Research Article

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The L_p dual Minkowski problem about $0 < p < 1$ and $q > 0$

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Abstract: The (p, q)-th dual curvature measures and the L_p dual Minkowski problem were recently introduced by Lutwak, Yang, and Zhang. In this paper, we give a solution to the existence part of the L_p dual Minkowski problem about $0 < p < 1$ and $q > 0$ for arbitrary measures. This fills up previously obtained results.

Keywords: (p, q)-th dual curvature measures, L_p dual Minkowski problem, Minkowski problem, convexity

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1 Introduction

A compact convex subset of $\mathbb{R}^n$ with nonempty interior is called a convex body. Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$, and $\mathcal{K}_o^n$ denote the set of convex bodies in $\mathbb{R}^n$ with the origin in their interiors. The unit sphere in $\mathbb{R}^n$ will be denoted by S^{n-1}.

For all $x \in \mathbb{R}^n$, the support function of $K \in \mathcal{K}^n$ is defined by

$$h(K, x) = h_K(x) = \max\{x \cdot y : y \in K\},$$

where $x \cdot y$ denotes the standard inner product of x and y.

For $K \in \mathcal{K}^n$ and $v \in S^{n-1}$, the supporting hyperplane $H(K, v)$ of K at v is defined by

$$H(K, v) = \{x \in \mathbb{R}^n : x \cdot v = h_K(v)\}.$$

The radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$, of $K \in \mathcal{K}^n$ is defined by

$$\rho(K, x) = \max\{\lambda : \lambda x \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

Let $K \in \mathcal{K}_o^n$ and $\eta \subset S^{n-1}$ be a Borel set. The reverse radial Gauss image of η, $\boldsymbol{\alpha}_K^*(\eta)$ is given by

$$\boldsymbol{\alpha}_K^*(\eta) = \{u \in S^{n-1} : \rho_K(u)u \in H(K, v) \text{ for some } v \in \eta\}.$$

Geometric measures and their associated Minkowski problems in the Brunn-Minkowski theory and its generalization are central to the study of convex geometric analysis. In regard to the dual Brunn-Minkowski theory, Huang et al. in [1] recently studied the q-th dual curvature measures: for $K \in \mathcal{K}_o^n$ and $q \in \mathbb{R}$, the q-th dual curvature measure, $\widetilde{C}_q(K, \cdot)$, defined for every Borel $\eta \subset S^{n-1}$ by

$$\widetilde{C}_q(K, \eta) = \frac{1}{n} \int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_K^q(u)\mathrm{d}u \tag{1.1}$$

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is the Borel measure on S^{n-1}. It is worth noting that the q-th dual curvature measure inconceivably connects the well-known cone volume measure ($q = n$) and Aleksandrov's integral curvature ($q = 0$). These measures have never been linked before.

Huang et al. [1] asked for necessary and sufficient conditions so that a given measure on the unit sphere is precisely the q-th dual curvature measure of a convex body in $\mathbb{R}^n$. This problem is called the dual Minkowski problem. The dual Minkowski problem contains critical problems such as the Aleksandrov problem ($q = 0$), see, e.g., [2–4] and the logarithmic Minkowski problem ($q = n$), see, e.g., [5–9] as special cases. The problem has been completely solved for $q < 0$ (see [10]), but critical case for $q > 0$ is still unsolved, see, e.g., [11–15].

Very recently, Lutwak et al. in [16] introduced a more general version of the q-th dual curvature measure called the (p, q)-th dual curvature measure. For $K \in \mathcal{K}_o^n$ and $p, q \in \mathbb{R}$, the (p, q)-th dual curvature measure $\widetilde{C}_{p,q}(K, \cdot)$ is defined by

$$\mathrm{d}\widetilde{C}_{p,q}(K, \cdot) = h_K^{-p}\mathrm{d}\widetilde{C}_q(K, \cdot). \tag{1.2}$$

It should be noted that this definition is slightly weaker than the one defined in [16]. Lutwak et al. [16] showed that special cases of the (p, q)-th dual curvature measure are the L_p surface area measure ($q = n$), the q-th dual curvature measure ($p = 0$), and the L_p integral curvature ($q = 0$). Regarding the new (p, q)-th dual curvature measure, the following L_p dual Minkowski problem was posed in [16].

The L_p dual Minkowski problem: Given a nonzero finite Borel measure μ on the unit sphere S^{n-1} and real numbers p, q, what are the necessary and sufficient conditions so that there is a convex body $K \in \mathcal{K}_o^n$ satisfying

$$\widetilde{C}_{p,q}(K, \cdot) = \mu?$$

When $p = 0$, the L_p dual Minkowski problem is just the dual Minkowski problem; when $q = 0$, it becomes the L_p Aleksandrov problem introduced and studied by Huang et al. [17]; when $q = n$, it reduces to the L_p Minkowski problem, proposed in [18], which has been extensively studied, see, e.g., [19–43].

When $p > 0$ and $q < 0$, a complete characterization to the existence part of the L_p dual Minkowski problem is given by Huang and Zhao [44].

Theorem 1.1. [44, Theorem 1.2] *Let $p > 0$, $q < 0$, and μ be a non-zero finite Borel measure on S^{n-1}. There is a convex body $K \in \mathcal{K}_o^n$ so that $\mu = \widetilde{C}_{p,q}(K, \cdot)$ if and only if μ is not contained on arbitrary closed hemisphere.*

The Orlicz extension of Theorem 1.1 was partially settled in [45], and later completely solved in [46].

When $p, q > 0$ and $p \neq q$, and the given measure is even, Huang and Zhao [44] also presented a complete solution to the existence part of the L_p dual Minkowski problem.

Theorem 1.2. [44, Theorem 1.3] *Let $p, q > 0$, $p \neq q$, and μ be a non-zero even Borel measure on S^{n-1}. There is an origin-symmetric convex body K in $\mathbb{R}^n$ so that $\mu = \widetilde{C}_{p,q}(K, \cdot)$ if and only if μ is not contained in arbitrary great subsphere.*

The Orlicz version of Theorem 1.2 was obtained in [46].

When $p > 1, q > 0$, and $p > q$, a sufficient condition on the existence of solutions to the L_p dual Minkowski problem is given by Böröczky and Fodor [47] and obtained the following result.

Theorem 1.3. [47, Theorem 1.2] *Let $p > 1$, $q > 0$, and $p > q$, and let μ be a finite Borel measure on S^{n-1} that is not contained on arbitrary closed hemisphere. Then there is a convex body $K \in \mathcal{K}_o^n$ so that $\mu = \widetilde{C}_{p,q}(K, \cdot)$.*

The Orlicz case of Theorem 1.3 is given in [46].

As we can see those theorems above, when $0 < p < 1$ and $q > 0$ there is no existence result concerning the L_p dual Minkowski problem in the general case (without the condition that the measure is even). The aim of this paper is to supplement the situation, which is motivated by the works of Zhu [41], Jian and Lu [48], Chen et al. [22], and Huang and Zhao [44]. Thus, the following result is obtained.

Theorem 1.4. *Let $0 < p < 1$, $q > 0$, and $q \neq p$. If μ is a finite Borel measure on S^{n-1} and is not contained on arbitrary closed hemisphere, then there is a convex body K in $\mathbb{R}^n$ such that $\mu = \widetilde{C}_{p,q}(K, \cdot)$.*

Theorem 1.4 contains as special cases the solution to the existence part of the L_p Minkowski problem for $0 < p < 1$ (see [22]) and of the discrete L_p Minkowski problem for $0 < p < 1$ (see [41]).

We remark that when $p, q < 0$ and the given measure is even, the existence part of the L_p dual Minkowski problem was independently solved by Huang and Zhao [44] and Gardner et al. [46]. However, this situation for $p < 0$ and $q > 0$ has not yet yielded any results, as far as we know.

In the next section, some preliminaries are given. In Section 3, we consider a minimizing problem and give its corresponding solution. In Section 4, we first discuss the discrete case of Theorem 1.4. Then the proof of Theorem 1.4 is completed by approximation.

2 Preliminaries

In this section, some basic facts about convex bodies are collected. The books of Schneider [49], Gardner [50], and Gruber [51] are excellent references regarding convex bodies.

The work is carried out in $\mathbb{R}^n$ equipped with the standard Euclidean norm. For any $x \in \mathbb{R}^n$, its Euclidean norm is denoted by $|x| = \sqrt{x \cdot x}$. The unit ball will be written by $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$. The set of continuous functions on S^{n-1} is denoted by $C(S^{n-1})$ and the set of positive functions in $C(S^{n-1})$ is written by $C^+(S^{n-1})$.

Let us define the Hausdorff distance of two convex bodies K, L in $\mathbb{R}^n$ as follows:

$$\delta(K, L) = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

Assume K_i is a sequence of convex bodies in $\mathbb{R}^n$. We claim that K_i converges to a convex body $K_0 \subset \mathbb{R}^n$ if

$$\delta(K_i, K_0) \to 0,$$

when $i \to \infty$.

For $g \in C^+(S^{n-1})$ and a closed subset $\Omega \subset S^{n-1}$ not concentrated on arbitrary closed hemisphere, the Aleksandrov body relevant to (g, Ω), written by $[g]$, is the convex body that is defined as follows:

$$[g] = \bigcap_{u \in \Omega} \{\xi \in \mathbb{R}^n : \xi \cdot u \leq g(u)\}. \tag{2.1}$$

Clearly, $h_{[g]} \leq g$ and $[h_K] = K$ if $\Omega = S^{n-1}$ and $K \in \mathcal{K}_o^n$. In fact, for any $v \in S^{n-1}$,

$$[g] \subset \{\xi \in \mathbb{R}^n : \xi \cdot v \leq g(v)\} =: E,$$

which implies

$$h_{[g]}(v) \leq h_E(v) = g(v).$$

Thus by the arbitrariness of $v \in S^{n-1}$,

$$h_{[g]} \leq g.$$

Moreover, on one hand,

$$h_{[h_K]} \leq h_K$$

for $K \in \mathcal{K}_o^n$. On the other hand, for any $u \in S^{n-1}$

$$K \subset \{\xi \in \mathbb{R}^n : \xi \cdot u \leq h_K(u)\} \subset \bigcap_{u \in S^{n-1}} \{\xi \in \mathbb{R}^n : \xi \cdot u \leq g(u)\}.$$

This has

$$h_{[h_K]} \geq h_K.$$

Namely,

$$h_{[h_K]} = h_K.$$

It was demonstrated that the (p, q)-th dual curvature measure is weakly convergent in [16]. Namely, if $p, q \in \mathbb{R}$, $K_i \in \mathcal{K}_o^n$, and $K_i \to K_0 \in \mathcal{K}_o^n$, then for every $f \in C(S^{n-1})$,

$$\lim_{i\to\infty} \int_{S^{n-1}} f(v) \mathrm{d}\widetilde{C}_{p,q}(K_i, v) = \int_{S^{n-1}} f(v) \mathrm{d}\widetilde{C}_{p,q}(K_0, v). \tag{2.2}$$

Moreover, it easily follows from (1.1) and (1.2) that for $K \in \mathcal{K}_o^n$ and $\lambda > 0$,

$$\widetilde{C}_{p,q}(\lambda K, \cdot) = \lambda^{q-p} \widetilde{C}_{p,q}(K, \cdot). \tag{2.3}$$

For $K \in \mathcal{K}^n$, the diameter of K is defined as follows:

$$D(K) = \max\{|x - y| : x, y \in K\}.$$

For $q \in \mathbb{R}$ and $K \in \mathcal{K}^n$, the q-th dual volume of K, denoted by $\widetilde{V}_q(K)$ and see [1], is

$$\widetilde{V}_q(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^q(u) \mathrm{d}u. \tag{2.4}$$

Let $\Omega \subset S^{n-1}$ represent a closed subset which is not concentrated on arbitrary closed hemisphere, and let $f : \Omega \to \mathbb{R}$ and $h_0 : \Omega \to (0, \infty)$ be continuous. For $t \in (-\delta, \delta)$ with $\delta > 0$, define continuous function $h_t : \Omega \to (0, \infty)$ by

$$\log h_t(v) = \log h_0(v) + tf(v) + o(t, v)$$

for $v \in \Omega$, where $o(t, \cdot) : \Omega \to \mathbb{R}$ is continuous and $\lim_{t\to 0} o(t, \cdot)/t = 0$.

The next variational formula, see [1, Theorem 4.5], is important in the proof of our main result.

Lemma 2.1. *Let $[h_t]$ be the Aleksandrov body associated with (h_t, Ω). Then for $q \neq 0$*

$$\lim_{t\to 0} \frac{\widetilde{V}_q([h_t]) - \widetilde{V}_q([h_0])}{t} = q \int_{\Omega} f(v) \mathrm{d}\widetilde{C}_q([h_0], v). \tag{2.5}$$

3 The minimization problem

To resolve the Minkowski problem by variational method, the first crucial step is to find an optimization problem whose optimizer is exactly the solution to the Minkowski problem. In this section, we consider a minimization problem and show the existence of a minimizer.

Let $g \in C^+(S^{n-1})$ and μ is a finite discrete measure on S^{n-1} which is not contained on arbitrary closed hemisphere of S^{n-1}. For the Aleksandrov body relevant to $(g, \mathrm{supp}(\mu))$, it is denoted by $[g]_\mu$. Therefore, for $0 < p < 1$, the function, $\Phi_{g,\mu} : [g]_\mu \to \mathbb{R}$, is defined as follows:

$$\Phi_{g,\mu}(\xi) = \int_{S^{n-1}} (g(u) - \xi \cdot u)^p \mathrm{d}\mu(u) = \int_{\mathrm{supp}(\mu)} (g(u) - \xi \cdot u)^p \mathrm{d}\mu(u). \tag{3.1}$$

Now, take into account the next minimizing problem:

$$\inf\left\{\sup_{\xi\in[g]_\mu}\Phi_{g,\mu}(\xi) : g \in C^+(S^{n-1}) \text{ and } \widetilde{V}_q([g]_\mu) = 1\right\}. \tag{3.2}$$

The solution of problem (3.2) will be given after the following Lemmas 3.1 and 3.2.

Lemma 3.1. *Let $0 < p < 1$, if μ is a finite discrete measure on S^{n-1} and is not contained on arbitrary closed hemisphere of S^{n-1}, then $\Phi_{g,\mu}$ is strictly concave on $[g]_\mu$ for $g \in C^+(S^{n-1})$.*

Proof. For $0 < p < 1$, t^p be strictly concave on $[0, +\infty)$, and for $\xi \in [g]_\mu$ and $u \in \operatorname{supp}(\mu)$, we have

$$g(u) - \xi\cdot u \geq h_{[g]_\mu}(u) - \xi\cdot u \geq 0.$$

Therefore, for $0 < \lambda < 1$ and $\xi_1, \xi_2 \in [g]_\mu$,

$$\begin{aligned}\Phi_{g,\mu}(\lambda\xi_1 + (1-\lambda)\xi_2) &= \int_{S^{n-1}} (g(u) - (\lambda\xi_1 + (1-\lambda)\xi_2)\cdot u)^p \mathrm{d}\mu(u)\\ &= \int_{S^{n-1}} (\lambda(g(u) - \xi_1\cdot u) + (1-\lambda)(g(u) - \xi_2\cdot u))^p \mathrm{d}\mu(u)\\ &\geq \lambda \int_{S^{n-1}} (g(u) - \xi_1\cdot u)^p \mathrm{d}\mu(u) + (1-\lambda)\int_{S^{n-1}} (g(u) - \xi_2\cdot u)^p \mathrm{d}\mu(u)\\ &= \lambda\Phi_{g,\mu}(\xi_1) + (1-\lambda)\Phi_{g,\mu}(\xi_2),\end{aligned}$$

this is also equivalent to proving that

$$g(u) - \xi_1\cdot u = g(u) - \xi_2\cdot u$$

for arbitrary $u \in \operatorname{supp}(\mu)$, namely,

$$(\xi_1 - \xi_2)\cdot u = 0.$$

Since μ is not contained in arbitrary closed hemisphere, and $\operatorname{supp}(\mu)$ spans the whole space $\mathbb{R}^n$. Therefore, we can conclude

$$\xi_1 = \xi_2,$$

this yields that $\Phi_{g,\mu}$ is strictly concave on $[g]_\mu$. □

Lemma 3.2. *Let $0 < p < 1$, if μ is a finite discrete measure on S^{n-1} and is not contained on arbitrary closed hemisphere of S^{n-1}, then for $g \in C^+(S^{n-1})$, there is a unique $\xi_g \in \operatorname{int}([g]_\mu)$, which ξ_g depends continuously on g so that*

$$\Phi_{g,\mu}(\xi_g) = \sup_{\xi\in[g]_\mu}\Phi_{g,\mu}(\xi).$$

Proof. Let $\Phi_{g,\mu}$ be strictly concave and continuous on $[g]_\mu$. Thus, there is a unique $\xi_g \in [g]_\mu$ so that

$$\Phi_{g,\mu}(\xi_g) = \sup_{\xi\in[g]_\mu}\Phi_{g,\mu}(\xi). \tag{3.3}$$

We will show that $\xi_g \in \operatorname{int}([g]_\mu)$. If not, ξ_g is on the boundary, $\partial([g]_\mu)$, $\xi_g \in \partial([g]_\mu)$.

Recalling the definition of $[g]_\mu$,

$$[g]_\mu = \bigcap_{u\in\operatorname{supp}(\mu)} \{\xi \in \mathbb{R}^n : \xi\cdot u \leq g(u)\}.$$

It is easy to see that there is $u \in \operatorname{supp}(\mu)$ so that

$$\xi_g\cdot u = g(u). \tag{3.4}$$

Otherwise, for any $u \in \text{supp}(\mu)$, there is $\xi_g \cdot u < g(u)$. Then, for some $\delta_1 > 0$ and any $u \in \text{supp}(\mu)$, we have

$$\xi_g \cdot u + \delta_1 < g(u),$$

i.e.,

$$(\xi_g + \delta_1 u) \cdot u < g(u).$$

This yields that $\xi_g \in \text{int}([g]_\mu)$, which is a contradiction.

Let

$$\text{supp}(\mu) = A \cup B, \tag{3.5}$$

where

$$A \coloneqq \{u \in \text{supp}(\mu) : \xi_g \cdot u = g(u)\}$$

and

$$B \coloneqq \{u \in \text{supp}(\mu) : \xi_g \cdot u < g(u)\}.$$

Then, from (3.4), and μ is not contained on arbitrary closed hemisphere of S^{n-1}, we can observe that A and B are two disjoint nonempty sets. According to the definition of set A, and noting $g \in C^+(S^{n-1})$, there exists a unit vector $u_0 \in S^{n-1}$ so that

$$u_0 \cdot u < 0, \tag{3.6}$$

for all $u \in A$. On the basis of these facts that B is a closed subset of S^{n-1} and $\xi_g \cdot u - g(u)$, $u \in S^{n-1}$ is continuous, it follows from that there is a positive constant $a > 0$ so that

$$\xi_g \cdot u + 2a < g(u), \tag{3.7}$$

for arbitrary $u \in B$. Therefore, for arbitrary $0 < \lambda < 2a$ and arbitrary $u \in \text{supp}(\mu)$, we obtain

$$(\xi_g + \lambda u_0) \cdot u < g(u).$$

This means that there is some $\delta_2 > 0$ so that

$$(\xi_g + \lambda u_0 + \delta_2 u) \cdot u < g(u),$$

for all $u \in \text{supp}(\mu)$, i.e.,

$$\xi(\lambda) \coloneqq \xi_g + \lambda u_0 \in \text{int}([g]_\mu).$$

By definitions (3.1) and (3.5), it follows that

$$\begin{aligned} \Phi_{g,\mu}(\xi(\lambda)) - \Phi_{g,\mu}(\xi_g) &= \int_{A \cup B} (g(u) - \xi(\lambda) \cdot u)^p \mathrm{d}\mu(u) - \int_{A \cup B} (g(u) - \xi_g \cdot u)^p \mathrm{d}\mu(u) \\ &= \int_A (g(u) - \xi(\lambda) \cdot u)^p \mathrm{d}\mu(u) + \int_B (g(u) - \xi(\lambda) \cdot u)^p - (g(u) - \xi_g \cdot u)^p \mathrm{d}\mu(u). \end{aligned} \tag{3.8}$$

For all $u \in A$ and some constant $\delta_3 > 0$, inequality (3.6) is strengthened as follows:

$$u_0 \cdot u < -\delta_3 < 0.$$

Thus, for all $u \in A$,

$$g(u) - \xi(\lambda) \cdot u = -\lambda u_0 \cdot u > \lambda \delta_3.$$

This has

$$\int_A (g(u) - \xi(\lambda) \cdot u)^p \mathrm{d}\mu(u) > \int_A (\lambda \delta_3)^p \mathrm{d}\mu(u) = (\lambda \delta_3)^p \mu(A). \tag{3.9}$$

From (3.7), we get that for arbitrary $u \in B$ and $0 < \lambda < a$.

$$g(u) - \xi(\lambda) \cdot u = g(u) - \xi_g \cdot u - \lambda u_0 \cdot u > 2a - \lambda > a.$$

For $0 < p < 1$, this has

$$|(g(u) - \xi(\lambda) \cdot u)^p - (g(u) - \xi_g \cdot u)^p| < pa^{p-1}| - \lambda u_0 \cdot u| \leq \lambda pa^{p-1}.$$

Thus,

$$\begin{aligned} \int_B (g(u) - \xi(\lambda) \cdot u)^p - (g(u) - \xi_g \cdot u)^p \mathrm{d}\mu(u) &\leq \left| \int_B (g(u) - \xi(\lambda) \cdot u)^p - (g(u) - \xi_g \cdot u)^p \mathrm{d}\mu(u) \right| \\ &\leq \int_B |(g(u) - \xi(\lambda) \cdot u)^p - (g(u) - \xi_g \cdot u)^p| \mathrm{d}\mu(u) \\ &< \lambda pa^{p-1}\mu(B). \end{aligned} \tag{3.10}$$

Associated with (3.8), (3.9) and (3.10), we have

$$\Phi_{g,\mu}(\xi(\lambda)) - \Phi_{g,\mu}(\xi_g) > (\lambda\delta_3)^p\mu(A) - \lambda pa^{p-1}\mu(B) = \lambda^p(\delta_3^p\mu(A) - p\lambda^{1-p}a^{p-1}\mu(B)).$$

We can choose $0 < \lambda_0 < a$ small enough so that $\xi(\lambda_0) \in \operatorname{Int}([g]_\mu)$ and

$$\Phi_{g,\mu}(\xi(\lambda_0)) > \Phi_{g,\mu}(\xi_g),$$

which is a contradiction since the maximum of $\Phi_{g,\mu}$ is achieved at the point ξ_g from (3.3). Therefore, $\xi_g \in \operatorname{int}([g]_\mu)$.

Let $g \in C^+(S^{n-1})$, $\{g_k\} \subset C^+(S^{n-1})$ be arbitrary sequence of functions and uniformly converging to g on S^{n-1}. We next show that ξ_{g_k} converges to ξ_g in $\mathbb{R}^n$.

Note that the fact, see [49, Lemma 7.5.2], that $[g_k]_\mu \to [g]_\mu$ as $g_k \to g$ uniformly on S^{n-1}. Since $\xi_{g_k} \in [g_k]_\mu$, ξ_{g_k} is bounded. Thus, we let $\{\xi_{g_{k_i}}\} \subset \{\xi_{g_k}\}$ be arbitrary convergent subsequence and $\xi_{g_{k_i}} \to \xi_0$ as $i \to +\infty$. We will prove $\xi_0 = \xi_g$.

Let $\xi \in [g]_\mu$. Thus, $[g_{k_i}]_\mu \to [g]_\mu$ as $i \to +\infty$, there is a sequence with $\xi_{k_i} \in [g_{k_i}]_\mu$ so that $\xi_{k_i} \to \xi$ as $i \to +\infty$. Then,

$$\begin{aligned} \Phi_{g,\mu}(\xi) &= \int_{S^{n-1}} (g(u) - \xi \cdot u)^p \mathrm{d}\mu(u) \\ &= \lim_{i\to+\infty} \int_{S^{n-1}} \big(g_{k_i}(u) - \xi_{k_i} \cdot u\big)^p \mathrm{d}\mu(u) \\ &\leq \lim_{i\to+\infty} \int_{S^{n-1}} \big(g_{k_i}(u) - \xi_{g_{k_i}} \cdot u\big)^p \mathrm{d}\mu(u) \\ &= \int_{S^{n-1}} (g(u) - \xi_0 \cdot u)^p \mathrm{d}\mu(u) \\ &= \Phi_{g,\mu}(\xi_0). \end{aligned}$$

This has

$$\sup_{\xi\in[g]_\mu} \Phi_{g,\mu}(\xi) = \Phi_{g,\mu}(\xi_0).$$

By the uniqueness of ξ_g, it follows that $\xi_g = \xi_0$, which proves $\xi_{g_k} \to \xi_g$. □

We are now ready to show the solution of problem (3.2).

Theorem 3.3. *Let $0 < p < 1$, if μ is a finite discrete measure on S^{n-1} and is not contained on arbitrary closed hemisphere of S^{n-1}, then there is a function $h \in C^+(S^{n-1})$ with $\xi_h = o$ and $\widetilde{V}_q([h]_\mu) = 1$ so that*

$$\Phi_{h,\mu}(o) = \inf\left\{\sup_{\xi\in[g]_\mu} \Phi_{g,\mu}(\xi) : g \in C^+(S^{n-1}) \text{ and } \widetilde{V}_q([g]_\mu) = 1\right\}. \tag{3.11}$$

Proof. Let $\{g_k\} \subset C^+(S^{n-1})$, $\widetilde{V}_q([g_k]_\mu) = 1$, and

$$\lim_{k\to+\infty} \Phi_{g_k,\mu}\left(\xi_{g_k}\right) = \inf\left\{\sup_{\xi\in[g]_\mu} \Phi_{g,\mu}(\xi) : g \in C^+(S^{n-1}) \text{ and } \widetilde{V}_q([g]_\mu) = 1\right\}. \tag{3.12}$$

Define $h_k = h_{[g_k]_\mu}$. We can observe that for $u \in \operatorname{supp}(\mu)$,

$$h_k(u) \le g_k(u),$$

where $[h_k]_\mu = [h_{[g_k]_\mu}]_\mu = [g_k]_\mu$. Note that $o \in \operatorname{int}([g_k]_\mu)$. Thus, $h_k \in C^+(S^{n-1})$. For any $\xi \in [h_k]_\mu = [g_k]_\mu$, we have

$$\Phi_{h_k,\mu}(\xi) = \int_{\operatorname{supp}(\mu)} (h_k(u) - \xi\cdot u)^p \mathrm{d}\mu(u) \le \int_{\operatorname{supp}(\mu)} (g_k(u) - \xi\cdot u)^p \mathrm{d}\mu(u) = \Phi_{g_k,\mu}(\xi).$$

This obtains

$$\sup_{\xi\in[h_k]_\mu} \Phi_{h_k,\mu}(\xi) \le \sup_{\xi\in[g_k]_\mu} \Phi_{g_k,\mu}(\xi).$$

Thus,

$$\lim_{k\to+\infty}\sup_{\xi\in[h_k]_\mu} \Phi_{h_k,\mu}(\xi) \le \lim_{k\to+\infty}\sup_{\xi\in[g_k]_\mu} \Phi_{g_k,\mu}(\xi). \tag{3.13}$$

Note that $h_k \in C^+(S^{n-1})$ and $\widetilde{V}_q([h_k]_\mu) = \widetilde{V}_q([g_k]_\mu) = 1$. Then,

$$\lim_{k\to+\infty}\sup_{\xi\in[h_k]_\mu} \Phi_{h_k,\mu}(\xi) \ge \lim_{k\to+\infty}\sup_{\xi\in[g_k]_\mu} \Phi_{g_k,\mu}(\xi). \tag{3.14}$$

Combining (3.13) with (3.14), we have

$$\lim_{k\to+\infty}\sup_{\xi\in[h_k]_\mu} \Phi_{h_k,\mu}(\xi) = \lim_{k\to+\infty}\sup_{\xi\in[g_k]_\mu} \Phi_{g_k,\mu}(\xi).$$

This, together with (3.12), has

$$\lim_{k\to+\infty} \Phi_{h_k,\mu}\left(\xi_{h_k}\right) = \inf\left\{\sup_{\xi\in[g]_\mu} \Phi_{g,\mu}(\xi) : g \in C^+(S^{n-1}) \text{ and } \widetilde{V}_q([g]_\mu) = 1\right\}.$$

From Lemma 3.2, we see $\xi_{h_k} \in \operatorname{int}([h_k]_\mu)$ and

$$\Phi_{h_k,\mu}\left(\xi_{h_k}\right) = \sup_{\xi\in[h_k]_\mu} \Phi_{h_k,\mu}(\xi).$$

Recalling $[h_k]_\mu = [g_k]_\mu$, we get $h_k = h_{[g_k]_\mu} = h_{[h_k]_\mu}$, namely, h_k is the support function of $[h_k]_\mu$ as well. For $x \in \mathbb{R}^n$, we calculate

$$\Phi_{h([h_k]_\mu+x),\mu}\left(\xi_{h_k} + x\right) = \Phi_{h_k,\mu}\left(\xi_{h_k}\right).$$

Therefore, we can find a sequence, again denoted by $\{h_k\} \subset C^+(S^{n-1})$, $\widetilde{V}_q([h_k]_\mu) = 1$, and $\xi_{h_k} = o$ such that

$$\lim_{k\to+\infty} \Phi_{h_k,\mu}(o) = \inf\left\{\sup_{\xi\in[g]_\mu} \Phi_{g,\mu}(\xi) : g \in C^+(S^{n-1}) \text{ and } \widetilde{V}_q([g]_\mu) = 1\right\}. \tag{3.15}$$

That is to say that $\{h_k\}$ is uniformly bounded on S^{n-1}. If not, there exists a subsequence of $\{h_k\}$, also written by $\{h_k\}$, such that

$$\lim_{k\to+\infty}\max_{u\in S^{n-1}} h_k(u) = +\infty.$$

Let $R_k = \max_{u\in S^{n-1}} h_k(u) = h_k(u_k)$ for $u_k \in S^{n-1}$. Since $\{u_k\} \subset S^{n-1}$, it can be seen from the compactness of S^{n-1} that there exists a convergent subsequence, say $\{u_k\}$, assuming

$$\lim_{k\to+\infty} u_k = u_0 \in S^{n-1}.$$

Since $\mathrm{supp}(\mu)$ is not contained in arbitrary closed hemisphere, thus, there is some $u' \in \mathrm{supp}(\mu)$ so that

$$u' \cdot u_0 > 0.$$

Let $b = \frac{1}{2}(u' \cdot u_0) > 0$. Then there is $k_0 \in \mathbb{N}$ so that as $k \geq k_0$,

$$u' \cdot u_k > b.$$

Note that $R_k u_k \in [h_k]_\mu$. Thus, when $k \geq k_0$,

$$h_k(u') \geq R_k(u' \cdot u_k) > R_k b.$$

It then follows from that μ be a finite discrete measure, we obtain that for $k \geq k_0$ and $0 < p < 1$,

$$\lim_{k\to+\infty} \Phi_{h_k,\mu}(o) = \lim_{k\to+\infty} \int_{S^{n-1}} h_k^p(u)\mathrm{d}\mu(u) \geq \lim_{k\to+\infty} h_k^p(u')\mu(u') > \lim_{k\to+\infty} (R_k b)^p \mu(u') = +\infty. \tag{3.16}$$

Let $h' \in C^+(S^{n-1})$ and $\widetilde{V}_q([h']_\mu) = 1$. Thus,

$$\lim_{k\to+\infty} \Phi_{h_k,\mu}(o) \leq \Phi_{h',\mu}\big(\xi_{h'}\big) = \int_{S^{n-1}} \big(h'(u) - \xi_{h'} \cdot u\big)^p \mathrm{d}\mu(u) < +\infty$$

this contradicts with (3.16). Consequently, $\{h_k\}$ is uniformly bounded.

According to the Blaschke selection theorem, $\{h_k\}$ has a convergent subsequence, also denoted by $\{h_k\}$, letting $h_k \to h$ on S^{n-1} as $k \to +\infty$. Thus, there are $h = h_{[h]_\mu}$ and $[h_k]_\mu \to [h]_\mu$. Moreover, we have $h \geq 0$ and $\widetilde{V}_q([h]_\mu) = 1$. From Lemma 3.2, we see

$$o = \lim_{k\to+\infty} \xi_{h_k} = \xi_h \in \mathrm{int}([h]_\mu).$$

Thus, $h > 0$, and associated with (3.15) we get

$$\Phi_{h,\mu}(o) = \inf\left\{\sup_{\xi\in[g]_\mu} \Phi_{g,\mu}(\xi) : g \in C^+(S^{n-1}) \text{ and } \widetilde{V}_q([g]_\mu) = 1\right\}.$$

The proof of Theorem 3.3 is completed. □

4 Solving the L_p dual Minkowski problem

In the following, we first prove that the solution h which is obtained in Theorem 3.3 is exactly the solution to the discrete case of Theorem 1.4. Then using approximation, Theorem 1.4 is proved.

Theorem 4.1. *Let $0 < p < 1$ and $q \neq 0$. If μ is a finite discrete measure on S^{n-1} and is not contained on arbitrary closed hemisphere of S^{n-1}, then there is a function $h \in C^+(S^{n-1})$ satisfying (3.11) and a positive constant $c > 0$ so that*

$$\mu = c\widetilde{C}_{p,q}([h]_\mu, \cdot), \quad \textit{where} \quad c = \int_{S^{n-1}} h^p(u)\mathrm{d}\mu(u).$$

Proof. By Theorem 3.3, we can see that there is a function $h \in C^+(S^{n-1})$ with $\xi_h = o$ and $\widetilde{V}_q([h]_\mu) = 1$ so that

$$\Phi_{h,\mu}(o) = \inf\left\{\sup_{\xi\in[g]_\mu} \Phi_{g,\mu}(\xi) : g \in C^+(S^{n-1}) \text{ and } \widetilde{V}_q([g]_\mu) = 1\right\}.$$

For arbitrary $f \in C(S^{n-1})$ and $t \in (-\delta, \delta)$ where $\delta > 0$ is small enough, we have

$$\varrho_t = he^{tf}.$$

Then,

$$\log \varrho_t = \log h + tf.$$

From Lemma 2.1, we have for $q \neq 0$,

$$\lim_{t\to 0} \frac{\widetilde{V}_q([\varrho_t]_\mu) - \widetilde{V}_q([h]_\mu)}{t} = q \int_{\mathrm{supp}(\mu)} f(u)\mathrm{d}\widetilde{C}_q([h]_\mu, u) = q \int_{S^{n-1}} f(u)\mathrm{d}\widetilde{C}_q([h]_\mu, u). \tag{4.1}$$

Let $g_t = \gamma(t)\varrho_t$, where

$$\gamma(t) = \widetilde{V}_q([\varrho_t]_\mu)^{-\frac{1}{q}}.$$

Then $g_t \in C^+(S^{n-1})$ and $\widetilde{V}_q([g_t]_\mu) = 1$. Since $\varrho_0 = h$, it follows from (4.1) that

$$\lim_{t\to 0} \frac{g_t - g_0}{t} = -h \int_{S^{n-1}} f(u)\mathrm{d}\widetilde{C}_q([h]_\mu, u) + hf. \tag{4.2}$$

Let $\xi(t) = \xi_{g_t}$ and

$$\Phi_\mu(t) = \sup_{\xi\in[g_t]_\mu} \int_{S^{n-1}} (g_t(u) - \xi\cdot u)^p \mathrm{d}\mu(u) = \int_{S^{n-1}} (g_t(u) - \xi(t)\cdot u)^p \mathrm{d}\mu(u). \tag{4.3}$$

This together with the fact that $\xi(t) \in \mathrm{int}([g_t]_\mu)$ has

$$\int_{S^{n-1}} (g_t(u) - \xi(t)\cdot u)^{p-1} u_i \mathrm{d}\mu(u) = 0 \tag{4.4}$$

for $i = 1, \ldots, n$, where $u = (u_1, \ldots, u_n)^T$. Note that $\xi(0) = \xi_h = o$. Then taking $t = 0$ in (4.4), we have

$$\int_{S^{n-1}} h^{p-1}(u) u_i \mathrm{d}\mu(u) = 0 \tag{4.5}$$

for $i = 1, \ldots, n$. Hence,

$$\int_{S^{n-1}} h^{p-1}(u) u \mathrm{d}\mu(u) = 0. \tag{4.6}$$

Let

$$F_i(t, \xi_1, \ldots, \xi_n) = \int_{S^{n-1}} (g_t(u) - (\xi_1 u_1 + \cdots + \xi_n u_n))^{p-1} u_i \mathrm{d}\mu(u)$$

for $i = 1, \ldots, n$. Then,

$$\frac{\partial F_i}{\partial \xi_j} = (1 - p) \int_{S^{n-1}} (g_t(u) - (\xi_1 u_1 + \cdots + \xi_n u_n))^{p-2} u_i u_j \mathrm{d}\mu(u).$$

Let $F = (F_1, \dots, F_n)$ and $\xi = (\xi_1, \dots, \xi_n)$. Thus,

$$\left(\frac{\partial F}{\partial \xi}\bigg|_{(0,\dots,0)}\right)_{n\times n} = (1-p)\int_{S^{n-1}} h^{p-2}(u)uu^T \mathrm{d}\mu(u),$$

and uu^T is an $n \times n$ matrix.

On account of μ is not contained in arbitrary closed hemisphere, and $\mathrm{supp}(\mu)$ spans the whole space $\mathbb{R}^n$. Then, for arbitrary $x \in \mathbb{R}^n$ with $x \neq 0$, there is a $u_{i_0} \in \mathrm{supp}(\mu)$ so that $u_{i_0} \cdot x \neq 0$. Consequently, for $0 < p < 1$ we get

$$\begin{aligned} x^T\left(\frac{\partial F}{\partial \xi}\bigg|_{(0,\dots,0)}\right)x &= x^T\left((1-p)\int_{S^{n-1}} h^{p-2}(u)uu^T\mathrm{d}\mu(u)\right)x \\ &= (1-p)\int_{S^{n-1}} h^{p-2}(u)(x\cdot u)^2\mathrm{d}\mu(u) \\ &\geq (1-p)h^{p-2}\big(u_{i_0}\big)\big(x\cdot u_{i_0}\big)^2\mu\big(u_{i_0}\big) > 0. \end{aligned}$$

This suggests that $\left(\frac{\partial F}{\partial \xi}\Big|_{(0,\dots,0)}\right)$ is positive definite, namely,

$$\det\left(\frac{\partial F}{\partial \xi}\bigg|_{(0,\dots,0)}\right) \neq 0.$$

From the implicit function theorem, the facts that $F_i(0,\dots,0) = 0$ follows by equation (4.5) for $i = 1,\dots,n$, and $\frac{\partial F_i}{\partial \xi_j}$ is continuous on a neighborhood of $(0,\dots,0)$ for all $1 \leq i, j \leq n$, we conclude that

$$\xi'(0) = (\xi_1'(0),\dots,\xi_n'(0))$$

exists.

Since $\Phi_\mu(0) = \Phi_{h,\mu}(o)$ and $\Phi_\mu(t) = \Phi_{g_t,\mu}(\xi_{g_t})$ and note that $g_t \in C^+(S^{n-1})$ and $\widetilde{V}_q([g_t]_\mu) = 1$, by Theorem 3.3 we have

$$\Phi_\mu(t) \geq \Phi_\mu(0),$$

i.e., $\Phi_\mu(0)$ is an extremum of $\Phi_\mu(t)$. Therefore, by (4.2) and (4.6) we get

$$\begin{aligned} 0 = \frac{1}{p}\Phi_\mu'(0) &= \int_{S^{n-1}} h^{p-1}(u)\left(-h(u)\int_{S^{n-1}} f(u)\mathrm{d}\widetilde{C}_q([h]_\mu, u) + h(u)f(u) - \xi'(0)\cdot u\right)\mathrm{d}\mu(u) \\ &= -\int_{S^{n-1}} h^p(u)\mathrm{d}\mu(u)\int_{S^{n-1}} f(u)\mathrm{d}\widetilde{C}_q([h]_\mu, u) + \int_{S^{n-1}} h^p(u)f(u)\mathrm{d}\mu(u) - \int_{S^{n-1}} \xi'(0)\cdot h^{p-1}(u)u\mathrm{d}\mu(u) \\ &= \int_{S^{n-1}} h^p(u)f(u)\mathrm{d}\mu(u) - c\int_{S^{n-1}} f(u)\mathrm{d}\widetilde{C}_q([h]_\mu, u), \end{aligned}$$

where

$$c = \int_{S^{n-1}} h^p(u)\mathrm{d}\mu(u) > 0.$$

That is, for all $f \in C(S^{n-1})$,

$$\int_{S^{n-1}} h^p(u)f(u)\mathrm{d}\mu(u) = c\int_{S^{n-1}} f(u)\mathrm{d}\widetilde{C}_q([h]_\mu, u).$$

Since $h = h_{[h]_\mu}$. Then,

$$\mathrm{d}\mu(u) = ch_{[h]_\mu}^{-p}(u)\mathrm{d}\widetilde{C}_q([h]_\mu, u).$$

Associated with (1.2), there is

$$\mathrm{d}\mu(u) = c\mathrm{d}\widetilde{C}_{p,q}([h]_\mu, u),$$

namely,

$$\mu = c\widetilde{C}_{p,q}([h]_\mu, \cdot). \qquad \square$$

Now, we have prepared enough to finish the proof of Theorem 1.4.

Proof of Theorem 1.4. The following proof is motivated by the work of [49, Theorem 8.2.2], for a given finite Borel measure μ on S^{n-1}, which is not contained on arbitrary closed hemisphere, then, we can find a sequence with finite discrete measure $\{\mu_j\}$ on S^{n-1} so that $\mu_j(S^{n-1}) = \mu(S^{n-1})$ and $\mu_j \to \mu$ when $j \to +\infty$. Especially, μ_j is not contained on arbitrary closed hemisphere, on the grounds of Theorem 4.1, for every μ_j there is a positive constant $c_j > 0$ and a function $h_j \in C^+(S^{n-1})$ so that

$$\mu_j = c_j\widetilde{C}_{p,q}\big([h_j]_{\mu_j}, \cdot\big), \tag{4.7}$$

where

$$c_j = \int_{S^{n-1}} h_j^p(u)\mathrm{d}\mu_j(u).$$

Moreover, h_j satisfies that $\xi_{h_j} = o$, $\widetilde{V}_q([h_j]_{\mu_j}) = 1$, and

$$\Phi_{h_j,\mu_j}(o) = \inf\left\{\sup_{\xi\in[g]_{\mu_j}} \Phi_{g,\mu_j}(\xi) : g \in C^+(S^{n-1}) \text{ and } \widetilde{V}_q\big([g]_{\mu_j}\big) = 1\right\},$$

where

$$[g]_{\mu_j} = \bigcap_{u\in\operatorname{supp}(\mu_j)} \{\xi \in \mathbb{R}^n : \xi\cdot u \le g(u)\}$$

and

$$\Phi_{g,\mu_j}(\xi) = \int_{S^{n-1}} (g(u) - \xi\cdot u)^p\mathrm{d}\mu_j(u) = \int_{\operatorname{supp}(\mu_j)} (g(u) - \xi\cdot u)^p\mathrm{d}\mu_j(u).$$

Let $m_j = \Phi_{h_j,\mu_j}(o)$. Let us prove that m_j is uniformly bounded. The Aleksandrov body is related to $(1, \operatorname{supp}(\mu_j))$, denoted by $[1]_{\mu_j}$. Let

$$\widehat{g}_j = \left(\frac{1}{\widetilde{V}_q([1]_{\mu_j})}\right)^{\frac{1}{q}}.$$

Then we see $[\widehat{g}_j]_{\mu_j} = \widehat{g}_j[1]_{\mu_j}$. Thus, there is $\widetilde{V}_q([\widehat{g}_j]_{\mu_j}) = 1$. Note that $\mu_j(S^{n-1}) = \mu(S^{n-1})$. Hence,

$$\begin{aligned} m_j = \Phi_{h_j,\mu_j}(o) &\le \sup_{\xi\in[\widehat{g}_j]_{\mu_j}} \int_{S^{n-1}} (\widehat{g}_j(u) - \xi\cdot u)^p\mathrm{d}\mu_j(u) \\ &\le \int_{S^{n-1}} D\left([\widehat{g}_j]_{\mu_j}\right)^p\mathrm{d}\mu_j(u) \\ &= D\left([\widehat{g}_j]_{\mu_j}\right)^p\mu(S^{n-1}) \\ &= \widehat{g}_j^{\,p}D\left([1]_{\mu_j}\right)^p\mu(S^{n-1}). \end{aligned} \tag{4.8}$$

We further show that $D([1]_{\mu_j})$ is uniformly bounded. Otherwise, then there is a sequence of $\{\xi_j\}$ so that $\xi_j \in [1]_{\mu_j}$ and

$$\lim_{j\to+\infty} |\xi_j| = +\infty.$$

Let $\overline{\xi}_j = \frac{\xi_j}{|\xi_j|} \in S^{n-1}$. By the compactness of S^{n-1}, we can assume

$$\lim_{j\to+\infty} \overline{\xi}_j \to \xi \in S^{n-1}.$$

On the other hand, supp(μ) is not contained on arbitrary closed hemisphere, there is $w \in \text{supp}(\mu)$ so that

$$\xi \cdot w > 0. \tag{4.9}$$

Let $U(w)$ be arbitrary neighborhood of w. Then there is

$$\liminf_{j\to+\infty} \mu_j(U(w)) \geq \mu(U(w)) > 0.$$

Now, choose j sufficiently large that satisfy

$$U(w)\bigcap \text{supp}(\mu_j) \neq \varnothing,$$

which means that we can find a sequence $\{w_{j_i}\}$ so that

$$w_{j_i} \in \text{supp}\left(\mu_{j_i}\right) \quad \text{and} \quad \lim_{i\to+\infty} w_{j_i} = w.$$

Note that $\xi_{j_i} \in [1]_{\mu_{j_i}}$. Therefore,

$$\xi_{j_i} \cdot w_{j_i} \leq h_{[1]_{\mu_{j_i}}}\left(w_{j_i}\right) \leq 1,$$

i.e.,

$$\overline{\xi}_{j_i} \cdot w_{j_i} \leq \frac{1}{|\xi_{j_i}|}.$$

Taking the limit, it follows that

$$\xi \cdot w \leq 0.$$

This contradicts (4.9). Hence, there is a positive constant $M > 0$ so that

$$D\left([1]_{\mu_j}\right) \leq M, \tag{4.10}$$

for all $j \in \mathbb{N}$.

By virtue of $B \subset [1]_{\mu_j}$ for each $j \in \mathbb{N}$, we have for $q > 0$

$$\widehat{g}_j \leq \left(\frac{1}{\widetilde{V}_q(B)}\right)^{\frac{1}{q}}. \tag{4.11}$$

Together with (4.8), (4.10), and (4.11), thus it can be seen that for $q > 0$ and all $j \in \mathbb{N}$,

$$m_j \leq M^p \widetilde{V}_q(B)^{-\frac{p}{q}} \mu(S^{n-1}), \tag{4.12}$$

namely, m_j is uniformly bounded.

Now, let us prove that $\{h_j\}$ is uniformly bounded on S^{n-1}. If this assertion is not true, there is a subsequence $\{h_{j_i}\} \subset \{h_j\}$ so that

$$\lim_{i\to+\infty} \max_{u\in S^{n-1}} h_{j_i}(u) = +\infty.$$

Let $R_{j_i} = \max_{u \in S^{n-1}} h_{j_i}(u) = h_{j_i}(u_{j_i})$, and $\{u_{j_i}\} \subset S^{n-1}$. Then, by the compactness of S^{n-1}, we can assume

$$\lim_{i\to+\infty} u_{j_i} = u_0 \in S^{n-1}.$$

Since $\operatorname{supp}(\mu)$ is not contained on arbitrary closed hemisphere, there is $v_0 \in \operatorname{supp}(\mu)$ so that

$$v_0 \cdot u_0 > 0.$$

Let $U(v_0)$ is a small neighborhood of v_0 so that for every $u \in U(v_0)$, there is

$$u \cdot u_0 > 0.$$

Let $\delta(u) = \frac{1}{2}(u \cdot u_0) > 0$ for $u \in U(v_0)$, and $R_{j_i}u_{j_i} \in [h_{j_i}]_{\mu_{j_i}}$. Then, we can choose sufficiently large i such that for all $u \in U(v_0)$,

$$u \cdot u_{j_i} > \delta(u),$$
$$h_{j_i}(u) \geq R_{j_i}\left(u \cdot u_{j_i}\right) > R_{j_i}\delta(u),$$

and

$$\mu_{j_i}(U(v_0)) \geq \mu(U(v_0)) > 0.$$

Therefore, for i sufficiently large we have

$$m_{j_i} = \int_{S^{n-1}} h_{j_i}^p(u)\mathrm{d}\mu_{j_i}(u) > R_{j_i}^p \int_{U(v_0)} \delta(u)^p \mathrm{d}\mu_{j_i}(u) \geq R_{j_i}^p \int_{U(v_0)} \delta(u)^p \mathrm{d}\mu(u),$$

which implies that $m_{j_i} \to +\infty$ when $i \to +\infty$. This contradicts (4.12). Hence, $\{h_j\}$ is uniformly bounded on S^{n-1}.

According to the Blaschke selection theorem, the sequence $\{h_j\}$ exits a convergent subsequence, again denoted by $\{h_j\}$, supposing that $h_j \to h$ on S^{n-1} as $j \to +\infty$. This implies $h \geq 0$, $[h_j]_{\mu_j} \to [h]_\mu$ as $j \to +\infty$, and

$$\lim_{j\to+\infty} c_j = \lim_{j\to+\infty} \int_{S^{n-1}} h_j^p(u)\mathrm{d}\mu_j(u) = \int_{S^{n-1}} h^p(u)\mathrm{d}\mu(u) =: c_0 \geq 0.$$

From this and (2.2), and taking the limit in (4.7), we see

$$\mu = c_0\widetilde{C}_{p,q}([h]_\mu, \cdot).$$

On the basis of this, we get $c_0 \neq 0$. Let $c_0 = \lambda^{q-p}$ with $q \neq p$ and $\lambda > 0$. Then from (2.3) we have

$$\mu = \widetilde{C}_{p,q}(\lambda[h]_\mu, \cdot).$$

This completes the proof of Theorem 1.4. □

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