



Research Article

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Almost factorizable weakly type B semigroups

<https://doi.org/10.1515/math-2021-0127>

received September 27, 2021; accepted November 22, 2021

Abstract: The definition of an almost left factorizable weakly type B semigroup is introduced by using the notion of a left factorizable weakly type B monoids, and some properties and characterizations on such semigroup are obtained. The main aim of this paper is to give characterizations of sufficient and necessary conditions for a weakly type B semigroup to be almost left factorizable. In particular, some properties of proper almost factorizable weakly type B semigroups are given.

Keywords: almost factorizable, permissible set, semidirect product, weakly type B semigroup

MSC 2020: 20M10, 06F05

1 Introduction

The relations $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$ are generalizations of Green's $*$ -relations \mathcal{L}^* and \mathcal{R}^* . The elements a and b of a semigroup S are $\tilde{\mathcal{R}}$ -related if and only if a and b have the same set of idempotent left identities (see [1]). The relation $\tilde{\mathcal{L}}$ is defined dually. The relation \mathcal{L}^* (resp., \mathcal{R}^*) is defined by the rule that $a\mathcal{L}^*b$ (resp., $a\mathcal{R}^*b$) for the elements a, b of a semigroup S if and only if a, b are related by Green's relation \mathcal{L} (resp., \mathcal{R}) in some oversemigroup of S . A semigroup S is *abundant* if each \mathcal{L}^* -class and each \mathcal{R}^* -class of S contains an idempotent. Fountain and others studied some classes of abundant semigroups and obtained many interesting results (see [2–15]). Following Gomes and Gould [1], a semigroup S is called a *left (resp., right) semi-abundant semigroup* if each $\tilde{\mathcal{R}}$ -class (resp., each $\tilde{\mathcal{L}}$ -class) of S contains an idempotent. A left (resp., right) semi-abundant semigroup is said to be *left (resp., right) semi-adequate semigroup* if its idempotents commute. A semigroup is *semi-abundant (semi-adequate)* if it is both left and right semi-abundant (semi-adequate). As usual, we denote by a^+ (resp., a^*) an idempotent $\tilde{\mathcal{R}}$ - (resp., $\tilde{\mathcal{L}}$ -) related to a ; $E(S)$ denotes the set of idempotents of S . Generally, \mathcal{L}^* and \mathcal{R}^* are a right and a left congruences on a semigroup S , respectively. However, $\tilde{\mathcal{L}}$ is not a right congruence and $\tilde{\mathcal{R}}$ is not a left congruence. A left semi-adequate semigroup is said to be *weakly left type B* [16] if the relation $\tilde{\mathcal{R}}$ is a left congruence, and it satisfies the following conditions:

- (1) $(\forall e, f \in E(S^1), a \in S) (aef)^+ = (ae)^+(af)^+$;
- (2) $(\forall e \in E(S), a \in S) e \leq a^+ \Rightarrow (\exists f \in E(S^1)) e = (af)^+$.

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The definition of a *factorizable semigroup* is that the semigroup S can be written as the set product AB of proper subsemigroups A and B of S . We call the pair (A, B) a *factorization* of S . In 1994, Lawson investigated almost factorizable inverse semigroups (see [17]). Since then, many authors have been engaged in extending the concepts and results of almost factorizable inverse semigroups to the broader framework of semigroups. For example, the notions of factorizable and almost factorizable IC quasi-adequate semigroups are introduced by Qiu et al. in [13]. It is proved that any factorizable IC quasi-adequate monoid is a good homomorphic image of a semidirect product of a band by a cancellative monoid, and *vice versa*. A notion of almost factorizable is introduced within the class of weakly ample semigroups by Gracinda and Mária in [18], and it is shown that a weakly ample semigroup is almost factorizable if and only if it is an idempotent separating $(2, 1, 1)$ -homomorphic image of a semigroup $W(T, Y)$, where Y is a semilattice and T is a unipotent monoid. It is well known that weakly type B semigroups are generalizations of inverse semigroups in the range of semi-abundant semigroups. Therefore, it is a natural thing to characterize an almost factorizable weakly type B semigroup.

The aim of this article is to study a class of weakly type B semigroups, that is, so-called almost factorizable weakly type B semigroups. In this paper, the concept of permissible sets in inverse semigroups could be generalized to weakly type B semigroups, and the definition of almost left factorizable weakly type B semigroups is introduced. It is shown that an arbitrary almost left factorizable weakly type B semigroup is an idempotent separating $(2, 1, 1)$ -homomorphism images of a weakly type B semigroup $W(T, Y)$, where Y is a semilattice and T is a unipotent monoid acting on Y . In particular, we give a characterization of a proper almost factorizable type B semigroup.

We proceed as follows: Section 2 provides some basic notions and some known results. In Sections 3 and 4, we give a new characterization and the concept of permissible sets for weakly type B semigroup, respectively. We denote by $C(S)$ the set of all permissible subsets of a weakly type B semigroup. It is proved that $C(S)$ is a weakly type B monoid, which S is embeddable. In Section 5, we investigate the definitions of a left factorizable weakly type B monoid and an almost left factorizable weakly type B semigroup. Furthermore, we verify that a weakly type B semigroup S is almost left factorizable if and only if S is an idempotent separating $(2, 1, 1)$ -homomorphism image of a semigroup $W(T, Y)$, where Y is a semilattice and T is a unipotent monoid acting on Y such that conditions **(B1)**–**(B6)** (see Definitions 3.1 and 3.2) hold. In Section 6, we consider the left-right symmetric versions of factorizability and almost factorizability.

2 Preliminaries

Throughout this paper, we shall use the notions and notations of (4, 17–21). In the first part of this section, we recall an equivalent definition of a weakly type B semigroup.

Definition 2.1. Let S be a left semi-abundant semigroup. Then, $(S; \cdot, +)$ is a weakly left type B semigroup as an algebra of type $(2, 1)$, if for all $a, b \in S$ and $e, f \in E(S)$, the following conditions **(WB0)**–**(WB5)**⁺ hold.

$$\text{(WB0)} \quad ef = fe;$$

$$\text{(WB1)}^+ \quad e^+ = e;$$

$$\text{(WB2)}^+ \quad a^+a = a;$$

$$\text{(WB3)}^+ \quad (ab)^+ = (ab^+)^+;$$

$$\text{(WB4)}^+ \quad (\forall e, f \in E(S^1), a \in S) (aef)^+ = (ae)^+(af)^+;$$

$$\text{(WB5)}^+ \quad (\forall e \in E(S), a \in S) e \leq a^+ \Rightarrow (\exists f \in E(S^1)) e = (af)^+, \text{ where } \leq \text{ is a natural partial-order on } E(S), \text{ i.e., for all } e, f \in E(S), e \leq f \Leftrightarrow e = ef = fe.$$

Remark 2.1. In our Definition 2.1, by condition **(WB0)**, $E(S)$ is a semilattice and $|\widetilde{\mathcal{R}}_a \cap E(S)| = 1$ for all $a \in S$, where $\widetilde{\mathcal{R}}_a$ denotes the $\widetilde{\mathcal{R}}$ -class containing a . On the other hand, it is easy to see that $\widetilde{\mathcal{R}}$ is a left congruence. In fact, let $a\widetilde{\mathcal{R}}b$. Then, $a^+\widetilde{\mathcal{R}}b^+$, and so $a^+ = b^+a^+ = a^+b^+ = b^+$. Moreover, by condition **(WB2)**⁺, $(ce)^+ca =$

$(ce)^+cea = cea = ca$ and $(ce)^+cb = (ce)^+ceb = ceb = cb$. Let $f \in E(S)$ such that $fca = ca$. Then, $(fca)^+ = (ca)^+$, and so $(fca^+)^+ = (ca^+)^+$ from **(WB3)**⁺. Note that $a^+ = b^+$. We have $(fcb^+)^+ = (cb^+)^+$, and so $(fcb)^+ = (cb)^+$ from **(WB3)**⁺. Hence,

$$\begin{aligned} fcb &= (fcb)^+fcb && \text{(by (WB2) }^+) \\ &= (fcb)^+fc^+cb && \text{(by (WB2) }^+) \\ &= (fcb)^+(fc^+)^+cb && \text{(by (WB1) }^+) \\ &= (fcb^+)^+(fc)^+cb = (fcb^+)^+(fc \cdot 1)^+cb && \text{(by (WB1) }^+) \\ &= (fcb^+ \cdot 1)^+cb && \text{(by (WB4) }^+) \\ &= (fcb^+)^+cb = (fcb)^+cb = (cb)^+cb = cb. \end{aligned}$$

Similarly, we can prove that $fcb = cb$ implies that $fca = ca$ for all $f \in E(S)$. That is, $ca\widetilde{\mathcal{R}}cb$. Thus, $\widetilde{\mathcal{R}}$ is a left congruence on S . These together with conditions **(WB4)**⁺ and **(WB5)**⁺ imply that S is a weakly left type B semigroup.

A weakly right type B semigroup S is defined dually and the conditions **(WBO)**–**(WB5)**^{*} are as follows:

- (WBO)** $(\forall e, f \in E(S)) ef = fe$;
- (WB1)**^{*} $(\forall e \in E(S)) e^* = e$;
- (WB2)**^{*} $(\forall a \in S) aa^* = a$;
- (WB3)**^{*} $(\forall a, b \in S) (ab)^* = (a^*b)^*$;
- (WB4)**^{*} $(\forall e, f \in E(S^1), a \in S) (efa)^* = (ea)^*(fa)^*$;
- (WB5)**^{*} $(\forall e \in E(S), a \in S) e \leq a^* \Rightarrow (\exists f \in E(S^1)) e = (fa)^*$.

A semigroup is said to be *weakly type B* if it is both a weakly left type B semigroup and a weakly right type B semigroup. Equivalently, we define a weakly type B semigroup $(S; \cdot, +, *)$ as an algebra of type $(2, 1, 1)$ such that $(S; \cdot)$ is a semigroup, a^+ and a^* are idempotents for all $a \in S$ and conditions **(WBO)**–**(WB5)** hold, where **(WBi)** is the conjunction of **(WBi)**⁺ and **(WBi)**^{*} ($i = 1, 2, 3, 4, 5$). We denote the relationship $\widetilde{\mathcal{R}} \cap \widetilde{\mathcal{L}}$ by $\widetilde{\mathcal{H}}$ on a weakly type B semigroup.

Here, we give two examples of semi-abundant semigroups, which are weakly type B.

Example 2.1. Let $S = \{2^n \mid n = 0, 1, 2, \dots\}$. Then, S is a unipotent semigroup including identity element 1 with respect to the general multiplication. In fact, it is easy to check that $\widetilde{\mathcal{R}} = \widetilde{\mathcal{L}} = S \times S$, and for all $a \in S$, $a^+ = a^* = 1$. Therefore, S is a semi-abundant semigroup. On the other hand, it is routine to check that S satisfies conditions **(WBO)**–**(WB5)**. This means that S is a weakly type B semigroup.

Example 2.2. Let $S = \{[x]_{2 \times 2} \mid x \in \mathbb{N}\} \cup \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2 \times 2} \right\}$, where \mathbb{N} is the set of all non-negative integers and $[x]_{2 \times 2}$ denotes the following second-order matrix

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

for all $x \in \mathbb{N} \cup \left\{ \frac{1}{2} \right\}$. Then, it is easy to see that S is a semi-abundant semigroup with respect to the general matrix multiplication, and that

$$E(S) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right\}.$$

In fact, it is easily seen that $\widetilde{\mathcal{L}}$ -classes and $\widetilde{\mathcal{R}}$ -classes of S are both $S \setminus \{[0]_{2 \times 2}\}$ and $\{[0]_{2 \times 2}\}$. Obviously, $E(S)$ is a semilattice and $[a]_{2 \times 2}^* = [a]_{2 \times 2}^+ = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2 \times 2}$ for all $a \in \mathbb{N}$ and $a \neq 0$. Thus, it is routine to check that S satisfies conditions **(WBO)**–**(WB5)**. Therefore, S is a weakly type B semigroup.

Definition 2.2. Let S be a weakly left type B semigroup. Define a partial-order relation “ \leq_r ” on S as follows:

$$(\forall a, b \in S) a \leq_r b \Leftrightarrow (\exists e \in E(S)) a = eb.$$

Equal definition

$$(\forall a, b \in S) a \leq_r b \Leftrightarrow a = a^+b$$

Dually, we can define a partial-order “ \leq_l ” on a weakly right type B semigroup.

In particular, we can define a partial-order “ \leq ” on a weakly type B semigroups by “ $\leq_r \cap \leq_l$.”

Lemma 2.1. [16] *Let S be a weakly left type B semigroup. Define the relation “ σ ” on S as follows:*

$$(\forall a, b \in S) a\sigma b \Leftrightarrow (\exists e \in E(S)) ae = be.$$

Then, the following statements are true:

- (1) σ is a congruence on S ;
- (2) If S is proper (i.e., $\widetilde{\mathcal{R}} \cap \sigma = I_S$, where I_S is the identity relation on S), then σ is the least unipotent congruence on S ;
- (3) $E(S) \subseteq 1\sigma$.

Definition 2.3. [22] Let S be a weakly type B semigroup. Define the relation σ on S as follows:

$$(a, b) \in \sigma \Leftrightarrow (\exists E \in E(S)) eae = ebe.$$

We call S proper if $\sigma \cap \widetilde{\mathcal{R}} = I_S$ and $\sigma \cap \widetilde{\mathcal{L}} = I_S$, where I_S is the identity relation on S .

Example 2.3. Let $S = \{2^n \mid n = 0, 1, 2, \dots\}$. Then, S is a unipotent semigroup with respect to the general multiplication. In fact, in our Example 2.1, we have proved that S is a weakly type B semigroup with $\widetilde{\mathcal{R}} = \widetilde{\mathcal{L}} = S \times S$ and $a^+ = a^* = 1$. On the other hand, it is easy to see that $\sigma = I_S$. Therefore, S is a proper weakly type B semigroup.

Following Howie [23], a semigroup S is *left E-unitary*, if for all $a \in S$, $e \in E(S)$, $ae \in E(S)$ imply $a \in E(S)$.

Lemma 2.2. [16] *Let S be a proper weakly type B semigroup. Then S is left E-unitary such that $E(S) = 1\sigma$.*

3 Definitions and properties

In this section, we consider a new characterization of a weakly type B semigroup by using the notion of a semidirect product.

Definition 3.1. Let T be a unipotent monoid and Y be a semilattice. Suppose that T acts on Y by endomorphisms, that is, the following statements are true:

- (B1) $(\forall a, b \in Y, t \in T) (ab)^t = a^t b^t$;
- (B2) $(\forall a \in Y, t, u \in T) (a^t)^u = (a)^{tu}$;
- (B3) $(\forall a \in Y) a^1 = a$;
- (B4) If for all $a, b \in Y$, $b = ab$. Then, for all $t \in T$, there is $c \in Y$ such that $b = c^t a$.

Then, the *semidirect product* of Y by T is a subset of the Cartesian product $T \times Y$, and we denote it by $T * Y$, where the multiplication on the set $T * Y$ is defined by the rule:

$$(\forall (t, a), (u, b) \in T * Y) (t, a)(u, b) = (tu, a^u b).$$

It is easy to check that $T * Y$ is a semigroup, where the idempotent set $E(T * Y) = \{(1, a) \mid a \in Y\}$ is isomorphic to Y .

Lemma 3.1. Let T be a unipotent monoid, Y be a semilattice and $T * Y$ be a semidirect product. Define a unary operation “ $*$ ” on $T * Y$ as follows:

$$(\forall (t, a) \in T * Y) (t, a)^* = (1, a).$$

Then, $T * Y$ is a weakly right type B semigroup.

Proof. It is clear that conditions $(\mathbf{WB1})^*$ and $(\mathbf{WB2})^*$ hold. Let $(t, a), (u, b) \in T * Y$. Then, $[(t, a)(u, b)]^* = [(tu, a^u b)]^* = (1, a^u b) = (u, a^u b)^* = [(1, a)(u, b)]^* = [(t, a)^*(u, b)]^*$. Therefore, $(\mathbf{WB3})^*$ holds. Next, we verify that $T * Y$ satisfies condition $(\mathbf{WB4})^*$. To see it, suppose $(t, a) \in T * Y, (1, b), (1, c) \in E(T * Y)$. Then, we have that,

$$(1, b)(1, c)(t, a) = (1, bc)(t, a) = (t, b^t c^t a)$$

and

$$\begin{aligned} [(1, b)(1, c)(t, a)]^* &= (t, b^t c^t a)^* = (1, b^t c^t a), \\ [(1, b)(t, a)]^* &= (t, b^t a)^* = (1, b^t a), \\ [(1, c)(t, a)]^* &= (t, c^t a)^* = (1, c^t a). \end{aligned}$$

Hence,

$$[(1, b)(t, a)]^* [(1, c)(t, a)]^* = (1, b^t a)(1, c^t a) = (1, b^t a c^t a) = (1, b^t c^t a) = [(1, b)(1, c)(t, a)]^*,$$

which implies the condition $(\mathbf{WB4})^*$ holds. Now, we show that $T * Y$ satisfies condition $(\mathbf{WB5})^*$. To see it, let $(t, a) \in T * Y, (1, b) \in E(T * Y)$ and $(1, b) \leq (t, a)^*$, that is, $(1, b) \leq (1, a)$. Then, $(1, b) = (1, b)(1, a) = (1, a)(1, b)$. Hence, $(1, b) = (1, ba) = (1, ab)$, that is, $b = ba = ab$. By $(\mathbf{B4})$, we have that for all $t \in T$, there exists $c \in Y$ such that $b = c^t a$. Therefore, $(1, b) = (1, ba) = (1, c^t a) = (t, c^t a)^* = [(1, c)(t, a)]^*$, where $(1, c) \in E(T * Y)$. This means that condition $(\mathbf{WB5})^*$ is true. Therefore, $T * Y$ is a weakly right type B semigroup. \square

Suppose that T acts on Y , and it satisfies the following conditions:

(B5) $(\forall a, b \in Y, t \in T) a^t = b^t \Rightarrow a = b$;

(B6) If for all $a, b \in Y, t \in T, a \leq b^t$, and then there is $c \in Y$ such that $a = c^t$.

This means that T acts on Y by injective endomorphisms such that the images of T are *order ideas* of Y . (A non-empty subset Y of a partial-order set (X, \leq) is an *order ideal* of X if for all $x \in X, y \in Y, x \leq y$ implies $x \in Y$.)

Definition 3.2. Put $W(T, Y) = \{(t, a^t) \in T * Y \mid a \in Y, t \in T\}$, where Y is a semilattice and T is a unipotent monoid acting on Y and it satisfies conditions $(\mathbf{B1})$ – $(\mathbf{B6})$.

Obviously, $W(T, Y)$ is a subsemigroup of $T * Y$ and $E(W(T, Y)) \subseteq E(T * Y)$, that is, $W(T, Y)$ is a full subsemigroup of $T * Y$. Therefore, it is easy to see that $W(T, Y)$ is a $(2, 1)$ subsemigroup. Thus, $W(T, Y)$ is a weakly right type B semigroup. From $(\mathbf{B5})$, we can get that “ $+$ ” is well defined. Next, we will show that $W(T, Y) = (W(T, Y); \cdot, +, *)$ is a weakly type B semigroup, and we will introduce some properties of $W(T, Y)$.

Definition 3.3. For all $(t, a^t) \in W(T, Y)$, put $(t, a^t)^+ = (1, a)$, where $(1, a)$ is an idempotent of $W(T, Y)$.

Proposition 3.2. Let Y be a semilattice and T be a unipotent monoid acting on Y , and it satisfies conditions $(\mathbf{B1})$ – $(\mathbf{B6})$. Then, the following statements are true:

- (1) $W(T, Y)$ is a weakly type B semigroup, where $E(W(T, Y)) = \{(1, a) \mid a \in Y\}$ is isomorphic to Y ;
- (2) The first projection $\pi: W(T, Y) \rightarrow T, (t, a^t) \mapsto t$ is a surjective homomorphism, and $\ker \pi = \sigma$ such that $W(T, Y)/\sigma$ is isomorphic to T ;
- (3) $W(T, Y)$ is proper;
- (4) $W(T, Y)$ is a monoid if and only if Y has an identity.

Proof.

(1) It is easily seen that conditions **(WB0)**, **(WB1)⁺** and **(WB2)⁺** hold. Next, we prove that condition **(WB3)⁺** holds. To see it, let $(t, a^t), (u, b^u) \in W(T, Y)$. Then, $a^{tu}b^u \leq a^{tu}$. By conditions **(B5)** and **(B6)**, we have that $[(t, a^t)(u, b^u)]^+ = (tu, a^{tu}b^u)^+ = (tu, c^{tu})^+ = (1, c)$, where c is the only one element on Y such that $a^{tu}b^u = c^{tu}$. On the other hand, $[(t, a^t)(u, b^u)]^+ = [(t, a^t)(1, b)]^+ = (t, a^t b)^+ = (t, d^t)^+ = (1, d)$, where t is the only one element on Y such that $a^t b = d^t$. From conditions **(B1)** and **(B2)**, we obtain that $a^{tu}b^u = d^{tu}$, that is, $c^{tu} = d^{tu}$. Again, by condition **(B5)**, we have that $c = d$ holds. Therefore, condition **(WB3)⁺** is true. Now, we show that condition **(WB4)⁺** is satisfied. Let $(t, a^t) \in W(T, Y)$ and $(1, b), (1, d) \in E(W(T, Y))$. Then, there is $c \in Y$ such that $a^t b = c^t$ and $a^t b \leq a^t$. Hence,

$$[(t, a^t)(1, b)]^+ = (t, a^t b)^+ = (t, c^t)^+ = (1, c).$$

We also obtain that there exists $e \in Y$ such that $a^t d = e^t$ since $a^t d \leq a^t$. Thus,

$$[(t, a^t)(1, d)]^+ = (t, a^t d)^+ = (t, e^t)^+ = (1, e).$$

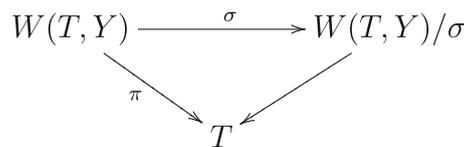
Similarly, since $a^t b d \leq a^t$, we have $a^t b d = f^t$ for some $f \in Y$. Therefore,

$$[(t, a^t)(1, b)(1, d)]^+ = [(t, a^t)(1, b d)]^+ = (t, a^t b d)^+ = (t, f^t)^+ = (1, f).$$

Note that Y is a semilattice and $a^t b = c^t, a^t d = e^t, a^t b d = f^t$. We have $a^t b d = f^t = c^t e^t = (ce)^t$. Again, by condition **(B5)**, we have that $f = ce$, that is, $(1, f) = (1, c)(1, e)$. In other words, $[(t, a)(1, b)(1, d)]^+ = [(t, a)(1, b)]^+ [(t, a)(1, d)]^+$, which implies that condition **(WB4)⁺** is true.

Finally, we explain that condition **(WB5)⁺** is also true. To see it, let $(t, a^t) \in W(T, Y), (1, b) \in E(W(T, Y))$ and $(1, b) \leq (t, a^t)^+$, that is, $(1, b) \leq (1, a)$. Then, $(1, b) = (1, b)(1, a) = (1, a)(1, b)$. This means that $b = ba = ab$. Therefore, $(1, b) = (1, ab) = (t, (ab)^t)^+ = [(t, a^t)(1, b^t)]^+$. But $(1, b^t) \in E(W(T, Y))$. Therefore, $W(T, Y)$ is a weakly type B semigroup.

(2) By the homomorphism theorem of $W(T, Y)$, we only show that $\sigma = \ker \pi$ holds. From the definition of π , it is clear that π is a surjective homomorphism. Hence, $\sigma \subseteq \ker \pi$ since σ is the least unipotent monoid congruence. Next, we prove the reverse inclusion relation. Let $(t, a^t)\pi = (u, b^u)\pi$. Then, $t = u$. Hence, there exists an idempotent element $(1, ab)$ such that $(1, ab)(t, a^t)(1, ab) = (1, ab)(t, b^t)(1, ab)$. Thus, $(t, a^t)\sigma(u, b^u)$, that is, the diagram commutes.



(3) Let $(t, a^t), (u, b^u) \in W(T, Y)$ such that $(t, a^t)[\widetilde{\mathcal{R}} \cap \sigma](u, b^u)$. Then, by (2), we have that $t = u$. On the other hand, $(t, a^t)^+ = (u, b^u)^+$, that is, $(1, a) = (1, b)$. Therefore, $(t, a^t) = (u, b^u)$. Similarly, let $(t, a^t)[\widetilde{\mathcal{L}} \cap \sigma](u, b^u)$. We obtain that $t = u$ and $(t, a^t)^* = (u, b^u)^*$. Therefore, $(1, a^t) = (1, b^u)$, that is, $(t, a^t) = (u, b^u)$.

(4) Straightforward. □

Remark 3.1. Let Y be a semilattice, T be a unipotent monoid. If T acts on Y by automorphisms satisfying conditions **(B1)**–**(B6)** and the following condition:

$$\mathbf{(B6)'} \quad (\forall t \in Y) \quad Y^t = Y,$$

then it is easy to check that $W(T, Y) = T * Y$.

4 Permissible sets

In this section, we first introduce the concept of permissible sets for a weakly type B semigroup as a compatible order ideal. The set of all permissible subsets of a weakly type B semigroup is denoted by $C(S)$. In particular, we prove that $C(S)$ is a weakly type B monoid in which S is embeddable.

Definition 4.1. Let S be a weakly type B semigroup. Then, a non-empty subset A of S is called permissible if the following conditions hold:

- (P1) A is an order ideal of S ;
- (P2) $(\forall a, b \in A) a^+b = b^+a$;
- (P3) $(\forall a, b \in A) ab^* = ba^*$.

Next, we will provide some properties about permissible sets.

Lemma 4.1. Let S be a weakly type B semigroup. Then, for every $A \in C(S)$, $a, b \in A$, the following statements are true:

- (1) If $a^+ = b^+$, then $a = b$;
- (2) If $a^* = b^*$, then $a = b$;
- (3) $a^*b^* = (a^+b)^*$;
- (4) $a^+b^+ = (ab^*)^+$.

Proof.

(1) By Definition 4.1 (P2), $a^+b = b^+a$, $a^+ = b^+$. Then, $a = a^+a = b^+a = a^+b = b^+b = b$. Dually, (2) holds.

(3) Obviously, we have that $a^+b = b^+a$ and $ab^* = ba^*$, from Definition 4.1 (P2) and (P3). Note that $E(S)$ is a semilattice and $\tilde{\mathcal{L}}$ is a right congruence with $|\tilde{\mathcal{L}}_a \cap E(S)| = 1$, where $\tilde{\mathcal{L}}_a$ is the $\tilde{\mathcal{L}}$ -class containing a . We have that $a^*b^* = b^*a^* = (ba^*)^* = (b^+ba)^* = (b^+ab^*)^* = (a^+bb^*)^* = (a^+b)^*$. Dually, (4) holds. \square

Proposition 4.2. Let S be a weakly type B semigroup. Then, $C(S)$ is a monoid with identity $E(S)$.

Proof. It is routine to check that $C(S)$ is closed with respect to the set product. This means that $C(S)$ is a semigroup. By Definition 4.1, it is clear that $E(S) \in C(S)$. Let $A \in C(S)$. Then, by (P1), we have $E(S)A \subseteq A$. By (WB2)⁺, we obtain $A \subseteq E(S)$. Hence, $A = E(S)A$, that is, $E(S)$ is a left identity element of $C(S)$. Dually, $E(S)$ is also a right identity of $C(S)$. This completes the proof. \square

Proposition 4.3. $E(C(S)) = \{E \subseteq E(S) \mid E \text{ is an order ideal of } E(S)\}$.

Proof. By Proposition 4.2, we only show that if $A \in C(S)$ and $A^2 = A$, then $A \subseteq E(S)$. To see it, let $A \in C(S)$, $A^2 = A$ and $a \in S$. Then, there exist $b, c \in A$ such that $a = bc$. By (P2), $a^+b = b^+a$. Hence, $(bc)^+b = b^+bc = bc$. On the other hand, we obtain that $(bc)^+b = (bc^+)^+b = b^+(bc^+) = b^+bc^+ = bc^+$ from conditions (WB3)⁺ and (P2). Therefore, $bc = bc^+$. Dually, we can prove that $bc = b^*c$. Thus, $a = bc = bb^*c^+c = bc^+b^*c = (bc)^2 = a^2$, which implies $a \in E(S)$, that is, $A \subseteq E(S)$. \square

Definition 4.2. Let $A \in C(S)$. Define A^+ and A^* as follows:

$$A^+ = \{a^+ \mid a \in A\}, \quad A^* = \{a^* \mid a \in A\}.$$

Obviously, $A^+, A^* \in E(C(S))$. In particular, it is routine to check that A^+ and A^* are order ideals of $E(C(S))$.

Theorem 4.4. The monoid $C(S) = (C(S); \cdot, +, *)$ is a weakly type B semigroup.

Proof. It suffices to show that conditions (WB0)–(WB5) in $C(S)$ hold. Let $A, B \in C(S)$ and $E, F \in E(C(S))$. It is easily seen that $EF = FE$ holds. Therefore, (WB0) is true. Obviously, $E^+ = E$ and $E^* = E$ hold. This means that (WB1) is also true. By (WB2)⁺ in S , we have $A \subseteq A^+A$. Furthermore, we obtain $A^+A \subseteq E(S)A = A$ from Proposition 4.3. Thus, $A^+A = A$. Similarly, $AA^* = A$. Therefore, condition (WB2) in $C(S)$ holds. By using condition (WB3)⁺ on S , it is easily seen that condition (WB3) in $C(S)$ is true. On the other hand, we have $(AEF)^+ \subseteq (AE)^+(AF)^+$ from condition (WB4)⁺ on S . Next, we will show the reverse inclusion. To see it, let $(ae)^+(bf)^+ \in (AE)^+(AF)^+$, where $a, b \in A$, $e \in E$, $f \in F$ and $b^+a \in A$. Then, we have $(ae)^+(bf)^+ = [(ae)^+bf]^+ = (b^+aef)^+ \in (AEF)^+$ from condition (P2), that is, $(AE)^+(AF)^+ \subseteq (AEF)^+$. Hence, $(AEF)^+ = (AE)^+(AF)^+$. Similarly,

$(AEF)^* = (AE)^*(AF)^*$. This means that $C(S)$ satisfies condition **(WB4)**. Finally, we explain condition **(WB5)** in $C(S)$ is also true. To see it, let $E \leq A^+$, that is, for all $e \in E$, $a^+ \in A^+$, we have $e \leq a^+$. Since S is a weakly type B semigroup, there exists $f \in E(C(S)) \subseteq E(S)$ such that $e = (af)^+$. Hence, there exists $F \in E(C(S))$ such that $E = (AF)^+$. Therefore, **(WB5)⁺** in $C(S)$ holds. Dually, condition **(WB5)^{*}** in $C(S)$ is also true. Summarizing the aforementioned arguments, $C(S)$ is a weakly type B semigroup. \square

Proposition 4.5. *Let $A, B \in C(S)$. Then, $A \leq B \Leftrightarrow A \subseteq B$.*

Proof.

(Necessity) If $A \leq B$ on $C(S)$, then $A = A^+B$. Since B is an order ideal of S , we have $A^+B \subseteq B$, and so $A \subseteq B$.

(Sufficiency) Let $A \subseteq B$. Then, $A = A^+A \subseteq A^+B$. On the other hand, for all $a \in A, b \in B$, we have $a, b \in B$ from $A \subseteq B$. Hence, $a^+b = b^+a \in A$, that is, $A^+B \subseteq A$, and so $A^+B = A$. Dually, $A = BA^+$. By Definition 2.2, $A \leq B$. This completes the proof. \square

Let $a \in S$. Put $(a] = \{b \in S | b \leq a\}$. Then, $(a]$ is the *principal order ideal* of S generated by a .

Theorem 4.6. *Let S be a weakly type B semigroup. Then, for all $a \in S$, there is $(a] \in C(S)$ such that the mapping $\tau_S: S \rightarrow C(S), a \mapsto (a]$ is a $(2, 1, 1)$ -embedding.*

Proof. It is easy to see that $(a]$ is an order ideal of S . That is, condition **(P1)** holds. Let $b, c \in (a]$. Then, there exist $e, f \in E(S)$ such that $b = ea$ and $c = fa$. Hence, $b^+c = (ea)^+c = ea^+fa = fa^+ea = (fa)^+ea = c^+b$. Thus, condition **(P2)** is true. Dually, condition **(P3)** is also true. Therefore, $(a] \in C(S)$ holds. Obviously, the mapping τ_S is injective. It is routine to check that τ_S is a $(2, 1, 1)$ homomorphism. This completes the proof. \square

Proposition 4.7. *Let S and T be two weakly left type B semigroups and $\theta: S \rightarrow T$ be a surjective $(2, 1)$ -homomorphism. Then,*

- (1) $E(S)\theta = E(T)$;
- (2) *If for all $s \in S, t \in T$, there is $t \leq s\theta$, then there exists $s' \in S$ such that $s' \leq_r s$ and $t = s'\theta$.*

Proof.

(1) We first prove $E(S)\theta \subseteq E(T)$ for all $e \in E(S)$. By **(WB1)⁺**, we have that $e\theta = e^+\theta = (e\theta)^+ \in E(T)$ since θ is a $(2, 1)$ homomorphism with respect to the unary operation “+”. That is, $E(S)\theta \subseteq E(T)$. On the other hand, let $f \in E(T)$. Then, there exists $e \in E(S)$ such that $f = e\theta \in E(S)\theta$ since θ is surjective. Therefore, $E(S)\theta = E(T)$.

(2) Let $s \in S, t \in T$ such that $t \leq s\theta$. Then, there exists $e \in E(T)$ with $t = e(s\theta)$. By (1), we have that for all $e \in E(T)$, there exists $f \in E(S)$ such that $e = f\theta$. That is, $t = f\theta(s\theta) = (fs)\theta$. Therefore, there exists $s' = fs \in S$. This means $s' \leq_r s$ with $t = s'\theta$ from Definition 2.2. \square

Theorem 4.8. *Let S, T be two weakly type B semigroups. Suppose that $\theta: S \rightarrow T$ is a $(2, 1, 1)$ -homomorphism such that its image is an order ideal in S . Then, the mapping $\hat{\theta}: C(S) \rightarrow C(T), A \mapsto A\theta = \{a\theta | a \in A\}$ is a $(2, 1, 1)$ -homomorphism satisfying $\tau_S\hat{\theta} = \theta\tau_T$. In particular, $\hat{\theta}$ is injective(bijective) if and only if θ is injective(bijective). Moreover, if $\hat{\theta}$ is surjective, then so is θ .*

Proof. First, we show that $(a]\theta = (a\theta)$ for all $a \in S$. To see it, let $t \in (a]\theta$. Then, there exists $b \in (a]$ such that $b \leq a$ and $b\theta = t$. Actually, θ is order preserving, that is, $b\theta \leq a\theta$. Thus, $t \leq a\theta$. Therefore, we have $(a]\theta \subseteq (a\theta)$. Next, we show that $(a\theta) \subseteq (a]\theta$. To see it, let $t \in T$ and $t \leq a\theta$. Since $a\theta$ is the image of θ and $a\theta$ is an order ideal in T , we have that t is the image of θ . Hence, there exists $a' \in (a]$ such that $t = a'\theta$ from Proposition 4.7. Therefore, $t \in (a]\theta$, that is, $(a\theta) \subseteq (a]\theta$, and so $(a\theta) = (a]\theta$. On the other hand, we have that $\tau_S\hat{\theta} = \theta\tau_T$ from the definitions of τ_S and τ_T . Moreover, for all $A \in C(S)$, $A\theta$ is an order ideal in T . It is clear that $A\theta$ satisfies conditions **(P2)** and **(P3)**, and so $A\theta \in C(T)$. This means that $\hat{\theta}$ is well defined. Now, we show that $\hat{\theta}$ is a $(2, 1)$ -homomorphism with respect to the unary operation “+”. In fact, since θ is a $(2, 1, 1)$ -homomorphism,

for all $A \in S$, we have that $A^+\theta = (A\theta)^+$. Hence, for all $A \in C(S)$, we obtain that $A \in S$ and $A^+\hat{\theta} = A^+\theta = (A\theta)^+ = (A\hat{\theta})^+$. Dually, $\hat{\theta}$ is a $(2, 1)$ -homomorphism with respect to the unary operation “ \cdot .” Therefore, $\hat{\theta}$ is a $(2, 1, 1)$ -homomorphism. Suppose that θ is injective. Then, $A\theta = B\theta$ imply $A = B$, and so $\hat{\theta}$ is injective. If θ is surjective, then $\theta^{-1}: T \rightarrow S$ is a $(2, 1, 1)$ -isomorphism since θ is a $(2, 1, 1)$ -isomorphism. Clearly, $\hat{\theta}^{-1}\hat{\theta}$ is the identity mapping of $C(T)$. Hence, $\hat{\theta}$ is also surjective. Conversely, if $\hat{\theta}$ is injective, we have that θ is injective since $\tau_S\hat{\theta} = \theta\tau_T$. Suppose $\hat{\theta}$ is surjective and let $t \in T$. Then, $\{t\} \in C(T)$. In fact, $\hat{\theta}$ is surjective, then there exists $A \in C(S)$ such that $\{t\} = A\theta$. This means that there is $a \in A \subseteq S$ such that $t = a\theta$. Therefore, θ is surjective. This completes the proof. \square

Proposition 4.9. *Let S be a proper weakly type B semigroup. Then, the following statements are true:*

- (1) *For all $a \in S$, $a\sigma_S \in C(S)$;*
- (2) *For all $A \in C(S)$ and $a \in A$, two elements A and $a\sigma_S$ of $C(S)$ are σ related on $C(S)$, and the element $a\sigma_S$ is the maximum in the $\sigma_{C(S)}$ class of A .*

Proof. (1) Let $a \in S, b \in a\sigma$. Suppose that $c \in S$ satisfying $c \leq b$. Then, there exists $e \in E(S)$ with $c = eb$, that is, $ece = ebe$. Hence, by the definition of σ , we have $c\sigma b$. Thus, $c \in b\sigma = a\sigma$. Therefore, $a\sigma$ is an order ideal of S . Suppose $b, c \in a\sigma$. Then, $b^+c\sigma c\sigma b^+c$, that is, $(b^+c)\sigma(c^+b)$. Furthermore, we have $(b^+c)^+ = b^+c^+ = c^+b^+(c^+b)^+$ from $\tilde{\mathcal{R}}$ is a left congruence. This means that $(b^+c)\tilde{\mathcal{R}}(c^+b)$. Therefore, $(b^+c) = (c^+b)$ since S is a proper weakly type B semigroup. Dually, $bc^* = cb^*$ holds. Therefore, $a\sigma \in C(S)$.

(2) Let $A \in C(S)$ and $a \in A$. If $b \in A$, then we have $a^+b = b^+a$, that is, $a^+\sigma b\sigma = b^+\sigma a\sigma$. Actually, since $E(S)$ is an identity element on $C(S)$, we have $a\sigma b$, and so $A \subseteq a\sigma$. Now, we prove the elements A and $a\sigma_S$ are $\sigma_{C(S)}$ -related. It is clear that $(a^+) \in E(C(S))$ and $(a^+)A \subseteq (a^+)a\sigma$. On the other hand, let $b \in a\sigma, e \in (a^+)$. Then, by Definition 2.2, $eb = ea^+b = ea^+a^+b = ea^+b^+a \in (a^+)A$. Therefore, $(a^+)A = (a^+)a\sigma$, and so $(a^+)A(a^+) = (a^+)a\sigma(a^+)$. In other words, $A\sigma_{C(S)}(a\sigma)$ holds. Put $A, B \in C(S)$ such that $A\sigma_{C(S)}B$. Then, there exists $E \in E(C(S))$ such that $EAE = EBE$. We obtain that $EAE = EBE \subseteq A, B$ since A, B are the order ideals in S . Therefore, $B\sigma_{C(S)}A\sigma_{C(S)}(a\sigma)$. This means that for all $a \in A, b \in B, a\sigma = b\sigma$. That is, $a\sigma_S$ is the maximum element in $A\sigma_{C(S)}$. This completes the proof. \square

Theorem 4.10. *Let S be a weakly type B semigroup. Then, the weakly type B monoid $C(S)$ is proper if and only if S is proper. Suppose that it is true. Then, there is a maximum element for all $\sigma_{C(S)}$ class, and the unipotent monoids S/σ_S and $C(S)/\sigma_{C(S)}$ are isomorphic.*

Proof.

(Necessity) Let $C(S)$ be proper. Then, by Theorem 4.6, $S \rightarrow C(S)$ is a $(2, 1, 1)$ -embedding. This means that S is proper.

(Sufficiency) Let S be proper. Next, we show that $C(S)$ is left proper. To see it, let $A, B \in C(S)$ with $A^+ = B^+$ and $A\sigma_S B$. By Proposition 4.9(2), we have that for all $a \in A, b \in B, a\sigma = b\sigma$. Suppose $a \in A$. Then, $a^+ \in A^+ = B^+$. Hence, there exists $b \in B$ such that $a^+ = b^+$. This together with $a\sigma = b\sigma$ implies that $a = b$ since S is left proper. Therefore, $a \in B$, that is, $A \subseteq B$. Symmetrically, the reverse inclusion $B \subseteq A$ holds, and so $A = B$. In other words, $C(S)$ is left proper. Dually, we show that $C(S)$ is right proper.

Define a mapping as follows:

$$\psi : S/\sigma \rightarrow C(S)/\sigma_{C(S)}, \quad a\sigma \mapsto (a\sigma)\sigma_{C(S)}.$$

Next, we prove that ψ is an isomorphism. Let $a\sigma, b\sigma \in S/\sigma$ with $(a\sigma)\psi = (b\sigma)\psi$. By Proposition 4.9(2), we obtain that $a\sigma$ and $b\sigma$ are in the same $\sigma_{C(S)}$ class and that $a\sigma$ and $b\sigma$ are both the maximum elements in it. Thus, $a\sigma = b\sigma$, that is, ψ is injective. Now, we verify that ψ is surjective. To see it, let $A \in C(S)$ and $a \in A$. Then, A and $a\sigma_S$ are $\sigma_{C(S)}$ related from Proposition 4.9. Hence, $A\sigma_{C(S)} = (a\sigma)\psi$. Put $a, b \in S$. It is easy to see that $a\sigma b\sigma \subseteq (ab)\sigma$ from Proposition 4.9(2). Hence, $a\sigma b\sigma$ and $(ab)\sigma$ are $\sigma_{C(S)}$ related, and so

$$(a\sigma b\sigma)\psi = ((ab)\sigma)\psi = ((ab)\sigma)\sigma_{C(S)} = (a\sigma b\sigma)\sigma_{C(S)} = (A\sigma)\sigma_{C(S)}(b\sigma)\sigma_{C(S)} = (a\sigma)\psi(b\sigma)\psi.$$

This means that ψ is an isomorphism. This completes the proof. \square

Theorem 4.11. *Let S be an arbitrary proper weakly type B semigroup. Then, S is a $(2, 1, 1)$ -embedded in a proper weakly type B monoid T such that T/σ_T is isomorphic to S/σ_S , and that there is a maximum element in each σ_T class.*

Proof. It follows from Theorems 4.6 and 4.10. □

5 Left factorizable and almost left factorizable weakly type B semigroups

In this section, we first introduce the definitions of left factorizable weakly type B monoids and almost left factorizable weakly type B semigroups. Furthermore, we investigate the relation between them. It is shown that an arbitrary weakly type B semigroup S is almost left factorizable if and only if it is an idempotent separating $(2, 1, 1)$ -homomorphism image of a semigroup $W(T, Y)$, where Y is a semilattice, T is a unipotent monoid acting on Y such that conditions **(B1)**–**(B6)** hold. In particular, we prove that the semigroup $W(T, Y)$ is a proper almost left factorizable weakly type B semigroup with respect to isomorphism.

We begin by giving some elementary facts about weakly type B semigroups.

Remark 5.1. Let S be a weakly type B semigroup. Then, for all $a, b \in S, e, f \in E(S)$, the following statements are true:

- (1) $(ea)^+ = ea^+$;
- (2) If $S = S^1$, then $\widetilde{\mathcal{R}}_1(S)$ is a unipotent submonoid of S .

According to the aforementioned conclusion, we have that if M is a weakly type B monoid, then $\widetilde{\mathcal{R}}_1(M)$ is a unipotent submonoid in M .

Definition 5.1. A weakly type B semigroup M is said to be left factorizable if $M = E(M)\widetilde{\mathcal{R}}_1(M)$. A right factorizable weakly type B monoid is defined dually.

Definition 5.2. A weakly type B semigroup S is said to be almost left factorizable if for all $a \in S$, there exists $A \in \widetilde{\mathcal{R}}_{E(S)}(C(S))$ such that $a \in S$, where $\widetilde{\mathcal{R}}_{E(S)}(C(S)) = \{A \in C(S) \mid A^+ = E(S)\}$. An almost right factorizable weakly type B semigroup is defined dually.

Lemma 5.1. *Let M be a weakly type B monoid. Then, $\widetilde{\mathcal{R}}_{E(M)}(C(M)) = \{[r] \mid r \in \widetilde{\mathcal{R}}_1(M)\}$.*

Proof. Let $r \in \widetilde{\mathcal{R}}_1(M)$. Then, $r^+ = 1$ since $\widetilde{\mathcal{R}}_1(M)$ is a unipotent monoid. Hence, $(r]^+ = (r^+] = (1] = E(M)$. This means that for every $r \in \widetilde{\mathcal{R}}_1(M)$, $[r] \in \widetilde{\mathcal{R}}_{E(M)}(C(M))$. Conversely, let $A \in \widetilde{\mathcal{R}}_{E(M)}(C(M))$. Then, $A^+ = E(M)$. Thus, there exists $r \in A$ with $r^+ = 1$. Note that $r^+b = b^+r$ for all $b \in A$. By $b = b^+r$, we obtain that $br^* = (b^+r)r^* = b^+r = b$, that is, $b = br^* = rb^*$. Summarizing the aforementioned arguments, we conclude that $b \leq r$, that is, $b \in [r]$, and so $A \subseteq [r]$. The reverse inclusion $[r] \subseteq A$ is clear since $r \in A$, and A is an order ideal. Therefore, $[r] = A$. This completes the proof. □

Proposition 5.2. *A weakly type B monoid is almost left factorizable if and only if it is left factorizable.*

Proof.

(Sufficiency) Suppose M is a left factorizable weakly type B monoid. Then, $M = E(M)\widetilde{\mathcal{R}}_1(M)$. Hence, for every $a \in M$, there is $e \in E(M), r \in \widetilde{\mathcal{R}}_1(M)$ such that $a = er$. By multiplying it on the right by r^* , we have that $ar^* = err^* = er = a$, that is, $a = ar^* = ra^*$. Thus, $a \leq r$, and so $a \in [r]$. By Lemma 5.1, we obtain $[r] \in \widetilde{\mathcal{R}}_{E(M)}(C(M))$. Therefore, M is almost left factorizable.

(Necessity) Suppose that M is an almost left factorizable weakly type B monoid and let $a \in M$. Then, there exists $r \in \widetilde{\mathcal{R}}_1(M)$ such that $a \in (r]$ from Lemma 5.1. In other words, there exists $e \in E(M)$ such that $a = er$. Thus, $M \subseteq E(M)\widetilde{\mathcal{R}}_1(M)$. It is easily seen that the reverse inclusion holds. Therefore, M is left factorizable. \square

Proposition 5.3. *Let M be a left factorizable weakly type B monoid. Then, $M \setminus \widetilde{\mathcal{R}}_1(M)$ is an almost left factorizable weakly type B semigroup.*

Proof. For convenience, we put $S = M \setminus \widetilde{\mathcal{R}}_1(M)$. Clearly, S is closed with respect to the multiplication. In fact, suppose $a, b \in S$ satisfying $ab \in \widetilde{\mathcal{R}}_1(M)$. Then, $1 = (ab)^+ = (a^+ab)^+ = a^+(ab)^+ = a^+1 = a^+$, and so $a \in \widetilde{\mathcal{R}}_1(M)$, a contradiction. Thus, $ab \notin \widetilde{\mathcal{R}}_1(M)$. This means that S is closed with respect to production. Next, we verify that S is closed with respect to the unary operations “+” and “*”. To see it, let $a \in S$. Then, $a \notin \widetilde{\mathcal{R}}_1(M)$, and so $a^+ \notin \widetilde{\mathcal{R}}_1(M)$ since $a\widetilde{\mathcal{R}}_1a^+$. Now, we show $a^* \notin \widetilde{\mathcal{R}}_1(M)$. To see it, suppose $a^* = 1$. Since M is left factorizable, we have that $e \in E(M), r \in \widetilde{\mathcal{R}}_1(M)$ such that $a = er$, and hence, by multiplying it on the right by r^* , we have that $ar^* = err^* = er = a$. Again, since (P3), we obtain $a = ar^* = ra^* = r \cdot 1 = r \in \widetilde{\mathcal{R}}_1(M)$, a contradiction. This means that $a^* \notin \widetilde{\mathcal{R}}_1(M)$. Therefore, S is a $(2, 1, 1)$ subsemigroup of M . That is, S is a weakly type B semigroup.

Finally, we explain that S is almost left factorizable. Let $a \in S$. Then, there exist $e \in E(M), r \in \widetilde{\mathcal{R}}_1(M)$ such that $a = er$, where $e \neq 1$ since M is left factorizable. Hence, $a \in (r] \setminus \{r\}$. Since $((r] \setminus \{r\})^+ = E(M) \setminus \{1\} = E(S)$, we have $(r] \setminus \{r\} \in \widetilde{\mathcal{R}}_{E(S)}(C(S))$. Therefore, S is an almost left factorizable weakly type B semigroup. \square

Theorem 5.4. *Let Y be a semilattice, T be a unipotent monoid acting on Y satisfying conditions (B1)–(B6). Then, $W(T, Y)$ is a proper almost left factorizable weakly type B semigroup.*

Proof. Obviously, $W(T, Y)$ is a proper weakly type B semigroup from Propositions 3.2(1) and (3). Let $(t, a^t) \in W(T, Y)$. Then, $A_t = \{(t, y^t) \mid y \in Y\}$ is a σ class including (t, a^t) from Proposition 3.2. Since $W(T, Y)$ is proper, we have that $A_t \in C(W(T, Y))$ from Remark 5.1. Note that $A_t^+ = \{(t, y^t) \mid y \in Y\} = \{(1, y) \mid y \in Y\} = E(W(T, Y))$, we have $A_t \in \widetilde{\mathcal{R}}_{E(W(T, Y))}(C(W(T, Y)))$, and so $W(T, Y)$ is almost left factorizable. This completes the proof. \square

Remark 5.2. From Theorem 5.4, $\{A_t \mid t \in T\} \subseteq \widetilde{\mathcal{R}}_{E(W(T, Y))}(C(W(T, Y)))$. In fact, the reverse inclusion is also true. Let $A \in \widetilde{\mathcal{R}}_{E(W(T, Y))}(C(W(T, Y)))$. Then, $A^+ = E(W(T, Y))$. By Theorem 4.6, there exists $t \in T$ such that $A \subseteq A_t$. Hence, $\widetilde{\mathcal{R}}_{E(W(T, Y))}(C(W(T, Y))) \subseteq \{A_t \mid t \in T\}$, and so $\widetilde{\mathcal{R}}_{E(W(T, Y))}(C(W(T, Y))) = \{A_t \mid t \in T\}$

Theorem 5.5. *Let S and T be two weakly type B semigroups such that the mapping $\theta: S \rightarrow T$ is a surjective $(2, 1, 1)$ -homomorphism. If S is almost left factorizable, then so is T .*

Proof. Let $t \in T$. Then, there exists $s \in S$ such that $t = s\theta$ since θ is surjective. Suppose that S is almost left factorizable. Then, $A \in \widetilde{\mathcal{R}}_{E(S)}(C(S))$ with $s \in A$. By Theorem 4.8, $A\theta \in C(T)$, and so $(A\theta)^+ = A^+\theta = (E(S))\theta = E(T)$ from θ is a $(2, 1, 1)$ -homomorphism. Hence, $A\theta \in \widetilde{\mathcal{R}}_{E(T)}(C(T))$. Again, since $t = s\theta \in A\theta$, we have that T is also almost left factorizable. This completes the proof. \square

Theorem 5.6. *Every almost left factorizable weakly type B semigroup is an idempotent separating $(2, 1, 1)$ -homomorphism image of $W(T, Y)$, where Y is a semilattice and T is a unipotent monoid acting on Y satisfying conditions (B1)–(B6).*

Proof. Suppose that S is an almost left factorizable weakly type B semigroup. Put $T = \widetilde{\mathcal{R}}_{E(S)}(C(S))$ and put $P = \{(a, A) \in S \times T \mid a \in A\}$ is the subset of the direct product $S \times T$. Obviously, P is closed with respect to the unary operation “+” and the componentwise multiplication. Therefore, P is a $(2, 1)$ subsemigroup of $S \times T$, that is, P is a weakly type B semigroup. Consider the following first projection $\pi: P \rightarrow S, (a, A) \mapsto a$. We have that π is surjective since S is almost left factorizable. Clearly, for all $(a, A) \in P$, there is $(a, A)^+\pi =$

$((a, A)\pi)^+$, and so π is a $(2, 1)$ -homomorphism. It is easily seen that $E(P) = \{(e, E(S)) \mid e \in E(S)\}$, and so $(e, E(S))\pi = (f, E(S))\pi$ implies $e = f$ for all $(e, E(S)), (f, E(S)) \in E(P)$. That is, $(e, E(S)) = (f, E(S))$. Thus, π is idempotent separating.

Next, we verify P is $(2, 1)$ isomorphic to a semigroup $W(T, Y)$, where $Y = E(S)$. To see it, define T acts on $E(S)$ by the rule: if $e \in E(S), A \in T$, then put $e^A = a^*$, where a is the only one element on A such that $a^+ = e$ holds. Since $A^+ = E(S)$, by Lemma 4.1, a is the unique. Let $e, f \in E(S), A, B \in T$. We show that T satisfies conditions **(B1)**–**(B6)**.

(B1): Let $a_1, a_2 \in A, e^A = a_1^*, f^A = a_2^*$ with $a_1^+ = e$ and $a_2^+ = f$. By Lemma 4.1(2), we obtain $e^A f^A = a_1^* a_2^* = (a_1^+ a_2^+)^*$, where $a_1^+ a_2^+ \in A$ and $(a_1^+ a_2^+)^+ = a_1^+ a_2^+ = ef$. Therefore, $(ef)^A = (a_1^+ a_2^+)^* = e^A f^A$.

(B2): Let $a \in A, e = a^+, e^A = a^*$. Put $(e^A)^B = b^*$, where $b \in B, e^A = b^+$. It is clear that $a^* = b^+$, and so $e = a^+ = (aa^*)^+ = (ab^+)^+ = (ab)^+$, $ab \in AB$. Thus, $e^{AB} = (ab)^*$. Therefore, $e^{AB} = (ab)^* = (a^*)^* = (b^+ b)^* = b^* = (e^A)^B$.

(B3): Since there exists the only e in $E(S)$ such that $e^+ = e$, we have $e^{E(S)} = e^* = e$ from the aforementioned definition.

(B4): Suppose that $f = ef, f^A = a_1^*$ and $e^A = a_2^*$. Then, there are $a_1, a_2 \in A, a_1^+ = f, a_2^+ = e$. Hence, $f = ef = a_2^+ a_1^+ = (a_2^+ a_1^+)^+ = a_1^* a_2^* = a_1^+ e$ from Lemma 4.1. Therefore, there exists $c = f$ such that $f = c^A e$.

(B5): Suppose that $e^A = f^A$. Let $e^A = a_1^*, f^A = a_2^*$. Then, there are $a_1, a_2 \in A$ such that $a_1^+ = e, a_2^+ = f$. By Lemma 4.1, $a_1 = a_2$. Therefore, $e = f$.

(B6): Suppose that $f \leq e^A$, where $e^A = a^*, a \in A, a^+ = e$. Then, $f = fe^A = fa^* = a^* f = (af)^*$. Clearly, $af \in A$, we have $f = ((af)^+)^A$.

Define a mapping φ as follows:

$$\varphi : P \longrightarrow W(T, Y), \quad (s, A) \mapsto (A, s^*).$$

Now, we show that φ is a $(2, 1)$ -isomorphism. Obviously, if $(s, A) \in P$, then $A \in T, s \in A$ and $(s^+)^A = s^*$, and so $(A, s^*) \in W(T, Y)$. This means that φ is well defined. By Lemma 4.1(2), φ is injective. Let $(A, e^A) \in W(T, Y)$. Then, there exists $s \in A$ satisfying $e = s^+$ and $e^A = s^*$. Hence, $(s, A)\varphi = (A, e^A)$, that is, φ is surjective. Suppose that $(s, A) \in P$. Then,

$$((s, A)^+)\varphi = (s^+, A^+)\varphi = (A^+, (s^+)^*) = (E(S), s^+) = (A, (s^+)^A)^+ = (A, s^*)^+ = ((s, A)\varphi)^+.$$

Therefore, φ is a $(2, 1)$ -homomorphism with respect to the unary operation “+”. Let $(s, A), (t, B) \in P$. Then, $((s, A)(t, B))\varphi = (s, A)\varphi(t, B)\varphi$ holds if and only if $(st)^* = (s^*)^B t^*$. By definition, $(s^*)^B = u^*$, where $u \in B$ and $u^+ = s^*$. By Lemma 4.1(3), $(s^*)^B t^* = u^* t^* = (u^+ t)^* = (s^+ t)^* = (st)^*$. Therefore, φ is a homomorphism with respect to the multiplication.

Put $\psi = \varphi^{-1}\pi : W(T, Y) \longrightarrow S$. Then, ψ is a $(2, 1)$ -homomorphism with respect to the unary operation “.”. In fact, for all $(A, s^*) \in W(T, Y)$, where $s \in S, A \in T$, we have $(A, s^*)\psi = s$. Again, since $((A, s^*)^*)\psi = (E(S), s^*)\psi = s^* = ((A, s^*)\psi)^*$, we have that ψ is an idempotent separating $(2, 1, 1)$ -homomorphism. This completes the proof. □

Theorem 5.7. *Let S be an arbitrary weakly type B semigroup S . Then, the following statements are equivalent:*

- (1) S is almost left factorizable;
- (2) S is an idempotent separating $(2, 1, 1)$ -homomorphism image of a weakly type B semigroup $W(T, Y)$, where Y is a semilattice and T is a unipotent monoid acting on Y satisfying conditions **(B1)**–**(B6)**;
- (3) S is a $(2, 1, 1)$ -homomorphism image of weakly type B semigroup $W(T, Y)$, where Y is a semilattice and T is a unipotent monoid acting on Y satisfying conditions **(B1)**–**(B6)**.

Proof. It follows from Proposition 5.3 and Theorems 5.5 and 5.6. □

Theorem 5.8. *A weakly type B semigroup S is $(2, 1)$ -isomorphic to a semigroup $W(T, Y)$, where Y is a semilattice and T is a unipotent monoid acting on Y satisfying conditions **(B1)**–**(B6)** if and only if S is a proper almost left factorizable weakly type B semigroup.*

Proof.

(Necessity) By hypothesis, the weakly type B semigroup $W(T, Y)$ is proper almost left factorizable from Theorem 5.4. $W(T, Y)$ is $(2, 1)$ -isomorphic to S . Therefore, S is a proper almost left factorizable weakly type B semigroup.

(Sufficiency) Suppose that S is a proper almost left factorizable weakly type B semigroup. Put $Y = E(S)$ and $T = \widetilde{\mathcal{R}}_{E(S)}(C(S))$. Consider the weakly type B semigroup $W(T, Y)$. Define $\psi: W(T, Y) \rightarrow S$ as in Theorem 5.6. By the proof of Theorem 5.6, ψ is a surjective $(2, 1, 1)$ -homomorphism. Let $(A, s^*), (B, t^*) \in W(T, Y)$ such that $s = t$. Then, we have that $A\sigma_{C(S)}B$ from Proposition 4.9(2) since $s \in A, t \in B$. On the other hand, let $A, B \in T$. Then, $A^+ = E(S) = B^+$. By Theorem 4.10, $C(S)$ is proper, and so $A = B$. Therefore, ψ is injective. This completes the proof. \square

6 Factorizable and almost factorizable weakly type B semigroups

In this section, we investigate the left-right symmetric versions of factorizability and almost factorizability of weakly type B semigroups. Obviously, for an arbitrary weakly type B semigroup M , the $\widetilde{\mathcal{H}}$ -class $\widetilde{\mathcal{H}}_1(M)$ is a unipotent monoid. In particular, a weakly type B semigroup M is called *factorizable*, if $M = E(M)\widetilde{\mathcal{H}}_1(M)$.

Proposition 6.1. *A weakly type B monoid M is factorizable if and only if M is both left factorizable and right factorizable.*

Proof.

(Necessity) Straightforward.

(Sufficiency) Let M be both left and right factorizable weakly type B monoid. Then, we have that $M = E(M)\widetilde{\mathcal{R}}_1(M) = E(M)\widetilde{\mathcal{L}}_1(M)$. Suppose that $a \in \widetilde{\mathcal{L}}_1(M)$. Then, there exist $e \in E(M), s \in \widetilde{\mathcal{R}}_1(M)$ such that $a = es$. Hence, $1 = a^* = (es)^* = (es^+s)^* = (es)^*(s^+s)^* = 1s^* = s^*$, and so $s \in \widetilde{\mathcal{H}}_1(M)$. This means that $\widetilde{\mathcal{L}}_1(M) \subseteq E(M)\widetilde{\mathcal{H}}_1(M)$. Thus, $M = E(M)\widetilde{\mathcal{L}}_1(M) \subseteq E(M)\widetilde{\mathcal{H}}_1(M)$. It is clear that the reverse inclusion is also true. Therefore, $M = E(M)\widetilde{\mathcal{H}}_1(M)$. In other words, M is factorizable. This completes the proof. \square

Definition 6.1. A weakly type B semigroup is almost factorizable if for all $a \in S$, there exists $A \in \widetilde{\mathcal{H}}_{E(S)}(C(S))$ satisfying $a \in A$.

Theorem 6.2. *Let S be a proper weakly type B semigroup. Then, S is almost factorizable if and only if it is both almost left factorizable and almost right factorizable.*

Proof.

(Necessity) Straightforward.

(Sufficiency) Suppose S is both almost left factorizable and almost right factorizable. Let $a \in S$. Then, there are $A, B \in C(S)$ such that $a \in A, A^+ = E(S)$ and $a \in B, B^* = E(S)$. Note that S is proper and $a \in A \cap B$. We have that $A, B \subseteq a\sigma \in C(S)$ from Proposition 4.9. Hence, $A \cap B \subseteq a\sigma \in C(S)$ and $A \cap B$ is an order ideal of S . Thus, $A \cap B \in C(S)$. On the other hand, $(A \cap B)^+ = E(S) = (A \cap B)^*$. Therefore, $a \in A \cap B$ and $A \cap B \in \widetilde{\mathcal{H}}_{E(S)}(C(S))$. This completes the proof. \square

Proposition 6.3. *Let M be a weakly type B monoid. Then,*

$$\widetilde{\mathcal{H}}_{E(M)}(C(S)) = \{[r] \mid r \in \widetilde{\mathcal{H}}_1(M)\}.$$

Proof. Suppose that $A \in \widetilde{\mathcal{H}}_{E(M)}(C(M))$. Then, by Lemma 5.1, there exists $r \in \widetilde{\mathcal{R}}_1(M)$ such that $A = [r]$. Dually, there is $s \in \widetilde{\mathcal{L}}_1(M)$ such that $A = [s]$. Therefore, $r = s$ and $r \cup \widetilde{\mathcal{H}}_1(M)$, that is, $A \subseteq [r]$. The converse is clear. This completes the proof. \square

Proposition 6.4. *A weakly type B monoid is almost factorizable if and only if it is factorizable.*

Proof. It follows from Proposition 5.2 and its dual. □

Lemma 6.5. *Let M be a factorizable weakly type B monoid M . Then,*

$$\widetilde{\mathcal{R}}_1(M) = \widetilde{\mathcal{H}}_1(M) = \widetilde{\mathcal{L}}_1(M).$$

Proof. Let $a \in \widetilde{\mathcal{R}}_1(M)$. Then, there exists $e \in E(M)$, $u \in \widetilde{\mathcal{H}}_1(M)$ such that $a = eu$ since M is factorizable. Hence, $1 = a^+ = (eu)^+ = eu^+ = e1 = e$, and so $a = u \in \widetilde{\mathcal{H}}_1(M)$. That is, $\widetilde{\mathcal{R}}_1(M) \subseteq \widetilde{\mathcal{H}}_1(M)$. Note that $\widetilde{\mathcal{H}} \subseteq \widetilde{\mathcal{R}}$. We have $\widetilde{\mathcal{R}}_1(M) = \widetilde{\mathcal{H}}_1(M)$. Dually, $\widetilde{\mathcal{L}}_1(M) = \widetilde{\mathcal{H}}_1(M)$. □

Theorem 6.6. *If M is a factorizable weakly type B monoid, then $M \setminus \widetilde{\mathcal{H}}_1(M)$ is an almost factorizable weakly type B semigroup.*

Proof. Let M be a factorizable weakly type B monoid. Then, $\widetilde{\mathcal{R}}_1(M) = \widetilde{\mathcal{H}}_1(M) = \widetilde{\mathcal{L}}_1(M)$ from Lemma 6.5. Since M is left factorizable, then $M \setminus \widetilde{\mathcal{H}}_1(M) = M \setminus \widetilde{\mathcal{R}}_1(M)$ is a weakly type B semigroup from Proposition 5.3. By the proof of Proposition 5.3, we have that for all $a \in S = M \setminus \widetilde{\mathcal{H}}_1(M)$, there exists $r \in \widetilde{\mathcal{R}}_1(M)$ such that $a \in (r] \setminus r \in \widetilde{\mathcal{R}}_{E(S)}(C(S))$. Since M is factorizable, there exists $r \in \widetilde{\mathcal{L}}_1(M)$ such that $(r] \setminus r \in \widetilde{\mathcal{L}}_{E(S)}(C(S))$. Summarizing the aforementioned arguments, we have that $a \in (r] \setminus r$ and $(r] \setminus r \in \widetilde{\mathcal{H}}_{E(S)}(C(S))$. Therefore, $M \setminus \widetilde{\mathcal{H}}_1(M)$ is almost factorizable. □

Theorem 6.7. *Let Y be a semilattice, T be a unipotent monoid, which acts on Y satisfying conditions (B1)–(B5) and (B6)'. Then, $T * Y$ is a proper almost factorizable weakly type B semigroup.*

Proof. By the proof of Theorem 5.4, we only prove that (B6)' implies $A_t^* = E(T * Y)$ for all $t \in T$. It is easy to check that $A_t^* = \{(t, a^t) \mid a \in Y\} = \{(1, a^t) \mid a \in Y\} = E(T * Y)$. This completes the proof. □

Theorem 6.8. *Let S and T be two weakly type B semigroups and let $\theta: S \rightarrow T$ be a surjective (2, 1, 1)-homomorphism. If S is almost factorizable, then so is T .*

Proof. It is similar to the proof of Theorem 5.5. □

Theorem 6.9. *Every almost factorizable weakly type B semigroup S is an idempotent separating (2, 1, 1)-homomorphism image of semigroup $T * Y$, where, Y is a semilattice and T is a unipotent monoid acting on Y and satisfying conditions (B1)–(B5) and (B6)'.*

Proof. Put $T = \widetilde{\mathcal{H}}_{E(S)}(C(S))$ and $P = \{(a, A) \in S \times T \mid a \in A\}$. It is clear that P is a weakly type B semigroup and the projection $\pi: P \rightarrow S, (a, A) \mapsto a$ is a surjective idempotent separating (2, 1, 1)-homomorphism from Theorem 5.6.

Let $Y = E(S)$. Then, conditions (B1)–(B6) hold from Theorem 5.6. Next, we verify condition (B6)' is also true. To see it, let $A \in T$. Then, for all $e \in E(S)$, there exists $a \in A$ such that $a^* = e$ since $A^* = E(S)$. Hence, $e = a^* = (a^+)^A \in Y^A$, and so $Y \subseteq Y^A$, that is, $Y^A \subseteq Y$. Therefore, condition (B6)' holds.

Consider the mapping $\varphi: P \rightarrow T * Y, (s, A) \mapsto (A, s^*)$. Obviously, φ is closed with respect to the unary operation “*.” Hence, φ is a (2, 1, 1)-isomorphism from Theorem 5.6. Furthermore, the mapping $\psi = \varphi^{-1}\pi: T * Y \rightarrow S, (A, s^*) \mapsto s$ is a surjective idempotent separating (2, 1, 1)-homomorphism, where π is defined by the rule $P \rightarrow S, (a, A) \mapsto a$. This completes the proof. □

Corollary 6.10. *A weakly type B semigroup S is (2, 1, 1)-isomorphic to the semigroup $T * Y$, where Y is a semilattice and T is a unipotent monoid acting on Y satisfying conditions (B1)–(B5) and (B6)' if and only if it is proper almost factorizable.*

Proof. It follows from Theorems 5.6 and 6.9. □

Acknowledgments: The authors are very grateful to the referees for their valuable suggestions which lead to an improvement of this paper.

Funding information: This work was supported by the NNSF (CN) (Nos. 11261018 and 11961026) and the NSF of Jiangxi Province (No. 20181BAB201002).

Conflict of interest: The authors state no conflict of interest.

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