

Research Article

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Poly-falling factorial sequences and poly-rising factorial sequences

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Abstract: In this paper, we introduce generalizations of rising factorials and falling factorials, respectively, and study their relations with the well-known Stirling numbers, Lah numbers, and so on. The first stage is to define poly-falling factorial sequences in terms of the polyexponential functions, reducing them to falling factorials if $k = 1$, necessitating a demonstration of the relations: between poly-falling factorial sequences and the Stirling numbers of the first and second kind, respectively; between poly-falling factorial sequences and the poly-Bell polynomials; between poly-falling factorial sequences and the poly-Bernoulli numbers; between poly-falling factorial sequences and poly-Genocchi numbers; and recurrence formula of these sequences. The later part of the paper deals with poly-rising factorial sequences in terms of the polyexponential functions, reducing them to rising factorial if $k = 1$. We study some relations: between poly-falling factorial sequences and poly-rising factorial sequences; between poly-rising factorial sequences and the Stirling numbers of the first kind and the power of x ; and between poly-rising factorial sequences and Lah numbers and the poly-falling factorial sequences. We also derive recurrence formula of these sequences and reciprocal formula of the poly-falling factorial sequences.

Keywords: falling factorials, rising factorials, modified polyexponential functions, Stirling numbers of the first and second kind, poly-Bernoulli polynomials, poly-Genocchi polynomials

MSC 2020: 11B73, 11B83, 05A19

1 Introduction

In the study of falling factorials and rising factorials, the following properties produce important characteristics for special numbers such as Stirling numbers and Lah numbers: the n th falling factorial of x is expressed in terms of the Stirling numbers of the first kind and the power of x ; the n th power of x is expressed in terms of the Stirling numbers of the second kind $S_2(n, l)$; the falling factorials of x ; the n th rising factorial of x is expressed in terms of Lah numbers and falling factorials; and the n th falling factorial of x can also be expressed in terms of Lah numbers and rising factorials [1–10]. Kim and Kim [11] introduced the polyexponential functions in the view of an inverse to the polylogarithm functions which were first studied by Hardy [12]. Many mathematicians have studied about “poly” for special polynomials and numbers arising from the polyexponential functions or the polylogarithm functions [4,8,10,13–16]. Recently, the degenerate poly-Bell polynomials and the degenerate poly-Lah-Bell polynomials derived from the degenerate polyexponential functions were introduced in [17,18], respectively. In addition, it is briefly mentioned that the degenerate Daehee numbers of order k expressed the degenerate polyexponential functions in [19]. In this paper, we intend to study the above-mentioned properties by generalizing falling factorials and rising factorials, respectively, using the polyexponential functions. In Section 2, we define

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poly-falling factorial sequences arising from the polyexponential functions, reducing them to falling factorial if $k = 1$. We demonstrate the relations: between poly-falling factorial sequences and the Stirling numbers of the first and second kind, respectively; between poly-falling factorial sequences and the poly-Bell polynomials; between poly-falling factorial sequences and the poly-Bernoulli numbers; between poly-falling factorial sequences and poly-Genocchi numbers; and recurrence formula of these sequences. In Section 3, we introduce poly-rising factorial sequences arising from the polyexponential functions, reducing them to rising factorial if $k = 1$. To elaborate, we analyze the relationships: between poly-falling factorial sequences and poly-rising factorial sequences; between poly-rising factorial sequences of x and the Stirling numbers of the first kind and the power of x ; between poly-rising factorial sequences and Lah numbers the poly-falling factorial sequences; recurrence formula of these sequences; and reciprocal formula of the poly-falling factorial sequences.

First, definitions and preliminary properties required in this paper are introduced.

For $x \in \mathbb{R}$, the falling factorials $(x)_n$ are defined by

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1), \quad (n \geq 1) \quad \text{and} \quad (x)_0 = 1 \quad (1)$$

(see [1,2,7,20–22]).

For $x \in \mathbb{R}$, the rising factorials $\langle x \rangle_n$ are defined by

$$\langle x \rangle_n = x(x+1)(x+2)\cdots(x+n-1), \quad (n \geq 1), \quad \text{and} \quad \langle x \rangle_0 = 1 \quad (2)$$

(see [2,5,6,20–22]).

It is well-known that for $|t| < 1$,

$$(1-t)^{-m} = \sum_{l=0}^{\infty} \binom{-m}{l} (-1)^l t^l = \sum_{l=0}^{\infty} \langle m \rangle_l \frac{t^l}{l!} \quad (3)$$

(see [2,17,18]).

For $n \geq 0$, the Stirling numbers of the first kind $S_1(n, l)$ are the coefficients of x^l in

$$(x)_n = \sum_{l=0}^n S_1(n, l) x^l \quad (4)$$

(see [2,9,22]).

From (4), it is easy to see that

$$\frac{1}{k!} (\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n, k) \frac{t^n}{n!}, \quad (|t| < 1) \quad (5)$$

(see [2,9,22]).

In the inverse expression to (4), for $n \geq 0$, the n th power of x can be expressed in terms of the Stirling numbers of the second kind $S_2(n, l)$ as follows:

$$x^n = \sum_{l=0}^n S_2(n, l) (x)_l, \quad (6)$$

(see [2,7,21,22]).

From (6), it is easy to see that for $|t| < 1$,

$$\frac{1}{k!} (e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}, \quad (t \in \mathbb{C}) \quad (7)$$

(see [2,7,9,22]).

The Bell polynomials $\text{bel}_n(x) = \sum_{k=0}^n S_2(n, k) x^k$ are natural extensions of the Bell numbers which are the number of ways to partition a set with n elements into nonempty subsets. It is well known that the generating function of the Bell polynomials is given by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \text{bel}_n(x) \frac{t^n}{n!}, \quad (t \in \mathbb{C})$$

(see [2,3,10,23,24]).

The unsigned Lah number $L(n, j)$ counts the number of ways a set of n elements can be partitioned into j nonempty linearly ordered subsets and has an explicit formula

$$L(n, j) = \binom{n-1}{j-1} \frac{n!}{j!}, \quad (j \geq 0) \quad (8)$$

(see [2,5,6,8,18,25,26]).

From (1), the generating function of $L(n, k)$ is given by

$$\frac{1}{j!} \left(\frac{t}{1-t} \right)^j = \sum_{n=j}^{\infty} L(n, j) \frac{t^n}{n!}, \quad (j \geq 0), \quad (t \in \mathbb{C} \setminus \{1\}) \quad (9)$$

(see [2,5,6,8,25,26]).

We observe that

$$\left(\frac{1}{1-t} \right)^x = \left(1 + \frac{t}{1-t} \right)^x = \sum_{j=0}^{\infty} \binom{x}{j} \left(\frac{t}{1-t} \right)^j = \sum_{n=0}^{\infty} \left(\sum_{j=0}^n L(n, j) \langle x \rangle_j \right) \frac{t^n}{n!}. \quad (10)$$

On the other hand, we get

$$\left(\frac{1}{1-t} \right)^x = (1-t)^{-x} = \sum_{n=0}^{\infty} \langle x \rangle_n \frac{t^n}{n!}. \quad (11)$$

From (10) and (11), we have

$$\langle x \rangle_n = \sum_{j=0}^n L(n, j) \langle x \rangle_j \quad (12)$$

(see [2,5]). Replacing x by $-x$, we get easily

$$\langle x \rangle_n = \sum_{j=0}^n (-1)^{n-j} L(n, j) \langle x \rangle_j \quad (13)$$

(see [5,6]).

Kim and Kim introduced the modified polyexponential functions as

$$\text{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)! n^k}, \quad (k \in \mathbb{Z}) \quad (14)$$

(see [11]). When $k = 1$, we see that $\text{Ei}_1(x) = e^x - 1$.

The poly-Bernoulli polynomials are defined by

$$\frac{\text{Ei}_k(\log(1+t))}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \quad (15)$$

(see [14,16]). When $x = 0$, $B_n^{(k)} = B_n^{(k)}(0)$, which are called the poly-Bernoulli numbers [14,16].

Since $\text{Ei}_1(\log(1+t)) = t$, $B_n^{(1)}(x)$ are the Bernoulli polynomials [1,5,13,19,22].

The poly-Genocchi polynomials are given by

$$\frac{2\text{Ei}_k(\log(1+t))}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!} \quad (16)$$

(see [27]), and $G_0^{(k)}(x) = 0$. When $x = 0$, $G_n^{(k)} = G_n^{(k)}(0)$, which are called the poly-Genocchi numbers.

When $k = 1$, $G_n^{(1)}(x)$ are the Genocchi polynomials [4,27–29].

We introduced the poly-Bell polynomials by

$$1 + \text{Ei}_k(x(e^t - 1)) = \sum_{n=0}^{\infty} \text{bel}_n^{(k)}(x) \frac{t^n}{n!} \quad (17)$$

(see [17]), and $\text{bel}_0^{(k)}(x) = 1$ (see [17]).

When $k = 1$, from (17), we note that

$$1 + \text{Ei}_1(x(e^t - 1)) = 1 + \sum_{n=1}^{\infty} \frac{(x(e^t - 1))^n}{(n-1)!n} = \exp(x(e^t - 1)) = \sum_{n=0}^{\infty} \text{bel}_n(x) \frac{t^n}{n!}. \quad (18)$$

Combining (17) with (18), we have the Bell polynomials

$$\text{bel}_n^{(1)}(x) = \text{bel}_n(x)$$

(see [2,3,22]).

As the multivariate version of the Stirling numbers $S(n, k)$ of the second kind, the generation function of the incomplete Bell polynomials $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ is given by

$$\frac{1}{k!} \left(\sum_{h=1}^{\infty} x_h \frac{t^h}{h!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(\alpha_1, \alpha_2, \dots, \alpha_{n-k+1}) \frac{t^n}{n!} \quad (19)$$

(see [2,23,25,30]).

From (19), we obtain

$$B_{n,k}(\alpha_1, \alpha_2, \dots, \alpha_{n-k+1}) = \sum_{h_1+2h_2+\dots+nh_n=n} \frac{n!}{h_1!h_2!\dots h_{n-k+1}!} \left(\frac{\alpha_1}{1}\right)^{h_1} \left(\frac{\alpha_2}{2!}\right)^{h_2} \dots \left(\frac{\alpha_{n-k+1}}{(n-k+1)!}\right)^{h_{n-k+1}} \quad (20)$$

(see [2,23,25,31]), where the sum over all nonnegative integers h_1, h_2, \dots, h_n satisfying $h_1 + h_2 + \dots + h_{n-k+1} = k$ and $h_1 + 2h_2 + \dots + (n-k+1)h_{n-k+1} = n$.

2 Poly-falling factorial sequences

In this section, we define the poly-falling factorial sequences by using the polyexponential functions. We also give some relations between them and special numbers, and derive recurrence formula of these sequences.

For $n \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}$, we consider the poly-falling factorial sequences $(x)_n^{(k)}$, which are arising from the polyexponential functions to be

$$1 + \text{Ei}_k(x \log(1+t)) = \sum_{n=0}^{\infty} (x)_n^{(k)} \frac{t^n}{n!} \quad \text{and} \quad (x)_0^{(k)} = 1 \quad (x \neq 0, |t| < 1). \quad (21)$$

Note that $\text{Ei}_1(x) = e^x - 1$.

When $k = 1$, since $\text{Ei}_1(x) = e^x - 1$, we note that

$$1 + \text{Ei}_1(x \log(1+t)) = 1 + e^{x \log(1+t)} - 1 = (1+t)^x = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!}. \quad (22)$$

Therefore, by (22), we have $(x)_n^{(1)} = (x)_n$.

Theorem 1. For $n \in \mathbb{N}$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}$, we have

$$(x)_n^{(k)} = \sum_{d=1}^n \frac{1}{d^{k-1}} S_1(n, d) x^d.$$

Proof. First, from (5), (14), and (21), we observe that

$$\begin{aligned}
 1 + \text{Ei}_k(x \log(1+t)) &= 1 + \sum_{d=1}^{\infty} \frac{x^d (\log(1+t))^d}{(d-1)! d^k} \\
 &= 1 + \sum_{d=1}^{\infty} \frac{x^d}{d^{k-1}} \frac{1}{d!} (\log(1+t))^d \\
 &= 1 + \sum_{d=1}^{\infty} \frac{x^d}{d^{k-1}} \sum_{n=d}^{\infty} S_1(n, d) \frac{t^n}{n!} \\
 &= 1 + \sum_{n=1}^{\infty} \left(\sum_{d=1}^n \frac{1}{d^{k-1}} S_1(n, d) x^d \right) \frac{t^n}{n!}.
 \end{aligned} \tag{23}$$

By comparing the coefficients of both sides of (23), we get the desired result. \square

Theorem 2. For $n \in \mathbb{N}$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}$, we have

$$n^{k-1} \sum_{d=1}^n S_2(n, d) (x)_d^{(k)} = x^n.$$

Proof. Substituting t by $e^t - 1$ in (21), the left-hand side of (21) is

$$1 + \text{Ei}_k(xt) = 1 + \sum_{n=1}^{\infty} \frac{(xt)^n}{(n-1)! n^k} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n^{k-1}} \frac{t^n}{n!}. \tag{24}$$

On the other hand, from (7), the right-hand side of (21) is

$$\sum_{d=0}^{\infty} (x)_d^{(k)} \frac{(e^t - 1)^d}{d!} = 1 + \sum_{d=1}^{\infty} (x)_d^{(k)} \sum_{n=d}^{\infty} S_2(n, d) \frac{t^n}{n!} = 1 + \sum_{n=1}^{\infty} \left(\sum_{d=1}^n (x)_d^{(k)} S_2(n, d) \right) \frac{t^n}{n!}. \tag{25}$$

Combining (24) with (25), we have the desired result. \square

In Theorems 1 and 2, when $k = 1$, we note that

$$(x)_n = \sum_{d=0}^n S_1(n, d) x^d \quad \text{and} \quad \sum_{d=0}^n S_2(n, d) (x)_d = x^n.$$

Theorem 3. For $n \in \mathbb{N}$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}$, we have

$$\text{bel}_n^{(k)}(x) = \sum_{m=1}^n \sum_{d=1}^m S_2(n, m) S_2(m, d) (x)_d^{(k)}.$$

Proof. Substituting t by $e^{(e^t-1)} - 1$ in (21), from (7), we observe that

$$\begin{aligned}
 1 + \text{Ei}_k(x(e^t - 1)) &= 1 + \sum_{d=1}^{\infty} (x)_d^{(k)} \frac{(e^{(e^t-1)} - 1)^d}{d!} \\
 &= 1 + \sum_{d=1}^{\infty} (x)_d^{(k)} \sum_{m=d}^{\infty} S_2(m, d) \frac{(e^t - 1)^m}{m!} \\
 &= 1 + \sum_{m=1}^{\infty} \left(\sum_{d=1}^m (x)_d^{(k)} S_2(m, d) \right) \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} \\
 &= 1 + \sum_{n=1}^{\infty} \left(\sum_{m=1}^n \sum_{d=1}^m S_2(n, m) S_2(m, d) (x)_d^{(k)} \right) \frac{t^n}{n!}.
 \end{aligned} \tag{26}$$

By comparing the coefficients of (17) and (26), for $n \geq 1$ we get

$$\text{bel}_n^{(k)}(x) = \sum_{m=1}^n \sum_{d=1}^m S_2(n, m) S_2(m, d) (x)_d^{(k)}. \quad (27)$$

□

From (4) and (5), we observe that

$$x^n = \sum_{l=0}^n S_2(n, l) (x)_l = \sum_{d=0}^n S_2(n, d) \sum_{j=0}^d S_1(d, j) x^j = \sum_{d=0}^n \sum_{j=0}^d S_2(n, d) S_1(d, j) x^j. \quad (28)$$

From (28), we have

$$\sum_{d=0}^n \sum_{j=0}^d S_2(n, d) S_1(d, j) = \delta_{n,j}. \quad (29)$$

In Theorem 3, for $n \geq 1$ when $k = 1$, by using (4), (29), and Theorem 1, we note that

$$\begin{aligned} \text{bel}_n(x) &= \sum_{m=1}^n \sum_{d=1}^m S_2(n, m) S_2(m, d) (x)_d \\ &= \sum_{m=1}^n \sum_{d=1}^m \sum_{j=0}^d S_2(n, m) S_2(m, d) S_1(d, j) x^j \\ &= \sum_{j=1}^n S_2(n, j) x^j \\ &= \sum_{j=0}^n S_2(n, j) x^j. \end{aligned} \quad (30)$$

Theorem 4. For $n \in \mathbb{N}$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}$, we have

$$(x)_n^{(k)} = \sum_{d=1}^n \sum_{m=d}^n S_1(n, m) S_1(m, d) \text{bel}_d^{(k)}(x).$$

Proof. First, we observe that

$$\begin{aligned} \frac{(\log(\log(1+t) + 1))^d}{d!} &= \sum_{m=d}^{\infty} S_1(m, d) \frac{(\log(1+t))^m}{m!} \\ &= \sum_{m=d}^{\infty} S_1(m, d) \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} \\ &= \sum_{n=d}^{\infty} \sum_{m=d}^n S_1(n, m) S_1(m, d) \frac{t^n}{n!}. \end{aligned} \quad (31)$$

Substituting t by $\log(\log(1+t) + 1)$ in (17), from (5), we observe that

$$\begin{aligned} 1 + \text{Ei}_k(x \log(1+t)) &= \sum_{d=0}^{\infty} \text{bel}_d^{(k)}(x) \frac{(\log(\log(1+t) + 1))^d}{d!} \\ &= 1 + \sum_{d=1}^{\infty} \text{bel}_d^{(k)}(x) \sum_{n=d}^{\infty} \sum_{m=d}^n S_1(n, m) S_1(m, d) \frac{t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \left(\sum_{d=1}^n \sum_{m=d}^n S_1(n, m) S_1(m, d) \text{bel}_d^{(k)}(x) \right) \frac{t^n}{n!}. \end{aligned} \quad (32)$$

Combining (21) with (32), for $n \geq 1$, we have the desired result.

□

Theorem 5. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$(x)_1^{(k-1)} = -(x)_1^{(k)}$$

and

$$(x)_n^{(k-1)} = \sum_{j=1}^n \binom{n}{j} (-1)^{j+1} (j-1)! ((x)_{n-j+1}^{(k)} + (n-j)(x)_{n-j}^{(k)}), \quad (n \geq 2).$$

Proof. Differentiating with respect to t in (21), the left-hand side of (21) is

$$\begin{aligned} \frac{\partial}{\partial t}(1 + \text{Ei}_k(x \log(1+t))) &= \sum_{n=1}^{\infty} \frac{x^n (\log(1+t))^{n-1}}{(n-1)! n^{k-1}} \frac{1}{1+t} \\ &= \frac{1}{(1+t) \log(1+t)} \sum_{n=1}^{\infty} \frac{(x \log(1+t))^n}{(n-1)! n^{k-1}} \\ &= \frac{1}{(1+t) \log(1+t)} \text{Ei}_{k-1}(x \log(1+t)) \\ &= \frac{1}{(1+t) \log(1+t)} \sum_{n=1}^{\infty} (x)_n^{(k-1)} \frac{t^n}{n!}. \end{aligned} \quad (33)$$

On the other hand, the right-hand side of (21) is

$$\frac{\partial}{\partial t} \left(\sum_{n=0}^{\infty} (x)_n^{(k)} \frac{t^n}{n!} \right) = \sum_{n=1}^{\infty} (x)_n^{(k)} \frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} (x)_{n+1}^{(k)} \frac{t^n}{n!}. \quad (34)$$

By (33) and (34), we get

$$\begin{aligned} \sum_{n=1}^{\infty} (x)_n^{(k-1)} \frac{t^n}{n!} &= (1+t) \log(1+t) \sum_{m=0}^{\infty} (x)_{m+1}^{(k)} \frac{t^m}{m!} \\ &= \left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} t^j \right) \left(\sum_{m=0}^{\infty} (x)_{m+1}^{(k)} \frac{t^m}{m!} \right) + \left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} t^j \right) \left(\sum_{m=0}^{\infty} (x)_{m+1}^{(k)} \frac{t^{m+1}}{m!} \right) \\ &= \left(\sum_{j=1}^{\infty} (-1)^{j+1} (j-1)! \frac{t^j}{j!} \right) \left(\sum_{m=0}^{\infty} (x)_{m+1}^{(k)} \frac{t^m}{m!} \right) + \left(\sum_{j=1}^{\infty} (-1)^{j+1} (j-1)! \frac{t^j}{j!} \right) \left(\sum_{m=1}^{\infty} (x)_m^{(k)} m \frac{t^m}{m!} \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{j=1}^n \binom{n}{j} (-1)^{j+1} (j-1)! (x)_{n-j+1}^{(k)} \right) \frac{t^n}{n!} + \sum_{n=2}^{\infty} \left(\sum_{j=1}^n \binom{n}{j} (-1)^{j+1} (j-1)! (n-j) (x)_{n-j}^{(k)} \right) \frac{t^n}{n!} \\ &= -(x)_1^{(k)} + \sum_{n=2}^{\infty} \left(\sum_{j=1}^n \binom{n}{j} (-1)^{j+1} (j-1)! ((x)_{n-j+1}^{(k)} + (n-j)(x)_{n-j}^{(k)}) \right) \frac{t^n}{n!}. \end{aligned} \quad (35)$$

By comparing the coefficients of both sides of (35), we get the desired result. \square

Theorem 6. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$(1)_n^{(k)} = \sum_{m=0}^n \binom{n}{m} B_m^{(k)},$$

where $B_n^{(k)}$ are the poly-Bernoulli numbers.

Proof. From (15), we get

$$\text{Ei}_k(\log(1+t)) = (e^t - 1) \sum_{m=0}^{\infty} B_m^{(k)} \frac{t^m}{m!} = \sum_{j=1}^{\infty} \frac{t^j}{j!} \sum_{m=0}^{\infty} B_m^{(k)} \frac{t^m}{m!} = \sum_{n=1}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} B_m^{(k)} \right) \frac{t^n}{n!}. \quad (36)$$

On the other hand, when $x = 1$ in (21), we observe that

$$1 + \text{Ei}_k(\log(1+t)) = \sum_{n=0}^{\infty} (1)_n^{(k)} \frac{t^n}{n!} = 1 + \sum_{n=1}^{\infty} (1)_n^{(k)} \frac{t^n}{n!}. \quad (37)$$

By (36) and (37), for $n \geq 1$, we attain the desired result. \square

Corollary 7. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$(1)_n^{(k)} = B_n^{(k)}(1) - B_n^{(k)}.$$

Proof. From (15), we observe that

$$\sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} = \frac{2\text{Ei}_k(\log(1+t))}{e^t + 1} e^{xt} = \sum_{j=0}^{\infty} B_j^{(k)} \frac{t^j}{j!} \sum_{m=0}^{\infty} x^m \frac{t^m}{m!} = \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} B_j^{(k)} x^{n-j} \frac{t^n}{n!}. \quad (38)$$

By comparing the coefficients of both sides of (38), we get

$$B_n^{(k)}(x) = \sum_{j=0}^n \binom{n}{j} x^{n-j} B_j^{(k)}. \quad (39)$$

From (15) and (39), we have

$$\text{Ei}_k(\log(1+t)) = (e^t - 1) \left(\sum_{d=0}^{\infty} B_d^{(k)} \frac{t^d}{d!} \right) = \sum_{n=0}^{\infty} \left(\sum_{d=0}^n \binom{n}{d} B_d^{(k)} - B_n^{(k)} \right) \frac{t^n}{n!} = \sum_{n=1}^{\infty} (B_n^{(k)}(1) - B_n^{(k)}) \frac{t^n}{n!}. \quad (40)$$

Combining (37) with (40), we have the desired result. \square

Theorem 8. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$(1)_n^{(k)} = \frac{1}{2} \left(\sum_{d=1}^{n-1} \binom{n}{d} G_d^{(k)} + 2G_n^{(k)} \right),$$

where $G_d^{(k)}$ are the poly-Genocchi numbers.

Proof. By (16) and $G_0^{(k)} = 0$, we observe that

$$\begin{aligned} 2\text{Ei}_k(\log(1+t)) &= (e^t + 1) \left(\sum_{d=0}^{\infty} G_d^{(k)} \frac{t^d}{d!} \right) \\ &= \left(\sum_{j=0}^{\infty} \frac{t^j}{j!} \right) \left(\sum_{d=0}^{\infty} G_d^{(k)} \frac{t^d}{d!} \right) + \sum_{d=0}^{\infty} G_d^{(k)} \frac{t^d}{d!} \\ &= \sum_{n=1}^{\infty} \sum_{d=0}^n \binom{n}{d} G_d^{(k)} \frac{t^n}{n!} + \sum_{n=1}^{\infty} G_n^{(k)} \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} \left(\sum_{d=1}^{n-1} \binom{n}{d} G_d^{(k)} + 2G_n^{(k)} \right) \frac{t^n}{n!}. \end{aligned} \quad (41)$$

From (37) and (41), we get what we want. \square

3 Poly-rising factorial sequences

In this section, we define poly-rising factorial sequences by using the polyexponential functions and give relations between poly-falling factorial sequences and poly-rising factorial sequences, and special

numbers. We also study recurrence formula of these sequences and a reciprocal formula of the poly-falling factorial sequences.

Now, we consider the poly-rising factorial sequences $\langle x \rangle_n^{(k)}$, which are arising from the polyexponential functions to be

$$1 + \text{Ei}_k(-x \log(1-t)) = \sum_{n=0}^{\infty} \langle x \rangle_n^{(k)} \frac{t^n}{n!} \quad \text{and} \quad \langle x \rangle_0^{(k)} = 1 \quad (x \neq 0, |t| < 1). \quad (42)$$

When $k = 1$, since $\text{Ei}_1(x) = e^x - 1$, we note that

$$1 + \text{Ei}_1(-x \log(1-t)) = 1 + \sum_{n=1}^{\infty} \frac{(-x \log(1+t))^n}{n!} = e^{-x \log(1-t)} = (1-t)^{-x} = \sum_{n=0}^{\infty} \langle x \rangle_n \frac{t^n}{n!}. \quad (43)$$

Therefore, we have $\langle x \rangle_n^{(1)} = \langle x \rangle_n$.

Theorem 9. For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$

$$\langle x \rangle_n^{(k)} = (-1)^n (-x)_n^{(k)} \quad \text{and} \quad (x)_n^{(k)} = (-1)^n \langle -x \rangle_n^{(k)}.$$

Proof. From (21) and (42), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \langle x \rangle_n^{(k)} \frac{t^n}{n!} &= 1 + \text{Ei}_k(-x \log(1-t)) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-x \log(1+(-t)))^n}{(n-1)! n^k} \\ &= \sum_{n=0}^{\infty} (-x)_n^{(k)} \frac{(-1)^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n (-x)_n^{(k)} \frac{t^n}{n!} \end{aligned} \quad (44)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} (x)_n^{(k)} \frac{t^n}{n!} &= 1 + \text{Ei}_k(x \log(1+t)) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-(-x) \log(1-(-t)))^n}{(n-1)! n^k} \\ &= \sum_{n=0}^{\infty} \langle -x \rangle_n^{(k)} \frac{(-1)^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \langle -x \rangle_n^{(k)} \frac{t^n}{n!}. \end{aligned} \quad (45)$$

By comparing the coefficients of both sides of (44) and (45), respectively, we have the desired result. \square

When $k = 1$, we note that $\langle x \rangle_n = (-1)^n (-x)_n$.

From Theorems 6, 7, 8, and 9, we observe the following identities, respectively.

Corollary 10. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

(1)

$$\langle -1 \rangle_n^{(k)} = \sum_{m=0}^n (-1)^m \binom{n}{m} B_m^{(k)},$$

(2)

$$\langle -1 \rangle_n^{(k)} = (-1)^n (B_n^{(k)}(1) - B_n^{(k)}),$$

(3)

$$\langle -1 \rangle_n^{(k)} = \frac{1}{2}(-1)^n \left(\sum_{d=1}^{n-1} \binom{n}{d} G_d^{(k)} + 2G_n^{(k)} \right),$$

where $B_n^{(k)}$ are poly-Bernoulli numbers and $G_n^{(k)}$ are poly-Genocchi numbers.

Next two theorems are relations of poly-rising factorial sequences and powers of x .

Theorem 11. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$\langle x \rangle_n^{(k)} = \sum_{d=1}^n \frac{(-1)^{n+d}}{d^{k-1}} S_1(n, d) x^d.$$

Proof. First, from (5) and (14), we observe that

$$\begin{aligned} 1 + \text{Ei}_k(-x \log(1-t)) &= 1 + \sum_{d=1}^{\infty} \frac{(-1)^d x^d (\log(1-t))^d}{(d-1)! d^k} \\ &= 1 + \sum_{d=1}^{\infty} \frac{(-1)^d x^d}{d^{k-1}} \frac{1}{d!} (\log(1-t))^d \\ &= 1 + \sum_{d=1}^{\infty} \frac{(-1)^d x^d}{d^{k-1}} \sum_{n=d}^{\infty} S_1(n, d) \frac{(-1)^n t^n}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \left(\sum_{d=1}^n \frac{(-1)^{n+d}}{d^{k-1}} S_1(n, d) x^d \right) \frac{t^n}{n!}. \end{aligned} \quad (46)$$

Combining (42) with (46), we get what we want. \square

Theorem 12. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$n^{k-1} \sum_{d=0}^n (-1)^{n+d} S_2(n, d) \langle x \rangle_d^{(k)} = x^n.$$

Proof. Substituting t by $1 - e^{-t}$ in (42), the left-hand side of (42) is

$$1 + \text{Ei}_k(xt) = 1 + \sum_{n=1}^{\infty} \frac{(xt)^n}{(n-1)! n^k} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n^{k-1}} \frac{t^n}{n!}. \quad (47)$$

On the other hand, from (7), the right-hand side (42) is

$$\sum_{d=0}^{\infty} \langle x \rangle_d^{(k)} \frac{(1 - e^{-t})^d}{d!} = \sum_{d=0}^{\infty} (-1)^d \langle x \rangle_d^{(k)} \sum_{n=d}^{\infty} S_2(n, d) \frac{(-1)^n t^n}{n!} = 1 + \sum_{n=1}^{\infty} \left(\sum_{d=0}^n (-1)^{n+d} S_2(n, d) \langle x \rangle_d^{(k)} \right) \frac{t^n}{n!}. \quad (48)$$

From (47) and (48), we attain the desired result. \square

To prove the next theorem, we observe that

$$\left(\frac{1}{1-t} \right)^x = (1-t)^{-x} = e^{-x \log(1-t)} = \sum_{d=0}^{\infty} (-1)^d x^d \frac{(\log(1-t))^d}{d!} = \sum_{n=0}^{\infty} \left(\sum_{d=0}^n (-1)^{n+d} S_1(n, d) x^d \right) \frac{t^n}{n!}. \quad (49)$$

From (10) and (49), we obtain

$$\sum_{j=0}^n L(n, j)(x)_j = \sum_{d=0}^n (-1)^{n+d} S_1(n, d) x^d. \quad (50)$$

From (6) and (50), we note that

$$\begin{aligned} \sum_{j=0}^n L(n, j)(x)_j &= \sum_{d=0}^n (-1)^{n+d} S_1(n, d) x^d \\ &= \sum_{d=0}^n (-1)^{n+d} S_1(n, d) \sum_{j=0}^d S_2(d, j)(x)_j \\ &= \sum_{j=0}^n \left(\sum_{d=j}^n (-1)^{n+d} S_1(n, d) S_2(d, j) \right) (x)_j. \end{aligned} \quad (51)$$

By comparing the coefficients of both sides of (51), we obtain

$$L(n, j) = \sum_{d=j}^n (-1)^{n+d} S_1(n, d) S_2(d, j). \quad (52)$$

Theorem 13. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$(x)_n^{(k)} = \sum_{v=0}^n (-1)^{v-n} L(n, v) \langle x \rangle_v^{(k)} \quad \text{and} \quad \langle x \rangle_n^{(k)} = \sum_{v=0}^n L(n, v) (x)_v^{(k)}.$$

Proof. By (52), Theorems 1 and 12, we obtain

$$\begin{aligned} (x)_n^{(k)} &= \sum_{d=1}^n \frac{1}{d^{k-1}} S_1(n, d) x^d \\ &= \sum_{d=1}^n \frac{1}{d^{k-1}} S_1(n, d) \sum_{v=0}^d d^{k-1} (-1)^{v+d} S_2(d, v) \langle x \rangle_v^{(k)} \\ &= \sum_{v=1}^n \sum_{d=1}^n (-1)^{v-n} (-1)^{n+d} S_1(n, d) S_2(d, v) \langle x \rangle_v^{(k)} \\ &= \sum_{v=1}^n (-1)^{v-n} L(n, v) \langle x \rangle_v^{(k)}. \end{aligned} \quad (53)$$

By $n \geq 1$, $L(n, 0) = 0$, and (53), we obtain

$$(x)_n^{(k)} = \sum_{v=0}^n (-1)^{v-n} L(n, v) \langle x \rangle_v^{(k)}. \quad (54)$$

Thus, substituting x by $-x$ in (54) and by using Theorem 9, we arrive at the result. \square

In Theorem 13, when $k = 1$, we note that

$$(x)_n = \sum_{v=0}^n (-1)^{v-n} L(n, v) \langle x \rangle_v \quad \text{and} \quad \langle x \rangle_n = \sum_{v=0}^n L(n, v) (x)_v.$$

Theorem 14. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$\langle x \rangle_1^{(k-1)} = -\langle x \rangle_1^{(k)}$$

and

$$\langle x \rangle_n^{(k-1)} = \sum_{j=1}^n \binom{n}{j} (j-1)! (\langle x \rangle_{n-j+1}^{(k)} - (n-j) \langle x \rangle_{n-j}^{(k)}) \quad (n \geq 2).$$

Proof. Differentiating with respect to t in (42), the left-hand side of (42) is

$$\begin{aligned} \frac{\partial}{\partial t} (1 + \text{Ei}_k(-x \log(1-t))) &= \sum_{n=1}^{\infty} \frac{(-x)^n (\log(1-t))^{n-1}}{(n-1)! n^{k-1}} \frac{-1}{1-t} \\ &= \frac{1}{(t-1) \log(1-t)} \sum_{n=1}^{\infty} \frac{(-x \log(1-t))^n}{(n-1)! n^{k-1}} \\ &= \frac{1}{(t-1) \log(1-t)} \text{Ei}_{k-1}(-x \log(1-t)) \\ &= \frac{1}{(t-1) \log(1-t)} \sum_{n=1}^{\infty} \langle x \rangle_n^{(k-1)} \frac{t^n}{n!}. \end{aligned} \quad (55)$$

On the other hand, the right-hand side of (42) is

$$\frac{\partial}{\partial t} \left(\sum_{n=0}^{\infty} \langle x \rangle_n^{(k)} \frac{t^n}{n!} \right) = \sum_{n=1}^{\infty} \langle x \rangle_n^{(k)} \frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \langle x \rangle_{n+1}^{(k)} \frac{t^n}{n!}. \quad (56)$$

By (56) and (57), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \langle x \rangle_n^{(k-1)} \frac{t^n}{n!} &= (t-1) \log(1-t) \sum_{m=0}^{\infty} \langle x \rangle_{m+1}^{(k)} \frac{t^m}{m!} \\ &= - \left(\sum_{j=1}^{\infty} \frac{1}{j} t^j \right) \left(\sum_{m=0}^{\infty} \langle x \rangle_{m+1}^{(k)} \frac{t^m}{m!} \right) + \left(\sum_{j=1}^{\infty} \frac{1}{j} t^j \right) \left(\sum_{m=0}^{\infty} \langle x \rangle_{m+1}^{(k)} \frac{t^{m+1}}{m!} \right) \\ &= - \left(\sum_{j=1}^{\infty} \frac{j!}{j} \frac{t^j}{j!} \right) \left(\sum_{m=0}^{\infty} \langle x \rangle_{m+1}^{(k)} \frac{t^m}{m!} \right) + \left(\sum_{j=1}^{\infty} \frac{j!}{j} \frac{t^j}{j!} \right) \left(\sum_{m=1}^{\infty} \langle x \rangle_m^{(k)} \frac{t^m}{(m-1)!} \right) \\ &= - \sum_{n=1}^{\infty} \left(\sum_{j=1}^n \binom{n}{j} (j-1)! \langle x \rangle_{n-j+1}^{(k)} \right) \frac{t^n}{n!} + \sum_{n=2}^{\infty} \left(\sum_{j=1}^n \binom{n}{j} (j-1)! (n-j) \langle x \rangle_{n-j}^{(k)} \right) \frac{t^n}{n!} \\ &= - \langle x \rangle_1^{(k)} + \sum_{n=2}^{\infty} \left(\sum_{j=1}^n \binom{n}{j} (j-1)! (\langle x \rangle_{n-j+1}^{(k)} - (n-j) \langle x \rangle_{n-j}^{(k)}) \right) \frac{t^n}{n!}. \end{aligned} \quad (57)$$

By comparing the coefficients of both sides of (58), we get what we want. \square

Now, we consider the reciprocal of the poly-falling factorial sequences given by

$$\frac{1}{1 + \text{Ei}_k(x \log(1+t))} = \sum_{n=0}^{\infty} R_n^{(k)}(x) \frac{t^n}{n!}. \quad (58)$$

In (59), when $k = 1$, by (1), (2), (11), and (12), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} R_n^{(1)}(x) \frac{t^n}{n!} &= \frac{1}{1 + \text{Ei}_1(x \log(1+t))} = \frac{1}{e^{x \log(1+t)}} = (1+t)^{-x} \\ &= \sum_{n=0}^{\infty} (-1)^n \langle x \rangle_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \left(\sum_{j=0}^n L(n, j) (x)_j \right) \frac{t^n}{n!}. \end{aligned} \quad (59)$$

From (59), we obtain

$$R_n^{(1)}(x) = (-1)^n \langle x \rangle_n = (-x)_n = (-1)^n \left(\sum_{j=0}^n L(n, j) (x)_j \right). \quad (60)$$

To give an explicit reciprocal formula of power series of the poly-falling factorial sequence $(x)_n^{(k)}$, when $k = 1$, we get the following theorem.

Theorem 15. Assume that

$$\sum_{n=0}^{\infty} (x)_n^{(1)} \frac{t^n}{n!} \sum_{m=0}^{\infty} R_m^{(1)}(x) \frac{t^m}{m!} = 1 \quad \text{with} \quad (x)_0^{(k)} = 1.$$

Then we have

$$R_n^{(1)}(x) = (-1)^n \langle x \rangle_n = (-x)_n = \sum_{j=0}^n (-1)^j L(n, j)(x).$$

The following theorem can be obtained simply by using Theorem 5 of [23].

Theorem 16. Assume that

$$\sum_{n=0}^{\infty} (x)_n^{(k)} \frac{t^n}{n!} \sum_{m=0}^{\infty} R_m^{(k)}(x) \frac{t^m}{m!} = 1 \quad \text{with} \quad (x)_0^{(k)} = 1.$$

Then we have

$$\begin{aligned} R_0^{(k)}(x) &= 1 \quad \text{and} \\ R_n^{(k)}(x) &= \sum_{d=1}^n B_{n,d}((x)_1^{(k)}, (x)_2^{(k)}, \dots, (x)_{n-d+1}^{(k)}) (-1)^d d! \\ &= \sum_{d=1}^n \sum_{h_1+2h_2+\dots+nh_n=n} \frac{n!}{h_1!h_2! \cdots h_{n-d+1}!} \left(\frac{x}{1!}\right)^{h_1} \left(\frac{\sum_{j=1}^2 \frac{1}{j^{k-1}} S_1(2, j)x^j}{2!}\right)^{h_2} \cdots \left(\frac{\sum_{j=1}^{n-d+1} \frac{1}{j^{k-1}} S_1(n-d+1, j)x^j}{(n-d+1)!}\right)^{h_{n-d+1}}. \end{aligned}$$

Proof. We observe that

$$\sum_{d=0}^{\infty} (x)_d^{(k)} \frac{t^d}{d!} \sum_{m=0}^{\infty} R_m^{(k)}(x) \frac{t^m}{m!} = \sum_{n=0}^{\infty} \sum_{d=0}^n \binom{n}{d} (x)_d^{(k)} R_{n-d}^{(k)}(x) \frac{t^n}{n!} = 1. \quad (61)$$

From (61), we have

$$R_0^{(k)}(x) = 1 \quad \text{and} \quad \sum_{d=0}^n \binom{n}{d} (x)_d^{(k)} R_{n-d}^{(k)}(x) = 0, \quad (n \geq 1). \quad (62)$$

From (62), we obtain

$$R_n^{(k)}(x) = - \sum_{d=1}^n \binom{n}{d} (x)_d^{(k)} R_{n-d}^{(k)}(x). \quad (63)$$

When $n = 1$, we have

$$R_1^{(k)}(x) = -(x)_1^{(k)} R_0^{(k)}(x) = -(x)_1^{(k)}. \quad (64)$$

When $n = 2$, we obtain

$$R_2^{(k)}(x) = - \binom{2}{1} (x)_2^{(k)} R_1^{(k)}(x) - (x)_2^{(k)} = 2((x)_1^{(k)})^2 - (x)_2^{(k)}. \quad (65)$$

On the other hand, from (19), we observe that

$$B_{2,1}(x_1, x_2) = \sum_{\substack{h_1+h_2=1 \\ h_1+2h_2=2}} \frac{2!}{h_1!h_2!} \left(\frac{x_1}{1!}\right)^{h_1} \left(\frac{x_2}{2!}\right)^{h_2} = \frac{2!}{0!1!} \left(\frac{x_1}{1!}\right)^0 \left(\frac{x_2}{2!}\right)^1 = x_2$$

and (66)

$$B_{2,2}(x_1) = \sum_{h_1=2} \frac{2!}{2!} \left(\frac{x_1}{1!}\right)^2 = x_1^2.$$

By (66), we have

$$\sum_{d=1}^2 B_{2,d}(x_1, x_2, \dots, x_{2-k+1}) (-1)^d d! = -B_{2,1}(x_1, x_2) + 2!B_{2,2}(x_1) = -x_2 + 2(x_1)^2. \quad (67)$$

From (65) and (67), we obtain

$$R_2^{(k)}(x) = -(x)_2^{(k)} + 2((x)_1^{(k)})^2 = \sum_{d=1}^n B_{2,d}((x)_1^{(k)}, (x)_2^{(k)}) (-1)^d d!. \quad (68)$$

When $n = 3$,

$$\begin{aligned} R_3^{(k)}(x) &= -\binom{3}{1} (x)_1^{(k)} R_2^{(k)}(x) - \binom{3}{2} (x)_2^{(k)} R_1^{(k)}(x) - (x)_3^{(k)} \\ &= -3(x)_1^{(k)} (2((x)_1^{(k)})^2 - (x)_2^{(k)}) - 3(x)_2^{(k)} (-x)_1^{(k)} - (x)_3^{(k)} \\ &= -6((x)_1^{(k)})^3 + 6(x)_2^{(k)} (x)_1^{(k)} - (x)_3^{(k)}. \end{aligned} \quad (69)$$

On the other hand, from (19), we observe that

$$\begin{aligned} B_{3,1}(x_1, x_2, x_3) &= \sum_{\substack{h_1+h_2+h_3=1 \\ h_1+2h_2+3h_3=3}} \frac{3!}{h_1!h_2!h_3!} \left(\frac{x_1}{1!}\right)^{h_1} \left(\frac{x_2}{2!}\right)^{h_2} \left(\frac{x_3}{3!}\right)^{h_3} = \frac{3!}{0!0!1!} \frac{x_3}{3!} = x_3, \\ B_{3,2}(x_1, x_2) &= 3x_1x_2, \quad \text{and} \quad B_{3,3}(x_1) = x_1^3. \end{aligned} \quad (70)$$

By (70), we have

$$\sum_{d=1}^3 B_{3,d}(x_1, x_2, \dots, x_{3-k+1}) (-1)^d d! = -B_{3,1}(x_1, x_2, x_3) + 2!B_{3,2}(x_1, x_2) - 3!B_{3,3}(x_1) = -x_3 + 6x_1x_2 - 3!(x_1)^3. \quad (71)$$

From (69) and (71), we obtain

$$R_3^{(k)}(x) = -6((x)_1^{(k)})^3 + 6(x)_2^{(k)} (x)_1^{(k)} - (x)_3^{(k)} = \sum_{d=1}^n B_{3,d}((x)_1^{(k)}, (x)_2^{(k)}, (x)_3^{(k)}) (-1)^d d!. \quad (72)$$

Continuing this process, we have

$$R_n^{(k)}(x) = \sum_{d=1}^n B_{n,d}((x)_1^{(k)}, (x)_2^{(k)}, \dots, (x)_{n-d+1}^{(k)}) (-1)^d d!. \quad (73)$$

By comparing (20), (73), and Theorem 1, we obtain the desired result. □

4 Conclusion

To conclude, note that in Section 2 we introduced poly-falling factorial sequences in terms of the poly-exponential functions, reducing them to rising factorial if $k = 1$. We demonstrated the following: the n th

poly-falling factorial sequences of x were expressed in terms of the Stirling numbers of the first kind and the power of x ; the n th power of x were expressed in terms of the Stirling numbers of the second kind and poly-falling factorial sequences of x ; the poly-Bell polynomials were represented in terms of the poly-falling factorial sequences; the poly-falling poly-falling factorial sequences were represented in terms of the poly-Bell polynomials; recurrence formula; and the poly-falling factorial sequences were expressed in terms of the poly-Bernoulli numbers and poly-Genocchi numbers, respectively. In addition, in Section 3 we introduced poly-rising factorial sequences in terms of the polyexponential functions, reducing them to rising factorial if $k = 1$. Thus, these relations between poly-falling factorial sequences and poly-rising factorial sequences were expressed as: the n th poly-rising factorial sequences of x were expressed in terms of the Stirling numbers of the first kind and the power of x ; the n th power of x were expressed in terms of the Stirling numbers of the second kind and poly-rising factorial sequences; the poly-falling factorial sequences are represented in terms of Lah numbers and the poly-rising factorial sequences; the poly-rising factorial sequences were represented in terms of Lah numbers and the poly-falling factorial sequences; a recurrence formula; and a reciprocal formula of the poly-falling factorial sequences.

In conclusion, in [8], one of the generalizations of Lah numbers, multi-Lah number was studied using the polylogarithms. From a similar point of view, we may generalize the results of the poly-falling factorial sequences and the poly-rising factorial sequences, respectively. There are various methods for studying special polynomials and numbers, including generating functions, combinatorial methods, umbral calculus, differential equations, and probability theory. The next academic project for our continuing research would be to examine the application of “poly” versions of certain special polynomials and numbers in the domains of physics, science, and engineering as well as mathematics of course.

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