

Research Article

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General decay rate for a viscoelastic wave equation with distributed delay and Balakrishnan-Taylor damping

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Abstract: A nonlinear viscoelastic wave equation with Balakrishnan-Taylor damping and distributed delay is studied. By the energy method we establish the general decay rate under suitable hypothesis.

Keywords: wave equation, exponential decay, distributed delay term, viscoelastic term, energy method

MSC 2020: 35B40, 35L70, 76Exx, 93D20

In memory of the mother of the second author: Ms. Fatma bent Tayeb Zeghdoud (1940–2021).

1 Introduction

Let $\mathcal{H} = \Omega \times (\tau_1, \tau_2) \times (0, \infty)$, in the present work, we consider the following wave equation:

$$\left\{ \begin{array}{l} u_{tt} - (\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u, \nabla u_t)_{L^2(\Omega)})\Delta u(t) + \alpha(t) \int_0^t h(t - \varrho)\Delta u(\varrho)d\varrho \\ \quad + \beta_1 |u_t(t)|^{m-2} u_t(t) + \int_{\tau_1}^{\tau_2} |\beta_2(s)| |u_t(t-s)|^{m-2} u_t(t-s) ds = 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in } \Omega, \\ u_t(x, -t) = f_0(x, t), \quad \text{in } \Omega \times (0, \tau_2), \\ u(x, t) = 0, \quad \text{in } \partial\Omega \times (0, \infty), \end{array} \right. \quad (1.1)$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $\zeta_0, \zeta_1, \sigma, \beta_1$ are positive constants, $m \geq 1$ for $N = 1, 2$, and $1 < m \leq \frac{N+2}{N-2}$ for $N \geq 3$.

$\tau_1 < \tau_2$ are non-negative constants such that $\beta_2 : [\tau_1, \tau_2] \rightarrow \mathbb{R}$ represents distributive time delay, h, α are positive functions.

The importance of the viscoelastic properties of materials has been realized because of the accelerated development of the rubber and plastics industry. Physically, the viscoelastic damping term is the relation-

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ship between the stress and strain history in the beam inspired by the Boltzmann theory, where the kernel of the term of memory is the function h , see [1–7].

In [8], Balakrishnan and Taylor proposed a new model of damping and called it the Balakrishnan-Taylor damping, as it relates to the span problem and the plate equation. For more details, the readers can refer to some papers that focused on the study of this damping [9–13].

On the other hand, the stability issue of systems with delay is of theoretical and practical great importance, whereas, the dynamic systems with delay terms have become a major research subject in differential equation since the last five decades. Recently, the stability and the asymptotic behavior of evolution systems with time delay especially the distributed delay effect have been studied by many authors, see [14–18].

Very recently, in [19] the authors considered our problem (1.1) but in the presence of the delay, they proved the general decay result of solutions by the energy method under suitable assumptions.

Based on all of the above, the combination of these terms of damping (memory term, Balakrishnan-Taylor damping, and the distributed delay) in one particular problem with the addition of $\alpha(t)$ to the term of memory and the distributed delay term $\left(\int_{\tau_1}^{\tau_2} |\beta_2(s)| |u_t(t-s)|^{m-2} u_t(t-s) ds \right)$ we believe that it constitutes a new problem worthy of study and research, different from the above that we will try to shed light on.

Our paper is divided into several sections: in Section 2 we lay down the hypotheses, concepts, and lemmas we need, and in Section 3 we prove our main result. Finally, we give a conclusion in Section 4.

2 Preliminaries

For studying our problem, in this section we need some materials.

First, introducing the following hypothesis for β_2 , h , and α :

(A1) $h, \alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are non-increasing C^1 functions satisfying

$$h(t) > 0, \quad \alpha(t) > 0, \quad l_0 = \int_0^\infty h(\varrho) d\varrho < \infty, \quad \zeta_0 - 2\alpha(t) \int_0^t h(\varrho) d\varrho \geq l > 0; \tag{2.1}$$

(A2) $\vartheta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-increasing C^1 function satisfying

$$\vartheta(t)h(t) + h'(t) \leq 0, \quad t \geq 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{-\alpha'(t)}{\vartheta(t)\alpha(t)} = 0; \tag{2.2}$$

(A3) $\beta_2 : [\tau_1, \tau_2] \rightarrow \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\beta_2(s)| ds < \beta_1. \tag{2.3}$$

Let us introduce

$$(h \circ \psi)(t) := \int_{\Omega} \int_0^t h(t-\varrho) |\psi(t) - \psi(\varrho)|^2 d\varrho dx$$

and

$$M(t) := (\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t)))_{L^2(\Omega)}.$$

Lemma 2.1. (Sobolev-Poincare inequality [20]). *Let $2 \leq q < \infty$ ($n = 1, 2$) or $2 \leq q < \frac{2n}{n-2}$ ($n \geq 3$). Then, $\exists c_* = c(\Omega, q) > 0$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega).$$

As in [18], taking the following new variables:

$$y(x, \rho, s, t) = u_t(x, t - s\rho),$$

which satisfy

$$\begin{cases} sy_t(x, \rho, s, t) + y_\rho(x, \rho, s, t) = 0, \\ y(x, 0, s, t) = u_t(x, t). \end{cases} \tag{2.4}$$

So, problem (1.1) can be written as

$$\begin{cases} u_{tt} - (\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u, \nabla u_t)_{L^2(\Omega)})\Delta u(t) + \alpha(t) \int_0^t h(t - \varrho)\Delta u(\varrho)d\varrho \\ \quad + \beta_1 |u_t(t)|^{m-2}u_t(t) + \int_{\tau_1}^{\tau_2} |\beta_2(s)| |y(x, 1, s, t)|^{m-2}y(x, 1, s, t)ds = 0, \\ sy_t(x, \rho, s, t) + y_\rho(x, \rho, s, t) = 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in } \Omega, \\ y(x, \rho, s, 0) = f_0(x, \rho s), \quad \text{in } \Omega \times (0, 1) \times (0, \tau_2), \\ u(x, t) = 0, \quad \text{in } \partial\Omega \times (0, \infty), \end{cases} \tag{2.5}$$

where

$$(x, \rho, s, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

Now, we give the energy functional.

Lemma 2.2. *The energy functional E , defined by*

$$\begin{aligned} E(t) = & \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \left(\zeta_0 - \alpha(t) \int_0^t h(\varrho)d\varrho \right) \|\nabla u(t)\|_2^2 + \frac{\zeta_1}{4} \|\nabla u(t)\|_2^4 + \frac{\alpha(t)}{2} (h \circ \nabla u)(t) \\ & + \frac{m-1}{m} \int_0^1 \int_{\tau_1}^{\tau_2} s |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho, \end{aligned} \tag{2.6}$$

satisfies

$$E'(t) \leq - \left(\beta_1 - \int_{\tau_1}^{\tau_2} |\beta_2(s)| ds \right) \|u_t(t)\|_m^m + \frac{\alpha(t)}{2} (h' \circ \nabla u)(t) - \frac{\alpha'(t)}{2} \left(\int_0^t h(\varrho)d\varrho \right) \|\nabla u(t)\|_2^2 - \frac{\sigma}{4} \left(\frac{d}{dt} \|\nabla u(t)\|_2^2 \right)^2. \tag{2.7}$$

Proof. Taking the inner product of (2.5)₁ with u_t , then integrating over Ω , we find

$$\begin{aligned} (u_{tt}(t), u_t(t))_{L^2(\Omega)} - (M(t)\Delta u(t), u_t(t))_{L^2(\Omega)} + \alpha(t) \int_0^t h(t - \varrho)\Delta u(\varrho)d\varrho, u_t(t)_{L^2(\Omega)} + \beta_1 (|u_t|^{m-2}u_t, u_t)_{L^2(\Omega)} \\ + \int_{\tau_1}^{\tau_2} |\beta_2(s)| (|y(x, 1, s, t)|^{m-2}y(x, 1, s, t), u_t(t))_{L^2(\Omega)} ds = 0. \end{aligned} \tag{2.8}$$

By computation, integration by parts and the last condition in (2.5), we get

$$(u_{tt}(t), u_t(t))_{L^2(\Omega)} = \frac{1}{2} \frac{d}{dt} \|u_t(t)\|_2^2, \quad (2.9)$$

by integration by parts, we find

$$\begin{aligned} -(M(t)\Delta u(t), u_t(t))_{L^2(\Omega)} &= -((\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)})\Delta u(t), u_t(t))_{L^2(\Omega)} \\ &= (\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}) \int_{\Omega} \nabla u(t) \cdot \nabla u_t(t) dx \\ &= (\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}) \frac{d}{dt} \left\{ \int_{\Omega} |\nabla u(t)|^2 dx \right\} \\ &= \frac{d}{dt} \left\{ \frac{1}{2} \left(\zeta_0 + \frac{\zeta_1}{2} \|\nabla u\|_2^2 \right) \|\nabla u(t)\|_2^2 \right\} + \frac{\sigma}{4} \frac{d}{dt} \{ \|\nabla u(t)\|_2^2 \}^2, \end{aligned} \quad (2.10)$$

and we have

$$\begin{aligned} \left(\int_0^t h(t-\varrho) \Delta u(\varrho) d\varrho, u_t(t) \right)_{L^2(\Omega)} &= \int_0^t h(t-\varrho) (\Delta u(\varrho), u_t(t))_{L^2(\Omega)} d\varrho \\ &= - \int_0^t h(t-\varrho) \left[\int_{\Omega} \nabla u(x, \varrho) \nabla u(x, t) dx \right] d\varrho \end{aligned} \quad (2.11)$$

and

$$-\nabla u(x, \varrho) \nabla u(x, t) = \frac{1}{2} \frac{d}{dt} \{ |\nabla u(x, \varrho) - \nabla u(x, t)|^2 \} - \frac{1}{2} \frac{d}{dt} |\nabla u(x, t)|^2, \quad (2.12)$$

then

$$\begin{aligned} - \int_0^t h(t-\varrho) (\nabla u(\varrho), \nabla u_t(t))_{L^2(\Omega)} d\varrho &= - \int_0^t h(t-\varrho) \int_{\Omega} \left[\frac{1}{2} \frac{d}{dt} \{ |\nabla u(x, \varrho) - \nabla u(x, t)|^2 \} \right] dx d\varrho, \\ - \int_0^t h(t-\varrho) \int_{\Omega} \left[\frac{1}{2} \frac{d}{dt} \{ |\nabla u(x, t)|^2 \} \right] dx d\varrho &= \frac{1}{2} \int_0^t h(t-\varrho) \left[\frac{d}{dt} \left\{ \int_{\Omega} |\nabla u(x, t) - \nabla u(x, \varrho)|^2 dx \right\} \right] d\varrho \\ &\quad - \frac{1}{2} \int_0^t h(t-\varrho) \left[\frac{d}{dt} \|\nabla u(x, t)\|_2^2 \right] dx d\varrho, \end{aligned} \quad (2.13)$$

by (2.1), we find

$$\begin{aligned} &\frac{1}{2} \int_0^t h(t-\varrho) \left[\frac{d}{dt} \left\{ \int_{\Omega} |\nabla u(x, t) - \nabla u(x, \varrho)|^2 dx \right\} \right] d\varrho \\ &= \frac{1}{2} \frac{d}{dt} \left\{ \int_0^t h(t-\varrho) \left[\int_{\Omega} |\nabla u(x, t) - \nabla u(x, \varrho)|^2 dx \right] d\varrho \right\} - \frac{1}{2} \int_0^t h'(t-\varrho) \left[\int_{\Omega} |\nabla u(x, t) - \nabla u(x, \varrho)|^2 dx \right] d\varrho \\ &= \frac{1}{2} \frac{d}{dt} (h \circ \nabla u)(t) - \frac{1}{2} (h' \circ \nabla u)(t) \end{aligned} \quad (2.14)$$

and

$$\begin{aligned}
 -\frac{1}{2} \int_0^t h(t-\varrho) \left[\frac{d}{dt} \|\nabla u(t)\|_2^2 \right] dx d\varrho &= -\frac{1}{2} \left(\int_0^t h(t-\varrho) d\varrho \right) \left(\frac{d}{dt} \|\nabla u(t)\|_2^2 \right) dx \\
 &= -\frac{1}{2} \left(\int_0^t h(\varrho) d\varrho \right) \left(\frac{d}{dt} \|\nabla u(t)\|_2^2 \right) dx \\
 &= -\frac{1}{2} \frac{d}{dt} \left\{ \left(\int_0^t h(\varrho) d\varrho \right) \|\nabla u(t)\|_2^2 \right\} + \frac{1}{2} h(t) \|\nabla u(t)\|_2^2,
 \end{aligned} \tag{2.15}$$

by inserting (2.14) and (2.15) into (2.13), we find

$$\begin{aligned}
 \left(\alpha(t) \int_0^t h(t-\varrho) \Delta u(\varrho) d\varrho, u_t(t) \right)_{L^2(\Omega)} &= \frac{d}{dt} \left\{ \frac{\alpha(t)}{2} (h \circ \nabla u)(t) - \frac{\alpha(t)}{2} \left(\int_0^t h(\varrho) d\varrho \right) \|\nabla u(t)\|_2^2 \right\} \\
 &\quad - \frac{\alpha(t)}{2} (h' \circ \nabla u)(t) + \frac{\alpha(t)}{2} h(t) \|\nabla u(t)\|_2^2 \\
 &\quad - \frac{\alpha'(t)}{2} (h \circ \nabla u)(t) + \frac{\alpha'(t)}{2} \left(\int_0^t h(\varrho) d\varrho \right) \|\nabla u(t)\|_2^2.
 \end{aligned} \tag{2.16}$$

Now, multiplying the equation (2.5)₂ by $-y|\beta_2(s)|$, integrating over $\Omega \times (0, 1) \times (\tau_1, \tau_2)$, and using (2.4)₂, we get

$$\begin{aligned}
 \frac{d}{dt} \frac{m-1}{m} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} s |\beta_2(s)| |y(x, \rho, s, t)|^m ds d\rho dx & \\
 &= -(m-1) \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(s)| |y|^{m-1} y_{\rho} ds d\rho dx \\
 &= -\frac{m-1}{m} \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} |\beta_2(s)| \frac{d}{d\rho} |y(x, \rho, s, t)|^m ds d\rho dx \\
 &= \frac{m-1}{m} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\beta_2(s)| (|y(x, 0, s, t)|^m - |y(x, 1, s, t)|^m) ds dx \\
 &= \frac{m-1}{m} \left(\int_{\tau_1}^{\tau_2} |\beta_2(s)| ds \right) \int_{\Omega} |u_t(t)|^m dx - \frac{m-1}{m} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\beta_2(s)| |y(x, 1, s, t)|^m ds dx \\
 &= \frac{m-1}{m} \left(\int_{\tau_1}^{\tau_2} |\beta_2(s)| ds \right) \|u_t(t)\|_m^m - \frac{m-1}{m} \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x, 1, s, t)\|_m^m ds,
 \end{aligned} \tag{2.17}$$

and by Young’s inequality, we have

$$\begin{aligned}
 &\int_{\tau_1}^{\tau_2} |\beta_2(s)| (|y(x, 1, s, t)|^{m-2} y(x, 1, s, t), u_t(t))_{L^2(\Omega)} ds \\
 &\leq \frac{1}{m} \left(\int_{\tau_1}^{\tau_2} |\beta_2(s)| ds \right) \|u_t(t)\|_m^m + \frac{m-1}{m} \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x, 1, s, t)\|_m^m ds.
 \end{aligned} \tag{2.18}$$

By inserting (2.9)–(2.10) and (2.16)–(2.18) into (2.8), we obtain (2.6) and (2.7).

Hence, by (2.2), we get the function E is a non-increasing $\forall t \geq t_1$. This completes the proof. \square

Now we state the local existence of problem (2.5).

Theorem 2.3. *Suppose that (2.1)–(2.3) are satisfied. Then, for any $u_0, u_1 \in H_0^1(\Omega) \cap L^2(\Omega)$, and $f_0 \in L^2(\Omega, (0, 1), (\tau_1, \tau_2))$, there exists a weak solution u of problem (2.5) such that*

$$\begin{aligned} u &\in C(]0, T[, H_0^1(\Omega)) \cap C^1(]0, T[, L^2(\Omega)), \\ u_t &\in C(]0, T[, H_0^1(\Omega)) \cap L^2(]0, T[, L^2(\Omega, (0, 1), (\tau_1, \tau_2))). \end{aligned}$$

3 General decay

In this section, we state and prove the asymptotic behavior of system (2.5). For this goal, we set

$$\Psi(t) := \int_{\Omega} u(t)u_t(t)dx + \frac{\sigma}{4}\|\nabla u(t)\|_2^4, \tag{3.1}$$

and

$$\Phi(t) := - \int_{\Omega} u_t \int_0^t h(t - \varrho)(u(t) - u(\varrho))d\varrho dx, \tag{3.2}$$

and

$$\Theta(t) := \int_0^1 \int_{\tau_1}^{\tau_2} se^{-\rho s} |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho. \tag{3.3}$$

Lemma 3.1. *The functional $\Psi(t)$ defined in (3.1) satisfies, for any $\varepsilon > 0$*

$$\begin{aligned} \Psi'(t) &\leq \|u_t\|_2^2 - (l - \varepsilon(c_1 + c_2))\|\nabla u\|_2^2 - \zeta_1\|\nabla u\|_2^4 + \frac{\alpha(t)}{4}(h \circ \nabla u)(t) \\ &\quad + c(\varepsilon) \left(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x, 1, s, t)\|_m^m ds \right). \end{aligned} \tag{3.4}$$

Proof. A differentiation of (3.1) and using (2.5)₁ gives

$$\begin{aligned} \Psi'(t) &= \|u_t\|_2^2 + \int_{\Omega} u_{tt}u dx + \sigma\|\nabla u\|_2^2 \int_{\Omega} \nabla u_t \nabla u dx \\ &= \|u_t\|_2^2 - \zeta_0\|\nabla u\|_2^2 - \zeta_1\|\nabla u\|_2^4 - \underbrace{\beta_1 \int_{\Omega} |u_t|^{m-2} u_t u dx}_{I_1} \\ &\quad + \underbrace{\alpha(t) \int_{\Omega} \nabla u(t) \int_0^t h(t - \varrho) \nabla u(\varrho) d\varrho dx}_{I_2} - \underbrace{\int_{\Omega} \int_{\tau_1}^{\tau_2} |\beta_2(s)| |y(x, 1, s, t)|^{m-2} y(x, 1, s, t) u ds dx}_{I_3}. \end{aligned} \tag{3.5}$$

We estimate the last three terms of the right-hand side (RHS) of (3.5). Applying Hölder’s, Sobolev-Poincaré’s, and Young’s inequalities, (2.1) and (2.6), we find

$$\begin{aligned} I_1 &\leq \varepsilon \beta_1^m \|u\|_m^m + c(\varepsilon) \|u_t\|_m^m \leq \varepsilon \beta_1^m c_p^m \|\nabla u\|_2^m + c(\varepsilon) \|u_t\|_m^m \\ &\leq \varepsilon \beta_1^m c_p^m \left(\frac{E(0)}{l}\right)^{(m-2)/2} \|\nabla u\|_2^2 + c(\varepsilon) \|u_t\|_m^m \leq \varepsilon c_1 \|\nabla u\|_2^2 + c(\varepsilon) \|u_t\|_m^m \end{aligned} \tag{3.6}$$

and

$$I_2 \leq 2\alpha(t) \left(\int_0^t h(\varrho) d\varrho \right) \|\nabla u\|_2^2 + \frac{\alpha(t)}{4} (h \circ \nabla u)(t) \leq (\zeta_0 - l) \|\nabla u\|_2^2 + \frac{\alpha(t)}{4} (h \circ \nabla u)(t). \tag{3.7}$$

Similar to I_1 , we have

$$I_3 \leq \varepsilon c_2 \|\nabla u\|_2^2 + c(\varepsilon) \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x, 1, s, t)\|_m^m ds. \tag{3.8}$$

Combining (3.6)–(3.8) and (3.5), we get

$$\begin{aligned} \Psi'(t) &\leq \|u_t\|_2^2 - (l - \varepsilon(c_1 + c_2)) \|\nabla u\|_2^2 - \zeta_1 \|\nabla u\|_2^4 + \frac{\alpha(t)}{4} (h \circ \nabla u)(t) \\ &\quad + c(\varepsilon) \left(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x, 1, s, t)\|_m^m ds \right). \end{aligned} \quad \square$$

Lemma 3.2. *The functional $\Phi(t)$ defined in (3.2) satisfies, for any $\delta > 0$*

$$\begin{aligned} \Phi'(t) &\leq - \left(\int_0^t h(\varrho) d\varrho - \delta \right) \|u_t\|_2^2 + \delta(\zeta_0 + 2(1 - l)^2 \alpha(t)) \|\nabla u\|_2^2 + \zeta_1 \delta \|\nabla u\|_2^4 \\ &\quad + \delta \frac{\sigma E(0)}{l} \left(\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 \right)^2 + \left(c(\delta) + \left(2\delta + \frac{1}{4\delta} \right) (\zeta_0 - l) \alpha(t) \right) (h \circ \nabla u)(t) \\ &\quad + c(\delta) \left(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x, 1, s, t)\|_m^m ds \right) - \frac{g(0)c_p^2}{4\delta} (h' \circ \nabla u)(t). \end{aligned} \tag{3.9}$$

Proof. A differentiation of (3.2) and using (2.5)₁ gives

$$\begin{aligned} \Phi'(t) &= - \int_{\Omega} u_{tt} \int_0^t h(t - \varrho)(u(t) - u(\varrho)) d\varrho dx - \int_{\Omega} u_t \int_0^t h'(t - \varrho)(u(t) - u(\varrho)) d\varrho dx - \left(\int_0^t h(\varrho) d\varrho \right) \|u_t\|_2^2 \\ &= \underbrace{(\zeta_0 + \zeta_1 \|\nabla u\|_2^2) \int_{\Omega} \nabla u \int_0^t h(t - \varrho)(\nabla u(t) - \nabla u(\varrho)) d\varrho dx}_{J_1} \\ &\quad + \underbrace{\sigma \int_{\Omega} \nabla u \nabla u_t dx \int_{\Omega} \nabla u \int_0^t h(t - \varrho)(\nabla u(t) - \nabla u(\varrho)) d\varrho dx}_{J_2} \end{aligned} \tag{3.10}$$

$$\begin{aligned}
 & \underbrace{-\alpha(t) \int_{\Omega} \left(\int_0^t h(t-\varrho) \nabla u(\varrho) d\varrho \right) \left(\int_0^t h(t-\varrho) (\nabla u(t) - \nabla u(\varrho)) d\varrho \right) dx}_{J_3} \\
 & \underbrace{-\beta_1 \int_{\Omega} |u_t|^{m-2} u_t \left(\int_0^t h(t-\varrho) (u(t) - u(\varrho)) d\varrho \right) dx}_{J_4} \\
 & \underbrace{- \int_{\Omega} \int_{\tau_1}^{\tau_2} |\beta_2(s)| |y(x, 1, s, t)|^{m-2} y(x, 1, s, t) \left(\int_0^t h(t-\varrho) (u(t) - u(\varrho)) d\varrho \right) d\varrho dx}_{J_5} \\
 & \underbrace{- \int_{\Omega} u_t \int_0^t h'(t-\varrho) (u(t) - u(\varrho)) d\varrho dx}_{J_6} - \left(\int_0^t h(\varrho) d\varrho \right) \|u_t\|_2^2.
 \end{aligned}$$

We estimate the terms J_i , $i = 1, \dots, 6$ of the RHS of (3.10). Applying Hölder’s, Sobolev-Poincaré’s, and Young’s inequalities, (2.1) and (2.6), we find

$$\begin{aligned}
 |J_1| & \leq (\zeta_0 + \zeta_1 \|\nabla u\|_2^2) \left(\delta \|\nabla u\|_2^2 + \frac{(\zeta_0 - l)}{4\delta} (h \circ \nabla u)(t) \right) \\
 & \leq \delta \zeta_0 \|\nabla u\|_2^2 + \delta \zeta_1 \|\nabla u\|_2^4 + \left(\frac{\zeta_0(\zeta_0 - l)}{4\delta} + \frac{\zeta_1(\zeta_0 - l)E(0)}{4l\delta} \right) (h \circ \nabla u)(t)
 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
 J_2 & \leq \delta \sigma \left(\int_{\Omega} \nabla u \nabla u_t dx \right)^2 \|\nabla u\|_2^2 + \frac{\sigma(\zeta_0 - l)}{4\delta} (h \circ \nabla u)(t) \\
 & \leq \delta \frac{\sigma E(0)}{l} \left(\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 \right)^2 + \frac{\sigma(\zeta_0 - l)}{4\delta} (h \circ \nabla u)(t),
 \end{aligned} \tag{3.12}$$

$$\begin{aligned}
 |J_3| & \leq \delta \alpha(t) \int_{\Omega} \left(\int_0^t h(t-\varrho) (|\nabla u(t)| - \nabla u(\varrho)| - \nabla |u(t)|) d\varrho \right)^2 dx \\
 & \quad + \frac{1}{4\delta} \alpha(t) \int_{\Omega} \left(\int_0^t h(t-\varrho) (\nabla u(t) - \nabla u(\varrho)) d\varrho \right)^2 dx \\
 & \leq 2\delta c(\zeta_0 - l)^2 \alpha(t) \|\nabla u\|_2^2 + c \left(2\delta + \frac{1}{4\delta} \right) (\zeta_0 - l) \alpha(t) (h \circ \nabla u)(t),
 \end{aligned} \tag{3.13}$$

$$\begin{aligned}
 |J_4| & \leq c(\delta) \|\nabla u_t\|_m^m + \delta \beta_1^m \int_{\Omega} \left(\int_0^t h(t-\varrho) (u(t) - u(\varrho)) d\varrho \right)^m dx \\
 & \leq c(\delta) \|\nabla u_t\|_m^m + \delta \beta_1^m (\zeta_0 - l)^{m-1} c_p^m \int_0^t h(t-\varrho) \|\nabla u(t) - \nabla u(\varrho)\|_2^m d\varrho \\
 & \leq c(\delta) \|\nabla u_t\|_m^m + \delta \left(\beta_1^m (\zeta_0 - l)^{m-1} c_p^m \left(\frac{E(0)}{l} \right)^{(m-2)/2} \right) (h \circ \nabla u)(t) \\
 & \leq c(\delta) \|\nabla u_t\|_m^m + \delta c_3 (h \circ \nabla u)(t).
 \end{aligned} \tag{3.14}$$

Similarly, we have

$$J_5 \leq c(\delta)\|y(x, 1, s, t)\|_m^m + \delta c_4(h \circ \nabla u)(t), \tag{3.15}$$

$$J_6 \leq \delta\|u_t\|_2^2 - \frac{h(0)c_p^2}{4\delta}(h' \circ \nabla u)(t). \tag{3.16}$$

By substituting of (3.11)–(3.16) into (3.10), we get

$$\begin{aligned} \Phi'(t) \leq & - \left(\int_0^t h(\varrho) d\varrho - \delta \right) \|u_t\|_2^2 + \delta(\zeta_0 + 2c(\zeta_0 - l)^2\alpha(t))\|\nabla u\|_2^2 + \zeta_1\delta\|\nabla u\|_2^4 \\ & + \delta \frac{\sigma E(0)}{l} \left(\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 \right)^2 + \left(c(\delta) + \left(2\delta + \frac{1}{4\delta} \right) (\zeta_0 - l)\alpha(t) \right) (h \circ \nabla u)(t) \\ & + c(\delta) \left(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x, 1, s, t)\|_m^m ds \right) - \frac{h(0)c_p^2}{4\delta}(h' \circ \nabla u)(t). \end{aligned} \quad \square$$

Lemma 3.3. *The functional $\Theta(t)$ defined in (3.3) satisfies*

$$\Theta'(t) \leq -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho - \eta_1 \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x, 1, s, t)\|_m^m ds + \beta_1 \|u_t(t)\|_m^m. \tag{3.17}$$

Proof. By differentiating $\Theta(t)$, and using (2.5)₂, we have

$$\begin{aligned} \Theta'(t) &= -m \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\beta_2(s)| |y|^{m-1} y_\rho(x, \rho, s, t) ds d\rho dx \\ &= - \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\beta_2(s)| |y(x, \rho, s, t)|^m ds d\rho dx \\ &\quad - \int_{\Omega} \int_{\tau_1}^{\tau_2} |\beta_2(s)| [e^{-s} |y(x, 1, s, t)|^m - |y(x, 0, s, t)|^m] ds dx. \end{aligned}$$

Applying $y(x, 0, s, t) = u_t(x, t)$, $e^{-s} \leq e^{-s\rho} \leq 1$, for any $0 < \rho < 1$, and $\eta_1 = e^{-\tau_2}$, we obtain

$$\Theta'(t) \leq -\eta_1 \int_{\Omega} \int_0^1 \int_{\tau_1}^{\tau_2} s |\beta_2(s)| |y(x, \rho, s, t)|^m ds d\rho dx - \eta_1 \int_{\Omega} \int_{\tau_1}^{\tau_2} |\beta_2(s)| |y(x, 1, s, t)|^m ds dx + \int_{\tau_1}^{\tau_2} |\beta_2(s)| ds \int_{\Omega} |u_t|^m(t) dx.$$

Using (2.3), we find (3.17). □

Now, we introduce the functional

$$\mathcal{G}(t) := E(t) + \varepsilon_1\alpha(t)\Psi(t) + \varepsilon_2\alpha(t)\Phi(t) + \varepsilon_3\alpha(t)\Theta(t), \tag{3.18}$$

for some positive constants ε_i , $i = 1, 2, 3$ to be determined.

Lemma 3.4. *There exist $\mu_1, \mu_2 > 0$, such that*

$$\mu_1 E(t) \leq \mathcal{G}(t) \leq \mu_2 E(t). \tag{3.19}$$

Proof. From (3.1)–(3.3), by using Hölder's, Young's, and Poincaré's inequalities, we get

$$\begin{aligned}
|\mathcal{G}(t) - E(t)| &\leq \varepsilon_1 \frac{|\alpha(t)|}{2} (\|u_t(t)\|_2^2 + c_p \|\nabla u(t)\|_2^2) + \varepsilon_1 \sigma \frac{|\alpha(t)|}{4} \|\nabla u(t)\|_2^4 \\
&\quad + \varepsilon_2 \frac{|\alpha(t)|}{2} \|u_t(t)\|_2^2 + \varepsilon_2 \frac{|\alpha(t)|}{2} c_p (\zeta_0 - l) (h \circ \nabla u)(t) \\
&\quad + \varepsilon_3 |\alpha(t)| \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-\rho s} |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho.
\end{aligned} \tag{3.20}$$

Using the fact that $0 < \alpha(t) \leq \alpha(0)$ and $e^{-\rho s} < 1$, we find

$$\begin{aligned}
|\mathcal{G}(t) - E(t)| &\leq \varepsilon_1 \frac{\alpha(0)}{2} (c_p \|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2) + \varepsilon_1 \sigma \frac{\alpha(0)}{4} \|\nabla u(t)\|_2^4 \\
&\quad + \varepsilon_2 \frac{\alpha(0)}{2} \|u_t(t)\|_2^2 + \varepsilon_2 \frac{\alpha(0)}{2} c_p (\zeta_0 - l) (h \circ \nabla u)(t) \\
&\quad + \varepsilon_3 \alpha(0) \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-\rho s} |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho \\
&\leq C(\varepsilon_1, \varepsilon_2, \varepsilon_3) E(t).
\end{aligned} \tag{3.21}$$

Choosing $\varepsilon_1, \varepsilon_2$, and ε_3 sufficiently small, then (3.19) follows from (3.21). \square

Lemma 3.5. *There exist $d_5, d_6, t_0 > 0$ satisfying*

$$\mathcal{G}'(t) \leq -d_5 \alpha(t) E(t) + d_6 \alpha(t) (h \circ \nabla u)(t), \quad t > t_0. \tag{3.22}$$

Proof. Since the function h is a positive and continuous, for all $t_0 > 0$, we have

$$\int_0^t h(\varrho) d\varrho \geq \int_0^{t_0} h(\varrho) d\varrho := h_0, \quad \forall t \geq t_0.$$

By differentiating (3.18), using (2.7) and Lemmas 3.1–3.3, we get

$$\begin{aligned}
\mathcal{G}'(t) &:= E'(t) + \varepsilon_1 \alpha'(t) \Psi(t) + \varepsilon_2 \alpha'(t) \Phi(t) + \varepsilon_3 \alpha'(t) \Theta(t) + \varepsilon_1 \alpha(t) \Psi'(t) + \varepsilon_2 \alpha(t) \Phi'(t) + \varepsilon_3 \alpha(t) \Theta'(t) \\
&\leq \alpha(t) (\varepsilon_1 - \varepsilon_2 (h_0 - \delta)) \|u_t\|_2^2 + \alpha(t) (\varepsilon_2 \delta (\zeta_0 + 2c(\zeta_0 - l)^2 \alpha(t)) - \varepsilon_1 (l - \varepsilon(c_1 + c_2))) \|\nabla u\|_2^2 \\
&\quad + \alpha(t) (\varepsilon_2 \zeta_1 \delta - \varepsilon_1 \zeta_1) \|\nabla u\|_2^4 + \alpha(t) \left(\varepsilon_2 \delta \frac{\sigma E(0)}{l} - \frac{\sigma}{\alpha(0)} \right) \left(\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 \right)^2 \\
&\quad + \alpha(t) \left(\varepsilon_1 \frac{\alpha(t)}{4} + \varepsilon_2 (c(\delta) + \left(2\delta + \frac{1}{4\delta} \right) (\zeta_0 - l) \alpha(t)) \right) (h \circ \nabla u)(t) \\
&\quad + \alpha(t) \left(\frac{1}{2} - \varepsilon_2 \frac{h(0)c_p^2}{4\delta} \right) (h' \circ \nabla u)(t) + \alpha(t) \left(\varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) + \varepsilon_3 \beta_1 - \frac{\eta_0}{\alpha(0)} \right) \|u_t\|_m^m \\
&\quad + \alpha(t) (\varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) - \eta_1 \varepsilon_3) \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x, 1, s, t)\|_m^m ds \\
&\quad - \alpha(t) \varepsilon_3 \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho - \frac{\alpha'(t)}{2} \left(\int_0^t h(\varrho) d\varrho \right) \|\nabla u(t)\|_2^2 \\
&\quad + \varepsilon_1 \alpha'(t) \int_{\Omega} u u_t dx + \varepsilon_2 \alpha'(t) \int_{\Omega} u_t \int_0^t h(t - \varrho) (u(t) - u(\varrho)) d\varrho dx \\
&\quad + \varepsilon_3 \alpha'(t) \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-\rho s} |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho.
\end{aligned} \tag{3.23}$$

By using the fact that $e^{-\rho s} < 1$, Young's and Sobolev-Poincare's inequalities, we find

$$\begin{aligned} & \alpha'(t) \int_{\Omega} uu_t dx + \alpha'(t) \int_{\Omega} u_t \int_0^t h(t - \varrho)(u(t) - u(\varrho))d\varrho dx + \alpha'(t) \int_0^1 \int_{\tau_1}^{\tau_2} se^{-\rho s} |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho \\ & \leq -\alpha'(t) \frac{c_p^2}{2} \|\nabla u\|_2^2 - \alpha'(t) \|u_t\|_2^2 - \alpha'(t) \frac{c_p^2}{2} \left(\int_0^t h(\varrho) d\varrho \right) h \circ \nabla u(t) - \alpha'(t) \int_0^1 \int_{\tau_1}^{\tau_2} s |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{G}'(t) & \leq \alpha(t) \left(\varepsilon_1 - \varepsilon_2(h_0 - \delta) - \frac{\alpha'(t)}{\alpha(t)} \right) \|u_t\|_2^2 \\ & + \alpha(t) \left(\varepsilon_2 \delta (\zeta_0 + 2c(\zeta_0 - l)^2 \alpha(t)) - \varepsilon_1(l - \varepsilon(c_1 + c_2)) - \frac{\alpha'(t)}{2\alpha(t)} \left(\int_0^t h(\varrho) d\varrho \right) - \varepsilon_1 \frac{\alpha'(t) c_p^2}{2\alpha(t)} \right) \|\nabla u\|_2^2 \\ & + \alpha(t) (\varepsilon_2 \zeta_1 \delta - \varepsilon_1 \zeta_1) \|\nabla u\|_2^4 + \alpha(t) \left(\varepsilon_2 \delta \frac{\sigma E(0)}{l} - \frac{\sigma}{\alpha(0)} \right) \left(\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 \right)^2 \\ & + \alpha(t) \left(\varepsilon_1 \frac{\alpha(t)}{4} + \varepsilon_2(c(\delta) + \left(2\delta + \frac{1}{4\delta} \right) (\zeta_0 - l)\alpha(t)) - \varepsilon_2 \frac{\alpha'(t) c_p^2}{2\alpha(t)} \left(\int_0^t h(\varrho) d\varrho \right) \right) (h \circ \nabla u)(t) \tag{3.24} \\ & + \alpha(t) \left(\frac{1}{2} - \varepsilon_2 \frac{h(0) c_p^2}{4\delta} \right) (h' \circ \nabla u)(t) + \alpha(t) \left(\varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) + \varepsilon_3 \beta_1 - \frac{\eta_0}{\alpha(0)} \right) \|u_t\|_m^m \\ & + \alpha(t) (\varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) - \eta_1 \varepsilon_3) \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x, 1, s, t)\|_m^m ds \\ & + \alpha(t) \varepsilon_3 \left(-\eta_1 - \frac{\alpha'(t)}{\alpha(t)} \right) \int_0^1 \int_{\tau_1}^{\tau_2} s |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho. \end{aligned}$$

Choosing δ, ε so small that

$$h_0 - \delta > \frac{1}{2} h_0, \quad \frac{\delta}{(l - \varepsilon(c_1 + c_2))} (\zeta_0 + 2(1 - l)^2) < \frac{1}{4} h_0.$$

For any fixed $\delta > 0$, we select $\varepsilon_1, \varepsilon_2$ so small satisfying

$$\frac{h_0}{4} \varepsilon_2 < \varepsilon_1 < \frac{h_0}{2} \varepsilon_2,$$

and

$$\begin{aligned} & \varepsilon_2(h_0 - \delta) - \varepsilon_1 > 0, \\ & \varepsilon_1(l - \varepsilon(c_1 + c_2)) - \varepsilon_2 \delta (\zeta_0 + 2c(\zeta_0 - l)^2 \alpha(t)) > 0. \end{aligned}$$

Then, we select $\varepsilon_1, \varepsilon_2$, and ε_3 so small that (3.19) and (3.24) remain valid, and further,

$$\begin{aligned} & \zeta_1(\varepsilon_1 - \varepsilon_2 \delta) > 0, \quad \frac{\sigma}{\alpha(0)} - \varepsilon_2 \delta \frac{\sigma E(0)}{l} > 0, \quad \frac{1}{2} - \varepsilon_2 \frac{h(0) c_p^2}{4\delta} > 0, \\ & \frac{\eta_0}{\alpha(0)} - \varepsilon_1 c(\varepsilon) - \varepsilon_2 c(\delta) - \varepsilon_3 \beta_1 > 0, \quad \eta_1 \varepsilon_3 - \varepsilon_1 c(\varepsilon) - \varepsilon_2 c(\delta) > 0, \end{aligned}$$

where $\eta_0 = \beta_1 - \int_{\tau_1}^{\tau_2} |\beta_2(s)| ds > 0$.

Therefore, (3.24) becomes, for positive constants d_1, d_2, d_3, d_6 ,

$$\begin{aligned} \mathcal{G}'(t) \leq & -\alpha(t) \left(d_1 + \frac{\alpha'(t)}{\alpha(t)} \right) \|u_t\|_2^2 - \alpha(t) d_3 \|\nabla u\|_2^4 - \alpha(t) \left(d_2 + \frac{\alpha'(t)}{2\alpha(t)} \left(\int_0^t h(\varrho) d\varrho \right) + \varepsilon_1 \frac{\alpha'(t) c_p^2}{2\alpha(t)} \right) \|\nabla u\|_2^2 \\ & + \alpha(t) \left(d_6 - \varepsilon_2 \frac{h_0 \alpha'(t) c_p^2}{2\alpha(t)} \right) (h \circ \nabla u)(t) - \alpha(t) \varepsilon_3 \left(\eta_1 - \frac{\alpha'(t)}{\alpha(t)} \right) \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho. \end{aligned} \tag{3.25}$$

According to (2.2), $\lim_{t \rightarrow \infty} \frac{\alpha'(t)}{\alpha(t)} = 0$, we can choose $t_1 > t_0$ so that (3.25) can be written as

$$\begin{aligned} \mathcal{G}'(t) \leq & -\alpha(t) \left(d_1 \|u_t\|_2^2 + d_3 \|\nabla u\|_2^4 + d_2 \|\nabla u\|_2^2 - d_6 (h \circ \nabla u)(t) + d_4 \int_0^1 \int_{\tau_1}^{\tau_2} s |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho \right) \\ \leq & -\alpha(t) d_5 E(t) + \alpha(t) d_6 (h \circ \nabla u)(t), \quad \forall t \geq t_1. \end{aligned} \tag{3.26}$$

□

Theorem 3.6. *Suppose that (2.1)–(2.3) are satisfied and $E(0) > 0$ for any (u_0, u_1, f_0) . Then, the energy $E(t)$ of (2.5) decays to zero exponentially. That is, $\exists \lambda_1, \lambda_2 > 0$ such that*

$$E(t) \leq \lambda_1 e^{-\lambda_2 \int_{t_1}^t \alpha(\varrho) \vartheta(\varrho) d\varrho}, \quad \forall t \geq t_1. \tag{3.27}$$

Proof. Multiplying (3.22) by $\vartheta(t)$, using (2.1) and (2.7), we find

$$\begin{aligned} \vartheta(t) \mathcal{G}'(t) & \leq -d_5 \vartheta(t) \alpha(t) E(t) + d_6 \alpha(t) \vartheta(t) (h \circ \nabla u)(t) \\ & \leq -d_5 \vartheta(t) \alpha(t) E(t) - d_6 \alpha(t) (h' \circ \nabla u)(t) \\ & \leq -d_5 \vartheta(t) \alpha(t) E(t) - d_6 \left\{ 2E'(t) - \alpha'(t) \left(\int_0^t h(\varrho) d\varrho \right) \|\nabla u(t)\|_2^2 \right\}. \end{aligned} \tag{3.28}$$

Since $\vartheta(t)$ is non-increasing function, we have

$$\frac{d}{dt} (\vartheta(t) \mathcal{G}(t) + 2d_6 E(t)) \leq -d_5 \vartheta(t) \alpha(t) E(t) - d_6 \alpha'(t) \left(\int_0^t h(\varrho) d\varrho \right) \|\nabla u(t)\|_2^2. \tag{3.29}$$

From (2.6) and (2.2) that $l \|\nabla u(t)\|_2^2 \leq E(t)$, we find

$$\begin{aligned} \frac{d}{dt} (\vartheta(t) \mathcal{G}(t) + 2d_6 E(t)) & \leq -d_5 \alpha(t) \vartheta(t) E(t) - d_6 \alpha'(t) \left(\int_0^t h(\varrho) d\varrho \right) \|\nabla u(t)\|_2^2 \\ & \leq -d_5 \alpha(t) \vartheta(t) E(t) - \frac{2d_6 \alpha'(t)}{l} E(t) \\ & \leq -\alpha(t) \vartheta(t) \left(d_5 + \frac{2d_6 l_0 \alpha'(t)}{l \vartheta(t) \alpha(t)} \right) E(t). \end{aligned} \tag{3.30}$$

Since $\lim_{t \rightarrow \infty} \frac{\alpha'(t)}{\vartheta(t) \alpha(t)} = 0$, we can choose $t_1 \geq t_0$ such that $d_5 + \frac{2d_6 l_0 \alpha'(t)}{l \alpha(t) \vartheta(t)} > 0$ for $t \geq t_1$.

Finally, let

$$\mathcal{K}(t) := \vartheta(t) \mathcal{G}(t) + 2d_6 E(t) \sim E(t). \tag{3.31}$$

Hence, for some $\lambda_2 > 0$, we obtain

$$\mathcal{K}'(t) \leq -\lambda_2 \vartheta(t) \alpha(t) \mathcal{K}(t), \quad \forall t \geq t_1. \tag{3.32}$$

By integrating (3.32) over (t_1, t) yields

$$\mathcal{K}(t) \leq \mathcal{K}(t_1) e^{-\lambda_2 \int_{t_1}^t \alpha(\varrho) \vartheta(\varrho) d\varrho}, \quad \forall t \geq t_1. \quad (3.33)$$

Hence, (3.27) is established by virtue of (3.31) and (3.33). The proof is complete. \square

4 Conclusion

The objective of this work is the study of the general decay of solutions for a viscoelastic wave equation with distributed delay and Balakrishnan-Taylor damping. This type of problem is frequently found in some mathematical models in applied sciences. Particularly in the theory of viscoelasticity. What interests us in this current work is the combination of these terms of damping (memory term, Balakrishnan-Taylor damping, and the distributed delay terms), which dictates the emergence of these terms in the problem.

In the next work, we will try to use the same method with the same problem, but with the addition of other damping (Dispersion, Source, and Logarithmic terms).

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