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Research Article

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Functional equations related to higher derivations in semiprime rings

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Abstract: We investigate the additivity and multiplicativity of centrally extended higher derivations and show that every centrally extended higher derivation of a semiprime ring with no nonzero central ideals is a higher derivation. Moreover, we study preservation of the center of the ring by a centrally extended higher derivation.

Keywords: functional equations, higher derivations, semiprime rings, central ideals

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1 Introduction

The purpose of the current article is to extend some results of Bell and Daif [1]. They defined the notion of *centrally extended derivation* on a ring R with center Z to be a map $d : R \to R$ satisfies $d(x + y) - d(x) - d(y) \in Z$ and $d(xy) - d(x)y - xd(y) \in Z$ for every $x, y \in R$. Recall that an additive map $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$, so one can see that the centrally extended derivation extends the concept of derivation. They showed that the notions of derivation and centrally extended derivation are coincident in semiprime rings with no nonzero central ideals. Motivated by Bell and Daif results, many algebraists studied centrally extended different kinds of maps (see [2–5]).

Let $\mathbb{N}_0$ be the set of all nonnegative integers and $D = (d_i)_{i\in\mathbb{N}_0}$ be a family of additive mappings of a ring R such that $d_0 = id_R$. Then D is said to be a *higher derivation* of R if for each $n \in \mathbb{N}_0$, $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ holds for all $x, y \in R$. There have been a lot of recent articles studying this interested notion of higher derivations based on different algebraic structures (see [6–10]). Also, extending results from derivations to higher derivations is a popular line of investigation, as an example, Ferrero and Haetinger [8] extended a very famous theorem by Herstein [11] on derivations to higher derivations.

In this article, we give the corresponding definition of a centrally extended higher derivation concept, as a natural extension to the centrally extended derivation, and prove that every centrally extended higher derivation of a semiprime ring with no nonzero central ideals is a higher derivation. Also, we study the influence of a centrally extended higher derivation on the center of the ring. Finally, proper examples are mentioned to show the existence of such families and to ensure that our conditions are not superfluous.

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2 Preliminaries

We begin by the definition

Definition 2.1. Let R be a ring. A family $D = (d_n)_{n\in\mathbb{N}_0}$ of mappings $d_n : R \to R$, with $d_0 = id_R$, is called a *centrally extended higher derivation*, if for all $x, y \in R$, $n \in \mathbb{N}_0$, the following two conditions hold
(i) $d_n(x + y) - d_n(x) - d_n(y) \in Z$,
(ii) $d_n(xy) - \sum_{i+j=n} d_i(x)d_j(y) \in Z$.

Clearly, every higher derivation is a centrally extended higher derivation. The converse in general need not be true. Of course, it happens that both are coincident in any commutative ring. In our main result it will be shown that this is also true in the case of a semiprime ring with no nonzero central ideals. The following example clarifies this point

Example 2.1. Let R be a ring with a nonzero central ideal J, let $g : R \to J$ be a nonzero constant mapping, and let $D = (d_n)_{n\in\mathbb{N}_0}$ be a higher derivation of R. Define a family $F = (f_n)_{n\in\mathbb{N}_0}$ of mappings $f_n : R \to R$, with $f_0 = id_R$ by $f_n(x) = d_n(x) + g(x)$ for all $x \in R$, $n \in \mathbb{N}_0 - \{0\}$. Then F is a centrally extended higher derivation which is not a higher derivation.

The following lemmas are of great importance in proving our main results.

Lemma 2.2. [12, Lemma 1.2] *Let R be a semiprime ring and $A, B : R \times R \to R$ be biadditive mappings. If $A(x, y)wB(x, y) = 0$ for all $x, y, w \in R$, then $A(x, y)wB(u, v) = 0$ for all $x, y, w, u, v \in R$.*

Lemma 2.3. [12, Lemma 1.3] *Let R be a semiprime ring and $a \in R$ be some fixed element. If $a[x, y] = 0$ for all $x, y \in R$, then there exists an ideal U of R such that $a \in U \subset Z$ holds.*

Lemma 2.4. [12, Lemma 1.1] *Let R be a semiprime ring. If $a, b \in R$ are such that $axb = 0$ for all $x \in R$, then $ab = ba = 0$.*

Lemma 2.5. [1, Theorem 2.4] *Let R be any ring with no nonzero central ideals. Then every centrally extended derivation D on R is additive.*

Lemma 2.6. [1, Theorem 2.5] *If R is a semiprime ring with no nonzero central ideals, then every centrally extended derivation D is a derivation.*

Lemma 2.7. [1, Theorem 3.3] *Let N be the set of all nilpotent elements of a ring R. If $N \cap Z = \{0\}$, then every centrally extended derivation D on R preserves Z.*

3 Additivity of centrally extended higher derivations

The next theorem, dealing with the additivity condition, is our first step.

Theorem 3.1. *Every centrally extended higher derivation on a ring R with no nonzero central ideals is additive.*

Proof. Assume that $D = (d_n)_{n\in\mathbb{N}_0}$ is a centrally extended higher derivation. By the definition, we have $d_0 = id_R$ is additive and by Lemma 2.5, d_1 is also additive. So we may suppose that d_m is additive, i.e., $d_m(x + y) = d_m(x) + d_m(y)$, for all $x, y \in R$ and for each $m < n$. For arbitrary elements $x, y \in R$, we have

$$d_n(x + y) - d_n(x) - d_n(y) \in Z. \tag{3.1}$$

So we can find an element $\lambda \in Z$, which depends on x and y, such that

$$d_n(x+y) = d_n(x) + d_n(y) + \lambda. \tag{3.2}$$

For arbitrary $z \in R$, the substitution of z for x and $(x+y)$ for y, in the second part of Definition (2.1), gives

$$d_n(z(x+y)) - \sum_{i+j=n} d_i(z)d_j(x+y) \in Z. \tag{3.3}$$

Again we can find an element $\mu \in Z$, which depends on z and $x+y$, such that

$$d_n(z(x+y)) = \sum_{i+j=n} d_i(z)d_j(x+y) + \mu = \sum_{\substack{i+j=n\\ i\neq 0, j\neq n}} d_i(z)d_j(x+y) + zd_n(x+y) + \mu. \tag{3.4}$$

Using (3.2), we can rewrite (3.4) as

$$d_n(z(x+y)) = \sum_{\substack{i+j=n\\ i\neq 0, j\neq n}} d_i(z)d_j(x+y) + z(d_n(x) + d_n(y) + \lambda) + \mu. \tag{3.5}$$

Furthermore, using the inductive assumption, (3.5) can be expanded to

$$d_n(z(x+y)) = \sum_{\substack{i+j=n\\ i\neq 0, j\neq n}} d_i(z)d_j(x) + \sum_{\substack{i+j=n\\ i\neq 0, j\neq n}} d_i(z)d_j(y) + zd_n(x) + zd_n(y) + z\lambda + \mu. \tag{3.6}$$

In the first part of Definition (2.1), the replacement of zx for x and zy for y yields

$$d_n(zx+zy) - d_n(zx) - d_n(zy) \in Z. \tag{3.7}$$

From Z we pick an element, ν, which depends on zx and zy to rewrite (3.7) as

$$d_n(zx+zy) = d_n(zx) + d_n(zy) + \nu. \tag{3.8}$$

By the definition, we know that

$$d_n(zx) - \sum_{i+j=n} d_i(z)d_j(x) \in Z \tag{3.9}$$

and

$$d_n(zy) - \sum_{i+j=n} d_i(z)d_j(y) \in Z. \tag{3.10}$$

So for appropriate central elements, ρ depends on z, x and ξ depends on z, y, we can expand equation (3.8) to

$$\begin{aligned} d_n(zx+zy) &= \sum_{i+j=n} d_i(z)d_j(x) + \rho + \sum_{i+j=n} d_i(z)d_j(y) + \xi + \nu \\ &= \sum_{\substack{i+j=n\\ i\neq 0, j\neq n}} d_i(z)d_j(x) + zd_n(x) + \rho + \sum_{\substack{i+j=n\\ i\neq 0, j\neq n}} d_i(z)d_j(y) + zd_n(y) + \xi + \nu. \end{aligned} \tag{3.11}$$

By comparing (3.6) and (3.11), we get $\rho + \xi + \nu = \mu + z\lambda$. So, we can conclude that $z\lambda \in Z$ for all $z \in R$, which implies that $R\lambda \subseteq Z$. It can be verified that $R\lambda$ is a central ideal of R. Hence, by our hypothesis, it must be zero ideal, i.e., $R\lambda = \{0\}$.

Now, consider the two-sided annihilator $A(R)$ of R, we have $\lambda \in A(R)$. But $A(R)$ is a central ideal of R, so $\lambda = 0$ and (3.2) yields the additivity of the centrally extended higher derivation. □

4 Multiplicativity of centrally extended higher derivations

In the next theorem, in order to show the complete equivalence between higher derivation and centrally extended higher derivation, we need also to prove the multiplicative property of the centrally extended higher derivation. Towards that goal, beside not allowing existence of central ideals in our ring, we require the ring to be semiprime.

Theorem 4.1. *Let R be a semiprime ring with no nonzero central ideals. Then every centrally extended higher derivation is a higher derivation.*

Proof. Let $D = (d_n)_{n\in\mathbb{N}_0}$ be a centrally extended higher derivation. We will proceed by induction on n. For $n = 0$, $d_0 = id_R$ satisfies the required result. By Lemma 2.6, d_1 is a derivation. So we may assume that $d_m(xy) = \sum_{i+j=m} d_i(x)d_j(y)$, for all $x, y \in R$ and for each $m < n$.

For arbitrary elements $x, y, z \in R$, we have

$$d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y) + \lambda, \ \ \lambda \in Z \tag{4.1}$$

and

$$d_n(yz) = \sum_{i+j=n} d_i(y)d_j(z) + \mu, \ \ \mu \in Z. \tag{4.2}$$

By the inductive assumption and associativity of our ring, we may use (4.1) and (4.2) to expand $d_n(xyz)$ in the following two different ways:

$$\begin{aligned} d_n((xy)z) &= \sum_{i+j=n} d_i(xy)d_j(z) + \nu \\ &= d_n(xy)z + \sum_{\substack{i+j=n \\ i\neq n, j\neq 0}} d_i(xy)d_j(z) + \nu \\ &= \sum_{i+j=n} d_i(x)d_j(y)z + \lambda z + \sum_{\substack{l+k+j=n \\ l+k\neq n, j\neq 0}} d_l(x)d_k(y)d_j(z) + \nu \\ &= \sum_{r+s+t=n} d_r(x)d_s(y)d_t(z) + \lambda z + \nu, \ \ \nu \in Z \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} d_n(x(yz)) &= \sum_{i+j=n} d_i(x)d_j(yz) + \rho \\ &= xd_n(yz) + \sum_{\substack{i+j=n \\ i\neq 0, j\neq n}} d_i(x)d_j(yz) + \rho \\ &= \sum_{i+j=n} xd_i(y)d_j(z) + x\mu + \sum_{\substack{i+l+k=n \\ i\neq 0, l+k\neq n}} d_i(x)d_l(y)d_k(z) + \rho \\ &= \sum_{r+s+t=n} d_r(x)d_s(y)d_t(z) + x\mu + \rho, \ \ \rho \in Z. \end{aligned} \tag{4.4}$$

Subtracting (4.3) from (4.4) gives $\lambda z + \nu = x\mu + \rho$. Hence,

$$-\lambda z + x\mu \in Z. \tag{4.5}$$

Therefore, $0 = [-\lambda z + x\mu, z] = [x, z]\mu$. Replacing x by $xr, r \in R$ and recalling the value of μ from (4.2), we get for all $x, y, z \in R$

$$[x, z]R\left(d_n(yz) - \sum_{i+j=n} d_i(y)d_j(z)\right) = \{0\}. \tag{4.6}$$

Putting $y = x$ in (4.6), we have

$$[x,z]R\left(d_n(xz) - \sum_{i+j=n} d_i(x)d_j(z)\right) = \{0\}. \tag{4.7}$$

By Theorem 3.1, D is additive and then by Lemma 2.2, we get for all $x, z, u, v \in R$, that

$$[x,z]R\left(d_n(uv) - \sum_{i+j=n} d_i(u)d_j(v)\right) = \{0\}. \tag{4.8}$$

By Lemma 2.4, since R is semiprime we get for all $x, z, u, v \in R$

$$[x,z]\left(d_n(uv) - \sum_{i+j=n} d_i(u)d_j(v)\right) = 0. \tag{4.9}$$

Using Lemma 2.3, there exists a central ideal I of R such that

$$d_n(uv) - \sum_{i+j=n} d_i(u)d_j(v) \in I, \quad \text{for all } u, v \in R. \tag{4.10}$$

But by our hypothesis, our ring has no nonzero central ideals, so

$$d_n(uv) = \sum_{i+j=n} d_i(u)d_j(v), \quad \text{for all } u, v \in R. \tag{4.11}$$

This completes the proof of our theorem. □

We conclude this section by the following example which shows the prominent role played by the absence of nonzero central ideals in Theorems (3.1) and (4.1) even if our ring is semiprime.

Example 4.1. Let S_1 be a noncommutative prime ring with higher derivation $D = (d_n)_{n\in\mathbb{N}_0}$, S_2 an integral domain and $S = S_1 \oplus S_2$. Define a family $F = (f_n)_{n\in\mathbb{N}_0}$ of mappings $f_n : S \to S$, with $f_0 = id_S$ by $f_n((x, y)) = (d_n(x), g(y))$, for all $x \in S_1$ and $y \in S_2$, where $g : S_2 \to S_2$ is any map which is not a derivation. Then straightforward calculation shows that S is a semiprime ring and F a centrally extended higher derivation which is not a higher derivation. Moreover, $\{0\} \oplus S_2$ is a central ideal of S.

5 On preservation of the center

Let S be any subset of a ring R. Recall that if map $f : R \to R$ satisfies the property that $f(S) \subseteq S$, then we say that S is preserved by f. The following theorem studies the condition under which centrally extended higher derivations preserve Z.

Theorem 5.1. *Let N be the set of all nilpotent elements of a ring R. If $N \cap Z = \{0\}$, then every centrally extended higher derivation $D = (d_n)_{n\in\mathbb{N}_0}$ on R preserves Z.*

Proof. By Lemma 2.7, we know that d_1 preserves Z. So we will suppose for $m < n$ that d_m preserves Z and show by induction on n that d_n also preserves Z. For $z \in Z$ and $r \in R$, we have

$$d_n(rz) - \sum_{i+j=n} d_i(r)d_j(z) \in Z, \quad d_n(rz) - d_n(r)z - rd_n(z) - \sum_{\substack{i+j=n\\ i\neq n, j\neq n}} d_i(r)d_j(z) \in Z, \tag{5.1}$$

and we also can expand $d_n(zr)$ in the following way:

$$d_n(zr) - \sum_{i+j=n} d_i(z)d_j(r) \in Z, \quad d_n(zr) - d_n(z)r - zd_n(r) - \sum_{\substack{i+j=n\\ i\neq n, j\neq n}} d_i(z)d_j(r) \in Z. \tag{5.2}$$

Subtracting (5.1) and (5.2) and using our induction hypothesis give

$$[r, d_n(z)] \in Z \quad \text{for all } r \in R. \tag{5.3}$$

Putting $r = rd_n(z)$ in (5.3) gives $[r, d_n(z)]d_n(z) \in Z$, so

$$\begin{aligned} [[r, d_n(z)]d_n(z), r] &= 0, \\ [r, d_n(z)]^2 &= 0 \quad \text{for all } r \in R. \end{aligned} \tag{5.4}$$

Since $N \cap Z = \{0\}$, (5.3) and (5.4) give $[r, d_n(z)] = 0$ for all $r \in R$. This completes the proof of our theorem. □

The following example shows that there exist centrally extended higher derivations of rings which do not preserve the center of these rings.

Example 5.1. Let R_1 be a noncommutative ring with the property that the product of any two elements of R_1 lies on the center of R_1. Let R_2 be any ring with the property that the product of any two elements of R_2 is zero and suppose that $(R_1, +) \cong (R_2, +)$. Define a ring R to be $R_1 \oplus R_2$. Let $f : (R_2, +) \to (R_1, +)$ be an isomorphism and let $D = (d_n)_{n \in \mathbb{N}_0}$ be a family of mappings $d_n : R \to R$, with $d_0 = id_R$ and $d_n((x, y)) = (f(y), 0)$, for all $x \in R_1$ and $y \in R_2$. A straightforward calculation shows that D is a centrally extended higher derivation on R. Moreover, $\{0\} \oplus R_2 \subseteq Z(R)$ and $d_n(\{0\} \oplus R_2) \nsubseteq Z$ for each nonnegative integer n. So D does not preserve Z.

The following corollary is immediate from the fact that zero element is the only nilpotent element in the center of any semiprime ring.

Corollary 5.2. *Every centrally extended higher derivation of a semiprime ring R preserves $Z(R)$.*

6 Conclusion

We construct Definition 2.1 of centrally extended higher derivation to be a family of additive mappings of our ring in which the zero member is the identity map and the first member is the centrally extended derivation defined recently by Bell and Daif [1]. This family of mappings is clearly an extension to both higher derivations and centrally extended derivations. We discuss the existence of such families by giving the proper Example 2.1 to a centrally extended higher derivation which is not a higher derivation. We state a series of extremely useful preliminary lemmas which play an important role in deducing our results.

In semiprime rings with no nonzero central ideals, we establish the coincidence between centrally extended higher derivations and higher derivations. Example 4.1 strengthens our results by emphasizing on the prominent role of the fact that our ring has no nonzero central ideals even if the ground ring still semiprime.

We study the problem of preserving the center of the ring by the centrally extended higher derivation and establish Theorem 5.1 which asserts that the center of the ring must not contain any nilpotent elements in order to prove that every centrally extended higher derivation of the ring preserves its center. We give Example 5.1 to show that there exists a centrally extended higher derivation of a ring which does not preserve its center. Finally, we use the fact that semiprime rings have no nilpotent elements in the center to give a corollary stating that every centrally extended higher derivation of a semiprime ring preserves its center.

The results obtained so far in this paper can be extended in various directions. It can be extended to generalized higher derivations, (α, β)-higher derivations and other types of higher derivations. Furthermore, one can ask a question of what happens if we replace semiprime rings with other classes of rings and algebras such as triangular algebras, near rings, and so on? We leave these discussions to future studies.

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References

[1] H. E. Bell and M. N. Daif, *On centrally-extended maps on rings*, Beitr. Algebra Geom. **57** (2016), no. 1, 129–136, DOI: https://doi.org/10.1007/s13366-015-0244-8.
[2] M. S. Tammam El-Sayiad, N. M. Muthana, and Z. S. Alkhamisi, *On rings with some kinds of centrally-extended maps*, Beitr. Algebra Geom. **57** (2016), no. 3, 579–588, DOI: https://doi.org/10.1007/s13366-015-0274-2.
[3] S. F. El-Deken and H. Nabiel, *Centrally-extended generalized $*$-derivations on rings with involution*, Beitr. Algebra Geom. **60** (2019), no. 2, 217–224, DOI: https://doi.org/10.1007/s13366-018-0415-5.
[4] N. Muthana and Z. Alkhamisi, *On centrally-extended multiplicative (generalized)-(α, β)-derivations in semiprime rings*, Hacet. J. Math. Stat. **49** (2020), no. 2, 578–585, DOI: https://doi.org/10.15672/hujms.568378.
[5] S. F. El-Deken and M. M. El-Soufi, *On centrally extended reverse and generalized reverse derivations*, Indian J. Pure Appl. Math. **51** (2020), no. 3, 1165–1180, DOI: https://doi.org/10.1007/s13226-020-0456-y.
[6] I. Kaygorodov, M. Khrypchenko, and F. Wei, *Higher derivations of finitary incidence algebras*, Algebr. Represent. Theor. **22** (2019), no. 6, 1331–1341, DOI: https://doi.org/10.1007/s10468-018-9822-4.
[7] Z. Xiao and F. Wei, *Jordan higher derivations on triangular algebras*, Linear Algebra Appl. **432** (2010), no. 10, 2615–2622, DOI: https://doi.org/10.1016/j.laa.2009.12.006.
[8] M. Ferrero and C. Haetinger, *Higher derivations and a theorem by Herstein*, Quaest. Math. **25** (2002), no. 2, 249–257, DOI: https://doi.org/10.2989/16073600209486012.
[9] M. S. Moslehian, *Superstability of higher derivations in multi-Banach algebras*, Tamsui Oxf. J. Math. Sci. **24** (2008), no. 4, 417–427.
[10] O. H. Ezzat, *A note on Jordan triple higher $*$-derivations on semiprime rings*, Int. Sch. Res. Notices **2014** (2014), 1–5, DOI: https://doi.org/10.1155/2014/365424.
[11] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), no. 6, 1104–1110, DOI: https://doi.org/10.1090/S0002-9939-1957-0095864-2.
[12] B. Zalar, *On centralizers of semiprime rings*, Comment. Math. Univ. Carolinae **32** (1991), no. 4, 609–614.