Research Article

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Entire solutions for several complex partial differential-difference equations of Fermat type in $\mathbb{C}^2$

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Abstract: By utilizing the Nevanlinna theory of meromorphic functions in several complex variables, we mainly investigate the existence and the forms of entire solutions for the partial differential-difference equation of Fermat type

$$\left(\alpha\frac{\partial f(z_1, z_2)}{\partial z_1} + \beta\frac{\partial f(z_1, z_2)}{\partial z_2}\right)^m + f(z_1 + c_1, z_2 + c_2)^n = 1$$

and

$$\left(\alpha\frac{\partial f(z_1, z_2)}{\partial z_1} + \beta\frac{\partial f(z_1, z_2)}{\partial z_2}\right)^2 + [\gamma_1 f(z_1 + c_1, z_2 + c_2) - \gamma_2 f(z_1, z_2)]^2 = 1,$$

where m, n are positive integers and $\alpha, \beta, \gamma_1, \gamma_2$ are constants in $\mathbb{C}$. We give some results about the forms of solutions for these equations, which are great improvements of the previous theorems given by Xu and Cao et al. Moreover, it is very satisfactory that we give the corresponding examples to explain the conclusions of our theorems in each case.

Keywords: existence, entire solution, partial differential-difference equation

MSC 2020: 35M30, 32W50, 39A45, 30D35

1 Introduction and main results

It is well known that Wiles and Taylor [1,2] in 1995 proved Fermat's last theorem. They pointed out that the Fermat equation $x^m + y^n = 1$ does not admit nontrivial solutions in rational numbers for $m = n \geq 3$, but there exists nontrivial rational solutions for this equation for $m = n = 2$. The study of the Fermat type equation can go back to Montel [3] and Gross [4], who had discussed the existence of solutions for the Fermat type functional equation $f^m + g^n = 1$ and showed that the entire solutions are $f = \cos a(z)$, $g = \sin a(z)$ for $m = n = 2$, where $a(z)$ is an entire function; there are no nonconstant entire solutions for $m = n > 2$.

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Of late, due to the establishment of difference analogues of Nevanlinna theory for meromorphic functions (see [5–7]), many scholars have paid an increasing interest in studying the properties of solutions for complex difference equations, complex differential-difference equations, and obtained a number of results about the existence and the form of solutions for some equations (including [7–17]).

Liu et al. [18] in 2012 studied the existence of solutions for some complex difference equations and obtained the following:

Theorem A. (see [18, Theorem 1.3]). *The transcendental entire solution with finite order of*

$$f'(z)^2 + f(z+c)^2 = 1$$

must satisfy $f(z) = \sin(z \pm Bi)$, *where* B *is a constant and* $c = 2k\pi$ *or* $c = (2k+1)\pi$, *and* k *is an integer.*

Theorem B. (see [18, Theorem 1.5]). *The transcendental entire solutions with finite order of*

$$f'(z)^2 + [f(z+c) - f(z)]^2 = 1$$

must satisfy $f(z) = 12\sin(2z + Bi)$, *where* $c = (2k+1)\pi$, k *is an integer, and* B *is a constant.*

In recent years, with the development of difference analogues of Nevanlinna theory for meromorphic functions with several complex variables, especially the difference analogue of the logarithmic derivative lemma given by Cao and Korhonen [19]. Very recently, Xu and Cao [20–22] investigated the existence of the entire and meromorphic solutions for some Fermat type partial differential-difference equations and obtained the following theorems.

Theorem C. (see [20, Theorem 3.2]). *Let* $c = (c_1, c_2) \in \mathbb{C}\backslash\{0\}$. *Suppose that* f *is a nontrivial meromorphic solution of the Fermat type partial difference equations*

$$\frac{1}{f(z_1+c_1, z_2+c_2)^m} + \frac{1}{f(z_1, z_2)^m} = A(z_1, z_2)f(z_1, z_2)^n$$

or

$$\frac{1}{f(z_1+c_1, z_2+c_2)^m} + \frac{1}{f(z_1+c_1, z_2)^m} + \frac{1}{f(z_1, z_2+c_2)^m} = A(z_1, z_2)f(z_1, z_2)^n,$$

where $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, *and* $A(z_1, z_2)$ *is a nonzero meromorphic function on* $\mathbb{C}^2$ *with respect to the solution* f, *that is,* $T(r, A) = o(T(r, f))$. *If* $\delta_f(\infty) > 0$, *then*

$$\limsup_{r\to\infty} \frac{\log T(r, f)}{r} > 0.$$

Remark 1.1. Let $n = 0$ and $A(z_1, z_2) = 1$, then the above equations become

$$\frac{1}{f(z_1+c_1, z_2+c_2)^m} + \frac{1}{f(z_1, z_2)^m} = 1$$

and

$$\frac{1}{f(z_1+c_1, z_2+c_2)^m} + \frac{1}{f(z_1+c_1, z_2)^m} + \frac{1}{f(z_1, z_2+c_2)^m} = 1,$$

which can be called as the partial difference equations of Fermat type.

Theorem D. (see [21, Theorem 1.1]). *Let* $c = (c_1, c_2) \in \mathbb{C}^2$. *Then the Fermat type partial differential-difference equation*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^n + f(z_1+c_1, z_2+c_2)^m = 1$$

does not have any transcendental entire solution with finite order, where m *and* n *are two distinct positive integers.*

Theorem E. (see [21, Theorem 1.2]). *Let* $c = (c_1, c_2) \in \mathbb{C}^2$. *Then any transcendental entire solution with finite order of the partial differential-difference equation*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1$$

has the form of $f(z_1, z_2) = \sin(Az_1 + B)$, *where A is a constant on* $\mathbb{C}$ *satisfying* $Ae^{iAc_1} = 1$, *and B is a constant on* $\mathbb{C}$; *in the special case whenever* $c_1 = 0$, *we have* $f(z_1, z_2) = \sin(z_1 + B)$.

Theorems D and E suggest a question naturally: *What will happen about the existence and the forms of transcendental entire solutions for the Fermat type partial differential-difference equation including both* $\frac{\partial f(z_1,z_2)}{\partial z_1}$ *and* $\frac{\partial f(z_1,z_2)}{\partial z_2}$?

In view of the above questions, this paper is concerned with the properties of the solutions for some Fermat type equations including both difference operator and two partial differential by utilizing the Nevanlinna theory and difference Nevanlinna theory of several complex variables [19,20,23]. We describe the existence and the forms of the transcendental entire solutions with finite order for the Fermat type partial differential-difference equations with more general forms, and some results are obtained to improve the previous theorems given by Xu and Cao [21] and Liu et al. [18]. Here and below, let $z + w = (z_1 + w_1, z_2 + w_2)$ for any $z = (z_1, z_2)$, $w = (w_1, w_2)$. The main results of this paper are listed below.

Theorem 1.1. *Let* $c = (c_1, c_2) \in \mathbb{C}^2$, m, n *be two distinct positive integers, and* α, β *be constants in* $\mathbb{C}$ *that are not equal to zero at the same time. If the Fermat type partial differential-difference equation*

$$\left(\alpha\frac{\partial f(z_1, z_2)}{\partial z_1} + \beta\frac{\partial f(z_1, z_2)}{\partial z_2}\right)^m + f(z_1 + c_1, z_2 + c_2)^n = 1 \tag{1}$$

satisfies one of the conditions

(*i*) $n > m$;

(*ii*) $m > n \geq 2$;

then equation (1) *does not have any transcendental entire solution with finite order.*

Remark 1.2. For $m > n = 1$, the transcendental entire solution of equation (1) with finite order can be found. Especially, let $m = 2$, $n = 1$. Then equation (1) becomes of the form

$$\left(\alpha\frac{\partial f(z_1, z_2)}{\partial z_1} + \beta\frac{\partial f(z_1, z_2)}{\partial z_2}\right)^2 + f(z_1 + c_1, z_2 + c_2) = 1. \tag{2}$$

Here, we will give the detail to get a solution of equation (2).

Set $F(z_1, z_2) = \alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}$, differentiating (2) for z_1, z_2, respectively, then it follows that

$$F(z_1, z_2)\left(\alpha\frac{\partial F(z_1, z_2)}{\partial z_1} + \beta\frac{\partial F(z_1, z_2)}{\partial z_2}\right) = -\frac{1}{2}F(z_1 + c_1, z_2 + c_2). \tag{3}$$

Assuming that

$$\alpha\frac{\partial F(z_1, z_2)}{\partial z_1} + \beta\frac{\partial F(z_1, z_2)}{\partial z_2} = -\frac{1}{2}, \tag{4}$$

$$F(z_1, z_2) = F(z_1 + c_1, z_2 + c_2). \tag{5}$$

For equation (4), one can find a finite order transcendental entire solution

$$F(z_1, z_2) = e^{\beta z_1 - \alpha z_2} - \frac{1}{4}\left(\frac{z_1}{\alpha} + \frac{z_2}{\beta}\right).$$

By combining with (5), we can deduce that $c_1 = \frac{\pi}{\beta}i$ and $c_2 = -\frac{\pi}{\alpha}i$.

In view of the assumption, it follows that

$$\alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2} = F(z_1,z_2) = e^{\beta z_1-\alpha z_2} - \frac{1}{4}\left(\frac{z_1}{\alpha} + \frac{z_2}{\beta}\right). \tag{6}$$

The characteristic equations for equation (6) are

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = \alpha, \quad \frac{\mathrm{d}z_2}{\mathrm{d}t} = \beta, \quad \frac{\mathrm{d}f}{\mathrm{d}t} = F(z_1,z_2).$$

Using the initial conditions: $z_1 = 0$, $z_2 = s$, and $f = f(0,s) := g(s)$ with a parameter s. Thus, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1 = \alpha t$, $z_2 = \beta t + s$,

$$f = \int_0^t F(\alpha t, \beta t + s)\mathrm{d}t + g(s) = \int_0^t \left[e^{-\alpha s} - \frac{1}{4}\left(2t + \frac{1}{\beta}s\right)\right]\mathrm{d}t + g(s) = te^{-\alpha s} - \frac{1}{4}t^2 - \frac{1}{4\beta}st + g(s),$$

where $g(s)$ is an entire function with finite order in s. Then, by combining with $t = \frac{z_1}{\alpha}$ and $s = z_2 - \frac{\beta}{\alpha}z_1$, the solution of equation (2) is of the form

$$f(z_1,z_2) = \frac{1}{\alpha}z_1e^{\beta z_1-\alpha z_2} - \frac{1}{4}\frac{z_1z_2}{\alpha\beta} + g\left(\frac{\alpha z_2 - \beta z_1}{\alpha}\right), \tag{7}$$

where $g(s)$ satisfies the following equation:

$$g\left(\frac{\alpha z_2 - \beta z_1}{\alpha} + \frac{\alpha c_2 - \beta c_1}{\alpha}\right) = 1 - \left[e^{\beta z_1-\alpha z_2} - \frac{1}{4}\left(\frac{z_1}{\alpha} + \frac{z_2}{\beta}\right)\right]^2 + \frac{z_1 + c_1}{\alpha}e^{\beta z_1-\alpha z_2} - \frac{1}{4}\frac{(z_1+c_1)(z_2+c_2)}{\alpha\beta}. \tag{8}$$

Let

$$z_1 = \frac{\alpha}{\alpha-\beta}s + \frac{2\alpha c_2}{\alpha+\beta}, \quad z_2 = \frac{\alpha}{\alpha-\beta}s - \frac{2\alpha c_2}{\alpha+\beta},$$

then

$$\frac{\alpha z_2 - \beta z_1}{\alpha} + 2c_2 = s.$$

Substituting z_1, z_2 into (8), and combining with $c_1 = \frac{\pi}{\beta}i$ and $c_2 = -\frac{\pi}{\alpha}i$, we conclude

$$\begin{aligned} g(s) = 1 &- \left(e^{-\alpha s} - \frac{1}{4}\frac{(\alpha+\beta)^2 s - 2(\alpha-\beta)^2 c_2}{\beta(\alpha^2-\beta^2)}\right)^2 + \left(\frac{s}{\alpha-\beta} - \frac{(\alpha-\beta)c_2}{\beta(\alpha+\beta)}\right)e^{-\alpha s} \\ &- \frac{1}{4}\left[\frac{s}{\alpha-\beta} - \frac{(\alpha-\beta)c_2}{\beta(\alpha+\beta)}\right]\left[\frac{\alpha s}{\beta(\alpha-\beta)} - \frac{(\alpha-\beta)c_2}{\alpha+\beta}\right]. \end{aligned} \tag{9}$$

Therefore, an entire solution of equation (2) is of the form

$$f(z_1,z_2) = \frac{1}{\alpha}z_1e^{\beta z_1-\alpha z_2} - \frac{1}{4}\frac{z_1z_2}{\alpha\beta} + g\left(\frac{\alpha z_2 - \beta z_1}{\alpha}\right),$$

where $g(s)$ is stated as in (9), $c_1 = \frac{\pi}{\beta}i$, and $c_2 = -\frac{\pi}{\alpha}i$.

For $m = n = 2$, we obtain the following result.

Theorem 1.2. *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and α, β be constants in $\mathbb{C}$ that are not equal to zero at the same time. Then any transcendental entire solution with finite order for the Fermat type partial differential-difference equation*

$$\left(\alpha\frac{\partial f(z_1, z_2)}{\partial z_1} + \beta\frac{\partial f(z_1, z_2)}{\partial z_2}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1 \tag{10}$$

has the following form:

$$f(z_1, z_2) = \frac{A_1 e^{a_1 z_1 + a_2 z_2 + H(z)} + A_2 e^{-(a_1 z_1 + a_2 z_2) - H(z)}}{2},$$

where $H(z) \coloneqq H(s_1)$ is a polynomial in s_1 satisfying $(\alpha c_2 - \beta c_1)H' \equiv 0$, $s_1 = c_2 z_1 - c_1 z_2$, $A_1 A_2 = 1$, A_1, A_2 are constants in $\mathbb{C}$, and $c, a_1, a_2, \alpha, \beta$ satisfy one of the following cases:
(*i*) $\alpha a_1 + \beta a_2 = i$ *and* $L(c) = 2k\pi i$, *where* $L(c) = a_1 c_1 + a_2 c_2$, *here and below* k *is an integer*;
(*ii*) $\alpha a_1 + \beta a_2 = -i$ *and* $L(c) = (2k + 1)\pi i$.

Remark 1.3. In view of Theorem 1.2, if $\alpha = 1$, $\beta = 0$, and $c_2 \neq 0$, then we can conclude that any finite order transcendental entire solution f of equation (10) has the form

$$f(z) = \frac{e^{L(z)+B} + e^{-(L(z)+B)}}{2}$$

satisfying $a_1 = i$ and $L(c) = 2k\pi i$, or $a_1 = -i$ and $L(c) = (2k + 1)\pi i$.

The following examples show the existence of solutions for equation (10).

Example 1.1. Let $a = (a_1, a_2) = (i, i)$ and $B = 0$. Thus, the function

$$f(z_1, z_2) = \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2}$$

satisfies equation (10) with $c = (c_1, c_2) = (\pi, \pi)$, $\alpha = 2$, $\beta = -1$.

Example 1.2. Let $a = (a_1, a_2) = (1, 1)$ and $B = 0$. Thus, the function

$$f(z_1, z_2) = \frac{e^{z_1+z_2} + e^{-(z_1+z_2)}}{2}$$

satisfies equation (10) with $c = (c_1, c_2) = (2\pi i, -\pi i)$, $\alpha = i$, $\beta = -2i$.

Example 1.3. Let $a = (a_1, a_2) = (3i, -2i)$, $H(z) = 4\pi^2(z_1 - z_2)^2$, and $B = 0$. Thus, the function

$$f(z_1, z_2) = \frac{2e^{i(3z_1-2z_2)+4\pi^2(z_1-z_2)^2} + \frac{1}{2}e^{-i(3z_1-2z_2)-4\pi^2(z_1-z_2)^2}}{2}$$

satisfies equation (10) with $c = (c_1, c_2) = (2\pi, 2\pi)$, $\alpha = 1$, $\beta = 1$.

Example 1.4. Let $a = (a_1, a_2) = (2i, -3i)$, $H(z) = \pi^4(z_1 - z_2)^4$, and $B = 0$. Thus, the function

$$f(z_1, z_2) = \frac{e^{i(2z_1-3z_2)+\pi^4(z_1-z_2)^4} + e^{-i(2z_1-3z_2)-\pi^4(z_1-z_2)^4}}{2}$$

satisfies equation (10) with $c = (c_1, c_2) = (-\pi, -\pi)$, $\alpha = 1$, $\beta = 1$.

Remark 1.4. From the conclusions of Theorems A and E, there only exist finite order transcendental entire solutions with growth order $\rho(f) = 1$. However, we can see that there exists transcendental entire solution of (10) with growth order $\rho(f) > 1$, that is, $\rho(f) = 2$ in Example 1.3 and $\rho(f) = 4$ in Example 1.4, these results are quite different from the previous one. Hence, our results are some improvements of the previous theorems given by Xu and Cao [21], and Liu et al. [18].

Theorem 1.3. *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $c_2, \alpha, \beta_1, \beta_2$ be nonzero constants in $\mathbb{C}$. Let f be a finite order transcendental entire solution of the Fermat type partial differential-difference equation*

$$\left(\alpha\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + [\beta_1 f(z_1 + c_1, z_2 + c_2) - \beta_2 f(z_1, z_2)]^2 = 1. \tag{11}$$

(*i*) *If $\frac{\partial f(z_1,z_2)}{\partial z_1}$ is not constant, then the solution $f(z)$ has the following form:*

$$f(z_1, z_2) = \frac{e^{a_1z_1+a_2z_2+B} - e^{-(a_1z_1+a_2z_2+B)}}{2a_1\alpha} + e^{\eta z_2}G(z_2),$$

where a_1, a_2, B, η are constants in $\mathbb{C}$, $G(z_2)$ is a finite order entire period function with period c_2. Moreover,

(*i₁*) *if $\beta_1 \neq \pm\beta_2$, then $\eta = \frac{\log\beta_2 - \log\beta_1}{c_2}$, and either $a_1\alpha = i(\beta_1 - \beta_2)$, $L(c) = 2k\pi i$, or $a_1\alpha = -i(\beta_2 + \beta_1)$, $L(c) = (2k + 1)\pi i$, where $L(c) = a_1c_1 + a_2c_2$;*

(*i₂*) *if $\beta_1 = \beta_2$, then $\eta \equiv 0$, $i\alpha a_1 = 2\beta_1$, and $L(c) = (2k + 1)\pi i$;*

(*i₃*) *if $\beta_1 = -\beta_2$, then $\eta = \frac{\log(-1)}{c_2}$, $i\alpha a_1 = -2\beta_1$, and $L(c) = 2k\pi i$;*

(*ii*) *If $\frac{\partial f(z_1,z_2)}{\partial z_1}$ is a constant, then the solution $f(z)$ has the following form:*

$$f(z) = \left[e^{\eta z_2}G(z_2) + \tau\right] + D_1z_1 + D_2z_2 + D_0,$$

where τ, D_0, D_1, D_2 are constants in $\mathbb{C}$. Moreover,

(*ii₁*) *if $\beta_1 = \beta_2$, then $\eta = 0$, $\tau \in \mathbb{C}$, and $[\beta_1(D_1c_1 + D_2c_2)]^2 = 1 - (\alpha D_1)^2$;*

(*ii₂*) *if $\beta_1 \neq \beta_2$, then $\eta = \frac{\log\beta_2 - \log\beta_1}{c_2}$, $D_1 = D_2 = 0$, and $\tau, \beta_1, \beta_2, D_0$ satisfy*

$$[(\beta_1 - \beta_2)(\tau - D_0)]^2 = \beta_1^2.$$

Next, we give some examples to explain the conclusions of Theorem 1.3 in each case.

Example 1.5. Let $a = (a_1, a_2) = (1, 1)$. Thus, the function

$$f(z_1, z_2) = \frac{e^{z_1+z_2} - e^{-(z_1+z_2)}}{2i} + e^{\frac{z_2\log 2}{\pi i}}\left(e^{2z_2} + 1\right)$$

satisfies equation (11) with $c = (c_1, c_2) = (\pi i, \pi i)$, $\alpha = i$, $\beta_1 = 2$, and $\beta_2 = 1$.

Example 1.6. Let $a = (a_1, a_2) = (-1, 1)$. Thus, the function

$$f(z_1, z_2) = \frac{e^{z_1-z_2} - e^{-(z_1-z_2)}}{-2i} + e^{\frac{z_2\log(-1.5)}{2\pi i}}\left(e^{z_2} - 1\right)$$

satisfies equation (11) with $c = (c_1, c_2) = (\pi i, 2\pi i)$, $\alpha = i$, $\beta_1 = 2$, and $\beta_2 = -3$.

Example 1.7. Let $a = (a_1, a_2) = (1, 2)$ and $B, D_0 \in \mathbb{C}$. Thus, the function

$$f(z_1, z_2) = \frac{e^{z_1+2z_2+B} - e^{-(z_1+2z_2+B)}}{2i} + e^{2z_2} + D_0$$

satisfies equation (11) with $c = (c_1, c_2) = (-\pi i, \pi i)$, $\alpha = i$, $\beta_1 = \beta_2 = \frac{1}{2}$.

Example 1.8. Let $a = (a_1, a_2) = \left(\frac{1}{2}, \pi i\right)$, $D_0 = 0$, and $B \in \mathbb{C}$. Thus, the function

$$f(z_1, z_2) = \frac{e^{\frac{1}{2}z_1+\pi iz_2+B} - e^{-\left(\frac{1}{2}z_1+\pi iz_2+B\right)}}{2i} + e^{\pi iz_2}e^{2\pi iz_2}$$

satisfies equation (11) with $c = (c_1, c_2) = (2\pi i, 1)$, $\alpha = 2i$, $\beta_1 = -\beta_2 = \frac{1}{2}$.

Example 1.9. Let $G(z_2) = \frac{e^{z_2}+e^{-z_2}}{2}$, $\tau = 0$, $D_0 = 1$, $D_1 = 0$ and $D_2 = \frac{1}{2\pi i}$. Thus, the function

$$f(z_1, z_2) = \frac{e^{z_2} + e^{-z_2}}{2} + \frac{1}{2\pi i} z_2 + 1$$

satisfies equation (11) with $c = (c_1, c_2) = (c_1, 2\pi i)$, $\alpha \in \mathbb{C}$, $\beta_1 = \beta_2 = 1$.

Example 1.10. Let $G(z_2) = e^{\pi i z_2} + e^{-\pi i z_2} + e^{2\pi i z_2}$, $\tau = 0$, $D_0 = 2$, $D_1 = \frac{\sqrt{3}}{4}$, and $D_2 = -\frac{1}{2}$. Thus, the function

$$f(z_1, z_2) = e^{\pi i z_2} + e^{-\pi i z_2} + e^{2\pi i z_2} + \frac{\sqrt{3}}{4} z_1 - \frac{1}{2} z_2 + 2$$

satisfies equation (11) with $c = (c_1, c_2) = (\sqrt{3}, 2)$, $\alpha = 2$, $\beta_1 = \beta_2 = 2$.

Example 1.11. Let $G(z_2) = e^{i\pi z_2} + e^{-i\pi z_2}$, $c = (c_1, c_2) = (c_1, 2)$, $\alpha \in \mathbb{C}$, $\beta_1 = 1$, $\beta_2 = e^2$, $D_0 = 1$, and $\tau = \frac{2-e^2}{e^2-1}$. Thus, the function

$$f(z_1, z_2) = e^{z_2}\left(e^{i\pi z_2} + e^{-i\pi z_2}\right) + \frac{1}{e^2 - 1}$$

satisfies the following equation:

$$\left(\alpha \frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + [f(z_1 + c_1, z_2 + 2) - e^2 f(z_1, z_2)]^2 = 1.$$

Remark 1.5. Examples 1.5–1.8 show that the forms of solutions are true for cases (i_1)–(i_3) in Theorem 1.3(i), and Examples 1.9–1.11 show that the forms of solutions are true for cases (ii_1)–(ii_2) in Theorem 1.3(ii).

Naturally, we should proceed to discuss the existence and form of solutions for the following Fermat type partial differential-difference equation:

$$\left(\alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2}\right)^2 + [\gamma_1 f(z_1 + c_1, z_2 + c_2) - \gamma_2 f(z_1, z_2)]^2 = 1, \tag{12}$$

where $\alpha, \beta, \gamma_1, \gamma_2$ are constants in $\mathbb{C}$.

Theorem 1.4. *Let $c = (c_1, c_2) \in \mathbb{C}^2$ and $c_1, c_2, \alpha, \beta, \gamma_1, \gamma_2$ be nonzero constants in $\mathbb{C}$ such that $\alpha c_2 - \beta c_1 \neq 0$. Let f be a finite order transcendental entire solution of the Fermat type partial differential-difference equation* (12)

(i) If $\alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2}$ is not constant, then the solution $f(z)$ has the following form:

$$f(z_1, z_2) = \frac{e^{a_1 z_1 + a_2 z_2 + B} - e^{-(a_1 z_1 + a_2 z_2 + B)}}{2(a_1\alpha + a_2\beta)} + e^{\eta(\alpha z_2 - \beta z_1)} G(\alpha z_2 - \beta z_1),$$

where $G(u)$ is a finite order entire period function with period $\alpha c_2 - \beta c_1$, a_1, a_2, B, η are constants in $\mathbb{C}$. Moreover,

(i_1) if $\gamma_1 \neq \pm\gamma_2$, then $\eta = \frac{\log \gamma_2 - \log \gamma_1}{\alpha c_2 - \beta c_1}$, and either $a_1\alpha + a_2\beta = i(\gamma_1 - \gamma_2)$, $L(c) = 2k\pi i$, or $a_1\alpha + a_2\beta = -i(\gamma_2 + \gamma_1)$, $L(c) = (2k+1)\pi i$, where $L(c) = a_1 c_1 + a_2 c_2$;

(i_2) if $\gamma_1 = \gamma_2$, then $\eta \equiv 0$, $i(a_1\alpha + a_2\beta) = 2\gamma_1$ and $L(c) = (2k+1)\pi i$;

(i_3) if $\gamma_1 = -\gamma_2$, then $\eta = \frac{\log(-1)}{\alpha c_2 - \beta c_1}$, $i(a_1\alpha + a_2\beta) = -2\gamma_1$ and $L(c) = 2k\pi i$;

(ii) If $\alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2}$ is a constant, then the solution $f(z)$ has the following form:

$$f(z_1, z_2) = D_1 z_1 + D_2(\alpha z_2 - \beta z_1) + \left[e^{\eta(\alpha z_2 - \beta z_1)} G(\alpha z_2 - \beta z_1) + \tau\right],$$

where τ, D_1, D_2 are constants in $\mathbb{C}$. Moreover,

(ii_1) if $\gamma_1 = \gamma_2$, then $\eta = 0$, $\tau \in \mathbb{C}$ and

$$\gamma_1^2 [D_1 c_1 + D_2(\alpha c_2 - \beta c_1)]^2 = 1 - (\alpha D_1)^2;$$

(ii_2) *if* $\gamma_1 \neq \gamma_2$, *then* $\eta = \frac{\log\gamma_2 - \log\gamma_1}{\alpha c_2 - \beta c_1}$, $D_1 = D_2 = 0$, *and* τ, γ_1, γ_2 *satisfy*

$$(\gamma_1 - \gamma_2)\tau = \pm 1.$$

Remark 1.6. Obviously, we can see that equation (12) is transformed into equation (10) when $\gamma_2 = 0$, and equation (11) for $\beta = 0$. Hence, we only consider the case that $\alpha, \beta, \gamma_1, \gamma_2$ are nonzero constants in $\mathbb{C}$ in Theorem 1.4.

Next, we give some examples to explain the conclusions of Theorem 1.4 in each case.

Example 1.12. Let $a_1 = a_2 = 1$ and $B \in \mathbb{C}$. Thus, the function

$$f(z_1, z_2) = \frac{e^{z_1+z_2+B} - e^{-(z_1+z_2+B)}}{2i} + e^{\frac{\log 2}{2\pi i}(z_2-z_1)}e^{z_1-z_2}$$

satisfies equation (12) with $\gamma_1 = 2$, $\gamma_2 = 1$, $\alpha = \beta = \frac{i}{2}$, and $c_1 = -c_2 = \pi i$.

Example 1.13. Let $a_1 = a_2 = i$ and $B \in \mathbb{C}$. Thus, the function

$$f(z_1, z_2) = \frac{e^{iz_1+iz_2+B} - e^{-(iz_1+iz_2+B)}}{-4i} + e^{2i(3z_1+z_2)}$$

satisfies equation (12) with $\gamma_1 = \gamma_2 = 1$, $\alpha = 1$, $\beta = -3$, and $c_1 = 2\pi$, $c_2 = -\pi$.

Example 1.14. Let $a_1 = a_2 = i$ and $B \in \mathbb{C}$. Thus, $\eta = \frac{i}{3}$ and the function

$$f(z_1, z_2) = \frac{e^{z_1+z_2+B} - e^{-(z_1+z_2+B)}}{4i} + e^{\frac{z_1-z_2}{3}}e^{2(z_1-z_2)}$$

satisfies equation (12) with $\gamma_1 = -\gamma_2 = 1$, $\alpha = \beta = i$ and $c_1 = \frac{5\pi i}{2}$, $c_2 = -\frac{\pi i}{2}$.

Example 1.15. If $\alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2} \equiv 0$, then from (12), it follows $\gamma_1 f(z_1 + c_1, z_2 + c_2) - \gamma_2 f(z_1, z_2) = \pm 1$. Thus, the function $f(z_1, z_2) = e^{\beta z_1 - \alpha z_2} + \beta z_1 - \alpha z_2$ satisfies the following equation:

$$\left(\alpha\frac{\partial f(z_1, z_2)}{\partial z_1} + \beta\frac{\partial f(z_1, z_2)}{\partial z_2}\right)^2 + \left[\frac{1}{2\pi i}f(z_1 + c_1, z_2 + c_2) - \frac{1}{2\pi i}f(z_1, z_2)\right]^2 = 1,$$

where α, β, c_1, c_2 are constants in $\mathbb{C}$ satisfying $\beta c_1 - \alpha c_2 = 2\pi i$.

Example 1.16. If $\alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2} = 1$, then from (12), it follows that $\gamma_1 f(z_1 + c_1, z_2 + c_2) - \gamma_2 f(z_1, z_2) = 0$. Let $\gamma_1 = \gamma_2 = 2$, $D_1 = 2$, $D_2 = -1$, $\alpha = -\frac{1}{2}$, $\beta = -1$, $c_1 = \pi i$, and $c_2 = -2\pi i$. Thus, the function $f(z_1, z_2) = e^{z_1 - \frac{1}{2}z_2} + 2z_1 - \left(-\frac{1}{2}z_2 + z_1\right)$ satisfies the following equation:

$$\left(\frac{1}{2}\frac{\partial f(z_1, z_2)}{\partial z_1} + \frac{\partial f(z_1, z_2)}{\partial z_2}\right)^2 + [2f(z_1 + \pi i, z_2 - 2\pi i) - 2f(z_1, z_2)]^2 = 1.$$

Example 1.17. Let $\gamma_1 = 2$, $\gamma_2 = 1$, $\tau = 1$, and α, β, c_1, c_2 be nonzero constants in $\mathbb{C}$ such that $\alpha c_2 - \beta c_1 = 2$. Then the function

$$f(z_1, z_2) = e^{-\frac{\log 2}{2}(\alpha z_2 - \beta z_1)}e^{\pi i(\alpha z_2 - \beta z_1)} + 1$$

satisfies the following equation:

$$\left(\alpha\frac{\partial f(z_1, z_2)}{\partial z_1} + \beta\frac{\partial f(z_1, z_2)}{\partial z_2}\right)^2 + [f(z_1 + c_1, z_2 + c_1) - 2f(z_1, z_2)]^2 = 1.$$

Remark 1.7. Examples 1.12–1.14 show that the forms of solutions are true for cases (i_1)–(i_3) in Theorem 1.4(i), and examples 1.15–1.17 show that the forms of solutions are true for cases (ii_1)–(ii_2) in Theorem 1.4(ii).

2 Proof of Theorem 1.1

To prove Theorem 1.1, we require the following lemmas.

Lemma 2.1. [24,25] *Let f be a nonconstant meromorphic function on $\mathbb{C}^n$ and let $I = (i_1, \ldots, i_n)$ be a multi-index with length $|I| = \sum_{j=1}^{n} i_j$. Assume that $T(r_0, f) \geq e$ for some r_0. Then*

$$m\left(r, \frac{\partial^I f}{f}\right) = S(r, f)$$

holds for all $r \geq r_0$ outside a set $E \subset (0, +\infty)$ of finite logarithmic measure $\int_E \frac{dt}{t} < \infty$, where $\partial^I f = \frac{\partial^{|I|} f}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}}$.

Lemma 2.2. [19,23] *Let f be a nonconstant meromorphic function with finite order on $\mathbb{C}^n$ such that $f(0) \neq 0, \infty$, and let $\varepsilon > 0$. Then for $c \in \mathbb{C}^n$,*

$$m\left(r, \frac{f(z)}{f(z+c)}\right) + m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f)$$

holds for all $r \geq r_0$ outside a set $E \subset (0, +\infty)$ of finite logarithmic measure $\int_E \frac{dt}{t} < \infty$.

Proof of Theorem 1.1. Suppose that f is a finite order transcendental entire solution of equation (1). Since α, β are constants and not equal to zero at the same time, then $\alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2}$ and $f(z_1 + c_1, z_2 + c_2)$ are transcendental. Here, we will consider two cases below.

Case 1. $n > m$. In view of Lemma 2.2, it yields that

$$m\left(r, \frac{f(z_1, z_2)}{f(z_1 + c_1, z_2 + c_2)}\right) = S(r, f) \tag{13}$$

holds for all $r > 0$ outside of a possible exceptional set $E \subset [1, +\infty)$ of finite logarithmic measure $\int_E \frac{dt}{t} < \infty$. Thus, it follows from (13) that

$$\begin{aligned} T(r, f(z_1, z_2)) &= m(r, f(z_1, z_2)) \\ &\leq m\left(r, \frac{f(z_1, z_2)}{f(z_1 + c_1, z_2 + c_2)}\right) + m(r, f(z_1 + c_1, z_2 + c_2)) + \log 2 \\ &= m(r, f(z_1 + c_1, z_2 + c_2)) + S(r, f) \\ &= T(r, f(z_1 + c_1, z_2 + c_2)) + S(r, f), \end{aligned} \tag{14}$$

for all $r \notin E$. In view of (14), Lemma 2.1, and the Mokhon'ko theorem in several complex variables [26, Theorem 3.4], it yields

$$\begin{aligned} nT(r, f(z_1, z_2)) &\leq nT(r, f(z_1 + c_1, z_2 + c_2)) + S(r, f) \\ &= T(r, f(z_1 + c_1, z_2 + c_2)^n) + S(r, f) \\ &= T\left(r, \left(\alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2}\right)^m - 1\right) + S(r, f) \\ &= mT\left(r, \alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2}\right) + S(r, f) \end{aligned} \tag{15}$$

$$
\begin{aligned}
&= mm\left(r, \alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}\right) + S(r,f)\\
&\le m\left(m\left(r, \frac{\alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}}{f_1(z_1,z_2)}\right) + m(r, f(z_1,z_2))\right) + S(r,f)\\
&= mT(r, f(z_1,z_2)) + S(r,f),
\end{aligned}
$$

for all $r \notin E$. That is,

$$
(n-m)T(r, f(z_1,z_2)) \le S(r,f), \quad r \notin E, \tag{16}
$$

which is a contradiction with the assumption that f is a transcendental entire function.

Case 2. $m > n \ge 2$. Then $\frac{1}{m} + \frac{1}{n} \le \frac{2}{n} < 1$ and $n > \frac{m}{m-1}$. In view of the Nevanlinna second fundamental theorem, Lemma 2.2, and equation (1), it follows that

$$
\begin{aligned}
&(m-1)T\left(r, \alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}\right)\\
&\le \overline{N}\left(r, \alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}\right) + \sum_{q=1}^{m}\overline{N}\left(r, \frac{1}{\alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2} - w_q}\right)\\
&\quad + S\left(r, \alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}\right)\\
&\le \overline{N}\left(r, \frac{1}{\left(\alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}\right)^m - 1}\right) + S\left(r, \alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}\right)\\
&\le \overline{N}\left(r, \frac{1}{f(z_1+c_1, z_2+c_2)}\right) + S(r,f)\\
&\le T(r, f(z_1+c_1, z_2+c_2)) + S(r,f_1) + S(r,f),
\end{aligned} \tag{17}
$$

where w_q is a root of $w^m - 1 = 0$.

On the other hand, in view of equation (1) and the Mokhon'ko theorem in several complex variables [26, Theorem 3.4], it follows

$$
\begin{aligned}
nT(r, f(z_1+c_1, z_2+c_2)) &= T(r, f(z_1+c_1, z_2+c_2)^n) + S(r,f)\\
&= T\left(r, \left(\alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}\right)^m - 1\right) + S(r,f)\\
&= mT\left(r, \alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}\right) + S(r,f).
\end{aligned} \tag{18}
$$

In view of (17)–(18) and $n > \frac{m}{m-1}$, it follows

$$
\left(n - \frac{m}{m-1}\right)T(r, f(z_1+c_1, z_2+c_2)) \le S(r,f),
$$

which is a contradiction with the assumption that f is a transcendental entire function.

Therefore, this completes the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

The following lemmas play the key role in proving Theorems 1.2–1.3.

Lemma 3.1. [27, Lemma 3.1] *Let $f_j(\not\equiv 0)$, $j = 1, 2, 3$, be meromorphic functions on $\mathbb{C}^m$ such that f_1 is not constant, and $f_1 + f_2 + f_3 = 1$, and such that*

$$\sum_{j=1}^{3}\left\{N_2\left(r, \frac{1}{f_j}\right) + 2\overline{N}(r, f_j)\right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda < 1$ is a positive number. Then either $f_2 = 1$ or $f_3 = 1$.

Remark 3.1. Here, $N_2\left(r, \frac{1}{f}\right)$ is the counting function of the zeros of f in $|z| \leq r$, where the simple zero is counted once, and the multiple zero is counted twice.

Lemma 3.2. [28,29] *For an entire function F on $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$. Then there exist a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.*

Remark 3.2. Here, denote $\rho(n_F)$ to be of the order of the counting function of zeros of F.

Lemma 3.3. [30] *If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*
(*a*) *the internal function h is a polynomial and the external function g is of finite order; or else*
(*b*) *the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.*

Now, we continue to prove Theorem 1.2.

Proof. Suppose that f is a finite order transcendental entire solution of equation (10). Since α, β are constants and not equal to zero at the same time, then it follows that $\alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}$ is transcendental. Otherwise, $f(z_1 + c_1, z_2 + c_2)$ is not transcendental, a contradiction. We first rewrite (10) as the following form:

$$\left[\left(\alpha\frac{\partial f(z)}{\partial z_1} + \beta\frac{\partial f(z)}{\partial z_2}\right) + if(z + c)\right]\left[\left(\alpha\frac{\partial f(z)}{\partial z_1} + \beta\frac{\partial f(z)}{\partial z_2}\right) - if(z + c)\right] = 1. \tag{19}$$

Since $\alpha\frac{\partial f(z_1,z_2)}{\partial z_1} + \beta\frac{\partial f(z_1,z_2)}{\partial z_2}$ and $f(z_1 + c_1, z_2 + c_2)$ are transcendental, then by Lemmas 3.2 and 3.3, it follows from (19) that

$$\begin{cases}\left(\alpha\dfrac{\partial f(z)}{\partial z_1} + \beta\dfrac{\partial f(z)}{\partial z_2}\right) + \text{ if }(z + c) = e^{p(z)},\\ \left(\alpha\dfrac{\partial f(z)}{\partial z_1} + \beta\dfrac{\partial f(z)}{\partial z_2}\right) - \text{ if }(z + c) = e^{-p(z)},\end{cases} \tag{20}$$

where $p(z)$ is a nonconstant polynomial in z. Thus, in view of (20), it yields

$$\begin{cases}\alpha\dfrac{\partial f(z_1, z_2)}{\partial z_1} + \beta\dfrac{\partial f(z_1, z_2)}{\partial z_2} = \dfrac{e^{p(z_1,z_2)} + e^{-p(z_1,z_2)}}{2},\\ f(z_1 + c_1, z_2 + c_2) = \dfrac{e^{p(z_1,z_2)} - e^{-p(z_1,z_2)}}{2i}.\end{cases} \tag{21}$$

In view of (21), we have

$$\frac{e^{p(z_1+c_1,z_2+c_2)}+e^{-p(z_1+c_1,z_2+c_2)}}{2}=\left(\alpha\frac{\partial p(z_1,z_2)}{\partial z_1}+\beta\frac{\partial p(z_1,z_2)}{\partial z_2}\right)\frac{e^{p(z_1,z_2)}+e^{-p(z_1,z_2)}}{2i}. \tag{22}$$

If $\alpha\frac{\partial p(z_1,z_2)}{\partial z_1}+\beta\frac{\partial p(z_1,z_2)}{\partial z_2}\equiv 0$, then from (22), it follows $e^{p(z_1+c_1,z_2+c_2)}+e^{-p(z_1+c_1,z_2+c_2)}\equiv 0$, that is, $e^{2p(z+c)}+1\equiv 0$, which is impossible because $p(z)$ is a nonconstant polynomial.

If $\alpha\frac{\partial p(z_1,z_2)}{\partial z_1}+\beta\frac{\partial p(z_1,z_2)}{\partial z_2}\not\equiv 0$, then (22) becomes

$$\frac{i}{\alpha\frac{\partial p(z)}{\partial z_1}+\beta\frac{\partial p(z)}{\partial z_2}}e^{p(z+c)+p(z)}+\frac{i}{\alpha\frac{\partial p(z)}{\partial z_1}+\beta\frac{\partial p(z)}{\partial z_2}}e^{p(z)-p(z+c)}-e^{2p(z)}\equiv 1. \tag{23}$$

Thus, by Lemma 3.1, it yields

$$\frac{i}{\alpha\frac{\partial p(z)}{\partial z_1}+\beta\frac{\partial p(z)}{\partial z_2}}e^{p(z)-p(z+c)}\equiv 1. \tag{24}$$

In view of (23) and (24), it follows

$$\frac{i}{\alpha\frac{\partial p(z)}{\partial z_1}+\beta\frac{\partial p(z)}{\partial z_2}}e^{p(z+c)-p(z)}\equiv 1. \tag{25}$$

Since $p(z)$ is a nonconstant polynomial, in view of (24) and (25), we conclude that $p(z)=L(z)+H(z)+B$, where $L(z)$ is a linear function as the form $L(z)=a_1z_1+a_2z_2$, $H(z):=H(s_1)$ is a polynomial in s_1, $s_1:=c_2z_1-c_1z_2$, and B is a constant in $\mathbb{C}$. Substituting $p(z)$ into (24), (25), we have

$$[\alpha a_1+\beta a_2+(\alpha c_2-\beta c_1)H']^2=-1,$$

which implies that $(\alpha c_2-\beta c_1)H'=\xi$, $\xi\in\mathbb{C}$. If $\alpha c_2-\beta c_1=0$, then $\xi=0$. If $\alpha c_2-\beta c_1\neq 0$, then H' is a constant, that is, $H(z)=b_1s_1+b_0=b_1c_2z_1-b_1c_1z_2+b_0$, where $b_1,b_0\in\mathbb{C}$. Thus, $p(z)=L(z)+H(z)+B=L_0(z)+B$ is still a linear form, this means $H(z)=0$. Hence, it follows that $(\alpha c_2-\beta c_1)H'=0$. Thus, it yields that

$$e^{L(c)}=1,\quad \alpha a_1+\beta a_2=i,\quad \text{or}\quad e^{L(c)}=-1,\quad \alpha a_1+\beta a_2=-i. \tag{26}$$

In view of (21), we get

$$f(z_1,z_2)=\frac{e^{p(z_1-c_1,z_2-c_2)}-e^{-p(z_1-c_1,z_2-c_2)}}{2i}=\frac{e^{L(z)+H(z)+B-L(c)}-e^{-L(z)-H(z)-B+L(c)}}{2i}=\frac{A_1e^{L(z)+H(z)}+A_2e^{-L(z)-H(z)}}{2}, \tag{27}$$

where $A_1,A_2\in\mathbb{C}$ satisfy $A_1A_2=1$.

Therefore, this completes the proof of Theorem 1.2. □

4 Proof of Theorem 1.3

Proof. Suppose that f is a finite order transcendental entire solution of equation (11). Next, we consider two cases as follows.

(i) Suppose that $\frac{\partial f}{\partial z_1}$ is not constant. Since the entire solutions of the Fermat type functional equation $f^2+g^2=1$ are $f=\cos a(z), g=\sin a(z)$, where $a(z)$ is an entire function, then $\frac{\partial f(z_1,z_2)}{\partial z_1}$ and $\beta_1 f(z_1+c_1,z_2+c_2)-\beta_2 f(z_1,z_2)$ are transcendental. Thus, equation (11) can be rewritten as the form:

$$\left\{\alpha\frac{\partial f(z)}{\partial z_1}+i[\beta_1 f(z+c)-\beta_2 f(z)]\right\}\left\{\alpha\frac{\partial f(z)}{\partial z_1}-i[\beta_1 f(z+c)-\beta_2 f(z)]\right\}=1.$$

Since f is a finite order transcendental entire function, then by Lemmas 3.2 and 3.3, it follows

$$\begin{cases} \dfrac{\partial f(z_1, z_2)}{\partial z_1} = \dfrac{e^{p(z_1,z_2)} + e^{-p(z_1,z_2)}}{2\alpha}, \\ \beta_1 f(z_1 + c_1, z_2 + c_2) - \beta_2 f(z_1, z_2) = \dfrac{e^{p(z_1,z_2)} - e^{-p(z_1,z_2)}}{2i}, \end{cases} \tag{28}$$

where $p(z_1, z_2)$ is a nonconstant polynomial in $\mathbb{C}^2$. In view of (28), it yields

$$\frac{e^{p(z)} + e^{-p(z)}}{2i}\frac{\partial p(z)}{\partial z_1} = \beta_1\frac{e^{p(z+c)} + e^{-p(z+c)}}{2\alpha} - \beta_2\frac{e^{p(z)} + e^{-p(z)}}{2\alpha},$$

that is,

$$\left(\frac{\beta_2}{\alpha} - i\frac{\partial p(z_1, z_2)}{\partial z_1}\right)\frac{e^{p(z_1,z_2)} + e^{-p(z_1,z_2)}}{2i} = \beta_1\frac{e^{p(z_1+c_1,z_2+c_2)} + e^{-p(z_1+c_1,z_2+c_2)}}{2\alpha}. \tag{29}$$

Obviously, $\frac{\partial p(z_1,z_2)}{\partial z_1} \neq -i\frac{\beta_2}{\alpha}$, otherwise, $e^{p(z_1+c_1,z_2+c_2)} + e^{-p(z_1+c_1,z_2+c_2)} \equiv 0$, which is a contradiction with $p(z)$ is a nonconstant polynomial. Hence, in view of (29), it follows

$$\frac{\beta_1}{\beta_2 - i\alpha\frac{\partial p(z)}{\partial z_1}}e^{p(z+c)+p(z)} + \frac{\beta_1}{\beta_2 - i\alpha\frac{\partial p(z)}{\partial z_1}}e^{p(z)-p(z+c)} - e^{2p(z)} \equiv 1. \tag{30}$$

Thus, by Lemma 3.1, it yields

$$\frac{\beta_1}{\beta_2 - i\alpha\frac{\partial p(z)}{\partial z_1}}e^{p(z)-p(z+c)} \equiv 1. \tag{31}$$

In view of (30) and (31), it follows

$$\frac{\beta_1}{\beta_2 - i\alpha\frac{\partial p(z)}{\partial z_1}}e^{p(z+c)-p(z)} \equiv 1. \tag{32}$$

Since $p(z)$ is a nonconstant polynomial, in view of (31) and (32), we conclude that $p(z) = L(z) + H(z) + B$, where $L(z)$ is a linear function as the form $L(z) = a_1z_1 + a_2z_2$, $H(z) := H(s_1)$ is a polynomial in s_1, $s_1 := c_2z_1 - c_1z_2$, and B is a constant in $\mathbb{C}$. Substituting $p(z)$ into (31) and (32), it follows that $\alpha c_2H' = 0$, which implies that $H(z)$ is a constant since $\alpha \neq 0$, $c_2 \neq 0$. Similar to the argument as in Theorem 1.2, $H(z)$ can be considered equal to zero, this does not affect the linear form of $p(z)$. Hence, we conclude that $p(z) = L(z) + B$. In view of (29), let

$$f(z_1, z_2) = \frac{e^{a_1z_1+a_2z_2+B} - e^{-(a_1z_1+a_2z_2+B)}}{2a_1\alpha} + g_2(z_2), \tag{33}$$

where $a_1(\neq 0)$, a_2, B, D_0 are constants in $\mathbb{C}$ and $g_2(x)$ is a finite order entire function in x.

Substituting (33) into the second equation of (28), it follows that

$$\frac{e^{L(z)+B} - e^{-(L(z)+B)}}{2i} = \beta_1\frac{e^{L(z)+B+L(c)} - e^{-(L(z)+B)-L(c)}}{2a_1\alpha} + \beta_1g_2(z_2 + c_2) - \beta_2\frac{e^{L(z)+B} - e^{-(L(z)+B)}}{2a_1\alpha} - \beta_2g_2(z_2),$$

which implies

$$\frac{e^{L(z)+B} - e^{-(L(z)+B)}}{2i} = \frac{(\beta_1e^{L(c)} - \beta_2)e^{L(z)+B} - (\beta_1e^{-L(c)} - \beta_2)e^{-(L(z)+B)}}{2a_1\alpha} \tag{34}$$

and

$$g_2(z_2 + c_2) = \frac{\beta_1}{\beta_2} g_2(z_2). \tag{35}$$

(i_1) If $\beta_1 \neq \pm\beta_2$, in view of (34), we conclude that

$$\beta_1 e^{L(c)} - \beta_2 = -ia_1\alpha, \quad \beta_1 e^{-L(c)} - \beta_2 = -ia_1\alpha,$$

this implies that

$$a_1\alpha = i(\beta_1 - \beta_2),\ e^{L(c)} = 1, \quad \text{or} \quad a_1\alpha = -i(\beta_1 + \beta_2),\ e^{L(c)} = -1. \tag{36}$$

In view of (35), it follows that

$$g_2(z_2) = e^{\frac{\log\beta_2 - \log\beta_1}{c_2} z_2} G(z_2), \tag{37}$$

where $G(z_2)$ is a finite order entire period function with period c_2. Hence, $f(z_1, z_2)$ is of the form

$$f(z_1, z_2) = \frac{e^{a_1z_1 + a_2z_2 + B} - e^{-(a_1z_1 + a_2z_2 + B)}}{2a_1\alpha} + e^{\frac{\log\beta_2 - \log\beta_1}{c_2} z_2} G(z_2),$$

where $G(z_2)$ is a finite order entire period function with period c_2, $a_1, a_2, \alpha, \beta_1, \beta_2, c$ satisfy (36) and (37).

(i_2) If $\beta_1 = \beta_2$, the equality (34) implies that $i\alpha a_1 = 2\beta_1$ and $e^{L(c)} = -1$. And (35) implies that $g_2(z_2)$ is a finite order entire period function with period c_2.

(i_3) If $\beta_1 = -\beta_2$, the equality (34) implies that $i\alpha a_1 = -2\beta_1$ and $e^{L(c)} = 1$. And (35) implies that $g_2(z_2) = e^{\frac{z_2 \log(-1)}{c_2}} G(z_2)$, where $G(z_2)$ is a finite order entire period function with period c_2.

Thus, this completes the proof of Theorem 1.3 (i).

(ii) Suppose that $\frac{\partial f}{\partial z_1}$ is constant, let

$$\frac{\partial f}{\partial z_1} = D_1, \quad D_1 \in \mathbb{C}. \tag{38}$$

Then in view of (11), it follows

$$\beta_1 f(z_1 + c_1, z_2 + c_2) - \beta_2 f(z_1, z_2) \equiv \kappa, \quad (\alpha D_1)^2 + \kappa^2 = 1, \quad \kappa \in \mathbb{C}. \tag{39}$$

Since f is a transcendental entire function of finite order, and in view of (39), we can assume that $f(z_1, z_2)$ is of the form:

$$f(z_1, z_2) = \psi(z_2) + \varphi(z_1, z_2) + D_0, \tag{40}$$

where $\psi(z_2)$ is a finite order entire function in z_2, $\varphi(z_1, z_2) = D_1z_1 + \varphi_1(z_2)$, $\varphi_1(z_2)$ is a polynomial in z_2, D_0 are constants in $\mathbb{C}$.

Substituting (40) into (39), we have

$$\begin{aligned}\kappa &= \beta_1 f(z_1 + c_1, z_2 + c_2) - \beta_2 f_2(z_1, z_2)\\ &= \beta_1\psi(z_2 + c_2) + \beta_1\varphi(z_1 + c_1, z_2 + c_2) + \beta_1 D_0 - \beta_2\psi(z_2) - \beta_2\varphi(z_1, z_2) - \beta_2 D_0.\end{aligned} \tag{41}$$

(ii_1) If $\beta_1 = \beta_2$, then in view of (41), it yields that $\varphi(z_1, z_2)$ is a linear function as the form $\varphi(z_1, z_2) = D_1z_1 + D_2z_2$ and $\psi(x)$ is a solution of equation

$$\psi(x + c_2) = \psi(x) + \theta_1,$$

where $\theta_1 = \frac{\kappa}{\beta_1} - (D_1c_1 + D_2c_2)$. Thus, $\psi(z_2)$ is a finite order entire periodic function with period c_2 and $\theta_1 \equiv 0$. Moreover, by combining with (39), we have

$$[\beta_1(D_1c_1 + D_2c_2)]^2 = 1 - (\alpha D_1)^2, \quad D_0 \in \mathbb{C}. \tag{42}$$

(ii_2) If $\beta_1 \neq \beta_2$, then in view of (41), it yields that $\varphi(z_1, z_2) \equiv 0$ and $\psi(z_2)$ is a solution of the difference equation

$$g(x + c_2) = \frac{\beta_2}{\beta_1} g(x) + \theta_2,$$

where $\theta_2 = \frac{\kappa - D_0(\beta_1 - \beta_2)}{\beta_1}$. Thus, $\psi(z_2)$ has the form

$$\psi(z_2) = e^{\frac{\log \beta_2 - \log \beta_1}{c_2} z_2} G(z_2) + \tau,$$

where $G(z_2)$ is a finite order entire periodic function with period c_2 and τ satisfies $\tau = \frac{\theta_2}{1 - \frac{\beta_2}{\beta_1}}$. Combining with (39) and $\theta_2 = \frac{\kappa - D_0(\beta_1 - \beta_2)}{\beta_1}$, we have

$$[(\beta_1 - \beta_2)(\tau + D_0)]^2 = 1.$$

Therefore, this completes the proof of Theorem 1.3. □

5 Proof of Theorem 1.4

Proof. Suppose that f is a finite order transcendental entire solution of equation (12). Here, two cases will be considered below.

(i) Suppose that $\alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2}$ is not constant. In view of the properties of the entire solutions of the Fermat type functional equation $f^2 + g^2 = 1$, then $\alpha \frac{\partial f(z_1, z_2)}{\partial z_1} + \beta \frac{\partial f(z_1, z_2)}{\partial z_2}$ and $\gamma_1 f(z_1 + c_1, z_2 + c_2) - \gamma_2 f(z_1, z_2)$ are transcendental. Thus, equation (12) can be rewritten as the form:

$$\left\{\alpha \frac{\partial f}{\partial z_1} + \beta \frac{\partial f}{\partial z_2} + i[\gamma_1 f(z + c) - \gamma_2 f(z)]\right\}\left\{\alpha \frac{\partial f}{\partial z_1} + \beta \frac{\partial f}{\partial z_2} - i[\gamma_1 f(z + c) - \gamma_2 f(z)]\right\} = 1.$$

Since f is a finite order transcendental entire function, then we have that $\alpha \frac{\partial f}{\partial z_1} + \beta \frac{\partial f}{\partial z_2} + i[\gamma_1 f(z + c) - \gamma_2 f(z)]$ and $\alpha \frac{\partial f}{\partial z_1} + \beta \frac{\partial f}{\partial z_2} + i[\gamma_1 f(z + c) - \gamma_2 f(z)]$ have no any zeros and poles. Thus, by Lemmas 3.2 and 3.3, it yields that

$$\begin{cases} \alpha \dfrac{\partial f(z_1, z_2)}{\partial z_1} + \beta \dfrac{\partial f(z_1, z_2)}{\partial z_2} = \dfrac{e^{p(z_1, z_2)} + e^{-p(z_1, z_2)}}{2}, \\ \gamma_1 f(z_1 + c_1, z_2 + c_2) - \gamma_2 f(z_1, z_2) = \dfrac{e^{p(z_1, z_2)} - e^{-p(z_1, z_2)}}{2i}, \end{cases} \tag{43}$$

where $p(z_1, z_2)$ is a nonconstant polynomial in $\mathbb{C}^2$. In view of (43), it yields

$$\frac{e^{p(z)} + e^{-p(z)}}{2i}\left(\alpha \frac{\partial p(z)}{\partial z_1} + \beta \frac{\partial p(z)}{\partial z_2}\right) = \gamma_1 \frac{e^{p(z+c)} + e^{-p(z+c)}}{2} - \gamma_2 \frac{e^{p(z)} + e^{-p(z)}}{2},$$

that is,

$$\left[\gamma_2 - i\left(\alpha \frac{\partial p(z)}{\partial z_1} + \beta \frac{\partial p(z)}{\partial z_2}\right)\right] \frac{e^{p(z)} + e^{-p(z)}}{2} = \gamma_1 \frac{e^{p(z+c)} + e^{-p(z+c)}}{2}. \tag{44}$$

Obviously, $\gamma_2 \neq i\left(\alpha \frac{\partial p(z)}{\partial z_1} + \beta \frac{\partial p(z)}{\partial z_2}\right)$, otherwise, $e^{p(z_1 + c_1, z_2 + c_2)} + e^{-p(z_1 + c_1, z_2 + c_2)} \equiv 0$, which is a contradiction with $p(z)$ is a nonconstant polynomial. Hence, equation (44) leads to

$$\frac{\gamma_1}{\gamma_2 - i\left(\alpha \frac{\partial p(z)}{\partial z_1} + \beta \frac{\partial p(z)}{\partial z_2}\right)} e^{p(z+c)+p(z)} + \frac{\gamma_1}{\gamma_2 - i\left(\alpha \frac{\partial p(z)}{\partial z_1} + \beta \frac{\partial p(z)}{\partial z_2}\right)} e^{p(z)-p(z+c)} - e^{2p(z)} \equiv 1. \tag{45}$$

Thus, by Lemma 3.1, it yields

$$\frac{\gamma_1}{\gamma_2 - i\left(\alpha\frac{\partial p(z)}{\partial z_1} + \beta\frac{\partial p(z)}{\partial z_2}\right)} e^{p(z)-p(z+c)} \equiv 1. \tag{46}$$

In view of (45) and (46), it follows

$$\frac{\gamma_1}{\gamma_2 - i\left(\alpha\frac{\partial p(z)}{\partial z_1} + \beta\frac{\partial p(z)}{\partial z_2}\right)} e^{p(z+c)-p(z)} \equiv 1. \tag{47}$$

Since $p(z)$ is a nonconstant polynomial, in view of (46) and (47), we conclude that $p(z) = L(z) + H(z) + B$, where $L(z)$ is a linear function as the form $L(z) = a_1z_1 + a_2z_2$, $H(z) := H(s_1)$ is a polynomial in s_1, B is a constant in $\mathbb{C}$. In view of $\alpha c_2 - \beta c_1 \neq 0$, similar to the argument as in Theorem 1.2, we can obtain that $H(z) \equiv 0$. Thus, in view of (46) and (47), we have

$$\gamma_1 e^{L(c)} - \gamma_2 = -i(a_1\alpha + a_2\beta), \quad \gamma_1 e^{-L(c)} - \gamma_2 = -i(a_1\alpha + a_2\beta). \tag{48}$$

In view of (43), we get the following equation:

$$\alpha\frac{\partial f(z_1, z_2)}{\partial z_1} + \beta\frac{\partial f(z_1, z_2)}{\partial z_2} = \frac{e^{L(z)+B} + e^{-L(z)-B}}{2}. \tag{49}$$

The characteristic equations for equation (49) are

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = \alpha, \quad \frac{\mathrm{d}z_2}{\mathrm{d}t} = \beta, \quad \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{e^{L(z)+B} + e^{-L(z)-B}}{2}.$$

Using the initial conditions: $z_1 = 0$, $z_2 = s$, and $f = f(0, s) := g(s)$ with a parameter s. Thus, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1 = \alpha t$, $z_2 = \beta t + s$,

$$\begin{aligned} f &= \int_0^t \frac{e^{(a_1\alpha+a_2\beta)t+a_2s+B} + e^{-[(a_1\alpha+a_2\beta)t+a_2s+B]}}{2}\mathrm{d}t + g(s) \\ &= \frac{e^{a_2s+B}}{2}\int_0^t e^{(a_1\alpha+a_2\beta)t}\mathrm{d}t + \frac{e^{-(a_2s+B)}}{2}\int_0^t e^{-(a_1\alpha+a_2\beta)t}\mathrm{d}t + g(s) \\ &= \frac{e^{a_2s+B}}{2(a_1\alpha + a_2\beta)}e^{(a_1\alpha+a_2\beta)t} - \frac{e^{-(a_2s+B)}}{2(a_1\alpha + a_2\beta)}e^{-(a_1\alpha+a_2\beta)t} + g_1(s), \end{aligned}$$

where $g(s)$ is an entire function with finite order in s, and

$$g_1(s) = g(s) + \frac{e^{-(a_2s+B)}}{2(a_1\alpha + a_2\beta)} - \frac{e^{a_2s+B}}{2(a_1\alpha + a_2\beta)}.$$

Substituting $t = \frac{z_1}{\alpha}$, $s = z_2 - \frac{\beta z_1}{\alpha}$ into f, it follows that

$$f(z_1, z_2) = \frac{e^{a_1z_2+a_2z_2+B} - e^{-(a_1z_2+a_2z_2+B)}}{2(a_1\alpha + a_2\beta)} + g_2(\alpha z_2 - \beta z_1), \tag{50}$$

where $g_2(s) = g_1(\alpha s)$. Substituting $f(z_1, z_2)$ into the second equation of (43), it yields

$$\begin{aligned} \frac{e^{a_1z_2+a_2z_2+B} - e^{-(a_1z_2+a_2z_2+B)}}{2i} = {} & \gamma_1\frac{e^{a_1z_2+a_2z_2+B+L(c)} - e^{-(a_1z_2+a_2z_2+B)-L(c)}}{2(a_1\alpha + a_2\beta)} + \gamma_1 g_2(\alpha z_2 - \beta z_1 + \alpha c_2 - \beta c_1) \\ & - \gamma_2\frac{e^{a_1z_2+a_2z_2+B} - e^{-(a_1z_2+a_2z_2+B)}}{2(a_1\alpha + a_2\beta)} - \gamma_2 g_2(\alpha z_2 - \beta z_1). \end{aligned} \tag{51}$$

This means that

$$\frac{e^{L(z)+B}-e^{-(L(z)+B)}}{2i}=\frac{(\gamma_1 e^{L(c)}-\gamma_2)e^{L(z)+B}-(\gamma_1 e^{-L(c)}-\gamma_2)e^{-(L(z)+B)}}{2(a_1\alpha+a_2\beta)} \tag{52}$$

and

$$\gamma_1 g_2(\alpha z_2-\beta z_1+\alpha c_2-\beta c_1)=\gamma_2 g_2(\alpha z_2-\beta z_1). \tag{53}$$

(i_1) If $\gamma_1\neq\pm\gamma_2$, in view of (48) and (52), it yields that

$$a_1\alpha+a_2\beta=i(\gamma_1-\gamma_2),\ e^{L(c)}=1,\quad \text{or}\quad a_1\alpha+a_2\beta=-i(\gamma_1+\gamma_2),\ e^{L(c)}=-1. \tag{54}$$

In view of (53), it follows that

$$g_2(u)=e^{u\frac{\log\beta_2-\log\beta_1}{c_2}}G(u), \tag{55}$$

where $G(u)$ is a finite order entire period function with period $\alpha c_2-\beta c_1$ and $u=\alpha z_2-\beta z_1$. Hence, $f(z_1,z_2)$ is of the form

$$f(z_1,z_2)=\frac{e^{a_1z_1+a_2z_2+B}-e^{-(a_1z_1+a_2z_2+B)}}{2(a_1\alpha+a_2\beta)}+e^{(\alpha z_2-\beta z_1)\frac{\log\gamma_2-\log\gamma_1}{\alpha c_2-\beta c_1}}G(\alpha z_2-\beta z_1),$$

where $G(u)$ is a finite order entire period function with period $\alpha c_2-\beta c_1$, $a_1,a_2,\alpha,\beta,\gamma_1,\gamma_2,c$ satisfy (54) and (55).

(i_2) If $\gamma_1=\gamma_2$, the equality (54) implies that $i(a_1\alpha+a_2\beta)=2\gamma_1$ and $e^{L(c)}=-1$. And (55) implies that $g_2(u)$ is a finite order entire period function with period $\alpha c_2-\beta c_1$.

(i_3) If $\gamma_1=-\gamma_2$, the equality (54) implies that $i(a_1\alpha+a_2\beta)=-2\gamma_1$ and $e^{L(c)}=1$. And (55) implies that $g_2(u)=e^{\frac{u\log(-1)}{\alpha c_2-\beta c_1}}G(u)$, where $G(u)$ is a finite order entire period function with period $\alpha c_2-\beta c_1$.

Thus, this completes the proof of Theorem 1.4(i).

(ii) Suppose that $\alpha\frac{\partial f(z_1,z_2)}{\partial z_1}+\beta\frac{\partial f(z_1,z_2)}{\partial z_2}$ is constant. Let

$$\alpha\frac{\partial f(z_1,z_2)}{\partial z_1}+\beta\frac{\partial f(z_1,z_2)}{\partial z_2}=D_1',\quad D_1'\in\mathbb{C}. \tag{56}$$

Then in view of (12), it follows

$$\gamma_1 f(z_1+c_1,z_2+c_2)-\gamma_2 f(z_1,z_2)\equiv\kappa,\quad D_1'^2+\kappa^2=1,\quad \kappa\in\mathbb{C}. \tag{57}$$

By using the same argument as in the proof of Theorem 1.4(i), $f(z_1,z_2)$ is of the form:

$$f(z_1,z_2)=D_1'\frac{z_1}{\alpha}+g_2(\alpha z_2-\beta z_1), \tag{58}$$

where $g_2(u)$ is a finite order entire function in u, and $u=\alpha z_2-\beta z_1$.

Substituting (58) into (57), we have

$$\begin{aligned}\kappa&=\gamma_1 f(z_1+c_1,z_2+c_2)-\gamma_2 f_2(z_1,z_2)\\&=\gamma_1 D_1'\frac{z_1+c_1}{\alpha}+\gamma_1 g_2(\alpha z_2-\beta z_1+\alpha c_2-\beta c_1)-\gamma_2 D_1'\frac{z_1}{\alpha}-\gamma_2 g_2(\alpha z_2-\beta z_1).\end{aligned} \tag{59}$$

(ii_1) If $\gamma_1=\gamma_2$, then (59) leads to

$$g_2(\alpha z_2-\beta z_1+\alpha c_2-\beta c_1)-g_2(\alpha z_2-\beta z_1)=\frac{\kappa}{\gamma_1}-\frac{D_1'c_1}{\alpha},$$

that is,

$$g_2(u+\alpha c_2-\beta c_1)-g_2(u)=\frac{\kappa}{\gamma_1}-\frac{D_1'c_1}{\alpha}, \tag{60}$$

for $u = \alpha z_2 - \beta z_1$. Thus, $g_2(u)$ can be represented as the form $g_2(u) = G_2(u) + D_2 u + \tau$, where $G_2(u)$ is a finite order entire period function with period $\alpha c_2 - \beta c_1$, and D_2, τ are constants in $\mathbb{C}$. Combining with (60), we obtain $D_2(\alpha c_2 - \beta c_1) = \frac{\kappa}{\gamma_1} - \frac{D_1' c_1}{\alpha}$. In view of (57), it yields

$$\gamma_1^2[D_1 c_1 + D_2(\alpha c_2 - \beta c_1)]^2 = 1 - (\alpha D_1)^2,$$

where $D_1 = \frac{D_1'}{\alpha}$. Thus, this completes the proof of Theorem 1.4(ii_1).

(ii_2) If $\gamma_1 \neq \gamma_2$, then equation (59) implies that $D_1' = 0$ and

$$g_2(u + \alpha c_2 - \beta c_1) = \frac{\gamma_2}{\gamma_1} g_2(u) + \frac{\kappa}{\gamma_1}, \tag{61}$$

for $u = \alpha z_2 - \beta z_1$. In view of (61), it follows that $g_2(u) = e^{u \frac{\log \gamma_2 - \log \gamma_1}{\alpha c_2 - \beta c_1}} G(u) + \tau$, where $G(u)$ is a finite order entire period function with period $\alpha c_2 - \beta c_1$ and τ satisfies

$$(\gamma_1 - \gamma_2)\tau = \pm 1.$$

This shows that the conclusion of Theorem 1.4(ii_2) holds.

Therefore, this completes the proof of Theorem 1.4. □

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