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Research Article

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A transient analysis to the $M(\tau)/M(\tau)/k$ queue with time-dependent parameters

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Abstract: This work considers the infinite multi-server Markovian queueing model with balking and catastrophes where the rates of arrivals, service, balking, and catastrophes are time dependent. The catastrophes arrive as negative customers to the system. The arrival of negative customers to a queueing system removes the positive customers. The catastrophes may come either from another service station or from outside the system. In this paper, we obtained the transient solution of this model using the approach of probability-generating function. Also, we derived an expression of transient probabilities in terms of Volterra equation of the second kind. Furthermore, we obtained a measure for time-dependent expected number of customers in the system.

Keywords: balking, catastrophes, generating function, multi-server queue, time-dependent queue, Volterra equation

MSC 2020: 60K20, 60K25, 60K30, 68M20, 90B22

1 Introduction

Queueing systems have been used effectively in computer networks, communication networks, hospitals, and manufacturing models. Queueing models with catastrophes attracted the attention of modelers. The catastrophes that happen randomly lead to the annihilation of most units in the queueing system. The catastrophes arrive as negative customers to the system. The catastrophes may come either from another service station or from outside the system. One of the characteristics of catastrophes is removable some or all of the regular customers in the system. For example, in computer networks, if a job infected with a virus, it transmits the virus to other processors and deactivates them [1].

Many researchers studied the models with constant time of arrival rate, service rate, balk, and catastrophe. However, in many situations, such as periodic phenomena or peak-hour traffic, it is normal for the arrival rate to depend on time. In addition, some cases include the emergency ambulance service, police patrols, ATMs, clearance of aircraft awaiting at airport, and waiting times at security checkpoints [2].

Margolius [3], Leese, and Boyd [4]; Rider [5]; Al-Seedy et al. [6]; and Massey [7] considered the queueing system $M_\tau/M_\tau/c$ for time-dependent rates. The multi-server Markovian queue for time-dependent rates was discussed by many papers such as those of Whitt [8], Zeifman et al. [9], and Margolius [10].

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Rakesh [11] studied the transient solution of the $M/M/c$ queueing system with balking and catastrophes, assuming that the arrival, departure, balk, and catastrophes rates are constants.

Sudhesh and Vaithiyanathan [12] discussed the single server queue with time-dependent arrival, in addition to discussion of the server rates with constant catastrophes rate. Zhang and Coyle [13] studied the model without balking and catastrophes, in which he obtained the boundary probability function $p_0(\tau)$ in the form of Volterra integral equation of the second kind, and presented the numerical solution of the Volterra integral equation using the Runge-Kutta algorithm [14]. Singh and Gupta [15] obtained the time-dependent and steady state solution explicitly with time-independent parameters. Jain and Singh [16] studied the transient model of the Markovian feedback queue subject to disaster and discouragement with other concepts of time-independent parameters.

This paper extends the presented system by Rakesh [11] to a more general setting in which the rates of balking and catastrophes are dependent on time. The generating function approach will be used to get the transient solution. Also, we will give an expression of transient probabilities in terms of Volterra equation of the second kind. Moreover, we will derive a measure for time-dependent expected number of customers in the system.

2 Mathematical model

Consider the $M(\tau)/M(\tau)/k$ queueing system characterized by a deterministic arrival rate function ($\lambda = \lambda(\tau)$), where

(1) λ is non-negative and integrable over the interval $(\tau_0, \tau]$.

(2) The number of arrivals (m) in the interval $(\tau_0, \tau]$ is Poisson with mean

$$a(\tau, \tau_0) = \int_{\tau}^{\tau} \lambda(s)\mathrm{d}s, \quad m < k \tag{1}$$

and

$$a_1(\tau, \tau_0) = \int_{\tau_0}^{\tau} b(s)\lambda(s)\mathrm{d}s, \quad m \geq k. \tag{2}$$

The customer who arrives at the system joins with the probability $b(\tau)$, $0 \leq b(\tau) < 1$, ($b(\tau) = 1$ when $m < k$) if the server was busy with k or more customers, and elsewhere balks have the probability $1 - b(\tau)$.

The random variable that represents the service time is an exponential type with varying parameter $\alpha(\tau)$ for multi-server queue.

Also, the service completions during the time interval $(\tau_0, \tau]$, when the queue is non-empty, gives a number that follows Poisson with mean:

$$a_2(\tau, \tau_0) = \int_{\tau_0}^{\tau} \alpha(s)\mathrm{d}s. \tag{3}$$

Furthermore, it is assumed that the catastrophe that occurs follows the Poisson process with function rate $\phi(\tau)$. When the system encounters a catastrophe, all k servers will be destroyed suddenly by the present customers and become momentarily inactivated. Then, the servers become available for service immediately after the catastrophe. Note that the service discipline is FIFO, starting with i customers in the model at $\tau = \tau_0$.

Assume that $\{Y(\tau), \tau > 0\}$ represents the number of customers present in the system exactly at time τ.

Let $p_m(\tau) = \Pr\{Y(\tau) = m\}$ represent probability that m customer occurrence at time τ in the system, $m = 0, 1, \ldots$, while $G(z, \tau)$ be the associated generating function.

The differential-difference system of state probabilities under transient state can be written as follows:

$$\frac{\mathrm{d}p_0(\tau)}{\mathrm{d}\tau} = -(\lambda(\tau) + \phi(\tau))p_0(\tau) + \alpha(\tau)p_1(\tau) + \phi(\tau), \tag{4}$$

$$\frac{\mathrm{d}p_m(\tau)}{\mathrm{d}\tau} = -(\lambda(\tau) + m\alpha(\tau) + \phi(\tau))p_m(\tau) + \lambda(\tau)p_{m-1}(\tau) + (m+1)\alpha(\tau)p_{m+1}(\tau),\ \ 1 \le m < k, \tag{5}$$

$$\frac{\mathrm{d}p_k(\tau)}{\mathrm{d}\tau} = -(b(\tau)\lambda(\tau) + k\alpha(\tau) + \phi(\tau))p_k(\tau) + \lambda(\tau)p_{k-1}(\tau) + k\alpha(\tau)p_{k+1}(\tau),\ \ m = k, \tag{6}$$

$$\frac{\mathrm{d}p_m(\tau)}{\mathrm{d}\tau} = -(b(\tau)\lambda(\tau) + k\alpha(\tau) + \phi(\tau))p_m(\tau) + b(\tau)\lambda(\tau)p_{m-1}(\tau) + k\alpha(\tau)p_{m+1}(\tau),\ \ m > k, \tag{7}$$

with $p_{i,m}(\tau_0) = \delta_{i,m}$; the Kronecker symbol, and $p_{i,m}(\tau) \equiv p_m(\tau)$, i.e., $p_{i,m}(\tau) = \Pr\{Y(\tau) = m|Y(\tau_0) = i\}$.

We define

$$G(z,\tau) = q_k(\tau) + \sum_{m=1}^{\infty} p_{m+k}(\tau)z^m, \tag{8}$$

where

$$q_k(\tau) = \sum_{m=0}^{k} p_m(\tau), \tag{9}$$

with

$$G(z,\tau_0) = \begin{cases} 1, & \text{for } i < k+1,\\ z^{i-k}, & \text{for } i \ge k+1. \end{cases} \tag{10}$$

By summing equations (4)–(6), we obtain

$$\begin{aligned}\frac{\mathrm{d}q_k(\tau)}{\mathrm{d}\tau} &= -\phi(\tau)\left[\sum_{m=0}^{k} p_m(\tau) - p_k(\tau)\right] + \phi(\tau) - \lambda(\tau)b(\tau)p_k(\tau) - \phi(\tau)p_k(\tau) + k\alpha(\tau)p_{k+1}(\tau)\\ &= -\lambda(\tau)b(\tau)p_k(\tau) + k\alpha(\tau)p_{k+1}(\tau) - \phi(\tau)q_k(\tau) + \phi(\tau).\end{aligned} \tag{11}$$

Multiplying equation (7) by z^m, and summing from $m = 1$ to ∞, we can have

$$\begin{aligned}\frac{\mathrm{d}\{\sum_{m=1}^{\infty} p_{k+m}(\tau)(\tau)z^m\}}{\mathrm{d}\tau} &= [-(\lambda(\tau)b(\tau) + \phi(\tau) + k\alpha(\tau)) + (\lambda(\tau)b(\tau)z + k\alpha(\tau)z^{-1})]\sum_{m=1}^{\infty} p_{k+m}(\tau)z^m\\ &\quad + \lambda(\tau)b(\tau)zp_k(\tau) - k\alpha(\tau)p_{k+1}(\tau).\end{aligned} \tag{12}$$

By summing equations (11) and (12), and using equation (8), we obtain

$$\begin{aligned}\frac{\partial G(z,\tau)}{\partial\tau} &= [b(\tau)\lambda(\tau)z - (b(\tau)\lambda(\tau) + k\alpha(\tau) + \phi(\tau)) + k\alpha(\tau)z^{-1}]G(z,\tau)\\ &\quad - [b(\tau)\lambda(\tau)z - (b(\tau)\lambda(\tau) + k\alpha(\tau)) + k\alpha(\tau)z^{-1}]q_k(\tau) + b(\tau)\lambda(\tau)(z-1)p_k(\tau) + \phi(\tau).\end{aligned} \tag{13}$$

Using the Lagrangian method, the solution of equation (13) can be given as follows:

$$\begin{aligned}G(z,\tau) &= \int_{\tau_0}^{\tau} \Psi_z(\tau,s)[b(s)\lambda(s)p_k(s)(z-1) + \phi(s) - \{b(s)\lambda(s)z - (b(s)\lambda(s) + k\alpha(s)) + k\alpha(s)z^{-1}\}q_k(s)]\mathrm{d}s\\ &\quad + \Psi_z(\tau,\tau_0)G(z,\tau_0),\end{aligned} \tag{14}$$

where

$$\Psi_z(\tau,s) = \exp\left\{\int_s^{\tau} (b(u)\lambda(u)z - (b(u)\lambda(u) + k\alpha(u) + \phi(u)) + k\alpha(u)z^{-1})\mathrm{d}u\right\}. \tag{15}$$

From equation (11) and definition of $q_k(\tau)$, ($q_{k+1}(\tau) = q_{k-1}(\tau) + p_k(\tau)$), we obtain the following equation:

$$\frac{dq_{k-1}(\tau)}{d\tau} = -\lambda(\tau)p_{k-1}(\tau) + k\alpha(\tau)p_k(\tau) + \phi(\tau) - \phi(\tau)q_{k-1}(\tau). \tag{16}$$

By partially differentiating of $\Psi_z(\tau, s)$ with respect to s, we obtain

$$\frac{\partial\Psi_z(\tau, s)}{\partial s} = -\{b(s)\lambda(s)z - (b(s)\lambda(s) + k\alpha(s) + \phi(s))k\alpha(s)z^{-1}\}\Psi_z(\tau, s). \tag{17}$$

By parts integration of the first term in equation (14) and using equations (16) and (17), then the solution of equation (14) can be given in the form:

$$G(z, \tau) = \int_{\tau_0}^{\tau} \Psi_z(\tau, s)[\lambda(s)p_{k-1}(s) - k\alpha(s)z^{-1}p_k(s)]ds + q_{k-1}(\tau) - \Psi_z(\tau, \tau_0)q_{k-1}(\tau_0) + \Psi_z(\tau, \tau_0)G(z, \tau_0) \tag{18}$$

Conducting equation (18) will be illustrated in the Appendix.

Also, we define

$$\tilde{I}_m(\tau, \eta) = \begin{cases} \sum_{r=0}^{\infty} \frac{a_1{}^{m+r}(\tau, \eta)}{(m+r)!} \frac{k^r a_2{}^r(\tau, \eta)}{r!} e^{-a_1(\tau,\eta)-ka_2(\tau,\eta)-a_3(\tau,\eta)}, & m \geq 0, \\ \sum_{r=0}^{\infty} \frac{a_1{}^r(\tau, \eta)}{r!} \frac{k^{r-m} a_2{}^{r-m}(\tau, \eta)}{(r-m)!} e^{-a_1(\tau,\eta)-ka_2(\tau,\eta)-a_3(\tau,\eta)}, & m < 0, \end{cases} \tag{19}$$

where

$$a_1(\tau, \eta) = \int_{\eta}^{\tau} b(u)\lambda(u)du, \tag{20}$$

$$a_2(\tau, \eta) = \int_{\eta}^{\tau} \alpha(u)du, \tag{21}$$

and

$$a_3(\tau, \eta) = \int_{\eta}^{\tau} \phi(u)du. \tag{22}$$

The relation between the function $\tilde{I}_m(\tau_0, \tau)$ and the nth modified Bessel function is given by:

$$\tilde{I}_m(\tau, \eta) = \left(\frac{a_1(\tau, \eta)}{a_2(\tau, \eta)}\right)^{m/2} I_m(2\sqrt{a_1(\tau, \eta)a_2(\tau, \eta)})e^{-a_1(\tau_0,\tau)-ka_2(\tau_0,\tau)-a_3(\tau,\eta)} \tag{23}$$

and

$$\tilde{I}_m(\tau, \tau) = \tilde{I}_0(0) = 1. \tag{24}$$

Thus, the function $\tilde{I}_m(\tau_0, \tau)$ satisfies the following properties:

$$\frac{\partial}{\partial\tau}\tilde{I}_m(\tau, \eta) = b(\tau)\lambda(\tau)\tilde{I}_{m-1}(\tau, \eta) - (b(\tau)\lambda(\tau) + k\alpha(\tau) + \phi(\tau))\tilde{I}_m(\tau, \eta) + k\alpha(\tau)\tilde{I}_{m+1}(\tau, \eta), \tag{25}$$

$$\frac{\partial}{\partial\eta}\tilde{I}_m(\tau, \eta) = -b(\tau)\lambda(\tau)\tilde{I}_{m-1}(\tau, \eta) + (b(\tau)\lambda(\tau) + k\alpha(\tau) + \phi(\tau))\tilde{I}_m(\tau, \eta) - k\alpha(\tau)\tilde{I}_{m+1}(\tau, \eta). \tag{26}$$

Referring to [13] we can show that

$$\Psi_z(\tau, \eta) = \sum_{m=-\infty}^{\infty} z^n\tilde{I}_m(\tau, \eta). \tag{27}$$

Using relation (27) in equation (18) and comparing the coefficients of z^n on two sides, it is found that, for $m = 1, 2, \ldots,$

$$p_{m+k}(\tau) = \int_{\tau_0}^{\tau} [\lambda(s)p_{k-1}(s)\tilde{I}_m(\tau, s) - k\alpha(\tau)p_k(s)\tilde{I}_{m+1}(\tau, s)]\mathrm{d}s + \tilde{I}_{m+k-i}(\tau, \tau_0)(1 - q_k(\tau_0)) + q_k(\tau_0)\tilde{I}_m(\tau, \tau_0), \quad (28)$$

and for $m = 0$,

$$q_k(\tau) = \int_{\tau_0}^{\tau} [\lambda(s)p_{k-1}(s)\tilde{I}_0(\tau, s) - k\alpha(s)p_k(s)\tilde{I}_1(\tau, s)]\mathrm{d}s + q_{k-1}(\tau) + \tilde{I}_{k-i}(\tau, \tau_0)(1 - q_k(\tau_0)) + q_k(\tau_0)\tilde{I}_0(\tau, \tau_0). \quad (29)$$

Simplifying, we obtain the following form:

$$p_k(\tau) = \int_{\tau_0}^{\tau} [\lambda(s)p_{k-1}(s)\tilde{I}_0(\tau, s) - k\alpha(s)p_k(s)\tilde{I}_1(\tau, s)]\mathrm{d}s + \tilde{I}_{k-i}(\tau, \tau_0)(1 - q_k(\tau_0)) + q_k(\tau_0)\tilde{I}_0(\tau, \tau_0). \quad (30)$$

The other probabilities $p_m(\tau)$, $m = 0, 1, \ldots, k - 1$, are acquired via solving equations (4) and (5). These equations are first put in the matrix form and given as:

$$\frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau} = A(\tau)P(\tau) + k\alpha(\tau)p_k(\tau)e_k + \phi(\tau)e_1, \quad (31)$$

where

$$P(\tau) = (p_0(\tau), p_1(\tau), \ldots, p_{k-1}(\tau))^T, \quad e_k = (0, 0, \ldots, 1)^T,$$

and $e_1 = (1, 0, \ldots, 0)^T$ are column vectors of order c.

Or

$$\frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau} = A(\tau)P(\tau) + H(\tau), \quad (32)$$

where

$$H(\tau) = (\phi(\tau), 0, \ldots, 0, k\alpha(\tau)p_k(\tau))^T. \quad (33)$$

With the initial condition:

$$P(\tau_0) = (p_0(\tau_0), p_1(\tau_0), \ldots, p_{k-1}(\tau_0))^T, \quad (34)$$

$$A(\tau) = \begin{pmatrix} -(\lambda(\tau) + \phi(\tau)) & \alpha(\tau) & 0 & \ldots & 0 \\ \lambda(\tau) & -(\lambda(\tau) + \alpha(\tau) + \phi(\tau)) & 2\alpha(\tau) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & (k-1)\alpha(\tau) \\ 0 & 0 & 0 & \ldots & -(\lambda(\tau) + (k-1)\alpha(\tau) + \phi(\tau)) \end{pmatrix}_{k\times k}. \quad (35)$$

Assume that $U(\tau_0, \tau)$ is a square matrix operator satisfying that $U(\tau, \tau) = I$, and

$$\frac{\partial}{\partial\tau}U(\tau_0, \tau) = A(\tau)U(\tau_0, \tau). \quad (36)$$

Thus, the solution of equation (31) can be given as follows:

$$P(\tau) = U(\tau_0, \tau)P(\tau_0) + \int_{\tau_0}^{\tau} U(\tau, s)G(s)\mathrm{d}s. \quad (37)$$

Also, the solution of $U(\tau_0, \tau)$ can be obtained by numerical scheme using efficient methods of solving system of linear ordinary differential equations [14]. When the values of τ_0 and τ are fixed, then $U(\tau_0, \tau)$ is a matrix with elements indexed from $0 \to k - 1$. More specifically, let $U_0(\tau_0, \tau)$ and $U_{k-1}(\tau_0, \tau)$ be the first and last rows of this matrix, and $U_{0,k-1}(\tau_0,\tau)$ be the elements in row 0, and column $k-1$, while $U_{k-1,k-1}(\tau_0,\tau)$ is the element in row $k-1$ and column $k-1$. Then

$$p_0(\tau) = \int_{\tau_0}^{\tau} (\phi(s)U_{0,0}(s,\tau) + k\alpha(s)p_k(\tau)U_{0,k-1}(s,\tau))\mathrm{d}s + U_0(\tau_0,\tau)P(\tau_0) \tag{38}$$

and

$$p_{k-1}(\tau) = \int_{\tau_0}^{\tau} (\phi(s)U_{k-1,0}(s,\tau) + k\alpha(s)p_k(s)U_{k-1,k-1}(s,\tau))\mathrm{d}s + U_{k-1}(\tau_0,\tau)P(\tau_0). \tag{39}$$

Therefore, $p_k(\tau)$ can be written as a Volterra equation of the second kind,

$$\begin{aligned} p_k(\tau) = &\int_{\tau_0}^{\tau} \left\{ \lambda(s)\tilde{I}_0(s,\tau)\left[\int_{t_0}^{s} k\alpha(v)p_k(v)U_{k-1,k-1}(s,v)dv + U_{k-1}(\tau_0,s)P(\tau_0) \right] - k\alpha(s)p_k(s)\tilde{I}_1(s,\tau) \right\}\mathrm{d}s \\ &+ \tilde{I}_{k-i}(\tau_0,\tau)(1-q_k(\tau_0)) + q_k(\tau_0)\tilde{I}_0(\tau_0,\tau). \end{aligned} \tag{40}$$

From equations (28), (37), and (40), we determined all the transient state probabilities.

3 Expressions for the expected queue size

The expected number of customers in the system, $L(\tau)$, can be given as follows:

$$L(\tau) = \sum_{m=1}^{k-1} mp_m(\tau) + \sum_{m=k}^{\infty} mp_m(\tau). \tag{41}$$

By differentiating equation (36) with respect to τ, we obtain

$$L'(\tau) = \sum_{m=1}^{k-1} mp'_m(\tau) + \sum_{m=k}^{\infty} mp'_m(\tau). \tag{42}$$

By multiplying equations (5), (6), and (7) by m, and summing from $m = 1 \to \infty$, we obtain

$$L'(\tau) = -\phi(\tau)L(\tau) + \lambda(\tau)q_{k-1}(\tau) - \mu(\tau)\sum_{m=1}^{k-1} mp_m(\tau) + (\lambda(\tau)b(\tau) - k\mu(\tau))\sum_{m=k}^{\infty} p_m(\tau). \tag{43}$$

Equation (43) is a linear differential equation, and its solution is

$$\begin{aligned} L(\tau) = &\int_{\tau_0}^{\tau} (\lambda(u)b(u) - k\mu(u))\mathrm{e}^{-a_3(\tau_0,u)}\mathrm{d}u + \sum_{m=0}^{k-1}\int_{\tau_0}^{\tau} \lambda(u)(1-b(u))p_m(u)\mathrm{e}^{-a_3(\tau_0,u)}\mathrm{d}u \\ &+ \sum_{m=1}^{k-1}\int_{\tau_0}^{\tau} (k-m)\mu(u)p_m(u)\mathrm{e}^{-a_3(\tau_0,u)}\mathrm{d}u + \int_{\tau_0}^{\tau} k\mu(u)p_0(u)\mathrm{e}^{-a_3(\tau_0,u)}\mathrm{d}u. \end{aligned} \tag{44}$$

4 Illustration examples

To illustrate the influence of the model's parameters on the system behavior, we give two examples for $M(\tau)/M(\tau)/3$ queue.

The first example in Figure 1 is the $M(\tau)/M(\tau)/3$ queue using constant parameters $\lambda = 5$, $\mu = 2$, $\tau_0 = 0$, $\phi(s) = 0$, $b(\tau) = 1$, and $k = 3$ in equations (37), (38), (39), and (44). The first part of Figure 1 shows probability $m = 0, 1$, or 2 in the queue. The system was empty at time 0. As $\tau \to \infty$, p_0 tends to about 0.05,

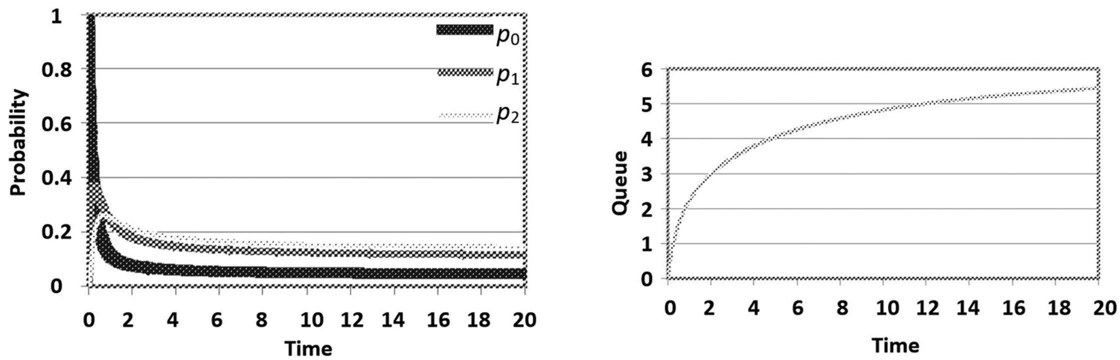

Figure 1: $M(\tau)/M(\tau)/3$ queue with constant parameters: $\mu(\tau) = 2$, and $\lambda(\tau) = 5$.

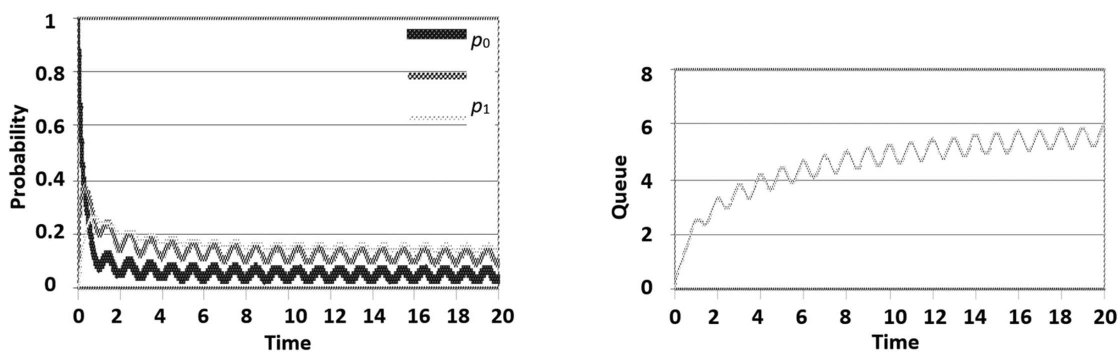

Figure 2: $M(\tau)/M(\tau)/3$ queue with variable service rate: $\mu(\tau) = 2 + \sin 2\pi\tau$, and $\lambda = 5$.

p_1 tends to 0.11, and p_2 tends to 0.14, to two decimal places accuracy. The second part of Figure 1 shows the expected number in the system. The asymptotic limit of the expected number in the queue for this example is about 6. Formulas for asymptotic limits of constant parameter multi-server queues can be found by Donald and Harris [17, pp. 87–88], and Zhang and Coyle [13].

The second example in Figure 2 is the $M(\tau)/M(\tau)/3$ queue using variable service rate $\mu(\tau) = 2 + \sin 2\pi\tau$, $\lambda = 5$, $\tau_0 = 0$, $\phi(\tau) = 0$, $b(\tau) = 1$, and $k = 3$ in equations (37), (38), (39), and (44). Both parts of Figure 2 show probability $m = 0$, 1, or 2 in the queue and the expected number in the system, respectively. Formulas for asymptotic limits of time-dependent parameters multi-server queues can be found by Margolius [3].

The graph in Figure 1 differs from Figure 2, which shows the form of parameters of queue effect in the graph of queue system.

5 Special cases

For a single-server and without catastrophes model, if we put $k = 1$ and $\phi(\tau) = 0$ in equations (37)–(40), then we obtain results consistent with those of Alseedy et al. [6]. When queue parameters are constants, i.e., take $\lambda(\tau) = \lambda$, $\mu(\tau) = \mu$, $\phi(\tau) = \phi$, and $b(\tau) = b$, the queue system agrees with that of Rakesh [11], which will be explained below.

If we take $\lambda(\tau) = \lambda$, $\alpha(\tau) = \alpha$, $b(\tau) = b$, and $\phi(\tau) = \phi$, then

$$a(\tau, \tau_0) = \lambda(\tau - \tau_0),$$
$$a_1(\tau, \tau_0) = b\lambda(\tau - \tau_0),$$
$$a_2(\tau, \tau_0) = \alpha(\tau - \tau_0),$$

and

$$a_3(\tau, \tau_0) = \phi(\tau - \tau_0).$$

Therefore, the probability generating function $P(z, \tau)$ for the transient probabilities of this model will be:

$$\begin{aligned} P(z,\tau) = {} & \exp\left\{\left[\left(\lambda pz + \frac{c\mu}{z}\right) - (\lambda p + \psi + c\mu)\tau\right]\right\} \\ & + \int_0^\tau \left\{\lambda p(z-1)P_c(u) - \left[\left(\lambda pz + \frac{c\mu}{z}\right) - (\lambda p + c\mu)\right]q_c(u)\right\} \\ & \times \exp\left\{\left[\left(\lambda pz + \frac{c\mu}{z}\right) - (\lambda p + \psi + c\mu)\right](\tau - u)\right\}\mathrm{d}u \\ & + \psi\int_0^\tau \exp\left\{\left[\left(\lambda pz + \frac{c\mu}{z}\right) - (\lambda p + \psi + c\mu)\right](\tau - u)\right\}\mathrm{d}u. \end{aligned} \tag{45}$$

Also, equation (15) becomes

$$\Psi_z(\tau, s) = \exp\left\{\left(\lambda pz - (\lambda p + k\alpha + \phi) + \frac{k\alpha}{z}\right)(\tau - s)\right\} = \sum_{-\infty}^{\infty}\left(\sqrt{\frac{\lambda p}{c\alpha}}z\right)^n I_n(2\sqrt{\lambda p\mu})\tau. \tag{46}$$

And equation (23) will be reduced to:

$$\tilde{I}_m(\tau, \eta) = \left(\frac{\lambda p}{c\alpha}\right)^{m/2} I_m(2\sqrt{\lambda p\mu})e^{-\{b\lambda - k\alpha - \phi\}(\tau - \eta)}. \tag{47}$$

Furthermore, equation (40) becomes equivalent to equations (14) and (16) in the study by Rakesh [11] as follows:

$$p_{n+k}(\tau) = n\beta^n \int_0^\tau \exp\{-(\lambda b + \psi + \gamma)(\tau - u)\}\frac{I_n(\alpha(\tau - u))}{(\tau - u)}P_k(u)\mathrm{d}u, \quad n = 1, 2, \ldots, \tag{48}$$

and

$$\begin{aligned} q_k(\tau) = {} & \exp\{-(\lambda b + \psi + \gamma)\tau\}I_0(\alpha\tau) \\ & + \lambda p\int_0^\tau \exp\{-(\lambda b + \psi + \gamma)(\tau - u)\}[I_1(\alpha(\tau - u))\beta^{-1} - I_0(\alpha(\tau - u))]p_k(u)\mathrm{d}u \\ & - \int_0^\tau \exp\{-(\lambda b + \psi + \gamma)(\tau - u)\}q_k(u)[\lambda bI_1(\alpha(\tau - u)) - (\lambda b + \gamma)I_0(\alpha(\tau - u))]\mathrm{d}u \\ & + \psi\int_0^\tau \exp\{-(\lambda b + \psi + \gamma)(\tau - u)\}I_0(\alpha(\tau - u))\mathrm{d}u. \end{aligned} \tag{49}$$

While equation (32) will be equivalent to equation (17) in the study by Rakesh [11], and it will be:

$$\frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau} = AP(\tau) + \gamma P_k(\tau)e_1 + \psi e_2, \tag{50}$$

where

$$A = \begin{bmatrix} -(\lambda+\psi) & \mu & \dots & 0 \\ \lambda & -(\lambda+\psi+\mu) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & (k-1)\mu \\ 0 & 0 & \dots & -(\lambda+\psi+(k-1)\mu) \end{bmatrix}_{k\times k},$$

$e_1 = (0, 0, \dots, 1)^T$, $e_2 = (1, 0, \dots, 0)^T$, and $P(\tau) = (p_0(\tau), p_1(\tau), \dots, p_{k-1}(\tau))^T$.

6 Conclusion

The varying parameters of multi-server Markovian queueing model with balking and catastrophes have been investigated. We obtained the transient solution of this model using the approach of probability-generating function. Also, we derived an expression of transient probabilities in terms of Volterra equation of the second kind. Furthermore, we obtained a measure for time-dependent expected number of customers in the system.

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Appendix

From equation (18), we have

$$G(z,\tau) = \int_{\tau_0}^{\tau} \Psi_z(\tau,s)[b(s)\lambda(s)p_k(s)(z-1) + \phi(s) - \{b(s)\lambda(s)z - (b(s)\lambda(s) + k\alpha(\tau)) + k\alpha(\tau)z^{-1}\}q_k(s)]\mathrm{d}s \\ + \Psi_z(\tau,\tau_0)G(z,\tau_0). \tag{51}$$

Using equations (16) and (17), we have

$$\frac{\mathrm{d}q_{k-1}(\tau)}{\mathrm{d}\tau} = -\lambda(\tau)p_{k-1}(\tau) + k\alpha(\tau)p_k(\tau) + \phi(\tau) - \phi(\tau)q_k(\tau) \tag{52}$$

and

$$\frac{\partial \Psi_z(\tau,s)}{\partial s} = -\{b(s)\lambda(s)z - (b(s)\lambda(s) + k\alpha(s) + \phi(s)) + k\alpha(s)z^{-1}\}\Psi_z(\tau,s). \tag{53}$$

We obtain:

$$G(z,\tau) = \int_{\tau_0}^{\tau} \Psi_z(\tau,s)[k\alpha(s)p_k(s)(1-z^{-1}) + \phi(s) - \phi(s)q_{k-1}(s) \\ - \{b(s)\lambda(s)z - (b(s)\lambda(s) + k\alpha(s) + \phi(s)) + k\alpha(s)z^{-1}\}q_{k-1}(s)]\mathrm{d}s + \Psi_z(\tau,\tau_0)G(z,\tau_0). \tag{54}$$

Then

$$G(z,\tau) = \int_{\tau_0}^{\tau} \Psi_z(\tau,s)[k\alpha(s)p_k(s)(1-z^{-1}) + \phi(s) - \phi(s)q_{k-1}(s)]\mathrm{d}s + \int_{\tau_0}^{\tau} \frac{\partial \Psi_z(\tau,s)}{\partial s} q_{k-1}(s)\mathrm{d}s \\ + \Psi_z(\tau,\tau_0)G(z,\tau_0). \tag{55}$$

Simplifying equation (55), we obtain

$$G(z,\tau) = \int_{\tau_0}^{\tau} \Psi_z(\tau,s)[\lambda(s)p_{k-1}(s) - k\alpha(s)z^{-1}p_k(s)]\mathrm{d}s + q_{k-1}(\tau) - \Psi_z(\tau,\tau_0)q_{k-1}(\tau_0) + \Psi_z(\tau,\tau_0)G(z,\tau_0). \tag{56}$$