Research Article

Wenjie Zhu and Chunfang Chen*

Ground state sign-changing solutions for a class of quasilinear Schrödinger equations

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Abstract: In this paper, we consider the following quasilinear Schrödinger equation:

$$-\Delta u + V(x)u + \frac{\kappa}{2}\Delta(u^2)u = K(x)f(u), \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $\kappa > 0$, $f \in C(\mathbb{R}, \mathbb{R})$, $V(x)$ and $K(x)$ are positive continuous potentials. Under given conditions, by changing variables and truncation argument, the energy of ground state solutions of the Nehari type is achieved. We also prove the existence of ground state sign-changing solutions for the aforementioned equation. Our results are the generalization work of M. B. Yang, C. A. Santos, and J. Z. Zhou, *Least action nodal solution for a quasilinear defocusing Schrödinger equation with supercritical nonlinearity*, Commun. Contemp. Math. **21** (2019), no. 5, 1850026, DOI: https://doi.org/10.1142/S0219199718500268.

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1 Introduction

Considering the existence of solitary wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + W(x)z - \varphi(|z|^2)z + \frac{\kappa}{2}\Delta l(|z|^2)l'(|z|^2)z, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $z : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential function, $l, \varphi : \mathbb{R} \to \mathbb{R}$ are suitable functions and κ is a real constant. For different forms of function l, the quasilinear equation (1.1) can be transformed into many models to reflect different physical phenomena. For example, when $l(s) \equiv 1$, equation (1.1) is transformed into the classical stationary semilinear Schrödinger equation; see [1]. Kurihura [2] studied the case of $l(s) = s$ for the superfluid membrane equation in hydrodynamics.

Set $z(t, x) = \exp(-iEt)u(x)$ and $l(s) = s$ in (1.1), where $E \in \mathbb{R}$ and u is a real function, and equation (1.1) can be reduced to elliptic equations:

$$-\Delta u + V(x)u + \frac{\kappa}{2}\Delta(u^2)u = K(x)f(u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $V(x) = W(x) - E$ and $f : \mathbb{R} \to \mathbb{R}$ is a new nonlinear term.

* **Corresponding author: Chunfang Chen,** Department of Mathematics, Nanchang University, Nanchang, Jiangxi, 330031, P. R. China, e-mail: ccfygd@sina.com
Wenjie Zhu: Department of Mathematics, Nanchang University, Nanchang, Jiangxi, 330031, P. R. China, e-mail: zwj5042@163.com

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In recent years, many authors studied the existence of positive solutions, ground state solutions and multiple solutions for quasilinear Schrödinger equations (see [3–5] and the references therein). Moreover, many other scholars have been paying attention to the existence of sign-changing solutions for quasilinear Schrödinger equations. For example, Deng et al. [6] obtained the multiplicity of sign-changing solutions for quasilinear Schrödinger equations via minimization argument. In [7], Yang et al. proved the existence of least-energy nodal solutions for quasilinear Schrödinger equations via Nehari manifold. Other results on sign-changing solutions for Schrödinger equations can be found in [8–11].

In this paper, we consider equation (1.2) with $\kappa > 0$. We need to deal with the following two problems:

(P_1) Owing to the appearance of non-convex term "$\Delta(u^2)u$," the energy functional of equation (1.2) is given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1 - \kappa u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} K(x)F(u)dx \tag{1.3}$$

which may be not well defined in usual Sobolev spaces.

(P_2) The unboundedness of the domain $\mathbb{R}^N$ leads to the lack of compactness. To overcome these difficulties, we will use the main methods of [12–14].

The aim of this paper is to establish the existence of sign-changing solutions and ground state solutions for the quasilinear Schrödinger equation. As far as we know, the case of the existence of ground state sign-changing solutions for quasilinear Schrödinger equation with $\kappa > 0$ is to be less concerned in pervious studies of quasilinear Schrödinger equation. Now, we assume that the potential $V(x)$, $K(x)$ and nonlinearity $f(t)$ satisfy the following conditions:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$, and meas $(\{x \in \mathbb{R}^N : V(x) \leq M\}) < \infty$ for each $M > 0$, where V_0 is a constant and meas denotes the Lebesgue measure in $\mathbb{R}^N$;

(K_1) $K \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$, $\frac{1}{K(x)} \in L^\infty(\mathbb{R}^N)$ and $K(x) > 0$ for all $x \in \mathbb{R}^N$;

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{t\to 0}\frac{f(t)}{t} = 0$;

(f_2) There exist constants $C > 0$ and $p \in (2, 2^*)$ such that $|f(t)| \leq C(1 + |t|^{p-1})$ for all $t \in \mathbb{R}$;

(f_3) There exists $\rho > 1$ close to 1 which satisfies $\rho < p - 1$ and $\lim_{|t|\to+\infty}\frac{f(t)}{t^\rho} = +\infty$.

To prove our results, we use the variable in [7]. Now, we consider the following elliptic equation:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = K(x)f(u), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $g : [0, +\infty) \to \mathbb{R}$ is given by

$$g(s) = \begin{cases} \sqrt{1 - \kappa s^2}, & \text{if } 0 \leq s < \frac{\sigma}{\sqrt{\kappa}}, \\ \frac{\sigma^3\sqrt{\kappa}}{\kappa\sqrt{1 - \sigma^2}s} + \frac{1}{\sqrt{\rho}}, & \text{if } s \geq \frac{\sigma}{\sqrt{\kappa}}, \\ g(-s), & \text{if } s < 0, \end{cases} \qquad \text{where } \sigma = \left[\frac{\left(4 - \frac{1}{\rho} - \sqrt{\frac{1}{\rho^2} + \frac{8}{\rho}}\right)}{8}\right]^{1/2}.$$

It follows that $g \in C^1\left(\mathbb{R}, \left(\frac{1}{\sqrt{\rho}}, 1\right]\right)$, g is an even function, which increases in $(-\infty, 0)$ and decreases in $[0, +\infty)$. The energy functional associated which equation (1.4) is given by

$$I_1(u) = \frac{1}{2}\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 - \int_{\mathbb{R}^N} K(x)F(u). \tag{1.5}$$

In what follows, let $G(t) = \int_0^t g(s)ds$ and we know that inverse function $G^{-1}(t)$ exists and it is an odd function. From the aforementioned variable, by setting $u = G^{-1}(v)$, then the energy functional I_1 reduces to the following functional:

$$I_\kappa(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 - \int_{\mathbb{R}^N} K(x)F(G^{-1}(v)). \tag{1.6}$$

To simplify the calculations, we rewrite equation (1.2) in the following form:

$$-\Delta v + V(x)v = K(x)\widetilde{f}(x, v), \quad x \in \mathbb{R}^N, \tag{1.7}$$

and the corresponding energy functional is given as follows:

$$J_\kappa(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|v|^2 - \int_{\mathbb{R}^N} K(x)\widetilde{F}(x, v), \tag{1.8}$$

where $\widetilde{F}(x, v) = \int_0^v \widetilde{f}(x, s)\mathrm{d}s$ with

$$\widetilde{f}(x, v) = \frac{f(G^{-1}(v))}{g(G^{-1}(v))} + \frac{V(x)}{K(x)}v - \frac{V(x)}{K(x)}\frac{G^{-1}(v)}{g(G^{-1}(v))}. \tag{1.9}$$

To achieve our results, we also need to make the following assumption:

(f_4) $\frac{\widetilde{f}(x,t)}{t^\rho}$ is non-decreasing on $\mathbb{R}\backslash\{0\}$.

Remark 1.1. The advantage of using this truncation argument is that we can transform the quasilinear Schrödinger equation into a semilinear case. That is, it makes the calculations easier. When $V = K \equiv 1$ and $g(t) = 1$, and

$$f(t) = |t|^\rho t.$$

Obviously, $\widetilde{f} = f$ satisfies (f_1)–(f_4).

Motivated by the aforementioned works, we will consider the following minimization problem:

$$m_0 := \inf_{v\in\mathcal{M}_0} J_\kappa(v) \quad \text{and} \quad c_0 = \inf_{v\in\mathcal{N}_0} J_\kappa(v), \tag{1.10}$$

where

$$\mathcal{M}_0 := \{v \in H : v^\pm \neq 0, \langle J_\kappa'(v), v^+\rangle = \langle J_\kappa'(v), v^-\rangle = 0\}, \tag{1.11}$$

and

$$\mathcal{N}_0 := \{v \in H : v \neq 0, \langle J_\kappa'(v), v\rangle = 0\}, \tag{1.12}$$

with $v^+ := \max\{v(x), 0\}$ and $v^- := \min\{v(x), 0\}$, which play an active role to seek sign-changing solutions and ground state solutions for problem (1.2).

In the following, let us state our results.

Theorem 1.2. *Suppose that* (V_1), (K_1) *and* (f_1)–(f_4) *hold. Then,* $c_0 > 0$ *is achieved.*

Theorem 1.3. *Suppose that* (V_1), (K_1) *and* (f_1)–(f_4) *hold. Then, there exists* $\kappa_* > 0$ *such that for any* $\kappa \in (0, \kappa_*]$, *problem* (1.7) *has a sign-changing solution* $v \in \mathcal{M}_0$ *satisfying* $\max_{x\in\mathbb{R}^N}|G^{-1}(v)| < \frac{\sigma}{\sqrt{\kappa}}$ *such that* $J_\kappa(v) = \inf_{\mathcal{M}_0} J_\kappa > 0$, *which has precisely two nodal domains.*

Theorem 1.4. *Suppose that* (V_1), (K_1) *and* (f_1)–(f_4) *hold. Then, problem* (1.7) *has a solution* $\bar{v} \in \mathcal{N}_0$ *satisfying* $\max_{x\in\mathbb{R}^N}|G^{-1}(\bar{v})| < \frac{\sigma}{\sqrt{\kappa}}$ *such that* $J_\kappa(\bar{v}) = \inf_{\mathcal{N}_0} J_\kappa$ *for* $\kappa \in (0, \kappa_*]$, *where* κ_* *is given in* Theorem 1.3. *Moreover,* $m_0 > 2c_0$.

Remark 1.5. By comparing with [7], we assume the nonlinearities f satisfy (f_3) weaker than Ambrosetti-Rabinowite condition. Furthermore, we seek the sign-changing solutions and ground state solutions of (1.2) via the non-Nehari method in [15,16]. Consequently, our results can be regarded as the generalization of [7].

Throughout this paper, let $H := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x < \infty\}$ with the norm $\|u\| = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\mathrm{d}x\right)^{\frac{1}{2}}$. Moreover, $|\cdot|_r$ denotes the norm in $L^r(\mathbb{R}^N)$. In most integrals, we omit the symbol "dx" and C denotes different constants.

2 Preliminaries

In this section, we will give the following two lemmas, which are essential to prove our results.

Lemma 2.1. *The functions g and G^{-1} satisfy*:
(1) $\lim_{t\to 0}\frac{G^{-1}(t)}{t} = 1$;
(2) $\lim_{t\to +\infty}\frac{G^{-1}(t)}{t} = \sqrt{\rho}$;
(3) $1 \le \frac{G^{-1}(t)}{t} \le \sqrt{\rho}$ *for all* $t \ne 0$;
(4) $-\frac{\sigma^2}{1-\sigma^2} \le \frac{t}{g(t)}g'(t) \le 0$ *for all* $t \in \mathbb{R}$;
(5) $\frac{t}{g(G^{-1}(t))} \ge G^{-1}(t)$ *for all* $t \ge 0$.

Proof. Conclusions (1)–(4) have been proved in [13]. Here, let us prove conclusion (5).

Define $m(t) = \frac{t}{g(G^{-1}(t))} - G^{-1}(t)$. Then, for $t \ge 0$, we have

$$m'(t) = \frac{g(G^{-1}(t)) - t\frac{g'(G^{-1}(t))}{g(G^{-1}(t))}}{g^2(G^{-1}(t))} - \frac{1}{g(G^{-1}(t))} = -\frac{t\frac{g'(G^{-1}(t))}{g(G^{-1}(t))}}{g^2(G^{-1}(t))}.$$

From (4) of Lemma 2.1, we have $m'(t) \ge 0$. Hence, $m(t) \ge m(0) = 0$, i.e., $\frac{t}{g(G^{-1}(t))} \ge G^{-1}(t)$ for all $t \ge 0$. □

Lemma 2.2. *Assume that* (f_1)–(f_3) *hold. Then, the function* $\tilde{f}(x,t)$ *has the following properties*:
$(\tilde{f}_1)$ $\tilde{f} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ *and* $\lim_{t\to 0}\frac{\tilde{f}(x,t)}{t} = 0$;
$(\tilde{f}_2)$ *there exist constants* $C_1 > 0$ *and* $p \in (2, 2^*)$ *such that* $|\tilde{f}(x,t)| \le C_1(1 + |t|^{p-1})$ *for all* $t \in \mathbb{R}$;
$(\tilde{f}_3)$ $\lim_{|t|\to +\infty}\frac{\tilde{f}(x,t)}{t^\rho} = +\infty$, *where* ρ *is given by* (f_3).

Proof. Since $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and functions V, K, g are continuous, $\tilde{f} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is obvious. Using (1) in Lemma 2.1, we have

$$\lim_{t\to 0}\frac{\tilde{f}(x,t)}{t} = \lim_{t\to 0}\frac{f(G^{-1}(t))}{g(G^{-1}(t))t} + \frac{V(x)}{K(x)}\lim_{t\to 0}\left(1 - \frac{G^{-1}(t)}{g(G^{-1}(t))t}\right) = 0 + \frac{V(x)}{K(x)}\left(1 - \frac{1}{g(0)}\right) = 0.$$

Then, $(\tilde{f}_1)$ holds. Next, using (2) in Lemma 2.1 and $p \in (2, 2^*)$, we have

$$\lim_{t\to +\infty}\frac{\tilde{f}(x,t)}{t^{p-1}} = \lim_{t\to +\infty}\frac{f(G^{-1}(t))}{(G^{-1}(t))^{p-1}}\left(\frac{G^{-1}(t)}{t}\right)^{p-1}\frac{1}{g(G^{-1}(t))} + \frac{V(x)}{K(x)}\lim_{t\to +\infty}\left(\frac{1}{t^{p-2}} - \frac{1}{t^{p-2}}\frac{G^{-1}(t)}{g(G^{-1}(t))t}\right) \le C_1,$$

where $C_1 > 0$ is a constant. Hence, $|\tilde{f}(x,t)| \le C_1(1 + |t|^{p-1})$, then $(\tilde{f}_2)$ holds. Next, by (f_3) and (4) and (5) in Lemma 2.1, we have

$$\lim_{|t|\to +\infty}\frac{\tilde{f}(x,t)}{t^\rho} = \lim_{|t|\to +\infty}\frac{f(G^{-1}(t))}{(G^{-1}(t))^\rho}\left(\frac{G^{-1}(t)}{t}\right)^\rho\frac{1}{g(G^{-1}(t))} + \frac{V(x)}{K(x)}\lim_{|t|\to +\infty}\left(\frac{1}{t^{\rho-1}} - \frac{G^{-1}(t)}{g(G^{-1}(t))t^\rho}\right) = +\infty,$$

and thus, $(\tilde{f}_3)$ holds. □

3 Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2. First, we recall the following using lemma, which was proved in [15].

Lemma 3.1. [15] *Suppose that* (V_1), (K_1), $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ *and* (f_4) *hold. Then, for* $\theta_0 \in (0,1)$, *we have that*

$$K(x)\left[\frac{1}{\rho+1}\tau\widetilde{f}(x,\tau)-\widetilde{F}(x,\tau)\right]+\frac{(\rho-1)\theta_0 V(x)}{2(\rho+1)}\tau^2 \geq 0,\ \ x\in\mathbb{R}^N,\ \ \tau\in\mathbb{R}. \tag{3.1}$$

Now, with the help of the aforementioned lemma, we prove Theorem 1.2.

Proof of Theorem 1.2. According to Lemma 4.4 in [15], we know that $\mathcal{N}_0 \neq \emptyset$. For any $v \in \mathcal{N}_0$, from the definition of $\mathcal{N}_0$ and (3.1), we have

$$\begin{aligned} J_\kappa(v) &= J_\kappa(v) - \frac{1}{\rho+1}\langle J_\kappa'(v), v\rangle \\ &= \frac{\rho-1}{2(\rho+1)}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{(\rho-1)}{2(\rho+1)}V(x)|v|^2 + \int_{\mathbb{R}^N}\left\{K(x)\left[\frac{1}{\rho+1}\widetilde{f}(x,v)v-\widetilde{F}(x,v)\right]\right\} \\ &\geq \frac{(\rho-1)(1-\theta_0)}{2(\rho+1)}\|v\|^2 + \int_{\mathbb{R}^N}\left\{K(x)\left[\frac{1}{\rho+1}\widetilde{f}(x,v)v-\widetilde{F}(x,v)\right]+\frac{(\rho-1)\theta_0 V(x)}{2(\rho+1)}|v|^2\right\} \\ &\geq \frac{(\rho-1)(1-\theta_0)}{2(\rho+1)}\|v\|^2. \end{aligned} \tag{3.2}$$

Since $\theta_0 \in (0,1)$, then (3.2) shows that $J_\kappa(v)$ is bounded from below on $\mathcal{N}_0$. Thus, $c_0 > 0$ is well defined.

Let $\{v_n\} \subset H$ satisfy $J_\kappa(v_n) \to c$ and $\|J_\kappa'(v_n)\|(1+\|v_n\|) \to 0$ as $n \to +\infty$, where $c \in (0, c_0]$. Then, we have $J_\kappa(v_n) = c + o_n(1)$ and $\langle J_\kappa'(v_n), v_n\rangle = o_n(1)$. From the aforementioned two equalities and (3.2), we have

$$\frac{(\rho-1)(1-\theta_0)}{2(\rho+1)}\|v_n\|^2 \leq c + o_n(1), \tag{3.3}$$

which shows that $\{v_n\}$ is bounded in H.

By the arguments similar to Lemma 2.4 in [8] and Lemma 2.8 in [16], we show that there exists a $v_\kappa \in H\backslash\{0\}$ such that $v_n \to v_\kappa$ in H and $J_\kappa'(v_\kappa) = 0$. Hence, $v_\kappa \in \mathcal{N}_0$ is a nontrivial solution of (1.7) and $J_\kappa(v_\kappa) \geq c_0$. By (3.1), the weak semicontinuity of norm and Fatou's lemma, we have

$$\begin{aligned} c_0 \geq c &= \lim_{n\to\infty}\left[J_\kappa(v_n) - \frac{1}{\rho+1}\langle J_\kappa'(v_n), v_n\rangle\right] \\ &= \lim_{n\to\infty}\left[\frac{\rho-1}{2(\rho+1)}\|v_n\|^2 + \int_{\mathbb{R}^N}K(x)\left(\frac{1}{\rho+1}\widetilde{f}(x,v_n)v_n-\widetilde{F}(x,v_n)\right)\right] \\ &\geq \liminf_{n\to\infty}\left[\frac{\rho-1}{2(\rho+1)}\int_{\mathbb{R}^N}|\nabla v_n|^2 + \frac{(\rho-1)(1-\theta_0)}{2(\rho+1)}\int_{\mathbb{R}^N}V(x)|v_n|^2\right] \\ &\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[K(x)\left(\frac{1}{\rho}\widetilde{f}(x,v_n)v_n-\widetilde{F}(x,v_n)\right)+\frac{(\rho-1)\theta_0 V(x)}{2(\rho+1)}|v_n|^2\right] \\ &\geq \frac{\rho-1}{2(\rho+1)}\int_{\mathbb{R}^N}|\nabla v_\kappa|^2 + \frac{(\rho-1)(1-\theta_0)}{2(\rho+1)}\int_{\mathbb{R}^N}V(x)|v_\kappa|^2 \\ &\quad + \int_{\mathbb{R}^N}\left[K(x)\left(\frac{1}{\rho}\widetilde{f}(x,v_\kappa)v_\kappa-\widetilde{F}(x,v_\kappa)\right)+\frac{(\rho-1)\theta_0 V(x)}{2(\rho+1)}|v_\kappa|^2\right] \end{aligned}$$

$$= \frac{\rho - 1}{2(\rho + 1)} \|v_\kappa\|^2 + \int_{\mathbb{R}^N} K(x)\left(\frac{1}{\rho + 1}\widetilde{f}(x, v_\kappa)v_\kappa - \widetilde{F}(x, v_\kappa)\right)$$
$$= J_\kappa(v_\kappa) - \frac{1}{\rho + 1}\langle J_\kappa'(v_\kappa), v_\kappa\rangle = J_\kappa(v_\kappa).$$

Hence, we have $J_\kappa(v_\kappa) \le c_0$, and so, $J_\kappa(v_\kappa) = c_0 = \inf_{\mathcal{N}_0} J_\kappa > 0$. The proof is completed. □

4 The proof of Theorems 1.3 and 1.4

Under the assumptions (V_1), (K_1), $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ and (f_4), according to the process of proof in [15,16], we have the following two theorems.

Theorem 4.1. *Suppose that* (V_1), (K_1), $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ *and* (f_4) *hold. Then, problem* (1.7) *has a sign-changing solution* $v \in \mathcal{M}_0$ *such that* $J_\kappa(v) = \inf_{\mathcal{M}_0} J_\kappa > 0$, *which has precisely two nodal domains.*

Theorem 4.2. *Suppose that* (V_1), (K_1), $(\widetilde{f}_1)$–$(\widetilde{f}_3)$ *and* (f_4) *hold. Then, problem* (1.7) *has a solution* $\bar{v} \in \mathcal{N}_0$ *such that* $J_\kappa(\bar{v}) = \inf_{\mathcal{N}_0} J_\kappa$. *Moreover,* $m_0 > 2c_0$.

By the truncation argument in Section 1, we know that if the solution v_0 of equation (1.7) satisfies $|u_0|_\infty = |G^{-1}(v_0)|_\infty < \frac{\sigma}{\sqrt{\kappa}}$, then u_0 is a sign-changing solution or ground state solution of original equation (1.2). Next, we present the following two lemmas.

Lemma 4.3. *If* v *is a critical point of* J_κ, *then there exists a constant* $\bar{C}$ *independent of* κ *such that* $\|v\| \le \bar{C}$.

Proof. The proof is similar to Theorem 1.2, and so we omit it. □

Lemma 4.4. *Let* v *be a solution of equation* (1.7), *then there exists a constant* $C^* > 0$ *independent of* κ *such that* $|v|_\infty \le C^*$.

Proof. For each $m \in \mathbb{N}$ and $\beta > 1$, set $A_m = \{x \in \mathbb{R}^N : |v|^{\beta-1} \le m\}$ and $B_m = \mathbb{R}^N \backslash A_m$. Define the following two sequences:

$$v_m = \begin{cases} v\,|v|^{2(\beta-1)} & \text{in } A_m, \\ m^2 v & \text{in } B_m, \end{cases} \quad \text{and} \quad w_m = \begin{cases} v\,|v|^{\beta-1} & \text{in } A_m, \\ mv & \text{in } B_m. \end{cases}$$

Observe that $v_m, w_m \in H$, $|v_m| \le |v|^{2\beta-1}$ and $|w_m|^2 = vv_m \le |v|^{2\beta}$. By direct calculation, we have

$$\nabla v_m = \begin{cases} (2\beta - 1)|v|^{2(\beta-1)}\nabla v & \text{in } A_m, \\ m^2\nabla v & \text{in } B_m, \end{cases} \quad \text{and} \quad \nabla w_m = \begin{cases} \beta\,|v|^{\beta-1}\nabla v & \text{in } A_m, \\ m\nabla v & \text{in } B_m. \end{cases}$$

Besides this, we have

$$\int_{\mathbb{R}^N} \nabla v \nabla v_m = (2\beta - 1)\int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2 + m^2\int_{B_m} |\nabla v|^2, \tag{4.1}$$

and

$$\int_{\mathbb{R}^N} (|\nabla w_m|^2 - \nabla v \nabla v_m) = (\beta - 1)^2\int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2. \tag{4.2}$$

Taking v_m as a text function, from (4.1) and (4.2), we obtain

$$\int_{\mathbb{R}^N} |\nabla w_m|^2 \le \left[\frac{(\beta-1)^2}{2\beta-1}+1\right]\int_{\mathbb{R}^N} \nabla v \nabla v_m \le \beta^2 \int_{\mathbb{R}^N} [\nabla v \nabla v_m + V(x) v v_m] = \beta^2 \int_{\mathbb{R}^N} K(x)\widetilde{f}(x,v)v_m. \tag{4.3}$$

From (K_1), Lemmas 2.1 and 2.2 and the fact that $w_m^2 = vv_m$, there exists a constant $C > 0$ such that

$$\widetilde{f}(x,v)v_m \le C(|w_m|^2 + |v|^{p-2}|w_m|^2). \tag{4.4}$$

By (4.3)–(4.4), Sobolev inequality and Hölder inequality, since $\frac{1}{r_1} + \frac{p-2}{2^*} = 1$, we have

$$\left(\int_{A_m} |w_m|^{2^*}\right)^{\frac{2}{2^*}} \le S^2\left(\int_{\mathbb{R}^N} |\nabla w_m|^2\right) \le C_1 S^2 \beta^2 \int_{\mathbb{R}^N} (|w_m|^2 + |v|^{p-2}w_m^2) \le C_1 S^2\beta^2(|w_m|_2^2 + |v|_{2^*}^{p-2}|w_m|_{2r_1}^2),$$

where $S > 0$ is the best Sobolev constant. According to the definition of w_m, and then, letting $m \to +\infty$ in the aforementioned inequality, we have

$$|v|_{\beta 2^*}^{2\beta} \le C_1 S^2\beta^2(|v|_{2\beta}^{2\beta} + |v|_{2^*}^{p-2}|v|_{2\beta r_1}^{2\beta}). \tag{4.5}$$

By interpolation inequality, we obtain

$$|v|_{\beta 2} \le |v|_2^{1-\xi}|v|_{2\beta r_1}^{\xi}, \tag{4.6}$$

where $\xi \in (0,1)$ satisfies $\frac{1}{2\beta} = \frac{1-\xi}{2} + \frac{\xi}{2\beta r_1}$, that is, $\xi = \frac{\beta r_1 - r_1}{\beta r_1 - 1}$.

In particular, by (4.6), we have

$$|v|_{2\beta}^{2\beta} \le |v|_2^{2\beta(1-\xi)}|v|_{2\beta r_1}^{2\beta\xi} \le (1 + |v|_2)^2|v|_{2\beta r_1}^{2\beta\xi}, \tag{4.7}$$

because $2\beta(1-\xi) = 2 + \frac{2(1-\beta)}{\beta r_1 - 1} < 2$. By (4.5) and (4.7), we have

$$|v|_{2^*\beta}^{2\beta} \le C_2\beta^2[(1 + |v|_2)^2|v|_{2\beta r_1}^{2\beta\xi} + |v|_{2^*}^{p-2}|v|_{2\beta r_1}^{2\beta}] \le 2C_2\beta^2(1 + |v|_2^2 + |v|_{2^*}^{p-2})|v|_{2\beta r_1}^{2\beta\tau}, \tag{4.8}$$

where $\tau \in \{1, \xi\}$. By (4.8), one has

$$|v|_{\beta 2^*} \le C^{1/2\beta}\beta^{1/\beta}(1 + |v|_2^2 + |v|_{2^*}^{p-2})^{1/2\beta}|v|_{2\beta r_1}^{\tau}. \tag{4.9}$$

Taking $\sigma = \frac{2^*}{2r_1} > 1$ and setting $\beta = \sigma$ in (4.9), we get

$$|v|_{\sigma 2^*} \le C^{1/2\sigma}\sigma^{1/\sigma}(1 + |v|_2^2 + |v|_{2^*}^{p-2})^{1/2\sigma}|v|_{2^*}^{\tau_1}, \tag{4.10}$$

where $\tau_1 \in \{1, \xi_1\}$ and $\xi_1 = \frac{\sigma r_1 - r_1}{\sigma r_1 - 1}$. Next, taking $\beta = \sigma^2$ in (4.9), we have

$$|v|_{\sigma^2 2^*} \le C^{1/2\sigma^2}\sigma^{2/\sigma^2}(1 + |v|_2^2 + |v|_{2^*}^{p-2})^{1/2\sigma^2}|v|_{2^*\sigma}^{\tau_2}, \tag{4.11}$$

where $\tau_2 \in \{1, \xi_2\}$ and $\xi_2 = \frac{\sigma^2 r_1 - r_1}{\sigma^2 r_1 - 1}$. Now taking $\beta = \sigma^j$ for $j \in \mathbb{N}$, we proceed the j times iterations and by combining (4.10) and (4.11), we deduce that

$$|v|_{\sigma^j 2^*} \le C^{\sum_{j=1}^{\infty}\frac{1}{2\sigma^j}}\sigma^{\sum_{j=1}^{\infty}\frac{j}{\sigma^j}}(1 + |v|_2^2 + |v|_{2^*}^{p-2})^{\sum_{j=1}^{\infty}\frac{1}{2\sigma^j}}|v|_{2^*}^{\tau_1\tau_2\ldots\tau_j}, \tag{4.12}$$

where $\tau_j \in \{1, \xi_j\}$ and $\xi_j = \frac{\sigma^j r_1 - r_1}{\sigma^j r_1 - 1}$. Next, we estimate the right side in (4.12). At this point, we analyze two cases: $|v|_{2^*} \ge 1$ or $|v|_{2^*} < 1$.

Case 1: If $|v|_{2^*} \ge 1$, then we have $|v|_{2^*}^{\tau_1\tau_2\ldots\tau_j} \le |v|_{2^*}$ due to $\tau_1\tau_2\ldots\tau_j \le 1$. Hence, we have

$$|v|_{\sigma^j 2^*} \le C^{\frac{1}{2(\sigma-1)}}\sigma^{\frac{\sigma}{(\sigma-1)^2}}(1 + |v|_2^2 + |v|_{2^*}^{p-2})^{\frac{1}{2(\sigma-1)}}|v|_{2^*}, \quad \forall j \in \mathbb{N}.$$

Letting $j \to +\infty$ in the last inequality, we have

$$|v|_\infty \le C^{\frac{1}{2(\sigma-1)}}\sigma^{\frac{\sigma}{(\sigma-1)^2}}(1 + |v|_2^2 + |v|_{2^*}^{p-2})^{\frac{1}{2(\sigma-1)}}|v|_{2^*}. \tag{4.13}$$

Case 2: If $|v|_{2^*} < 1$, then for any $j \in \mathbb{N}$, we have that $0 < \xi_1\xi_2 \dots \xi_j \le \tau_1\tau_2 \dots \tau_j \le 1$, and

$$\sum_{k=1}^{j} \ln \xi_k \le \sum_{k=1}^{j} \ln \tau_k = \ln(\tau_1\tau_2 \dots \tau_j) \le 0, \tag{4.14}$$

where $\xi_j = \frac{\sigma^j r_1 - r_1}{\sigma^j r_1 - 1} = 1 - \frac{r_1 - 1}{\sigma^j r_1 - 1} < 1$, $\tau_j = \{1, \xi_j\}$. By $\ln(1 - s) \ge -\frac{s}{1-s}$ for all $s \in (0, 1)$, we have

$$\sum_{k=1}^{j} \ln \xi_k = \sum_{k=1}^{j} \ln\left(1 - \frac{r_1 - 1}{r_1\sigma^k - 1}\right) \ge -\frac{r_1 - 1}{r_1} \sum_{k=1}^{j} \frac{1}{\sigma^k - 1}.$$

Setting $\nu := \sum_{k=1}^{\infty} \frac{1}{\sigma^k - 1}$, by (4.14) and the last inequality, we get $\ln(\tau_1\tau_2 \dots \tau_j) \ge -\frac{r_1-1}{r_1}\nu$, and setting $\omega := -\frac{r_1-1}{r_1}\nu$, then we have $\omega < 0$; hence, $\tau_1\tau_2 \dots \tau_j \ge \exp(\omega)$, $\forall j \in \mathbb{N}$.

According to $|v|_{2^*} < 1$, we have that $|v|_{2^*}^{\tau_1\tau_2\dots\tau_j} \le |v|_{2^*}^{\exp(\omega)}$. By (4.12), we get that

$$|v|_{\sigma^j 2^*} \le C^{\frac{1}{2(\sigma-1)}} \sigma^{\frac{\sigma}{(\sigma-1)^2}} (1 + |v|_2^2 + |v|_{2^*}^{p-2})^{\frac{1}{2(\sigma-1)}} |v|_{2^*}^{\exp(\omega)}, \quad \forall j \in \mathbb{N}.$$

Letting $j \to +\infty$ in the aforementioned inequality, we have

$$|v|_{\infty} \le C^{\frac{1}{2(\sigma-1)}} \sigma^{\frac{\sigma}{(\sigma-1)^2}} (1 + |v|_2^2 + |v|_{2^*}^{p-2})^{\frac{1}{2(\sigma-1)}} |v|_{2^*}^{\exp(\omega)}. \tag{4.15}$$

Combining (4.13) and (4.15), we have

$$|v|_{\infty} \le C^{\frac{1}{2(\sigma-1)}} \sigma^{\frac{\sigma}{(\sigma-1)^2}} (1 + |v|_2^2 + |v|_{2^*}^{p-2})^{\frac{1}{2(\sigma-1)}} |v|_{2^*}^{\varsigma},$$

where $\varsigma = 1$ or $\varsigma = \exp(\omega)$. From Lemma 4.3 and $H \hookrightarrow L^s(\mathbb{R}^N)$ for $s \in [2, 2^*]$, we have

$$|v|_{\infty} \le C^{\frac{1}{2(\sigma-1)}} \sigma^{\frac{\sigma}{(\sigma-1)^2}} (1 + \|v\|^2 + \|v\|^{p-2})^{\frac{1}{2(\sigma-1)}} \|v\|^{\varsigma} \le C^*,$$

where C^* is a real constant independent of $\kappa > 0$. □

Proof of Theorems 1.3 and 1.4. Combining Theorem 4.1 and Lemma 4.4, we deduce that the solution v of (1.7) satisfying $|v|_{\infty} \le C_*$. Hence, there exists $\kappa_1 > 0$ such that

$$|G^{-1}(v)|_{\infty} \le \sqrt{\rho}\, |v|_{\infty} < \frac{\sigma}{\sqrt{\kappa}}, \quad \forall \kappa \in (0, \kappa_1].$$

Similarly, combining Theorem 4.2 and Lemma 4.4, there exists $\kappa_2 > 0$ such that

$$|G^{-1}(\bar{v})|_{\infty} \le \sqrt{\rho}\, |\bar{v}|_{\infty} < \frac{\sigma}{\sqrt{\kappa}}, \quad \forall \kappa \in (0, \kappa_2].$$

Choosing $\kappa_* \le \min\{\kappa_1, \kappa_2\}$, since $\kappa \in (0, \kappa_*]$, we have

$$|G^{-1}(v)|_{\infty} < \frac{\sigma}{\sqrt{\kappa}} \quad \text{and} \quad |G^{-1}(\bar{v})|_{\infty} < \frac{\sigma}{\sqrt{\kappa}}.$$

Therefore, $u = G^{-1}(v)$ is a sign-changing solution and $\bar{u} = G^{-1}(\bar{v})$ is a ground state solution of equation (1.2). Therefore, Theorems 1.3 and 1.4 are completed. □

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