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Research Article

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Multiple solutions for a quasilinear Choquard equation with critical nonlinearity

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Abstract: In the present work, we are concerned with the multiple solutions for quasilinear Choquard equation with critical nonlinearity in $\mathbb{R}^N$. We show multiplicity results for this problem, which are characterized, respectively, by the new version of symmetric mountain-pass theorem and the mountain-pass theorem for even functionals. The novelty of our work is the appearance of the convolution terms as well as critical nonlinearities.

Keywords: quasilinear Choquard equation, Hardy-Littlewood-Sobolev critical exponent, Mountain-pass theorem, concentration-compactness principle

MSC 2020: 35A15, 35J60, 35J20

1 Introduction

In the present paper, we are interested in the existence of multiple solutions for the following quasilinear Choquard equation with critical nonlinearity in $\mathbb{R}^N$:

$$-\left(a + b\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)\Delta u - a[\Delta(u^2)]u = \alpha k(x)|u|^{p-2}u + \beta\left(\int\limits_{\mathbb{R}^N} \frac{|u|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{22^*_\mu-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $a > 0$, $b \geq 0$, $0 < \mu < 4$, $1 < p \leq 4$, $N \geq 3$, α, and β are real parameters, $2^*_\mu = \frac{2N-\mu}{N-2}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, and $k(x) \in L^r(\mathbb{R}^N)$ with $r = \frac{22^*}{22^*-p}$.

First, we make a quick overview of the literature. To begin with, we note that the following Choquard equation

$$-\Delta u + V(x)u = (|x|^{-\mu} * F(u))f(u) \quad x \in \mathbb{R}^N \tag{1.2}$$

was introduced by Choquard in 1976, to study an electron trapped in its hole. Equation (1.2) can be used to describe many physical models. For instance, the quantum theory of a polaron [1], the modeling of an electron, a certain approximation to Hartree-Fock theory of one-component plasma [2], and the self-gravitational collapse of a quantum mechanical wave-function, etc. Moreover, the existence and qualitative properties of solutions to equation (1.2) have been widely studied in the last few decades, see e.g. [3–8] for the work of Choquard-type equations.

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Once we turn our attention to the Kirchhoff-type problems with critical nonlinearity, we immediately see that the literature is relatively scarce. In this case, we can cite the recent works of [9–17]. We call attention to [18] in which work the authors have dealt with the following Kirchhoff-type equation

$$-\left(a + b\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V_\lambda(x)u = (\mathcal{K}_\mu * u^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

By using the Nehari manifold and the concentration-compactness principle, the author obtained the existence of ground state solutions for equation (1.3) if the parameter λ is large enough.

On the other hand, another important reference is [19], where the authors have considered the following quasilinear Choquard equation:

$$-\Delta u + V(x)u - [\Delta(u^2)]u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\mu \in (0, (N+2)/2)$, $p \in (2, (4N - 4\mu)/(N-2))$. By a changing variable and perturbation method, the existence of positive solutions, negative solutions, and high energy solutions was obtained. Moreover, we also cite previous studies [20–24] with no attempt to provide the full list of references.

From the above mentioned papers, it is natural to ask what results can be recovered with this kind of quasilinear Choquard equation with critical nonlinearity in $\mathbb{R}^N$. Compared to the above papers, some difficulties arise in our paper when dealing with problem (1.1), because of the appearance of the convolution terms as well as critical nonlinearities which provokes some mathematical difficulties, and these make the study of problem (1.1) particularly interesting.

Our main results are as follows.

Theorem 1.1. *Let* $0 < \mu < 4$ *and* $1 < p < 4$. *Suppose that* $\Omega := \{x \in \mathbb{R}^N : k(x) > 0\}$ *is an open subset of* $\mathbb{R}^N$ *and that* $0 < |\Omega| < \infty$. *Then, for each* $\beta > 0$ *there exists* $\overline{\Lambda} > 0$ *such that if* $\alpha \in (0, \overline{\Lambda})$ *or for each* $\alpha > 0$ *there exists* $\underline{\Lambda} > 0$ *such that if* $\beta \in (0, \underline{\Lambda})$, *problem* (1.1) *has a sequence of solutions* $(u_n)_n$ *and* $u_n \to 0$ *as* $n \to \infty$ *in* $D^{1,2}(\mathbb{R}^N)$.

Theorem 1.2. *Let* $0 < \mu < 4$, $p = 4$, *and* $\beta = 1$. *Then, there exists a positive constant* a^* *such that, for each* $a > a^*$ *and* $\alpha \in \left(0, \frac{1}{2}aS\|k\|_r^{-1}\right)$, *problem* (1.1) *has at least k pairs of nontrivial weak solutions.*

Remark 1.1. The difficulties of this paper mainly lie in two aspects: one of difficulties of the problem (1.1) stems from that there is no suitable working space on which the energy functional enjoys both smoothness and compactness, so the standard critical point theory cannot be applied directly. In order to overcome this difficulty, we use the method in [25–27]. The other is caused by the convolution terms as well as critical nonlinearities, which leads to some estimates about nonlocal term that are likely to be confronted some difficulties. In order to prove the compactness condition, we use the second concentration-compactness principle and concentration-compactness principle at infinity to prove that the $(PS)_c$ condition holds.

2 Preliminaries

In this section, we first recall the following well-known Hardy-Littlewood-Sobolev inequality, which will be used in the sequel.

Proposition 2.1. [28] *Let* $t, r > 1$ *and* $0 < \mu < N$ *with* $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$, *and* $h \in L^r(\mathbb{R}^N)$. *There exists a sharp constant* $C(t, r, \mu, N)$ *independent of* f, h, *such that*

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \leq C(t, r, \mu, N)\|f\|_{L^t}\|h\|_{L^r}. \tag{2.1}$$

Let S be the best constant for the embedding $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$, that is,

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\left\{\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x : \int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x = 1\right\}. \tag{2.2}$$

Consequently, we define

$$S_{H,L} = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\left\{\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = 1\right\}. \tag{2.3}$$

Remark 2.1. From [4], we know that the constant $S_{H,L}$ defined in (2.3) is achieved, and

$$\|\cdot\|_{NL} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\cdot|^{2^*_\mu}|\cdot|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{22^*_\mu}}$$

defines a norm on $L^{2^*}(\mathbb{R}^N)$.

Problem (1.1) corresponding to the energy functional $J : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\begin{aligned} J(u) &:= \frac{a}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2 + a\int_{\mathbb{R}^N}|u|^2|\nabla u|^2\mathrm{d}x - \frac{\alpha}{p}\int_{\mathbb{R}^N}k(x)|u|^p\mathrm{d}x \\ &\quad - \frac{\beta}{22^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{22^*_\mu}|u(y)|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\ &= \frac{a}{2}\int_{\mathbb{R}^N}(1+2|u|^2)|\nabla u|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2 - \frac{\alpha}{p}\int_{\mathbb{R}^N}k(x)|u|^p\mathrm{d}x - \frac{\beta}{22^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{22^*_\mu}|u(y)|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Note that the functional J is not well defined in $D^{1,2}(\mathbb{R}^N)$. In order to overcome this difficulty, we make the changing of variables $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}}, \quad \text{and} \quad f(0) = 0$$

on $[0, +\infty)$ and by $f(t) = -f(-t)$ on $(-\infty, 0]$.

We have collected some properties of the function f.

Lemma 2.1. [25,29] *The function f satisfies the following properties*:

(f_0) *f is uniquely defined C^∞ and invertible.*
(f_1) $|f'(t)| \le 1$ *for all* $t \in \mathbb{R}$.
(f_2) $\frac{f(t)}{t} \to 1$ *as* $t \to 0$.
(f_3) $\frac{f(t)}{\sqrt{t}} \to 2^{\frac{1}{4}}$ *as* $t \to \infty$.
(f_4) $\frac{1}{2}f(t) \le tf'(t) \le f(t)$ *for all* $t \ge 0$.
(f_5) $\frac{1}{2}f^2(t) \le f(t)f'(t)t \le f^2(t)$ *for all* $t \in \mathbb{R}$.
(f_6) $|f(t)| \le t$ *for all* $t \in \mathbb{R}$.
(f_7) $|f(t)| \le 2^{\frac{1}{4}}|t|^{\frac{1}{2}}$ *for all* $t \in \mathbb{R}$.
(f_8) *The function $f^2(t)$ is strictly convex.*

(f_9) *There exists a positive constant C such that*

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{\frac{1}{2}}, & |t| \geq 1. \end{cases}$$

(f_{10}) *There exist positive constants C_1 and C_2 such that*

$$|t| \leq C_1|f(t)| + C_2|f(t)|^2 \text{ for all } t \in \mathbb{R}.$$

(f_{11}) $|f(t)f'(t)| \leq \frac{1}{\sqrt{2}}$ *for all* $t \in \mathbb{R}$.

So after the change of variables, we can write $J(u)$ as

$$\begin{aligned} J(v) := \frac{a}{2}\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N}|f'(v)|^2|\nabla v|^2\mathrm{d}x\right)^2 - \frac{\alpha}{p}\int_{\mathbb{R}^N}k(x)|f(v)|^p\mathrm{d}x \\ - \frac{\beta}{22^*_\mu}\iint_{\mathbb{R}^{2N}}\frac{|f(v(x))|^{22^*_\mu}|f(v(y))|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{2.4}$$

By Proposition 2.1 and Lemma 2.1, we know that the functional $J \in C^1(D^{1,2}(\mathbb{R}^N), \mathbb{R})$. As in [25], we note that if f is a nontrivial critical point of J, then v is a nontrivial solution of problem

$$-a\Delta v - b\int_{\mathbb{R}^N}|f'(v)|^2|\nabla v|^2\mathrm{d}x \cdot (f'(v)f''(v)|\nabla v|^2 + |f'(v)|^2\Delta v) = g(x, v), \tag{2.5}$$

where

$$g(x, s) = f'(s)\left(\alpha k(x)|f(s)|^{p-2}f(s) + \beta\left(\int_{\mathbb{R}^N}\frac{|f(v)|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}y\right)|f(v)|^{22^*_\mu-2}f(s)\right).$$

Therefore, let $u = f(v)$ and since $(f^{-1})'(t) = [f'(f^{-1}(t))]^{-1} = \sqrt{1+2t^2}$, we conclude that u is a nontrivial solution of the problem

$$-\left(a + b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u - a[\Delta(u^2)]u = \alpha k(x)|u|^{p-2}u + \beta\left(\int_{\mathbb{R}^N}\frac{|u|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{22^*_\mu-2}u.$$

3 The Palais-Smale condition

In this section, we will use the concentration-compactness principle for studying the critical Choquard equation [30] which is due to Lions [31] to prove the $(PS)_c$ condition.

Lemma 3.1. *Let* $1 < p < 4$. *Then any* $(PS)_c$ *sequence* $\{v_n\}$ *is bounded in* $D^{1,2}(\mathbb{R}^N)$.

Proof. Let $\{v_n\}$ be a $(PS)_c$ sequence in $D^{1,2}(\mathbb{R}^N)$ such that

$$\begin{aligned} c + o(1) = J(v_n) = \frac{a}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N}|f'(v_n)|^2|\nabla v_n|^2\mathrm{d}x\right)^2 \\ - \frac{\alpha}{p}\int_{\mathbb{R}^N}k(x)|f(v_n)|^p\mathrm{d}x - \frac{\beta}{22^*_\mu}\iint_{\mathbb{R}^{2N}}\frac{|f(v_n(x))|^{22^*_\mu}|f(v_n(y))|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{3.1}$$

$$o(1)\|w\| = \langle J'(v_n), w\rangle = a\int_{\mathbb{R}^N} \nabla v_n \nabla w \mathrm{d}x - \alpha \int_{\mathbb{R}^N} k(x)|f(v_n)|^{p-2} f(v_n) f'(v_n) w \mathrm{d}x$$
$$+ b\left(\int_{\mathbb{R}^N} \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\mathrm{d}x\right)\left(\int_{\mathbb{R}^N} \frac{\nabla v_n \nabla w(1+2f^2(v_n)) - 2|\nabla v_n|^2 f(v_n) f'(v_n) w}{[1+2f^2(v_n)]^2}\mathrm{d}x\right) \tag{3.2}$$
$$- \beta \iint_{\mathbb{R}^{2N}} \frac{|f(v_n(x))|^{22^*_\mu} |f(v_n(y))|^{22^*_\mu - 2} f(v_n(y)) f'(v_n(y)) w(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.$$

Choose $w = w_n = \sqrt{1+2f^2(v_n)}f(v_n)$, we have $w_n \in D^{1,2}(\mathbb{R}^N)$. From (f_4) and since

$$|\nabla w_n| = \left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|,$$

we deduce that $\|w_n\| \leq c\|v_n\|$. By (3.2) we have

$$o(1)\|v_n\| = \langle J'(v_n), w_n\rangle = a\int_{\mathbb{R}^N}\left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2\mathrm{d}x + b\left(\int_{\mathbb{R}^N} \frac{|\nabla v_n|^2}{1+2f^2(v_n)}\mathrm{d}x\right)^2$$
$$- \alpha\int_{\mathbb{R}^N} k(x)|f(v_n)|^p\mathrm{d}x - \beta\iint_{\mathbb{R}^{2N}} \frac{|f(v_n(x))|^{22^*_\mu}|f(v_n(y))|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \tag{3.3}$$

Thus, using Hölder's inequality and Sobolev embedding, and together with (3.1), (3.2), and (3.3), we have

$$c + o(1)\|v_n\| = J(v_n) - \frac{1}{22^*_\mu}\langle J'(v_n), w_n\rangle$$
$$= a\int_{\mathbb{R}^N}\left[\frac{1}{2} - \frac{1}{22^*_\mu}\left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)\right]|\nabla v_n|^2\mathrm{d}x$$
$$+ \left(\frac{1}{4} - \frac{1}{22^*_\mu}\right)b\left(\int_{\mathbb{R}^N}|f'(v_n)|^2|\nabla v_n|^2\mathrm{d}x\right)^2 + \left(\frac{1}{22^*_\mu} - \frac{1}{p}\right)\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^p\mathrm{d}x$$
$$\geq \left(\frac{1}{2} - \frac{1}{2^*_\mu}\right)a\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x - \left(\frac{1}{p} - \frac{1}{22^*_\mu}\right)\alpha\left(\int_{\mathbb{R}^N}|k(x)|^r\mathrm{d}x\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^N}|f(v_n)|^{22^*}\mathrm{d}x\right)^{\frac{p}{22^*}}$$
$$\geq \left(\frac{1}{2} - \frac{1}{2^*_\mu}\right)a\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x - \left(\frac{1}{p} - \frac{1}{22^*_\mu}\right)\alpha\left(\int_{\mathbb{R}^N}|k(x)|^r\mathrm{d}x\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^N}|\nabla f^2(v_n)|^2\mathrm{d}x\right)^{\frac{p}{4}}$$
$$\geq \left(\frac{1}{2} - \frac{1}{2^*_\mu}\right)a\|v_n\|^2 - c\|v_n\|^{\frac{p}{2}},$$

which implies that $\{v_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$ since $2 < 2^*_\mu$ and $1 < p < 4$. □

Lemma 3.2. *Let $c < 0$, $0 < \mu < 4$, and $1 < p < 4$. The next two properties hold.*
(*i*) *For each $\beta > 0$ there exists $\overline{\Lambda} > 0$ such that J satisfies the $(PS)_c$ condition for all $\alpha \in (0, \overline{\Lambda})$.*
(*ii*) *For each $\alpha > 0$ there exists $\underline{\Lambda} > 0$ such that J satisfies the $(PS)_c$ condition for any $\beta \in (0, \underline{\Lambda})$.*

Proof. Let $\{v_n\} \subset D^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$-sequence. By Lemma 3.1, $\{v_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$. Then $\{f(v_n)\}$ is also bounded in $D^{1,2}(\mathbb{R}^N)$. Therefore, we can assume that $v_n \rightharpoonup v$ in $D^{1,2}(\mathbb{R}^N)$, $v_n \to v$ a.e. in $\mathbb{R}^N$, since $f \in C^\infty$, then $f^2(v_n) \to f^2(v)$ a.e. in $\mathbb{R}^N$ and then $f^2(v_n) \rightharpoonup f^2(v)$ in $D^{1,2}(\mathbb{R}^N)$. Hence, we can assume that

$$|\nabla f^2(v_n)|^2 \rightharpoonup \omega, \quad |f(v_n)|^{22^*} \to \zeta, \quad \left(\int_{\mathbb{R}^N} \frac{|f(v_n(y))|^{22^*_\mu}}{|x-y|^\mu} \mathrm{d}y\right)|f(v_n)|^{22^*_\mu} \rightharpoonup \nu,$$

where ω, ζ, and ν are bounded nonnegative measures on $\mathbb{R}^N$. By the concentration-compactness principle in [30], there exist at most countable sets I, sequences of points $\{x_i\}_{i\in I} \subset \mathbb{R}^N$, and families of positive numbers $\{\nu_i : i \in I\}$, $\{\omega_i : i \in I\}$, and $\{\zeta_i : i \in I\}$ such that

$$\nu = \left(\int_{\mathbb{R}^N} \frac{|f(v(y))|^{22^*_\mu}}{|x-y|^\mu} \mathrm{d}y\right)|f(v(y))|^{22^*_\mu} + \sum_{i\in I} \nu_i \delta_{x_i}, \tag{3.4}$$

$$\omega \geq |\nabla f^2(v)|^2 + \sum_{i\in I} \omega_i \delta_{x_i}, \quad \zeta \geq |f(v(y))|^{22^*} + \sum_{i\in I} \zeta_i \delta_{x_i}, \tag{3.5}$$

$$S_{H,L} \nu_i^{\frac{1}{2^*_\mu}} \leq \omega_i \quad \text{and} \quad \nu_i \leq C(N,\mu)\zeta_i^{\frac{2N-\mu}{N}}, \tag{3.6}$$

where δ_{x_i} is the Dirac mass at x_i. Now, we take a smooth cut-off function $\varphi_{\varepsilon,i}$ centered at x_i such that

$$0 \leq \varphi_{\varepsilon,i}(x) \leq 1, \quad \varphi_{\varepsilon,i}(x) = 1 \text{ in } B\left(x_i, \frac{\varepsilon}{2}\right), \quad \varphi_{\varepsilon,i}(x) = 0 \text{ in } \mathbb{R}^N \backslash B(x_i, \varepsilon), \quad |\nabla \varphi_{\varepsilon,i}(x)| \leq \frac{4}{\varepsilon},$$

for any $\varepsilon > 0$ small. Let $w_n = \sqrt{1 + 2f^2(v_n)} f(v_n)$, then $\{w_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and $\langle J'(v_n), w_n\varphi_{\varepsilon,i}\rangle \to 0$. Thus,

$$\begin{aligned}
&-a \int_{\mathbb{R}^N} \sqrt{1+2f^2(v_n)} f(v_n) \nabla v_n \nabla \varphi_{\varepsilon,i} \mathrm{d}x - b\left(\int_{\mathbb{R}^N} \frac{|\nabla v_n|^2}{1+2f^2(v_n)} \mathrm{d}x\right)\left(\int_{\mathbb{R}^N} \frac{f(v_n)\nabla v_n \nabla \varphi_{\varepsilon,i}}{\sqrt{1+2f^2(v_n)}} \mathrm{d}x\right) \\
&= a \int_{\mathbb{R}^N} \left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2 \varphi_{\varepsilon,i} \mathrm{d}x + b\left(\int_{\mathbb{R}^N} \frac{|\nabla v_n|^2}{1+2f^2(v_n)} \mathrm{d}x\right)\left(\int_{\mathbb{R}^N} \frac{|\nabla v_n|^2 \varphi_{\varepsilon,i}}{1+2f^2(v_n)} \mathrm{d}x\right) \\
&\quad - \alpha \int_{\mathbb{R}^N} k(x)|f(v_n)|^p \varphi_{\varepsilon,i} \mathrm{d}x - \beta \iint_{\mathbb{R}^{2N}} \frac{|f(v_n(x))|^{22^*_\mu} |f(v_n(y))|^{22^*_\mu} \varphi_{\varepsilon,i}(y)}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y + o_n(1).
\end{aligned} \tag{3.7}$$

The Hölder inequality and (f_4) imply that

$$\begin{aligned}
0 &\leq \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left| a \int_{\mathbb{R}^N} \sqrt{1+2f^2(v_n)} f(v_n) \nabla v_n \nabla \varphi_{\varepsilon,i} \mathrm{d}x \right| \\
&\leq c \lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} v_n \nabla v_n \nabla \varphi_{\varepsilon,i} \mathrm{d}x \\
&\leq C \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left[\left(\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |v_n \nabla \varphi_{\varepsilon,i}|^2 \mathrm{d}x\right)^{\frac{1}{2}}\right] \\
&\leq C \lim_{\varepsilon\to 0} \left(\int_{B(x_i,2\varepsilon)} |v|^{2^*} \mathrm{d}x\right)^{\frac{1}{2^*}} = 0.
\end{aligned} \tag{3.8}$$

Similarly, we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left[b\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^2}{1+2f^2(v_n)}dx\right)\left(\int_{\mathbb{R}^N}\frac{f(v_n)\nabla v_n\nabla\varphi_{\varepsilon,i}}{\sqrt{1+2f^2(v_n)}}dx\right)\right]=0. \tag{3.9}$$

From the definition of $\varphi_{\varepsilon,i}$ that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}k(x)|f(v_n)|^p\varphi_{\varepsilon,i}dx=0.$$

By (3.7)–(3.9), we get

$$\begin{aligned}0&=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{a\int_{\mathbb{R}^N}\left(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2\varphi_{\varepsilon,i}dx+b\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^2}{1+2f^2(v_n)}dx\right)\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^2\varphi_{\varepsilon,i}}{1+2f^2(v_n)}dx\right)\right.\\&\quad\left.-\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^p\varphi_{\varepsilon,i}dx-\beta\iint_{\mathbb{R}^{2N}}\frac{|f(v_n(x))|^{22^*_\mu}|f(v_n(y))|^{22^*_\mu}\varphi_{\varepsilon,i}(y)}{|x-y|^\mu}dxdy\right\}\\&\ge\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{\frac{a}{2}\int_{\mathbb{R}^N}\varphi_{\varepsilon,i}|\nabla f^2(v_n)|^2dx-\beta\iint_{\mathbb{R}^{2N}}\frac{|f(v_n(x))|^{22^*_\mu}|f(v_n(y))|^{22^*_\mu}\varphi_{\varepsilon,i}(y)}{|x-y|^\mu}dxdy\right\}\\&=\frac{a}{2}\omega_i-\beta v_i.\end{aligned}\tag{3.10}$$

This fact implies that $a\omega_i\le 2\beta v_i$. Together with (3.6), we obtain

$$\omega_i\ge(2^{-1}\beta^{-1}aS_{H,L}^{2^*_\mu})^{\frac{1}{2^*_\mu-1}}\quad\text{or}\quad\omega_i=0. \tag{3.11}$$

If $w_{i_0}\ge(2^{-1}\beta^{-1}aS_{H,L}^{2^*_\mu})^{\frac{1}{2^*_\mu-1}}$ for $i_0\in I$. From the Hölder inequality, the Sobolev embedding, and the Young inequality, we have

$$\begin{aligned}\alpha\int_{\mathbb{R}^N}k(x)|f(v)|^pdx&\le\alpha\|k\|_rS^{-\frac{p}{2}}\|f(v)\|^p\\&=\left(\left[\left(\frac12-\frac{1}{2^*_\mu}\right)\frac{a}{2}\left(\frac1p-\frac{1}{22^*_\mu}\right)^{-1}\right]^{\frac{p}{2}}\|f(v)\|^p\right)\left(\left[\left(\frac12-\frac{1}{2^*_\mu}\right)\frac{a}{2}\left(\frac1p-\frac{1}{22^*_\mu}\right)^{-1}\right]^{\frac{-p}{2}}\alpha\|k\|_rS^{-\frac{p}{2}}\right)\\&\le\left(\frac12-\frac{1}{2^*_\mu}\right)\frac{a}{2}\left(\frac1p-\frac{1}{22^*_\mu}\right)^{-1}\|f(v)\|^2+\frac{2-p}{2}\left[\left(\frac12-\frac{1}{2^*_\mu}\right)^{-1}\frac{2}{aS}\left(\frac1p-\frac{1}{22^*_\mu}\right)\right]^{\frac{p}{2-p}}\|k\|_r^{\frac{2}{2-p}}\alpha^{\frac{2}{2-p}}.\end{aligned}\tag{3.12}$$

According to this fact, we have

$$\begin{aligned}0>c&=\lim_{n\to+\infty}\left(J(v_n)-\frac{1}{22^*_\mu}\langle J'(v_n),\sqrt{1+2f^2(v_n)}f(v_n)\rangle\right)\\&=\lim_{n\to+\infty}\left\{a\int_{\mathbb{R}^N}\left[\frac12-\frac{1}{22^*_\mu}\left(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right)\right]|\nabla v_n|^2dx+\left(\frac14-\frac{1}{22^*_\mu}\right)b\left(\int_{\mathbb{R}^N}|f'(v_n)|^2|\nabla v_n|^2dx\right)^2\right.\\&\quad\left.+\left(\frac{1}{22^*_\mu}-\frac1p\right)\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^pdx\right\}\end{aligned}\tag{3.13}$$

$$\geq \lim_{n\to+\infty}\left\{\left(\frac{1}{2}-\frac{1}{2_\mu^*}\right)a\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x-\left(\frac{1}{p}-\frac{1}{22_\mu^*}\right)\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^p\mathrm{d}x\right\}$$
$$\geq \lim_{n\to+\infty}\left\{\left(\frac{1}{2}-\frac{1}{2_\mu^*}\right)\frac{a}{2}\int_{\mathbb{R}^N}|\nabla f^2(v_n)|^2\mathrm{d}x-\left(\frac{1}{p}-\frac{1}{22_\mu^*}\right)\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^p\mathrm{d}x\right\}$$
$$\geq\left(\frac{1}{2}-\frac{1}{2_\mu^*}\right)\frac{a}{2}\left(\|f(v)\|^2+\sum_{i\in I}w_i\right)-\left(\frac{1}{p}-\frac{1}{22_\mu^*}\right)\alpha\int_{\Omega}k(x)|f(v)|^p\mathrm{d}x$$
$$\geq\left(\frac{1}{2}-\frac{1}{2_\mu^*}\right)\frac{a}{2}w_{i_0}-\frac{2-p}{2}\left[\left(\frac{1}{2}-\frac{1}{2_\mu^*}\right)^{-1}\frac{2}{aS}\right]^{\frac{p}{2-p}}\left(\frac{1}{p}-\frac{1}{22_\mu^*}\right)^{\frac{2}{2-p}}\|k\|_r^{\frac{2}{2-p}}\alpha^{\frac{2}{2-p}}$$
$$\geq\left(\frac{1}{2}-\frac{1}{2_\mu^*}\right)(2^{-1}aS_{H,L})^{\frac{2_\mu^*}{2_\mu^*-1}}\beta^{-\frac{1}{2_\mu^*-1}}-\frac{2-p}{2}\left[\left(\frac{1}{2}-\frac{1}{2_\mu^*}\right)^{-1}\frac{2}{aS}\right]^{\frac{p}{2-p}}\left(\frac{1}{p}-\frac{1}{22_\mu^*}\right)^{\frac{2}{2-p}}\|k\|_r^{\frac{2}{2-p}}\alpha^{\frac{2}{2-p}}.$$

Thus, for any $\beta>0$, we can choose $\alpha_1>0$ so small such that for every $0<\alpha<\alpha_1$, the last term on the right-hand side above is greater than zero, which is a contradiction.

Similarly, if $\alpha>0$ is given, we take $\beta_1>0$ so small that for every $\beta\in(0,\beta_1)$ again the right-hand side of (3.13) is greater than zero. This gives the required contradiction. Consequently, $\omega_i=0$ for all $i\in I$ in (3.11).

To obtain the possible concentration of mass at infinity, similarly, we define a cut-off function ψ_R in $C^\infty(\mathbb{R}^N)$ such that $\psi_R=0$ in $B_R(0)$, $\psi_R=1$ in $\mathbb{R}^N\backslash B_{R+1}(0)$, and $|\nabla\psi_R|\leq 2/R$ in $\mathbb{R}^N$. Let

$$\omega_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}|\nabla u_n|^2\mathrm{d}x,$$
$$\zeta_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}|u_n|^{2^*}\mathrm{d}x,$$

and

$$\nu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}(\mathcal{K}_\mu*|u_n|^{2_\mu^*})|u_n|^{2_\mu^*}\mathrm{d}x.$$

Thus, the Hardy-Littlewood-Sobolev and the Hölder inequalities give

$$\nu_\infty=\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|f(v_n(y))|^{22_\mu^*}}{|x-y|^\mu}\mathrm{d}y\right)|f(v_n(x))|^{22_\mu^*}\psi_R(y)\mathrm{d}x$$
$$\leq C(N,\mu)\lim_{R\to\infty}\lim_{n\to\infty}|f(v_n)|_{2^*}^{2_\mu^*}\left(\int_{\mathbb{R}^N}|f(v_n(x))|^{2^*}\psi_R(y)\mathrm{d}x\right)^{\frac{2_\mu^*}{2^*}}\leq\hat{C}\zeta_\infty^{\frac{2_\mu^*}{2^*}}.$$

Therefore,

$$0=\lim_{R\to\infty}\lim_{n\to\infty}\langle J'(v_n),\psi_R w_n\rangle$$
$$=\lim_{R\to\infty}\lim_{n\to\infty}\left\{a\int_{\mathbb{R}^N}\left(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2\psi_R\mathrm{d}x+b\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^2}{1+2f^2(v_n)}\mathrm{d}x\right)\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^2\psi_R}{1+2f^2(v_n)}\mathrm{d}x\right)\right.$$
$$\left.-\alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^p\psi_R\mathrm{d}x-\beta\iint_{\mathbb{R}^{2N}}\frac{|f(v_n(x))|^{22_\mu^*}|f(v_n(y))|^{22_\mu^*}\psi_R(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right\}\tag{3.14}$$

$$\geq \lim_{R\to\infty}\lim_{n\to\infty}\left\{\frac{a}{2}\int_{\mathbb{R}^N}\psi_R|\nabla f^2(v_n)|^2\mathrm{d}x - \alpha\int_{\mathbb{R}^N}k(x)|f(v_n)|^p\psi_R\mathrm{d}x - \beta\iint_{\mathbb{R}^{2N}}\frac{|f(v_n(x))|^{22^*_\mu}|f(v_n(y))|^{22^*_\mu}\psi_R(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right\}$$
$$\geq \frac{a}{2}\omega_\infty - \beta\hat{C}\zeta_\infty^{\frac{2^*_\mu}{2^*}}.$$

Therefore, $\frac{a}{2}\omega_\infty \leq \beta\hat{C}\zeta_\infty^{\frac{2^*_\mu}{2^*}}$. As the discussion in [30], we obtain

$$\omega_\infty \geq \left(2^{-1}aS^{\frac{2^*_\mu}{2}}\hat{C}^{-1}\beta^{-1}\right)^{\frac{2}{2^*_\mu-2}} \text{ or } \omega_\infty = 0. \tag{3.15}$$

As in (3.8) and (3.9), we have

$$0 > c \geq \left(\frac{1}{2}-\frac{1}{2^*_\mu}\right)(2^{-1}aS)^{\frac{2^*_\mu}{2^*_\mu-2}}\hat{C}^{-\frac{2}{2^*_\mu-2}}\beta^{-\frac{2}{2^*_\mu-2}} - \frac{2-p}{2}\left[\left(\frac{1}{2}-\frac{1}{2^*_\mu}\right)^{-1}\frac{2}{aS}\right]^{\frac{p}{2-p}}\left(\frac{1}{p}-\frac{1}{22^*_\mu}\right)^{\frac{2}{2-p}}\|k\|_r^{\frac{2}{2-p}}\alpha^{\frac{2}{2-p}}. \tag{3.16}$$

Thus, for any $\beta > 0$, we choose $\alpha_2 > 0$ so small that for every $\alpha \in (0, \alpha_2)$ the right-hand side of (3.16) is greater than zero, which is a contradiction.

Similarly, if $\alpha > 0$ is given, we select $\beta_2 > 0$ so small that for every $\beta \in (0, \beta_2)$ the right-hand side of (3.12) is greater than zero. This gives the required contradiction. Therefore, $\omega_\infty = 0$ in (3.11).

From the arguments above, put

$$\overline{\Lambda} = \min\{\alpha_1, \alpha_2\} \quad \text{and} \quad \underline{\Lambda} = \min\{\beta_1, \beta_2\}.$$

Then, for any $c < 0$ and $\beta > 0$ we have

$$\omega_i = 0 \text{ for all } i \in I \quad \text{and} \quad \omega_\infty = 0$$

for all $\alpha \in (0, \overline{\Lambda})$.

Similarly, for any $c < 0$ and $\alpha > 0$ we again have

$$\omega_i = 0 \text{ for all } i \in I \quad \text{and} \quad \omega_\infty = 0$$

for any $\beta \in (0, \overline{\Lambda})$.

Hence, as $n \to \infty$

$$\iint_{\mathbb{R}^{2N}}\frac{|v_n(x)|^{2^*_\mu}|v_n(y)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \to \iint_{\mathbb{R}^{2N}}\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y$$

and

$$\int_{\mathbb{R}^N}k(x)(|v_n|^q - |v|^q)\mathrm{d}x \leq \|k\|_r\||v_n|^q - |v|^q\|_{\frac{2^*_\mu}{q}} \to 0.$$

Since $(\|v_n\|)_n$ is bounded and $J'(v) = 0$, the weak lower semicontinuity of the norm and the Brézis-Lieb lemma yield as $n \to \infty$

$$\begin{aligned} o(1)\|v_n\| = \langle J'(v_n), w_n\rangle &= a\int_{\mathbb{R}^N}\left(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right)|\nabla v_n|^2\mathrm{d}x + b\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^2}{1+2f^2(v_n)}\mathrm{d}x\right)^2 \\ &\quad - \alpha\int_{\mathbb{R}^N}k(x)|f(v_n]^p\mathrm{d}x - \beta\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(v_n(x))|^{22^*_\mu}|f(v_n(y))|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\ &= a\|v_n\| + a\int_{\mathbb{R}^N}\frac{2f^2(v_n)}{1+2f^2(v_n)}|\nabla v_n|^2\mathrm{d}x + b\left(\int_{\mathbb{R}^N}\frac{|\nabla v_n|^2}{1+2f^2(v_n)}\mathrm{d}x\right)^2 \end{aligned}$$

$$
\begin{aligned}
&- \alpha \int_{\mathbb{R}^N} k(x)|f(v_n)|^p \mathrm{d}x - \beta \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(v(x))|^{22^*_\mu}|f(v(y))|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\
&\geq a\|v_n - v\|^2 + a\|v\|^2 + a\int_{\mathbb{R}^N} \frac{2f^2(v)}{1+2f^2(v)}|\nabla v|^2\mathrm{d}x + b\left(\int_{\mathbb{R}^N}\frac{|\nabla v|^2}{1+2f^2(v)}\mathrm{d}x\right)^2 \\
&- \alpha \int_{\mathbb{R}^N} k(x)|f(v)|^p \mathrm{d}x - \beta \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(v(x))|^{22^*_\mu}|f(v(y))|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\
&= \|v_n - v\|^2 + o(1)\|v\|.
\end{aligned}
$$

This fact implies that $\{v_n\}$ strongly converges to v in $D^{1,2}(\mathbb{R}^N)$. This completes the proof of Lemma 3.2. □

4 Proof of Theorems 1.1

In this section, we will use the following version of the symmetric mountain-pass lemma to prove the existence of infinitely many solutions of (1.1) which tend to zero.

Lemma 4.1. [32] *Let E be an infinite-dimensional Banach space and $J \in C^1(E, \mathbb{R})$. Suppose that the following properties hold.*

(J_1) *J is even, bounded from below in E, $J(0) = 0$ and J satisfies the local Palais-Smale condition.*

(J_2) *For each $n \in \mathbb{N}$ there exists $A_n \in \Sigma_n$ such that $\sup_{u\in A_n} J(u) < 0$, where*

$$\Sigma_n := \{A : A \subset E \text{ is closed symmetric}, \ 0 \notin A, \gamma(A) \geq n\}$$

and $\gamma(A)$ is a genus of A.

Then J admits a sequence of critical points $(u_n)_n$ such that $J(u_n) \leq 0$, $u_n \neq 0$ for each n and $(u_n)_n$ converges to zero as $n \to \infty$.

Note that

$$
\begin{aligned}
J(v) &\geq \frac{a}{2}\|v\|^2 - \frac{\alpha}{p}\left(\int_{\mathbb{R}^N}|k(x)|^r\mathrm{d}x\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^N} f^{2(2^*)}(v)\mathrm{d}x\right)^{\frac{p}{2(2^*)}} - \frac{S_{H,L}^{-1}}{22^*_\mu}\beta\|f(v)\|^{22^*_\mu} \\
&\geq \frac{a}{2}\|v\|^2 - \alpha c_1\|f(v)\|^{\frac{p}{2}} - \beta c_2\|f(v)\|^{22^*_\mu} \\
&\geq \frac{a}{2}\|v\|^2 - \alpha c_1\|v\|^{\frac{p}{2}} - \beta c_2\|v\|^{22^*_\mu} \\
&=: Q(\|v\|),
\end{aligned}
$$

where c_1 and c_2 are some positive constants and $Q(t) := \frac{a}{2}t^2 - \alpha c_1 t^{\frac{p}{2}} - \beta c_2 t^{22^*}$. Obviously, fixed $\beta > 0$ there exists $\alpha_1 > 0$ so small that for every $0 < \alpha < \alpha_1$, there exists $0 < t_0 < t_1$, $Q(t) > 0$ for $t_0 < t < t_1$, $Q(t) < 0$ for $t > t_1$ and $0 < t < t_0$. Similarly, fixed $\alpha > 0$, we can choose $\beta_1 > 0$ with the property that t_0, t_1 as above exist for each $0 < \beta < \beta_1$. Clearly, $Q(t_0) = 0 = Q(t_1)$. Following the same idea as in [34], we consider the truncated functional

$$
\begin{aligned}
\widetilde{J}(v) = &\frac{a}{2}\int_{\mathbb{R}^N}|\nabla v|^2\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N}|f'(v)|^2|\nabla v|^2\mathrm{d}x\right)^2 - \frac{\alpha}{p}\int_{\mathbb{R}^N}k(x)|f(v)|^p\mathrm{d}x \\
&- \frac{\beta}{22^*_\mu}\varphi(v)\iint_{\mathbb{R}^{2N}}\frac{|f(v(x))|^{22^*_\mu}|f(v(y))|^{22^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y,
\end{aligned}
\tag{4.1}
$$

where $\varphi(v) = \chi(\|v\|)$ and $\chi : \mathbb{R}^+ \to [0,1]$ is a nonincreasing C^∞ function such that $\chi(t) = 1$ if $t \le T_0$ and $\chi(t) = 0$ if $t \ge T_1$. Thus,

$$\widetilde{J}(v) \ge \overline{Q}(\|v\|),$$

where $\overline{Q}(t) = \frac{a}{2}t^2 - \alpha c_1 t^{\frac{p}{2}} - \beta c_2 t^{22^*}\varphi(t)$. It is clear that $\widetilde{J}(v) \in C^1$ and is bounded from below in $D^{1,2}(\mathbb{R}^N)$.

From the above arguments, we have the next results for functional $\widetilde{J}(v)$.

Lemma 4.2. *Let $\widetilde{J}(v)$ be defined as in* (4.1). *Then*

(i) *If $\widetilde{J}(v) < 0$, then $\|v\| \le T_0$ and $\widetilde{J}(v) = J(v)$.*

(ii) *Let $c < 0$. Then, for any $\beta > 0$ there exists $\overline{\Lambda} > 0$ such that $\widetilde{J}$ satisfies the $(PS)_c$ condition for all $\alpha \in (0, \overline{\Lambda})$.*

(iii) *Let $c < 0$. Then, for any $\alpha > 0$ there exists $\underline{\Lambda} > 0$ such that $\widetilde{J}$ satisfies the $(PS)_c$ condition for all $\beta \in (0, \underline{\Lambda})$.*

Proof of Theorem 1.1. Clearly, $\widetilde{J}(0) = 0$, and $\widetilde{J}$ is of class $C^1(D^{1,2}(\mathbb{R}^N))$, even, coercive, and bounded from below in $D^{1,2}(\mathbb{R}^N)$. Furthermore, $\widetilde{J}$ satisfies the $(PS)_c$ condition in $D^{1,2}(\mathbb{R}^N)$, with $c < 0$, by Lemma 4.2.

For any $n \in \mathbb{N}$, we take n disjointing open sets X_i such that $\cup_{i=1}^n X_i \subset \Omega$, where Ω is the nonempty open set introduced in the statement of Theorem 1.1. For each $i = 1, 2, \cdots, n$, take $v_i \in (D^{1,2}(\mathbb{R}^N) \cap C_0^\infty(X_i))\backslash\{0\}$, with $\|v_i\| = 1$. Put $E_n = \text{span}\{v_1, v_2, \cdots, v_n\}$.

Thus, for any $v \in E_n$, with $\|v\| = \rho$, we have

$$\begin{aligned}\widetilde{J}(v) &\le \frac{a}{2}\|v\|^2 + \frac{b}{4}\|v\|^4 - \frac{\alpha}{p}\int_\Omega k(x)|v|^p \mathrm{d}x - \frac{\beta}{2\cdot 2_\mu^*}\|v\|_{NL}^{2\cdot 2_\mu^*} \\ &\le \frac{a}{2}\rho^2 + \frac{b}{4}\rho^4 - C_1\rho^p - C_2\rho^{2\cdot 2_\mu^*},\end{aligned}$$

where C_1 and C_2 are some positive constants, since all the norms are equivalent in the finite dimensional space E_n. Hence, $\widetilde{J}(v) < 0$ provided that $\rho > 0$ is sufficiently small, being $1 < q < 4$. Therefore,

$$\{u \in E_n : \|v\| = \rho\} \subset \{v \in E_n : \widetilde{J}(v) < 0\}.$$

Moreover,

$$\gamma(\{v \in E_n : \|v\| = \rho\}) = n.$$

Hence by the monotonicity of the genus γ, we have

$$\gamma(\{v \in E_n : \widetilde{J}(v) < 0\}) \ge n.$$

Choosing $A_n = \{v \in E_n : \widetilde{J}(v) < 0\}$, we have $A_n \in \sum_n$ and $\sup_{v\in A_n}\widetilde{J}(v) < 0$. Therefore, all the assumptions of Lemma 4.1 are satisfied, since $D^{1,2}(\mathbb{R}^N)$ is a real infinite Hilbert space. Thus, there exists a sequence $\{v_n\}$ in $D^{1,2}(\mathbb{R}^N)$ such that

$$\widetilde{J}(v_n) \le 0, \quad v_n \ne 0, \quad \widetilde{J}'(v_n) = 0 \text{ for each } n \quad \text{and} \quad \|v_n\| \to 0 \text{ as } n \to \infty.$$

Combining with Lemma 4.2 and taking n so large that $\|v_n\| \le \rho$ is small enough, then these infinitely many nontrivial functions v_n are solutions of (2.5). This completes the proof of Theorem 1.1, since $u_m = f(v_m) \ne u_n = f(v_n)$ if $v_m \ne v_n$ and $f \in C^\infty$. □

5 Proof of Theorem 1.2

In this section, we give the following general version of the mountain pass lemma in [33], which will be used to prove Theorem 1.2.

Proposition 5.1. [33] *Let X be an infinite dimensional Banach space with $X = V \oplus Y$, where V is finite dimensional and let $J \in C^1(X, \mathbb{R})$ be an even functional with $J(0) = 0$ such that the following conditions hold:*

(I_1) *There exist positive constants $\varrho, \rho > 0$ such that $J(u) \geq \varrho$ for all $u \in \partial B_\rho(0) \cap Y$.*

(I_2) *There exists $c^* > 0$ such that J satisfies the $(PS)_c$ condition for $0 < c < c^*$.*

(I_3) *For each finite dimensional subspace $\widehat{X} \subset X$, there exists $R = R(\widehat{X})$ such that $J(u) \leq 0$ for all $u \in \widehat{X} \backslash B_R(0)$.*

Suppose that V is k dimensional and $V = \operatorname{span}\{e_1, e_2, \ldots, e_k\}$. For $n \geq k$, inductively choose $e_{n+1} \notin X_n := \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. Let $R_n = R(X_n)$ and $D_n = B_{R_n}(0) \cap X_n$. Define

$$G_n := \{h \in C(D_n, X) : h \text{ is odd and } h(u) = u, \ \forall \partial B_{R_n}(0) \cap X_n\}$$

and

$$\Gamma_j := \{h(\overline{D_n \backslash E}) : h \in G_n, n \geq j, E \in \sum \text{ and } \gamma(E) \leq n - j\}.$$

For each $j \in \mathbb{N}$, let

$$c_j := \inf_{K \in \Gamma_j} \max_{u \in K} J(u).$$

Then, $0 < \varrho \leq c_j \leq c_{j+1}$ for $j > k$, and if $j > k$ and $c_j < c^*$, then we conclude that c_j is the critical value of J. Moreover, if $c_j = c_{j+1} = \ldots = c_{j+l} = c < c^*$ for $j > k$, then $\gamma(K_c) \geq l + 1$, where

$$K_c := \{u \in E : J(u) = c \text{ and } J'(u) = 0\}.$$

Lemma 5.1. *Let $\alpha \in \left(0, \frac{1}{2} a S^2 \|k\|_{L^r}^{-1}\right)$. Then J satisfies $(PS)_c$ condition, for all $c \in (0, c^*)$, where*

$$c^* := \min\left\{\left(\frac{1}{4} - \frac{1}{22_\mu^*}\right)(2^{-1} a S_{H,L})^{\frac{2_\mu^*}{2_\mu^*-1}}, \left(\frac{1}{4} - \frac{1}{22_\mu^*}\right)(2^{-1} a S)^{\frac{2_\mu^*}{2_\mu^*-2}} \hat{C}^{-\frac{2}{2_\mu^*-2}}\right\}. \tag{5.1}$$

Proof. On the one hand, from Hölder's inequality, Sobolev embedding theorem and (f_7), we get

$$\int_{\mathbb{R}^N} k(x)|f(v)|^4 \mathrm{d}x \leq 2S^{-2} \|k\|_{L^r} \|v\|^2. \tag{5.2}$$

Together $\alpha \in \left(0, \frac{1}{2} a S^2 \|k\|_{L^r}^{-1}\right)$ with (5.2), and proceeding as in proof of Lemma 3.1, we have

$$\begin{aligned}
c + o(1)\|v_n\| &= J(v_n) - \frac{1}{22_\mu^*}\langle J'(v_n), w_n\rangle \\
&\geq \left(\frac{1}{2} - \frac{1}{2_\mu^*}\right) a \int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x - \left(\frac{1}{4} - \frac{1}{22_\mu^*}\right) \alpha \int_{\mathbb{R}^N} k(x)|f(v_n)|^4 \mathrm{d}x \\
&\geq \left(\frac{1}{2} - \frac{1}{2_\mu^*}\right) a \|v_n\|^2 - \left(\frac{1}{2} - \frac{1}{2_\mu^*}\right) \alpha S^{-2} \|k\|_{L^r} \|v_n\|^2 \\
&> \left(\frac{1}{4} - \frac{1}{22_\mu^*}\right) a \|v_n\|^2.
\end{aligned}$$

This fact implies that $\{v_n\}$ is bounded since $2 < 2_\mu^*$. As similar discussion in Lemma 3.2, we deduce that (3.11) and (3.15) hold. By contradiction, we assume that $w_{i_0} \geq (2^{-1} a S_{H,L}^{2_\mu^*})^{\frac{1}{2_\mu^*-1}}$ for $i_0 \in I$ and $\omega_\infty \geq \left(2^{-1} a S^{\frac{2_\mu^*}{2}} \hat{C}^{-1}\right)^{\frac{2}{2_\mu^*-2}}$ hold. Similar to Lemma 3.2, we deduce

$$c = \lim_{n \to +\infty} \left(J(v_n) - \frac{1}{22_\mu^*}\langle J'(v_n), \sqrt{1 + 2f^2(v_n)} f(v_n)\rangle\right) \tag{5.3}$$

$$\geq \lim_{n\to+\infty}\left\{\left(\frac{1}{2}-\frac{1}{2^*_\mu}\right)a\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x-\left(\frac{1}{2}-\frac{1}{2^*_\mu}\right)\alpha S^{-2}\|k\|_{L^r}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right\}$$
$$\geq \lim_{n\to+\infty}\left(\frac{1}{4}-\frac{1}{22^*_\mu}\right)\frac{a}{2}\int_{\mathbb{R}^N}|\nabla f^2(v_n)|^2\mathrm{d}x$$
$$\geq \left(\frac{1}{4}-\frac{1}{22^*_\mu}\right)\frac{a}{2}w_{i_0}\geq\left(\frac{1}{4}-\frac{1}{22^*_\mu}\right)(2^{-1}aS_{H,L})^{\frac{2^*_\mu}{2^*_\mu-1}}$$

and

$$c\geq\left(\frac{1}{4}-\frac{1}{22^*_\mu}\right)(2^{-1}aS)^{\frac{2^*_\mu}{2^*_\mu-2}}\hat{C}^{-\frac{2}{2^*_\mu-2}}. \tag{5.4}$$

Then, for any $c\in(0,c^*)$, (5.3) and (5.4) cannot happen. Thus, we have

$$\omega_i=0\ \text{ for all }\ i\in I\quad\text{ and }\quad\omega_\infty=0.$$

The rest of the proof is the same as in the proof to Lemma 3.2. Therefore, the compactness of the Palais-Smale sequence holds. □

Remark 5.1. It is easy to verify that the functional J satisfies the hypotheses (I_1) and (I_3) for $\alpha\in\left(0,\frac{1}{2}aS^2\|k\|_{L^r}^{-1}\right)$.

Lemma 5.2. *There exists a sequence $\{M_n\}\subset(0,+\infty)$ independent of α, with $M_n\leq M_{n+1}$, such that for any $\alpha>0$,*

$$c_n^\alpha:=\inf_{K\in\Gamma_n}\max_{u\in K}J(v)<M_n.$$

Proof. Our proof is similar to that presented in [35, Lemma 5]. The definition of c_n^α implies that

$$c_n^\alpha=\inf_{K\in\Gamma_n}\max_{v\in K}J(v)\leq\inf_{K\in\Gamma_n}\max_{v\in K}\left\{\frac{a}{2}\|v\|^2+\frac{b}{4}\|v\|^4-\frac{1}{22^*_\mu}\iint_{\mathbb{R}^{2N}}\frac{|f(v(x))|^{2^*_\mu}|f(v(y))|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right\}.$$

Let

$$M_n:=\inf_{K\in\Gamma_n}\max_{v\in K}\left\{\frac{a}{2}\|v\|^2+\frac{b}{4}\|v\|^4-\frac{1}{22^*_\mu}\iint_{\mathbb{R}^{2N}}\frac{|f(v(x))|^{2^*_\mu}|f(v(y))|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right\},$$

then we conclude that $M_n<+\infty$ and $M_n\leq M_{n+1}$ by the definition of Γ_n. □

Proof of Theorem 1.2. Taking $a^*>0$ large enough such that for any $a>a^*$, we have

$$0<c_1^\alpha\leq c_2^\alpha\leq\cdots\leq c_k^\alpha<M_k<c^*.$$

From Proposition 5.1, the levels $c_1^\alpha\leq c_2^\alpha\leq\cdots\leq c_k^\alpha$ are critical values of J. So, if $c_1^\alpha<c_2^\alpha<\cdots<c_k^\alpha$, the functional J has at least k critical points. Now, if $c_j^\alpha=c_{j+1}^\alpha$ for some $j=1,2,\dots,k-1$, again Proposition 5.1 implies that $K_{c_j^\alpha}$ is an infinite set (see [33, Chapter 7]) and hence in this case, problem (5.1) has infinitely many weak solutions. Consequently, problem (5.1) has at least k pair of weak solutions. Therefore, problem (2.5) has at least k pairs of solutions and $u=f(v)$ must solve problem (1.1). □

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