

Research Article

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Note on structural properties of graphs

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Abstract: In this paper, we establish sufficient and necessary conditions for the existence of abelian subgroups of maximal order of a finite group G , by means of its commuting graph. The order of these subgroups attains the bound $c = |[x_1]| + \dots + |[x_m]|$, where $[x_i]$ denotes the conjugacy class of x_i in G and m is the smallest integer j such that $|[x_1]| + \dots + |[x_j]| \geq |C_G(x_j)|$, where $C_G(x_j)$ is the centralizer of x_j in G .

Keywords: Abelian subgroup, commuting graph, clique, chromatic number, weakly perfect graph

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1 Introduction

Over the years, graphs generated from a group or a semigroup have been extensively studied. For example, in 1964, Bosak [1] studied graphs induced by semigroups; in [2] the *intersection graphs* of non-trivial subgroups of abelian groups with finite order were studied by Zelinka; in [3–5] the directed graph defined over the elements of a group, the well-known *Cayley digraph*, was studied; in the books in [6–8], numerous valuable applications of this kind of graph are presented, hence the importance of researching them. Another best-known graph is the so-called *directed power graph*, whose definition was given by Kelarev and Quinn [9], which is defined in such a way that it is possible to apply it also to semigroups, and thus, it was in [10–12] that power graphs of semigroups were considered for the first time. These papers use only the short-term “power graph.” However, it makes sense for both directed and undirected power graphs. In addition, Kelarev and Quinn [11] defined an interesting class of directed graphs over semigroups, the *semigroup divisibility graphs*. There is also one more example, the well-known hyperbolic graphs, mentioned in the works [13–15], whose main objects of study were initially Cayley graphs associated with finite groups, nowadays these graphs have several applications in Physics and Geometry. For more information on structural properties associated with a graph, see [16–18].

It is said that group theory started with Galois (1811–1832), who showed that the best way to understand polynomials is by relating them to certain groups of permutations of their roots. From then onward, group theory has become a useful tool for different fields of mathematics, such as combinatorics, geometry, logic, number theory, and topology.

At the end of the nineteenth century, there were two main streams of group theory: on the one hand topological groups (especially Lie groups); on the other hand, finite groups. The latter since its beginnings

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in the nineteenth century has grown to become an extensive and diverse part of algebra, in particular, the theory of locally finite groups and the theory of nilpotent, solvable groups [19,20]. In the early 1980s, this development culminated in the classification of finite simple groups, impressively and convincingly demonstrating the strength of its methods and results. Thanks to those works, all finite groups that are constructible from these simple groups are now known.

Graph theory is an increasing mathematical discipline containing deep and strong results of high applicability. Its rapid development in the last few decades is not only due to its status as the main structure on which currently applied mathematics (computer science, combinatorial optimization, and operations research) is supported, but also due to its increasing connections in the applied sciences.

In this paper, we are concerned with the *commuting graph* \mathcal{G}_G of a group G . It is defined as the graph with G as the set of vertices and where two of them form an edge if and only if they commute. This graph has been studied from several perspectives, for example, in [21] the chromatic and clique numbers are obtained for the commuting graphs of the dihedral-type groups; in [22] it is proved that the commuting graph of a finite minimal non-solvable group has diameter ≥ 3 ; in [23] the authors obtain the number of spanning trees of the commuting graph for some specific groups, as well as the classification of finite groups for which the power graph and the commuting graph coincide; and in [24] properties of this graph, as coloring and independence number, are used to prove results about finite groups.

The existence of abelian subgroups of maximal order of a given group G is a topic widely studied in different papers. For example, in [25] it was shown that if m is the maximal order of an abelian subgroup of a finite group G , then $|G|$ divides $m!$ and in [26] some results are presented on m . Moreover, in [27] a classification of the abelian subgroups of maximal order of finite irreducible Coxeter groups is obtained, the geometry of these subgroups is studied and some applications of such classification to statistical physics are given.

In [24] Bertram, by using the commuting graph \mathcal{G}_G , gave an upper bound for the order of an abelian subgroup of a finite group G . In addition, he proposed to find necessary and sufficient conditions for the existence of an abelian subgroup whose order attains the bound $||[x_1]| + |[x_2]| + \dots + |[x_m]|$. In this paper, we give a solution to this problem and present some related results.

In [28] the following result appears.

Lemma 1.1. *If, for some q the number of vertices of degree $\geq q$ is $\leq q$, then the graph \mathcal{G} can be q -colored.*

And it is used in [24] to prove the following theorem.

Theorem 1.2. *Let G be a finite group with conjugacy classes indexed by cardinality:*

$$1 = |[x_1]| \leq |[x_2]| \leq \dots,$$

and let $C_G(x)$ denote the centralizer of x . If m is the smallest number i satisfying

$$|[x_1]| + |[x_2]| + \dots + |[x_i]| \geq |C_G(x_i)|,$$

then each abelian subgroup A of G satisfies

$$|[x_1]| + |[x_2]| + \dots + |[x_m]| \geq |A|. \quad (1)$$

The problem proposed in [24] is the following:

Find necessary and sufficient conditions on G in order that inequality (1) in Theorem 1.2 becomes an equality for some abelian subgroup $A < G$.

In this paper, we use the following notation and known facts. Let \mathcal{G} be an undirected simple graph, whose vertex set is $V = \{x_1, x_2, \dots, x_n\}$. The *degree* of a vertex x is the number of edges incident with x and it is denoted by $d(x)$. We say that the vertices of \mathcal{G} can be k -colored when there exists a partition of V into k

subsets, with no two vertices in the same subset connected by an edge of \mathcal{G} . The minimum number k for which \mathcal{G} can be k -colored is called the *chromatic number* of \mathcal{G} and it is denoted by $\chi(\mathcal{G})$. A *complete* subgraph of \mathcal{G} is a subgraph when every pair of vertices is connected by an edge. A maximal complete subgraph of \mathcal{G} is called a *clique*; we also use the term k -clique for a clique consisting of k vertices. The *clique number* $\omega(\mathcal{G})$ is the maximal size of a clique contained in \mathcal{G} . Note that if \mathcal{G} can be k -colored, then the number of vertices in each clique (as well as in each complete subgraph) has cardinality $\leq k$; hence, we always have $\omega(\mathcal{G}) \leq \chi(\mathcal{G})$. A graph \mathcal{G} is called *weakly perfect* if $\omega(\mathcal{G}) = \chi(\mathcal{G})$ (see [29]).

A subset X of a finite group G is called a *commuting set* if $xy = yx$ for any $x, y \in X$. The *commuting number* $\kappa(G)$ of a finite group G is the maximum cardinality of a commuting set. A subset $Y \subseteq G$ is called an *anti-commuting set* if $xy = yx$ implies $x = y$ for any $x, y \in Y$. The Λ -number of G is the minimal number k such that G can be partitioned into k anti-commuting subsets; it is denoted by $\Lambda(G)$.

From the proof of Theorem 1.2, we get the following remark.

Remark 1.3. The graph \mathcal{G}_G is $[|x_1|] + [x_2] + \cdots + [x_m]$ -colorable, thus,

$$\chi(\mathcal{G}_G) \leq [x_1] + [x_2] + \cdots + [x_m].$$

2 Results

Throughout this paper we denote by c the sum $[x_1] + [x_2] + \cdots + [x_m]$, the left hand side of inequality (1). Note that

$$\omega(\mathcal{G}_G) \leq \chi(\mathcal{G}_G) \leq c.$$

2.1 \mathcal{G}_G parameters and the bound c

Theorem 2.1. A finite group G contains an abelian subgroup of order c if and only if \mathcal{G}_G contains a c -clique.

Corollary 2.2. A finite group G contains an abelian subgroup A of order c if and only if \mathcal{G}_G is a weakly perfect graph with $\chi(\mathcal{G}_G) = c$.

Proof. If G contains an abelian subgroup of order c , then $c \leq \omega(\mathcal{G}_G) \leq \chi(\mathcal{G}_G) \leq c$ by Theorem 2.1, and hence $\chi(\mathcal{G}_G) = c$.

For the converse, since $\omega(\mathcal{G}_G) = \chi(\mathcal{G}_G) = c$, the graph \mathcal{G}_G contains a c -clique, which implies that G contains an abelian subgroup A of order c . \square

Corollary 2.3. A finite group G contains an abelian subgroup A of order c if and only if $\kappa(G) = \Lambda(G) = c$.

Proof. If \mathcal{G}_G is the commuting graph of G , then $\kappa(G) = \omega(\mathcal{G}_G)$, $\Lambda(G) = \chi(\mathcal{G}_G)$ and Theorem 2.2 applies. \square

Proposition 2.4. Let G be a finite group. Then the following statements are equivalent.

- (1) G contains a commuting set with c elements.
- (2) G contains an abelian subgroup of order c .
- (3) \mathcal{G}_G contains a c -clique.
- (4) \mathcal{G}_G is a weakly perfect graph and $\chi(\mathcal{G}_G) = c$.
- (5) \mathcal{G}_G contains a complete subgraph on c vertices.
- (6) $\chi(\mathcal{G}_G) = \kappa(G) = c$.
- (7) $\omega(\mathcal{G}_G) = \Lambda(G) = c$.

Proof.

- (1) \Rightarrow (2) Note that if X is a commuting set of G , then $\langle X \rangle$ is a complete subgraph of \mathcal{G}_G on c vertices. $\langle X \rangle$ is in fact a c -clique and we have already seen that this implies that X is an abelian subgroup.
- (2) \Rightarrow (3) If G contains a commuting set with c elements, \mathcal{G}_G contains a c -clique. By Theorem 2.1, G contains an abelian subgroup of order c .
- (3) \Rightarrow (4) If \mathcal{G}_G contains a c -clique, G contains an abelian subgroup of order c , by Corollary 2.2 \mathcal{G}_G is weakly perfect and $\chi(\mathcal{G}_G) = c$.
- (4) \Rightarrow (5) \mathcal{G}_G weakly perfect implies $\omega(\mathcal{G}_G) = \chi(\mathcal{G}_G)$, thus there is a c -clique.
- (5) \Rightarrow (6) \mathcal{G}_G can be c -colored and does not contain a complete subgraph in more than c vertices and therefore G does not contain a commutative set with more than c vertices. From (5), \mathcal{G}_G cannot be k -colored for $k < c$ and G contains a commutative set with c elements. It follows at once that $\chi(\mathcal{G}_G) = \kappa(G) = c$.
- (6) \Rightarrow (7) Suppose $\chi(\mathcal{G}_G) = \kappa(G) = c$, since $\kappa(G) = \omega(\mathcal{G}_G)$ and $\Lambda(G) = \chi(\mathcal{G}_G)$ we get the result.
- (7) \Rightarrow (1) $\omega(\mathcal{G}_G) = c$ implies the existence of a c -clique, thus, there is commuting set of cardinality c . \square

The following lemma given in [28] together with the fact that the degree of a vertex $x \in \mathcal{G}_G$ equals the order of its centralizer in G minus 1 lead to Proposition 2.6.

Lemma 2.5. *Let \mathcal{G} be a graph with chromatic number $\chi(\mathcal{G}) = q + 1$ and without any $(q + 1)$ -cliques. Let $T = \{x \in \mathcal{G} : d(x) > q\}$, then*

$$\sum_{x \in T} (d(x) - q) \geq q - 2. \quad (2)$$

Proposition 2.6. *Let G be a finite group. If $\chi(\mathcal{G}_G) = c$ and the set $S = \{x \in G : |C_G(x)| > c\}$ satisfies*

$$\sum_{x \in S} (|C_G(x)| - c) < c - 3, \quad (3)$$

then G contains an abelian subgroup of order c .

Proof. If G does not contain such an abelian subgroup, Theorem 2.1 says that \mathcal{G}_G does not contain any c -clique, hence we may apply Lemma 2.5 with $q = c - 1$, if $T = \{x \in V(\mathcal{G}_G) : d(x) > c - 1\}$, then

$$\sum_{x \in T} (d(x) - c + 1) \geq c - 3. \quad (4)$$

But $d(x) = |C_G(x)| - 1$ for each x . Hence, $S = T$ and inequality (4) becomes

$$\sum_{x \in S} (|C_G(x)| - c) \geq c - 3. \quad \square$$

Proposition 2.7. *Let G be a finite group. If G satisfies the hypothesis of Proposition 2.6, then G is abelian of order > 3 .*

Proof. If $S = \emptyset$, then left hand side of inequality (3) is 0 and therefore we must have $c > 3$. Moreover, note that

$$|G| = |C_G(1)| \leq c = \chi(\mathcal{G}) \leq |V(\mathcal{G})|$$

and G is abelian (otherwise some pair of vertices of \mathcal{G}_G could be colored with the same color).

If S is non-empty, then $1 \in G$ must be in S and hence $|C_G(1)| - c = |G| - c < c - 3$. This implies that $|G| < 2c - 3$. On the other hand, since G contains an abelian subgroup of order c , we have $|G| = ck$ for some integer k and $c(k - 2) < -3$. But this inequality holds just for $k = 1$ and $c > 3$, in particular we must have $k = 1$ and hence G is abelian of order $c > 3$. \square

From Propositions 2.6 and 2.7, we obtain the following result.

Theorem 2.8. *Let G be a finite group of order >3 with $\Lambda(G) = c$ and consider $S = \{x \in G : |C_G(x)| > c\}$. Then G is abelian if and only if*

$$\sum_{x \in S} (|C_G(x)| - c) < c - 3. \quad (5)$$

Proof. If G is abelian, then $|C_G(x)| = |G| = \Lambda(G) = c$ for all $x \in G$, hence the left hand side in inequality (3) is 0, while the right hand side is $|G| - 3$, a number greater than 0.

For the converse, suppose that G is a non-abelian group of order >3 . According to Proposition 2.7, the hypothesis of Proposition 2.6 is not satisfied by G , so we must have $\chi(\mathcal{G}_G) \neq c$ or

$$\sum_{x \in S} (|C_G(x)| - c) \geq c - 3.$$

Since $\chi(\mathcal{G}_G) = \Lambda(G) = c$, the result follows. \square

2.2 Groups for which the bound is attained

The following results show a relation between the expression of the order of finite groups and the existence of abelian subgroups of order c . In order to prove Theorems 2.9, 2.10 and 2.13 we use some known results from group theory.

Theorem 2.9. *Let p be a prime and G a group of order p^3 , then G has an abelian subgroup of order c .*

Proof. If G is abelian, we are done. If not, it is known that $|Z(G)| = p$ and the number N of conjugacy classes is

$$N = p + \frac{1}{p^2} \sum |C_G(x)|,$$

where the sum $\sum |C_G(x)|$ is taken over the non-central elements.

Note that $|C_G(x)| = p^2$ for each non-central element, otherwise there exists some non-central elements such that $|C_G(x)| = p$. But $Z(G) \leq C_G(x)$ also has order p ; hence, $Z(G) = C_G(x)$. Since $g \in C_G(x)$, $x \in Z(G)$ and this is a contradiction. Therefore, the number of conjugacy classes is $N = p + (p^2 - 1)/p$ with just one element and $(p^2 - 1)/p$ with p elements.

Finally, if $m = 2p - 1$ we have $c = 1 + 1 + \dots + 1 + p + \dots + p = p^2$. Note that G contains a subgroup of order p^2 , which is necessarily an abelian subgroup. \square

Theorem 2.10. *If p is a prime number and P is a non-abelian group of order p^3 then, $P \times \mathbb{Z}_p$ contains an abelian subgroup of order c .*

Proof. First note that $Z(P \times \mathbb{Z}_p) = Z(P) \times Z(\mathbb{Z}_p) = Z(P) \times \mathbb{Z}_p$. Now, since P is non-abelian, $|Z(P)| = p$ and therefore $|Z(P \times \mathbb{Z}_p)| = p^2$. Also, we have $C_{P \times \mathbb{Z}_p}(x, y) = C_P(x) \times C_{\mathbb{Z}_p}(y) = C_P(x) \times \mathbb{Z}_p$ for each $(x, y) \in P \times \mathbb{Z}_p$. If $x \in P$ is a non-central element, we have already seen in the proof of Theorem 2.9 that $|C_P(x)| = p^2$, in particular, $C_P(x)$ is an abelian subgroup of P . Hence, if $(x, y) \in P \times \mathbb{Z}_p$ is a non-central element, then $C_{P \times \mathbb{Z}_p}(x, y)$ is an abelian subgroup of order p^3 .

According to Burnside's lemma, there are p^2 conjugacy classes of order 1 and $p^3 - p$ conjugacy classes of order p . Hence, $m = 2p^2 - p$ in Theorem 1.2 and consequently $c = p^2 + (p^2 - p)p = p^3$. If $x \in P - Z(P)$ and $y \in \mathbb{Z}_p$ is any element, then (x, y) is a non-central element of $P \times \mathbb{Z}_p$ and $C_{P \times \mathbb{Z}_p}(x, y)$ is an abelian subgroup of order c . \square

Next we analyze special cases when the order is $p^n q$. The following lemmas (see [30] and [31]) will be useful.

Lemma 2.11. *If G is a group of order $p^n q$, where $p > q$ are primes, then G contains a unique Sylow p -subgroup P .*

Lemma 2.12. *If G is a group which is not abelian, then $G/Z(G)$ is not cyclic. In particular, $[G : Z(G)]$ can never be a prime number.*

Theorem 2.13. *Let G be a non-abelian group of order $p^n q$, where p and q are prime numbers:*

- (1) *If $p > q$ and the unique Sylow p -subgroup of G is abelian, then $c = p^n$ and G contains an abelian subgroup of order c . In particular, for $n = 1, 2$ the group G contains an abelian subgroup of order c .*
- (2) *If $p < q$ and $n = 2$, then either $c = 4$, $c = pq$ or $c = q$. In all these cases G contains an abelian subgroup of order c .*

Moreover, if p^2 does not divide $q - 1$, the case $c = q$ is not possible.

Proof. (1) Let P be the unique Sylow p -subgroup of G , note that all elements of this Sylow p -subgroup have the smallest conjugacy classes with one or q elements and note also that actually there exist elements $x \in P$ with $|\langle x \rangle| = q$ (otherwise $Z(G) = G$ or $|Z(G)| = p^n$ contradicting hypothesis or Lemma 2.12 in any case).

Finally, let $[x_1], [x_2], \dots, [x_k]$ be the conjugacy classes of the elements in P ordered increasingly by size. Since P is the union of these conjugacy classes we have

$$p^n = |P| = \sum_{i=1}^k |[x_i]| = |C_G(x_k)| = c.$$

If $n = 1, 2$ the unique Sylow p -subgroup of G is abelian.

(2) First, we examine the possible numbers of Sylow q -subgroups and Sylow p -subgroups. Let n_q and n_p be the numbers of Sylow q -subgroups and Sylow p -subgroups, respectively. Since $q > p$, $q \nmid (p - 1)$, it follows that $n_q \neq p$. Now, if $n_q = p^2$, then necessarily $q \mid (p + 1)$, and hence $q = p + 1$. But the only prime numbers satisfying this condition are $p = 2$ and $q = 3$. Consequently, $|G| = 12$ and then $G \cong A_4, D_{12}, \mathbb{Z}_3 \rtimes \mathbb{Z}_4$. In the first case $c = 4$ and in the other cases $c = 6$. A glance to the subgroups of G shows the existence of an abelian subgroup of order c .

The other possibility is $n_q = 1$, let Q be such a Sylow q -subgroup. Let us now consider the possible values for n_p . Note that such a number cannot be equal to 1, otherwise G is the direct product of these Sylow subgroups, and being each abelian, G must be abelian. This yields to $n_p = q$.

For analyzing the order of $Z(G)$, Lemma 2.12 ensures that $|Z(G)| \neq pq, p^2$. On the other hand, if $Z(G)$ has order q , then this is the unique Sylow q -subgroup, so all the elements of the different Sylow p -subgroups, except for the identity, are non-central elements. Hence, their centralizers have order p^2 . But $|Z(G)|$ divides the order of such centralizers, which is impossible. The only remaining possible values for $|Z(G)|$ are p or 1. Suppose that $|Z(G)| = p$. Since each Sylow p -subgroup is a centralizer, $Z(G)$ is contained in each Sylow p -subgroup. Thus, the sum of $|Z(G)|$ and the number of non-central elements in each Sylow $(p$ or $q)$ -subgroup is $p + q(p^2 - p) + (q - 1) = p^2 q - pq + p + q - 1$. Subtracting this sum from $p^2 q$ we get $pq - p - q + 1$, which corresponds to the number of non-central elements of G that do not belong to any Sylow $(p$ or $q)$ -subgroup.

Now, the non-central elements of Sylow p -subgroups and Sylow q -subgroups have centralizers of order p^2 and pq , respectively. The remaining $pq - p - q + 1$ elements have centralizers of order pq . Applying Burnside's lemma we get:

$$N = p + \frac{1}{p^2 q} ((q - 1)pq + (pq - p - q + 1)pq + q(p^2 - p)p^2) = p + (q - 1) + p(p - 1),$$

where p , $q - 1$ and $p(p - 1)$ are the number of conjugacy classes of cardinality 1, p and q , respectively. Thus, the sum of the cardinals of the first p conjugacy classes and the following $q - 1$ conjugacy classes whose cardinals are p is $c = pq$. If x is an element not belonging to any Sylow $(p$ or $q)$ -subgroup, then $|\langle x \rangle| = pq$.

Suppose $|Z(G)| = 1$, for $x \in Q$, $x \neq 1$, we may observe that $Q \leq C_G(x)$, thus, $|C_G(x)| = q$ or pq . If $|C_G(x)| = pq$ there exists $y \in C_G(x)$ of order p and, therefore, it must belong to a Sylow p -subgroup P , which implies that $|P \cap Q| \geq 2$, a contradiction. Hence, $|C_G(x)| = q$ and $Q = C_G(x)$. Moreover, the intersection of all Sylow p -subgroups is trivial, otherwise we may obtain a subgroup of order p contained in all of them and the centralizer of a non-trivial element of this subgroup would have order $> p^2$ (it commutes with every element of these Sylow p -subgroups). Now, if we take the sum over the number of elements in all Sylow subgroups we obtain $q(p^2 - 1) + (q - 1) + 1 = p^2q$, which means that G is the union of its Sylow subgroups. Finally, calculating N we obtain

$$N = 1 + \frac{1}{p^2q}((q - 1)q + q(p^2 - 1)p^2) = 1 + \frac{q - 1}{p^2} + (p^2 - 1).$$

So G has one conjugacy class with a unique element, $\frac{q-1}{p^2}$ conjugacy classes each with p^2 elements and $p^2 - 1$ conjugacy classes each with q elements. This yields $c = q$ and the order of the unique Sylow q -subgroup attains the bound. Notice that this case is not possible if p^2 does not divide $q - 1$. \square

3 Comparative analysis and conclusion

The main goal of our research was to determine under what conditions a finite group contains an abelian subgroup of maximal order. This interest is motivated by [24], where Bertram, relying on the commuting graph of a finite group, establishes that the order of every abelian subgroup of a finite group has an upper bound c . A question that immediately arises is “under what conditions equality is attained?”. Precisely, Bertram proposed it in his work without finding the solution, he only mentioned as an example the solvable groups with a number of conjugation classes less than or equal to 7, except for $G = \text{Sym}(4)$ (the symmetric group on 4 letters), where the bound c coincides with the order of a centralizer, in fact, the largest centralizer other than G . In this paper, we give a solution to this problem, that is, we find necessary and sufficient conditions on a finite group G to contain an abelian subgroup with order the bound c . In addition, we find some results that relate structural properties of the commuting graph \mathcal{G}_G to those of the underlying group G , as well as some families of non-abelian groups attaining the upper bound c .

Over the years, other works have studied, in a certain way, abelian subgroups of maximal order of a finite group. For example, in [25] and [26] some arithmetic properties on the maximum order of an abelian subgroup are established, as well as their relation to the order of the group. In [25], it is shown that for any upper bound k for the order of the abelian subgroups of a finite group G occurs that $|G|$ divides to $k!$. If no number less than c satisfies this property we conclude that G contains an abelian subgroup of order c . In [27], abelian subgroups of maximal order of finite irreducible Coxeter groups are classified. Based on the development of our research, from the commuting graph of such Coxeter groups, it could be known which of these maximal order subgroups attains the bound c .

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