &\preccurlyeq n^{\kappa\left(1-\frac{2}{p}\right)}.
\end{aligned}$$

Combining the last three estimates, in case of $\gamma < 0$ for $z \in \widehat{\Omega}(\delta)$, we have:

$$\sum_{k=1}^{3} J_2^k(z) \preccurlyeq n^{\kappa\left(1-\frac{2}{p}\right)}. \qquad (41)$$

Then, for $\gamma < 0$, from (39)–(41), we obtain:

$$\sum_{k=1}^{3} J_2^k(z) \preccurlyeq \begin{cases} n^{m(1+\kappa) - \left[1+\left(1-\frac{2}{p}\right)\kappa - \frac{1}{p}\right]}, & z \in \Omega(\delta), \\ n^{\kappa\left(1-\frac{2}{p}\right)}, & z \in \widehat{\Omega}(\delta), \end{cases}$$

and, consequently, in this case from (24), we have:

$$A_n(z) \preccurlyeq n^{\frac{1}{p}} \|P_n\|_p \begin{cases} n^{m(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & z \in \Omega(\delta), \\ n^{\kappa\left(1-\frac{2}{p}\right)}, & z \in \widehat{\Omega}(\delta). \end{cases} \tag{42}$$

Therefore, combining (30) and (42), for any $\gamma > -2$, $p \geq 2$, we obtain:

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)(1+\kappa)}, & p \geq 2, & m \geq 2, \ \gamma > -2, \\ n^{\left(\frac{\gamma+2}{p}+m-1\right)(1+\kappa)}, & 2 \leq p < 1+(2+\gamma)(1+\kappa), & m = 1, \ \gamma \geq 0, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 1+(2+\gamma)(1+\kappa), & m = 1, \ \gamma \geq 0, \\ n^{1-\frac{1}{p}}, & p > 1+(2+\gamma)(1+\kappa), & m = 1, \ \gamma \geq 0, \\ n^{\left(\frac{2}{p}+m-1\right)(1+\kappa)}, & p \geq 2, & m \geq 1, \ -2 < \gamma < 0, \end{cases} \tag{43}$$

if $z \in \Omega(\delta)$, and

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+m-1\right)(1+\kappa)}, & 2 \leq p < \frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma > \frac{1}{1+\kappa}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma > \frac{1}{1+\kappa}, \\ n^{1-\frac{1}{p}}, & p > \frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, & \gamma > \frac{1}{1+\kappa}, \\ n^{1-\frac{1}{p}}, & p \geq 2, & 0 \leq \gamma \leq \frac{1}{1+\kappa}, \\ n^{\kappa\left(1-\frac{2}{p}\right)+\frac{1}{p}}, & p \geq 2, & -2 < \gamma < 0, \end{cases} \tag{44}$$

if $z \in \widehat{\Omega}(\delta)$.

Case 2. Let $q > 2\,(p < 2)$. Then, $2 - q < 0$, and so

$$\begin{aligned} \left(I\left(E^{1k}_{R_1,1}\right)\right)^q &:= \begin{cases} \displaystyle\int_{E^{1k}_{R_1,1}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}}, & \text{if } \gamma \geq 0, \\ \displaystyle\int_{E^{1k}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)}|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{qm}}, & \text{if } \gamma < 0, \end{cases} \\ \left(I\left(E^{1k}_{R_1,2}\right)\right)^q &:= \int_{E^{1k}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+q}}, \quad k = 1, 2, \\ (J_2^3(z))^q &:= \int_{E^{13}_{R_1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}|\Psi(\tau)-\Psi(w)|^{qm}}. \end{aligned} \tag{45}$$

2.1. If $\gamma \geq 0$ and $z \in \Omega(\delta)$, applying Lemmas 1 and 2 to (45), we obtain:

$$\begin{aligned} \left(I\left(E^{11}_{R_1,1}\right)\right)^q &\preccurlyeq \int_{E^{11}_{R_1,1}} \left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \preccurlyeq n^{\kappa(q-2)} \int_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{[\gamma(q-1)+qm](1+\kappa)}} \\ &\preccurlyeq n^{[\gamma(q-1)+qm](1+\kappa)+\kappa(q-2)} \mathrm{mes} E^{11}_{R_1,1} \preccurlyeq n^{[\gamma(q-1)+qm](1+\kappa)+\kappa(q-2)-1}, \\ I\left(E^{11}_{R_1,1}\right) &\preccurlyeq n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}, \end{aligned} \tag{46}$$

$$\begin{aligned}\left(I\left(E^{11}_{R_1,2}\right)\right)^q &\preccurlyeq \int\limits_{E^{11}_{R_1,2}}\left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+qm}} \preccurlyeq n^{\kappa(q-2)}\int\limits_{E^{11}_{R_1,2}}\frac{|d\tau|}{|\tau-w_1|^{\gamma(q-1)+qm}}\\ &\preccurlyeq n^{[\gamma(q-1)+qm](1+\kappa)+\kappa(q-2)}\mathrm{mes}E^{11}_{R_1,2} \preccurlyeq n^{[\gamma(q-1)+qm](1+\kappa)+\kappa(q-2)-1},\\ I\left(E^{11}_{R_1,2}\right) &\preccurlyeq n^{(\frac{\gamma}{p}+m)(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]},\end{aligned} \tag{47}$$

$$\begin{aligned}\left(I\left(E^{12}_{R_1,1}\right)\right)^q &\preccurlyeq n^{\kappa(q-2)}\int\limits_{E^{12}_{R_1,1}}\frac{|d\tau|}{|\tau-w|^{[\gamma(q-1)+qm](1+\kappa)}} \preccurlyeq n^{\kappa(q-2)+[\gamma(q-1)+qm](1+\kappa)-1},\\ I\left(E^{12}_{R_1,1}\right) &\preccurlyeq n^{(\frac{\gamma}{p}+m)(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]},\\ \left(I\left(E^{12}_{R_1,2}\right)\right)^q &\preccurlyeq \int\limits_{E^{12}_{R_1,2}}\frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+qm}} \preccurlyeq n^{\kappa(q-2)}\int\limits_{E^{12}_{R_1,2}}\frac{|d\tau|}{|\tau-w_1|^{[\gamma(q-1)+qm](1+\kappa)}}\\ &\preccurlyeq n^{\kappa(q-2)+[\gamma(q-1)+qm](1+\kappa)-1},\\ I\left(E^{12}_{R_1,2}\right) &\preccurlyeq n^{(\frac{\gamma}{p}+m)(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}.\end{aligned} \tag{48}$$

For $\tau \in E^{13}_{R_1}$, we see that $\eta < |\tau - w_1| < 2\pi\dot{R}_1$. Therefore, from Lemma 1, we have $|\Psi(\tau)-\Psi(w_1)| \succcurlyeq 1$. For $w \in \Delta(w_1,\eta)|\Psi(\tau)-\Psi(w)| \succcurlyeq |\tau-w|^{1+\kappa}$. Then, applying Lemma 2, we obtain:

$$\begin{aligned}(J^3_2(z))^q &\preccurlyeq n^{\kappa(q-2)}\int\limits_{E^{13}_{R_1}}\frac{|d\tau|}{|\tau-w|^{qm(1+\kappa)}} \preccurlyeq n^{\kappa(q-2)+qm(1+\kappa)-1},\\ J^3_2(z) &\preccurlyeq n^{m(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}.\end{aligned} \tag{49}$$

If $z \in \widehat{\Omega}(\delta)$, then $|w-w_1| \geq \eta$, from (18), we have:

$$\begin{aligned}(J^1_2(z))^q &\preccurlyeq \int\limits_{E^{11}_{R_1}}\left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq n^{\kappa(q-2)}\int\limits_{E^{11}_{R_1}}\frac{|d\tau|}{|\tau-w_1|^{\gamma(q-1)(1+\kappa)}}\\ &\preccurlyeq n^{\kappa(q-2)+\gamma(q-1)(1+\kappa)}\mathrm{mes}E^{11}_{R_1} \preccurlyeq n^{\kappa(q-2)+\gamma(q-1)(1+\kappa)-1},\\ J^1_2(z) &\preccurlyeq n^{\frac{\gamma}{p}(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]},\\ (J^2_2(z))^q &\preccurlyeq \int\limits_{E^{12}_{R_1}}\left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq n^{\kappa(q-2)}\int\limits_{E^{12}_{R_1}}\frac{|d\tau|}{|\tau-w_1|^{\gamma(q-1)(1+\kappa)}}\\ &\preccurlyeq n^{\kappa(q-2)}\begin{cases} n^{\gamma(q-1)(1+\kappa)-1}, & \gamma(q-1)(1+\kappa)>1,\\ \ln n, & \gamma(q-1)(1+\kappa)=1,\\ 1, & \gamma(q-1)(1+\kappa)<1,\end{cases}\\ J^2_2(z) &\preccurlyeq \begin{cases} n^{\frac{\gamma}{p}(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}, & \gamma > \dfrac{p-1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)}(\ln n)^{1-\frac{1}{p}}, & \gamma = \dfrac{p-1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)}, & \gamma < \dfrac{p-1}{1+\kappa},\end{cases}\\ (J^3_2(z))^q &= \int\limits_{E^{13}_{R_1}}\frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}|\Psi(\tau)-\Psi(w)|^{qm}}\\ &\preccurlyeq \int\limits_{E^{13}_{R_1}}\frac{|d\tau|}{|\Psi'(\tau)|^{q-2}} \preccurlyeq \int\limits_{E^{13}_{R_1}}\left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2}|d\tau| \preccurlyeq n^{\kappa(q-2)},\\ J^3_2(z) &\preccurlyeq n^{\kappa\left(\frac{2}{p}-1\right)}.\end{aligned}$$

From (46)–(49) and (24), for $\gamma \geq 0$, $1 < p < 2$, $m \geq 1$, we have:

$$A_n(z) \preccurlyeq \|P_n\|_p n^{\left(\frac{\gamma+2}{p}+m-1\right)(1+\kappa)}, \tag{50}$$

if $z \in \Omega(\delta)$ and

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}+m-1)(1+\kappa)}, & \gamma > \dfrac{p-1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & \gamma = \dfrac{p-1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & \gamma < \dfrac{p-1}{1+\kappa}, \end{cases} \tag{51}$$

if $z \in \widehat{\Omega}(\delta)$.

2.2. Let $\gamma < 0$. For $z \in \Omega(\delta)$, according to Lemma 1, we have:

$$\begin{aligned} \left(I\left(E^{11}_{R_1,1}\right)\right)^q &= \int\limits_{E^{11}_{R_1,1}} \left|\frac{\Psi(\tau)-\Psi(w_1)}{\Psi(\tau)-\Psi(w)}\right|^{(-\gamma)(q-1)} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{qm+\gamma(q-1)}} \\ &\preccurlyeq n^{\kappa(q-2)} \int\limits_{E^{11}_{R_1,1}} \left(\frac{|\tau-w_1|}{|\tau-w|}\right)^{(-\gamma)(q-1)\frac{1}{c_1}} \frac{|d\tau|}{|\tau-w|^{[qm+\gamma(q-1)](1+\kappa)}} \\ &\preccurlyeq n^{\kappa(2-q)+[\gamma(q-1)+qm](1+\kappa)} \mathrm{mes} E^{11}_{R_1,1} \preccurlyeq n^{\kappa(2-q)+[\gamma(q-1)+qm](1+\kappa)-1}, \\ I\left(E^{11}_{R_1,1}\right) &\preccurlyeq n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}, \end{aligned} \tag{52}$$

$$\begin{aligned} \left(I\left(E^{11}_{R_1,2}\right)\right)^q &\preccurlyeq n^{\kappa(q-2)} \int\limits_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+qm}} \preccurlyeq n^{\kappa(q-2)} \int\limits_{E^{11}_{R_1}} \frac{|d\tau|}{|\tau-w_1|^{[qm+\gamma(q-1)](1+\kappa)}} \\ &\preccurlyeq n^{\kappa(q-2)+[qm+\gamma(q-1)](1+\kappa)} \mathrm{mes} E^{11}_{R_1,2} \preccurlyeq n^{\kappa(q-2)+[qm+\gamma(q-1)](1+\kappa)-1}, \\ I\left(E^{11}_{R_1,2}\right) &\preccurlyeq n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}, \end{aligned}$$

$$\begin{aligned} \left(I\left(E^{12}_{R_1,1}\right)\right)^q &\preccurlyeq \int\limits_{E^{12}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)}|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{qm}} \preccurlyeq \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{qm}} \\ &\preccurlyeq n^{\kappa(q-2)} \int\limits_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{qm(1+\kappa)}} \preccurlyeq n^{\kappa(q-2)+qm(1+\kappa)-1}, \\ I\left(E^{12}_{R_1,1}\right) &\preccurlyeq n^{m(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}, \end{aligned} \tag{53}$$

$$\begin{aligned} \left(I\left(E^{12}_{R_1,2}\right)\right)^q &\preccurlyeq \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{qm}} \preccurlyeq n^{\kappa(q-2)} \int\limits_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\tau-w|^{qm(1+\kappa)}} \\ &\preccurlyeq n^{\kappa(q-2)+qm(1+\kappa)-1}, \\ I\left(E^{12}_{R_1,2}\right) &\preccurlyeq n^{m(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}, \end{aligned}$$

$$\begin{aligned} (J_2^3(z))^q &\preccurlyeq \int\limits_{E^{13}_{R_1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)}|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{qm}} \preccurlyeq n^{\kappa(q-2)} \int\limits_{E^{13}_{R_1}} \frac{|d\tau|}{|\tau-w|^{qm(1+\kappa)}} \\ &\preccurlyeq n^{\kappa(q-2)+qm(1+\kappa)-1}, \\ J_2^3(z) &\preccurlyeq n^{m(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}. \end{aligned} \tag{54}$$

For $z \in \widehat{\Omega}(\delta)$, analogously we obtain:

$$J_2^i(z) \preccurlyeq n^{\left(\frac{2}{p}-1\right)\kappa}, \;\; i = 1, 2, 3. \tag{55}$$

So, for $\gamma < 0$, from (24), we have:

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{2}{p}+m\right)(1+\kappa)}, & \text{if } z \in \Omega(\delta), \\ n^{\left(\frac{2}{p}-1\right)\kappa+\frac{1}{p}}, & \text{if } z \in \widehat{\Omega}(\delta). \end{cases} \tag{56}$$

Therefore, for any $\gamma \geq -2$, $1 < p < 2$, $m \geq 1$, from (50), (51), and (56), we obtain:

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}+m-1)(1+\kappa)}, & \gamma \geq 0, \\ n^{(\frac{2}{p}+m-1)(1+\kappa)}, & \gamma < 0, \end{cases} \tag{57}$$

if $z \in \Omega(\delta)$, and

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}+m-1)(1+\kappa)}, & \gamma > \dfrac{p-1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & \gamma = \dfrac{p-1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & -2 < \gamma < \dfrac{p-1}{1+\kappa}, \end{cases} \tag{58}$$

if $z \in \widehat{\Omega}(\delta)$.

(**B**) Now, we begin to estimate the $\left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right|$.

Since $\Phi(\infty) = \infty$, then Cauchy integral representation for the region Ω_R gives:

$$\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} = -\frac{1}{2\pi i}\int\limits_{L_{R_1}} \frac{1}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta - z)^{j+1}}, \;\; z \in \Omega_R.$$

If we go from both sides to the module, we obtain:

$$\left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right| \leq \frac{1}{2\pi}\int\limits_{L_{R_1}} \frac{1}{|\Phi^{n+1}(\zeta)|}\frac{|d\zeta|}{|\zeta - z|^{j+1}} \leq \frac{1}{2\pi}\int\limits_{L_{R_1}} \frac{|d\zeta|}{|\zeta - z|^{j+1}}.$$

Replacing the variable $\tau = \Phi(\zeta)$ and according to (18), we obtain:

$$\begin{aligned} \left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right| &\preccurlyeq \int\limits_{|\tau|=R_1} \frac{d(\Psi(\tau), L)}{|\tau| - 1}\frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{j+1}} \preccurlyeq n \int\limits_{|\tau|=R_1} \frac{|d\tau|}{|\tau - w|^{j(1+\kappa)}} \\ &\preccurlyeq \begin{cases} n^{j(1+\kappa)}, & \text{if } z \in \Omega(\delta), \\ n^{\kappa}, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases} \quad j = \overline{1, m}. \end{aligned} \tag{59}$$

Combining estimates (19)–(24), (43), (44), (57), (58), and (59), we obtain:

$$|P_n^{(m)}(z)| \leq |\Phi^{n+1}(z)| \left\{ \frac{A_n(z)}{d(z, L)} + \sum_{j=1}^{m} C_m^j |P_n^{(m-j)}(z)| \begin{cases} n^{j(1+\kappa)}, & \text{if } z \in \Omega(\delta), \\ n^{\kappa}, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases} \right\} \tag{60}$$

where for any $\gamma > -2$, $p \geq 2$, $m \geq 1$

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & p \geq 2, \quad m \geq 2, \ \gamma > -2, \\ n^{\left(\frac{\gamma}{p}+m\right)(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & 2 \leq p < 1+(2+\gamma)(1+\kappa), \quad m=1, \ \gamma \geq 0, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{1-\frac{1}{p}}, & p = 1+(2+\gamma)(1+\kappa), \quad m=1, \ \gamma \geq 0, \\ n^{1-\frac{2}{p}}, & p > 1+(2+\gamma)(1+\kappa), \quad m=1, \ \gamma \geq 0, \\ n^{m(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & p \geq 2, \quad m \geq 1, \ \gamma < 0, \end{cases}$$

if $z \in \Omega(\delta)$ and

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{\frac{\gamma}{p}(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & 2 \leq p < \frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{\left(1-\frac{1}{p}\right)}, & p = \frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{1-\frac{2}{p}}, & p > \frac{1}{2+\kappa}+(\gamma+2)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{1-\frac{2}{p}}, & p \geq 2, \quad 0 \leq \gamma \leq \frac{1}{1+\kappa}, \\ n^{\kappa\left(1-\frac{2}{p}\right)}, & p \geq 2, \quad \gamma < 0, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$; for any $\gamma \geq -2$, $1 < p < 2$, $m \geq 1$,

$$A_n(z) \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma^*}{p}+m\right)(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}, & \gamma > -2, \quad \text{if } z \in \Omega(\delta), \\ n^{\frac{\gamma}{p}(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}, & \gamma > \frac{p-1}{1+\kappa}, \quad \text{if } z \in \widehat{\Omega}(\delta), \\ n^{\kappa\left(\frac{2}{p}-1\right)}(\ln n)^{\left(1-\frac{1}{p}\right)}, & \gamma = \frac{p-1}{1+\kappa}, \quad \text{if } z \in \widehat{\Omega}(\delta), \\ n^{\kappa\left(\frac{2}{p}-1\right)}, & -2 < \gamma < \frac{p-1}{1+\kappa}, \quad \text{if } z \in \widehat{\Omega}(\delta), \end{cases}$$

and $\gamma^* := \max\{0; \gamma\}$.

(C) $|P_n^{(m-j)}(z)|$. Estimate $|P_n^{(m-j)}(z)|$ for $m = 2$, $j = 1, 2$, and we carry out separately as follows. Therefore, the proof of Theorems 2 and 3 is completed. □

Proof of Theorem 4. Now let us start the evaluations of $|P_n(z)|$. For this, we will make the necessary evaluations by writing the above proof for $m = 0$. Since the function $H_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}$, $H_n(\infty) = 0$, is analytic in Ω, continuous on $\overline{\Omega}$, then Cauchy integral representation for the region Ω_{R_1} gives

$$H_n(z) = -\frac{1}{2\pi i} \int\limits_{L_{R_1}} H_n(\zeta) \frac{d\zeta}{\zeta - z}, \quad z \in \Omega_R.$$

Then,

$$\left| \frac{P_n(z)}{\Phi^{n+1}(z)} \right| \leq \frac{1}{2\pi} \int\limits_{L_{R_1}} \left| \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)} \right| \frac{|d\zeta|}{|\zeta - z|} \leq \frac{1}{2\pi d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)||d\zeta|, \tag{61}$$

and so,

$$|P_n(z)| \preccurlyeq \frac{|\Phi^{n+1}(z)|}{d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)||d\zeta|.$$

We denote

$$A_n := \int_{L_{R_1}} |P_n(\zeta)||d\zeta|, \tag{62}$$

and estimate this integral.

To estimate $A_n(z)$, first replacing the variable $\tau = \Phi(\zeta)$ and multiplying the numerator and denominator of the integrant by $|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}} |\Psi'(\tau)|^{\frac{2}{p}}$ and applying the Hölder inequality, we obtain:

$$\begin{aligned}
A_n &= \int_{L_{R_1}} |P_n(\zeta)||d\zeta| = \sum_{i=1}^{2} \int_{F_{R_1}^i} \frac{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma_j}{p}} \left| P_n(\Psi(\tau))(\Psi'(\tau))^{\frac{2}{p}} \right| |\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}} |d\tau| \\
&\leq \sum_{i=1}^{2} \left(\int_{F_{R_1}^i} |\Psi(\tau) - \Psi(w_1)|^{\gamma} |P_n(\Psi(\tau))|^p |\Psi'(\tau)|^2 |d\tau| \right)^{\frac{1}{p}} \left(\int_{F_{R_1}^i} \left(\frac{|\Psi'(\tau)|^{1-\frac{2}{p}}}{|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}} \right)^q |d\tau| \right)^{\frac{1}{q}} \\
&=: \sum_{i=1}^{2} A_n^i,
\end{aligned}$$

where $F_{R_1}^1 := \Phi(L_{R_1}^1) = \Delta_1' \cap \{\tau : |\tau| = R_1\}$, $F_{R_1}^2 := \Phi(L_{R_1}) \backslash F_{R_1}^1$ and

$$A_n^i := \left(\int_{F_{R_1}^i} |f_{n,p}(\tau)|^p |d\tau| \right)^{\frac{1}{p}} \left(\int_{F_{R_1}^i} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} |d\tau| \right)^{\frac{1}{q}} =: J_{n,1}^i \cdot J_{n,2}^i,$$

$$f_{n,p}(\tau) := (\Psi(\tau) - \Psi(w_1))^{\frac{\gamma}{p}} P_n(\Psi(\tau))(\Psi'(\tau))^{\frac{2}{p}}, \quad |\tau| = R_1.$$

Applying Lemma 3, we We denote:

$$J_{n,1}^i \preceq n^{\frac{1}{p}} \|P_n\|_p, \quad i = 1, 2.$$

Therefore, we need to evaluate the following integrals:

$$(J_{n,2}^i)^q = \int_{F_{R_1}^i} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}} |d\tau|, \quad i = 1, 2.$$

For the estimation of the integral $J_{n,2}^i$, for $i = 1, 2$, we use notations (23) and (24) and, consequently, we need to evaluate the following statement:

$$A_n = n^{\frac{1}{p}} \|P_n\|_p (J_2^1 + J_2^2 + J_2^3), \tag{63}$$

where

$$(J_2^k)^q := \int_{E_{R_1}^{1k}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)}}, \quad k = 1, 2, 3.$$

So, for any $k = 1, 2, 3$, we will estimate the integrals J_2^k.

Given the possible values q ($q > 2$ and $q < 2$) and γ ($-2 < \gamma < 0$ and $\gamma \geq 0$), we will consider the cases separately.

Case 1. Let $1 < q \le 2\,(p \ge 2)$.

1.1. Let $\gamma \ge 0$. Applying Lemma 2, we We denote:

$$
\begin{aligned}
(J_2^1)^q &= \int\limits_{E_{R_1}^{11}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq \int\limits_{E_{R_1,2}^{11}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \\
&\preccurlyeq n^{(2-q)} \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau-w_1|^{[\gamma(q-1)-(2-q)](1+\kappa)}} \preccurlyeq n^{(2-q)+[\gamma(q-1)-(2-q)](1+\kappa)} \mathrm{mes} E_{R_1,2}^{11} \\
&\preccurlyeq n^{(2-q)+[\gamma(q-1)-(2-q)](1+\kappa)-1}, \\
J_2^1 &\preccurlyeq n^{\frac{\gamma}{p}(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}.
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
(J_2^2)^q &\le \int\limits_{E_{R_1}^{12}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq n^{(2-q)} \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau-w_1|^{[\gamma(q-1)-(2-q)](1+\kappa)}} \\
&\preccurlyeq n^{(2-q)} \begin{cases} n^{[\gamma(q-1)-(2-q)](1+\kappa)-1}, & [\gamma(q-1)-(2-q)](1+\kappa) > 1, \\ \ln n, & [\gamma(q-1)-(2-q)](1+\kappa) = 1, \\ 1, & [\gamma(q-1)-(2-q)](1+\kappa) < 1, \end{cases} \\
J_2^2 &\preccurlyeq \begin{cases} n^{\frac{\gamma}{p}(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & 2 \le p < \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{\left(1-\frac{1}{p}\right)}, & p = \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{\left(1-\frac{2}{p}\right)}, & p > \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{\left(1-\frac{2}{p}\right)}, & p \ge 2, \quad -2 < \gamma \le \frac{1}{1+\kappa}. \end{cases}
\end{aligned}
\tag{65}
$$

For $\tau \in E_{R_1}^{13}$, we see that $\eta < |\tau - w_1| < 2\pi R_1$. Therefore, $|\Psi(\tau)-\Psi(w_1)| \succcurlyeq 1$, from Lemma 1 and applying (18), we We denote:

$$
\begin{aligned}
(J_2^3)^q &\preccurlyeq \int\limits_{E_{R_1}^{13}} |\Psi'(\tau)|^{2-q}|d\tau| \preccurlyeq \int\limits_{E_{R_1}^{13}} \left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q} |d\tau| \preccurlyeq n^{\kappa(2-q)}, \\
J_2^3 &\preccurlyeq n^{\kappa\left(1-\frac{2}{p}\right)}.
\end{aligned}
\tag{66}
$$

Combining (64)–(66), for $p \ge 2$, $\gamma \ge 0$ and $z \in \Omega_R$, we denote:

$$
\sum_{k=1}^{3} J_2^k \preccurlyeq \begin{cases} n^{\frac{\gamma}{p}(1+\kappa)-\left[1+\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]}, & 2 \le p < \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{\left(1-\frac{2}{p}\right)}(\ln n)^{1-\frac{1}{p}}, & p = \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{1-\frac{2}{p}}, & p > \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{1-\frac{2}{p}}, & p \ge 2, \quad 0 < \gamma \le \frac{1}{1+\kappa}. \end{cases}
\tag{67}
$$

From (63)–(67), we obtain:

$$
A_n \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)(1+\kappa)}, & 2 \le p < \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{\left(1-\frac{1}{p}\right)}(\ln n)^{1-\frac{1}{p}}, & p = \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{1-\frac{1}{p}}, & p > \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma > \frac{1}{1+\kappa}, \\ n^{1-\frac{1}{p}}, & p \ge 2, \quad -2, 0 < \gamma \le \frac{1}{1+\kappa}. \end{cases}
\tag{68}
$$

1.2. If $\gamma < 0$, analogously we have:

$$\begin{aligned}(J_2^1)^q &\preccurlyeq \int\limits_{E_{R_1}^{11}} |\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)}\left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q}|d\tau| \preccurlyeq n^{(2-q)}\int\limits_{E_{R_1}^{11}}|\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)+(2-q)}|d\tau|\\ &\preccurlyeq n^{(2-q)}\int\limits_{E_{R_1}^{11}}|\tau-w_1|^{[(-\gamma)(q-1)+(2-q)](1-\kappa)}|d\tau| \preccurlyeq n^{(2-q)+[\gamma(q-1)-(2-q)](1-\kappa)}\mathrm{mes}E_{R_1}^{11}\\ &\preccurlyeq n^{(2-q)+[\gamma(q-1)-(2-q)](1-\kappa)-1},\\ J_2^1 &\preccurlyeq n^{\frac{\gamma}{p}(1-\kappa)-\left[1-\left(1-\frac{2}{p}\right)\kappa-\frac{1}{p}\right]} \preccurlyeq 1.\end{aligned} \tag{69}$$

For $\tau \in E_{R_1}^{12}$, we denote:

$$\begin{aligned}(J_2^2)^q &\preccurlyeq \int\limits_{E_{R_1}^{12}} |\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)}\left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q}|d\tau| \preccurlyeq n^{\kappa(2-q)},\\ J_2^2 &\preccurlyeq n^{\kappa\left(1-\frac{2}{p}\right)},\end{aligned} \tag{70}$$

$$\begin{aligned}(J_2^3)^q &\preccurlyeq \int\limits_{E_{R_1}^{12}} |\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)}\left(\frac{d(\Psi(\tau),L)}{|\tau|-1}\right)^{2-q}|d\tau| \preccurlyeq n^{\kappa(2-q)},\\ J_2^3 &\preccurlyeq n^{\kappa\left(1-\frac{2}{p}\right)}.\end{aligned} \tag{71}$$

Therefore, combining (69)–(71) in case of $\gamma < 0$ for $z \in \Omega_R$, we have:

$$\sum_{k=1}^{3} J_2^k \preccurlyeq n^{\kappa\left(1-\frac{2}{p}\right)},$$

and, consequently, in this case from (63), we have:

$$A_n \preccurlyeq \|P_n\|_p \cdot n^{\kappa\left(1-\frac{2}{p}\right)+\frac{1}{p}},\ z \in \Omega_R. \tag{72}$$

Therefore, combining (67) and (72), for any $\gamma > -2$, $p \geq 2$, $z \in \Omega_R$, we obtain:

$$A_n \preccurlyeq \|P_n\|_p \begin{cases} n^{(\frac{\gamma+2}{p}-1)(1+\kappa)}, & 2 \leq p < \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa},\\ (n\ln n)^{1-\frac{1}{p}}, & p = \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa},\\ n^{1-\frac{1}{p}}, & p > \frac{1}{2+\kappa} + (2+\gamma)\frac{1+\kappa}{2+\kappa}, \quad \gamma \geq \frac{1}{1+\kappa},\\ n^{1-\frac{1}{p}}, & p \geq 2, \quad 0 \leq \gamma < \frac{1}{1+\kappa},\\ n^{\kappa\left(1-\frac{2}{p}\right)+\frac{1}{p}}, & p \geq 2, \quad -2 < \gamma < 0.\end{cases} \tag{73}$$

Case 2. Let $q > 2$ ($p < 2$). Then, $2 - q < 0$, and so

$$(J_2^k(z))^q := \int\limits_{E_{R_1}^{1k}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}}, \quad k = 1, 2, 3. \tag{74}$$

2.1. If $\gamma \geq 0$, applying Lemmas 1 and 2 and (18), we obtain:

$$\begin{aligned}(J_2^1)^q &\preccurlyeq \int\limits_{E_{R_1,2}^{11}}\left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq n^{\kappa(q-2)}\int\limits_{E_{R_1,2}^{11}}\frac{|d\tau|}{|\tau-w_1|^{\gamma(q-1)}}\\ &\preccurlyeq n^{\gamma(q-1)(1+\kappa)+\kappa(q-2)}\mathrm{mes}E_{R_1,2}^{11} \preccurlyeq n^{\gamma(q-1)(1+\kappa)+\kappa(q-2)-1},\\ J_2^1 &\preccurlyeq n^{\frac{\gamma}{p}(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]},\end{aligned} \tag{75}$$

$$(J_2^2)^q \preccurlyeq \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq n^{\kappa(q-2)} \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau-w_1|^{\gamma(q-1)(1+\kappa)}}$$

$$\preccurlyeq \begin{cases} n^{\gamma(q-1)(1+\kappa)+\kappa(q-2)-1}, & \gamma(q-1)(1+\kappa)>1, \\ n^{\kappa(q-2)}\ln n, & \gamma(q-1)(1+\kappa)=1, \\ n^{\kappa(q-2)}, & \gamma(q-1)(1+\kappa)<1, \end{cases}$$

$$J_2^2 \preccurlyeq \begin{cases} n^{\frac{\gamma}{p}(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}, & 1<p<1+\gamma(1+\kappa), \quad 0<\gamma<\frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)}(\ln n)^{1-\frac{1}{p}}, & p=1+\gamma(1+\kappa), \quad 0<\gamma<\frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)}, & 1+\gamma(1+\kappa)<p<2, \quad 0\le\gamma<\frac{1}{1+\kappa}, \\ n^{\frac{\gamma}{p}(1+\kappa)-\left[1-\left(\frac{2}{p}-1\right)\kappa-\frac{1}{p}\right]}, & 1<p<2, \quad \gamma\ge\frac{1}{1+\kappa}. \end{cases} \tag{76}$$

For $\tau \in E_{R_1}^{13}\eta < |\tau - w_1| < 2\pi\dot{R}_1$ and from Lemma 1 $|\Psi(\tau)-\Psi(w_1)| \asymp 1$. Then, we obtain:

$$(J_2^3)^q = \int\limits_{E_{R_1}^{13}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}} \preccurlyeq n^{\kappa(q-2)} \int\limits_{E_{R_1}^{13}} |d\tau| \preccurlyeq n^{\kappa(q-2)},$$
$$J_2^3 \preccurlyeq n^{\kappa\left(\frac{2}{p}-1\right)}. \tag{77}$$

From (74)–(77) and (63), for $\gamma \ge 0$, $1<p<2$, $z \in \Omega_R$, we have:

$$A_n \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)(1+\kappa)}, & 1<p<1+\gamma(1+\kappa), \quad 0<\gamma<\frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=1+\gamma(1+\kappa), \quad 0<\gamma<\frac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & 1+\gamma(1+\kappa)<p<2, \quad 0\le\gamma<\frac{1}{1+\kappa}, \\ n^{\left(\frac{\gamma+2}{p}-1\right)(1+\kappa)}, & 1<p<2, \quad \gamma\ge\frac{1}{1+\kappa}. \end{cases} \tag{78}$$

2.2. Let $\gamma < 0$. For $z \in \Omega_R$, according to Lemma 1, we have:

$$(J_2^1)^q \preccurlyeq \int\limits_{E_{R_1}^{11}} |\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)} \left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2} |d\tau|$$
$$\preccurlyeq n^{\kappa(q-2)} \int\limits_{E_{R_1}^{11}} |\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)}|d\tau| \preccurlyeq n^{\kappa(q-2)} \int\limits_{E_{R_1}^{11}} |\tau-w_1|^{(-\gamma)(q-1)(1-\kappa)}|d\tau| \tag{79}$$
$$\preccurlyeq n^{\kappa(q-2)+\gamma(q-1)(1-\kappa)} \operatorname{mes} E_{R_1}^{11} \preccurlyeq n^{\kappa(q-2)+\gamma(q-1)(1-\kappa)-1} \preccurlyeq n^{\kappa(q-2)-1},$$
$$J_2^1 \preccurlyeq n^{\left(\frac{2}{p}-1\right)\kappa-1+\frac{1}{p}},$$

$$(J_2^2)^q \preccurlyeq \int\limits_{E_{R_1}^{12}} |\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)} \left(\frac{|\tau|-1}{d(\Psi(\tau),L)}\right)^{q-2} |d\tau|$$
$$\preccurlyeq n^{\kappa(q-2)} \int\limits_{E_{R_1}^{12}} |\Psi(\tau)-\Psi(w_1)|^{(-\gamma)(q-1)}|d\tau| \preccurlyeq n^{\kappa(q-2)}, \tag{80}$$
$$J_2^2 \preccurlyeq n^{\left(\frac{2}{p}-1\right)\kappa},$$

$$(J_2^3)^q \preccurlyeq \int\limits_{E_{R_1}^{13}} \frac{|\Psi(\tau) - \Psi(w_1)|^{(-\gamma)(q-1)}|d\tau|}{|\Psi'(\tau)|^{q-2}} \preccurlyeq n^{\kappa(q-2)},$$
$$J_2^3 \preccurlyeq n^{\left(\frac{2}{p}-1\right)\kappa}. \tag{81}$$

So, for $\gamma < 0$, from (63), we have:

$$A_n \preccurlyeq n^{\left(\frac{2}{p}-1\right)\kappa+\frac{1}{p}} \, \|P_n\|_p, \quad z \in \Omega_R. \tag{82}$$

Therefore, for any $\gamma \geq -2$, $1 < p < 2$, from (78) and (82), we obtain:

$$A_n \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)(1+\kappa)}, & 1 < p < 1+\gamma(1+\kappa), \ \ 0 < \gamma < \dfrac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p = 1+\gamma(1+\kappa), \ \ 0 < \gamma < \dfrac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & 1+\gamma(1+\kappa) < p < 2, \ \ 0 \leq \gamma < \dfrac{1}{1+\kappa}, \\ n^{\left(\frac{\gamma+2}{p}-1\right)(1+\kappa)}, & 1 < p < 2, \ \ \gamma \geq \dfrac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & 1 < p < 2, \ \ -2 < \gamma < 0. \end{cases} \tag{83}$$

Combining estimates (19)–(63), (73), and (83), we get:

$$|P_n(z)| \preccurlyeq \frac{|\Phi^{n+1}(z)|}{d(z, L_{R_1})} \|P_n\|_p A_n,$$

where for $p \geq 2$

$$A_n \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)(1+\kappa)}, & 2 \leq p < \dfrac{1}{2+\kappa} + (2+\gamma)\dfrac{1+\kappa}{2+\kappa}, \quad \gamma \geq \dfrac{1}{1+\kappa}, \\ n^{\left(1-\frac{1}{p}\right)}(\ln n)^{\left(1-\frac{1}{p}\right)}, & p = \dfrac{1}{2+\kappa} + (2+\gamma)\dfrac{1+\kappa}{2+\kappa}, \quad \gamma \geq \dfrac{1}{1+\kappa}, \\ n^{\left(1-\frac{1}{p}\right)}, & p > \max\left\{2; \dfrac{1}{2+\kappa} + (2+\gamma)\dfrac{1+\kappa}{2+\kappa}\right\}, \quad \gamma \geq 0, \\ n^{\kappa\left(1-\frac{2}{p}\right)+\frac{1}{p}}, & p \geq 2, \quad -2 < \gamma < 0, \end{cases}$$

and for $1 < p < 2$

$$A_n \preccurlyeq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}-1\right)(1+\kappa)}, & 1 < p < \min\{2; 1+\gamma(1+\kappa)\}, \quad \gamma > 0, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p = 1+\gamma(1+\kappa), \quad 0 < \gamma < \dfrac{1}{1+\kappa}, \\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}}, & \max\{1; 1+\gamma(1+\kappa)\} < p < 2, \quad -2 < \gamma < \dfrac{1}{1+\kappa}, \end{cases}$$

and therefore, the proof of Theorem 4 is completed. □

Proof of Theorems 7 and 8. According to Theorems 2, 4, 5 and estimates (20), (21), (43), (44), (60), for $m = 2$ and $p \geq 2$ from (19), we have:

$$\begin{aligned} |P_n''(z)| &\leq \left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)''\right| + \sum_{j=1}^{2} C_2^j \left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right| |P_n^{(2-j)}(z)| \\ &\leq |\Phi^{n+1}(z)| \left[\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)''\right| + C_2^1 B_{n,1}^1 |P_n'(z)| + C_2^2 B_{n,2}^1 |P_n(z)|\right] \\ &\preccurlyeq |\Phi^{n+1}(z)| \left[\frac{\|P_n\|_p}{d(z,L)} A_n^1(z,2) + C_2^1 B_{n,1}^1 |P_n'(z)| + C_2^2 B_{n,2}^1 |P_n(z)|\right]. \end{aligned}$$

Substituting estimates for the $B^1_{n,j}$, $j=1,2$, $|P_n(z)|$ and $|P'_n(z)|$ from Theorems 2, 4, and 5 correspondingly, we obtain:

$$|P''_n(z)| \preccurlyeq \frac{|\Phi^{3(n+1)}(z)|}{d(z,L_{R_1})}\|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & 2\le p<\frac{1}{2+\kappa}+(2+\gamma)\frac{1+\kappa}{2+\kappa},\quad \gamma\ge\frac{1}{1+\kappa},\\ n^{1-\frac{1}{p}+2(1+\kappa)}(\ln n)^{1-\frac{1}{p}}, & p=\frac{1}{2+\kappa}+(2+\gamma)\frac{1+\kappa}{2+\kappa},\quad \gamma\ge\frac{1}{1+\kappa},\\ n^{1-\frac{1}{p}+2(1+\kappa)}, & p>\frac{1}{2+\kappa}+(2+\gamma)\frac{1+\kappa}{2+\kappa},\quad \gamma\ge\frac{1}{1+\kappa},\\ n^{1-\frac{1}{p}+2(1+\kappa)}, & 2\le p<1+(2+\gamma)(1+\kappa),\quad 0\le\gamma<\frac{1}{1+\kappa},\\ n^{2+\frac{1}{p}+\left(3-\frac{2}{p}\right)\kappa}, & p\ge 2,\quad -2<\gamma<0,\end{cases}$$

for $z\in\Omega(\delta)$, and

$$|P''_n(z)| \preccurlyeq \frac{|\Phi^{3(n+1)}(z)|}{d(z,L_{R_1})}\|P_n\|_p \begin{cases} n^{\frac{\gamma+2}{p}(1+\kappa)+\kappa}, & 2\le p<\frac{1}{2+\kappa}+(2+\gamma)\frac{1+\kappa}{2+\kappa},\quad \gamma\ge\frac{1}{1+\kappa},\\ n^{1+2\kappa-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=\frac{1}{2+\kappa}+(2+\gamma)\frac{1+\kappa}{2+\kappa},\quad \gamma\ge\frac{1}{1+\kappa},\\ n^{1+2\kappa-\frac{1}{p}}, & p>\frac{1}{2+\kappa}+(2+\gamma)\frac{1+\kappa}{2+\kappa},\quad \gamma\ge\frac{1}{1+\kappa},\\ n^{1+2\kappa-\frac{1}{p}}, & p\ge 2,\quad 0\le\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(2-\frac{2}{p}\right)+\frac{1}{p}+\kappa}, & p\ge 2,\quad -2<\gamma<0,\end{cases}$$

for $z\in\widehat{\Omega}(\delta)$,

Analogously, from Theorems 3, 4, and 6, for $1<p<2$, we have:

$$|P''_n(z)| \preccurlyeq \frac{|\Phi^{3(n+1)}(z)|}{d(z,L_{R_1})}\|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & 1<p<2,\quad \gamma\ge\frac{1}{1+\kappa},\\ n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & 1<p<1+\gamma(1+\kappa),\quad 0<\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}+1\right)+\frac{1}{p}+2}(\ln n)^{1-\frac{1}{p}}, & p=1+\gamma(1+\kappa),\quad 0<\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}+1\right)+\frac{1}{p}+2}, & 1+\gamma(1+\kappa)<p<2,\quad 0\le\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}+1\right)+\frac{1}{p}+2}, & 1<p<2,\quad -2<\gamma<0,\end{cases}$$

for $z\in\Omega(\delta)$, and

$$|P''_n(z)| \preccurlyeq \frac{|\Phi^{3(n+1)}(z)|}{d(z,L_{R_1})}\|P_n\|_p \begin{cases} n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & 1<p<2,\quad \gamma\ge\frac{1}{1+\kappa},\\ n^{\left(\frac{\gamma+2}{p}+1\right)(1+\kappa)}, & p<1+\gamma(1+\kappa),\quad 0<\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+2\kappa}(\ln n)^{1-\frac{1}{p}}, & p=1+\gamma(1+\kappa),\quad 0<\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+2\kappa}, & p>1+\gamma(1+\kappa),\quad 0<\gamma<\frac{1}{1+\kappa},\\ n^{\kappa\left(\frac{2}{p}-1\right)+\frac{1}{p}+2\kappa}, & 1<p<2,\quad -2<\gamma\le 0,\end{cases}$$

for $z\in\widehat{\Omega}(\delta)$. Therefore, the proof of Theorems 7 and 8 completed.

The proofs of Theorems 5 and 6 are similarly carried out to the proofs of Theorems 7 and 8, using the corresponding estimates (11)–(13).

In conclusion, note that in proofs, everywhere there is a quantity $d(z, L_{R_1})$. Let us show that $d(z, L_{R_1}) \succcurlyeq d(z, L)$ holds for all $z \in \Omega_R$. For the points $z \notin \Omega(L_{R_1}, d(L_{R_1}, L_R))$, we have: $d(z, L_{R_1}) \succcurlyeq \delta \succcurlyeq d(z, L)$. Now, let $z \in \Omega(L_{R_1}, d(L_{R_1}, L_R))$. Denote by $\xi_1 \in L_{R_1}$ the point such that $d(z, L_{R_1}) = |z - \xi_1|$, and point $\xi_2 \in L$, such that $d(z, L) = |z - \xi_2|$, and for $w = \Phi(z)$, $t_1 = \Phi(\xi_1)$, $t_2 = \Phi(\xi_2)$, we have: $|w - w_1| \geq ||w - w_2| - |w_2 - w_1|| \geq \left||w - w_2| - \frac{1}{2}|w - w_2|\right| \geq \frac{1}{2}|w - w_2|$. Then, according to Lemma 1, we obtain $d(z, L_{R_1}) \succcurlyeq d(z, L)$. □

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