Research Article

Jia Li, Xiaoling Hao*, Kun Li, and Siqin Yao

The finite spectrum of Sturm-Liouville problems with n transmission conditions and quadratic eigenparameter-dependent boundary conditions

Abstract: For any positive integer n and a set of positive integers m_i, $i = 1, 2, \ldots, n + 1$, we construct a class of quadratic eigenparameter-dependent boundary Sturm-Liouville problems with n transmission conditions, which have at most $\sum_{i=1}^{n+1} m_i + n + 5$ eigenvalues. The key to this analysis is still the division of intervals and an iterative construction of the characteristic function. Further, some examples are given for a simple explanation.

Keywords: Sturm-Liouville problems, finite spectrum, eigenvalue, transmission conditions, quadratic eigenparameter-dependent

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1 Introduction

The Sturm-Liouville problems with transmission conditions occupy an important position for their widely use in mathematical physics, engineering, etc. Such problems are related to heat and mass transfer and vibrating string problems (see [1–4]). It is worth mentioning that the theory of fractional calculus is often used to study these problems, and some results about fractional differential equations can be referred to [5–9]. In recent years, the research of Sturm-Liouville problems includes not only in one interior point, but also in two or infinite many interior points. The Sturm-Liouville problems with an eigenparameter contained in the boundary conditions were early introduced in [10], and the author mainly studied an expansion theorem of eigenfunctions, which was identical with the regular Sturm-Liouville problem in all respects except for the appearance of the parameter λ in the boundary condition. Further developments of such problems can be found in [2,10–12].

It is well known that the classical Sturm-Liouville theory states that the spectrum of regular or singular, self-adjoint Sturm-Liouville problems is unbounded and therefore infinite, but Atkinson [13] proposed that if the coefficients of Sturm-Liouville problems satisfy some conditions, the problems may have finite eigenvalues. For this reason, in 2001, Kong et al. constructed a class of Sturm-Liouville problems with exactly n eigenvalues for any positive integer n in [14], and in 2009, they gave the matrix representations of

* **Corresponding author: Xiaoling Hao,** School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China, e-mail: xlhao1883@163.com
Jia Li: School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China, e-mail: jli6036@163.com
Kun Li: School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China, e-mail: qslikun@163.com
Siqin Yao: School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China,
e-mail: siqinyao_math@126.com

these problems with self-adjoint boundary conditions in [15]. They further indicated the fact that Sturm-Liouville problems have a finite spectrum. Since then, Ao et al. extended the problems to the case of Sturm-Liouville problems with transmission conditions and eigenparameter-dependent boundary conditions; moreover, they also gave the corresponding matrix representation (see, for example, [1,16–21]). In particular, Ao et al. obtained that the Sturm-Liouville problems with transmission conditions have at most $m + n + 2$ eigenvalues in [16], and the Sturm-Liouville problems with transmission conditions and eigenparameter-dependent boundary conditions have at most $m + n + 4$ eigenvalues in [18]. In 2018, Xu et al. [22] generalized these results to the case of the Sturm-Liouville problems with n transmission conditions, which have at most $\sum_{i=1}^{n+1} m_i + n + 1$ eigenvalues.

However, the following questions remain: For the Sturm-Liouville problems with n transmission conditions and quadratic eigenparameter-dependent boundary conditions, will the upper limit of the number of eigenvalues change? What factors will affect the number of eigenvalues? In this article, we conclude the Sturm-Liouville problems with n transmission conditions and quadratic eigenparameter-dependent boundary conditions have at most $\sum_{i=1}^{n+1} m_i + n + 5$ eigenvalues, where m_i are connected with the partition of the considered interval. In other words, the division of intervals, the number of transmission conditions and the boundary conditions of spectral parameters will all have an impact on the number of eigenvalues.

Following this introduction, in Section 2, we give some preliminaries, and main results on the finite spectrum of Sturm-Liouville problems with n transmission conditions and quadratic eigenparameter-dependent boundary conditions are given in Section 3; moreover, some examples are given to illustrate the problems of this article more concisely.

2 Notation and preliminaries

We study the Sturm-Liouville problem consisting of the following equation:

$$-(py')' + qy = \lambda wy, \quad \text{on } J = (a, c_1) \cup (c_1, c_2) \cup \cdots \cup (c_n, b), \tag{2.1}$$

where

$$-\infty < a < b < +\infty, \quad c_i \in (a, b), \quad i = 1, 2, \ldots, n,$$

with the eigenparameter-dependent boundary conditions

$$A_\lambda Y(a) + B_\lambda Y(b) = 0, \quad Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \tag{2.2}$$

and the transmission conditions

$$C_i Y(c_i-) + D_i Y(c_i+) = 0, \quad i = 1, 2, \ldots, n, \tag{2.3}$$

where

$$A_\lambda = \begin{pmatrix} \alpha_1\lambda^2 + \beta_1\lambda + \gamma_1 & \alpha_1'\lambda^2 + \beta_1'\lambda + \gamma_1' \\ \alpha_2\lambda^2 + \beta_2\lambda + \gamma_2 & \alpha_2'\lambda^2 + \beta_2'\lambda + \gamma_2' \end{pmatrix}, \quad B_\lambda = \begin{pmatrix} \alpha_3\lambda^2 + \beta_3\lambda + \gamma_3 & \alpha_3'\lambda^2 + \beta_3'\lambda + \gamma_3' \\ \alpha_4\lambda^2 + \beta_4\lambda + \gamma_4 & \alpha_4'\lambda^2 + \beta_4'\lambda + \gamma_4' \end{pmatrix},$$

with $\alpha_i, \beta_i, \gamma_i, \alpha_i', \beta_i', \gamma_i' \in \mathbb{R}$, $i = 1, 2$ satisfying $|\alpha_i| + |\alpha_i'| \neq 0$, $\det(A_\lambda) \neq 0$, $\det(B_\lambda) \neq 0$,

$$\text{rank}\begin{pmatrix} \alpha_1 & \alpha_1' & \alpha_3 & \alpha_3' \\ \beta_1 & \beta_1' & \beta_3 & \beta_3' \\ \gamma_1 & \gamma_1' & \gamma_3 & \gamma_3' \end{pmatrix} \geq 2, \quad \text{rank}\begin{pmatrix} \alpha_2 & \alpha_2' & \alpha_4 & \alpha_4' \\ \beta_2 & \beta_2' & \beta_4 & \beta_4' \\ \gamma_2 & \gamma_2' & \gamma_4 & \gamma_4' \end{pmatrix} \geq 2, \tag{2.4}$$

and C_i and D_i are real valued 2×2 matrices satisfying $\det(C_i) = \rho_i > 0$, $\det(D_i) = \theta_i > 0$. Here, λ is the so-called spectral parameter, and the coefficients satisfy the conditions:

$$r = \frac{1}{p}, \, q, \, w \in L(J, \mathbb{C}), \tag{2.5}$$

where $L(J, \mathbb{C})$ denotes the complex-valued functions, which are Lebesgue integrable on J. Condition (2.5) ensures the uniqueness of the solutions on $[a, b]$ for all initial value problem of equation (2.1) (see [10]).

Let $r = \frac{1}{p}$, $u = y$, $v = py'$. Then, equation (2.1) is equivalent to the system

$$u' = rv, v' = (q - \lambda w)u, \text{ on } J. \tag{2.6}$$

Remark 1. As usual, the self-adjoint extension of Sturm-Liouville problems with n transmission conditions needs additional restrictions on C_i, D_i and a new weighted Hilbert space defined as in [23–25]. With this weighted Hilbert space, the operator associated with Sturm-Liouville problems with n transmission conditions is self-adjoint if and only if the associated new operator is self-adjoint, and they consist of the same eigenvalues and satisfy the condition $\theta_1 \cdots \theta_n AE^{-1}A^* = \rho_1 \cdots \rho_n BE^{-1}B^*$, with $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For further details of the self-adjointness of Sturm-Liouville problems with n transmission conditions, please see [23]. The construction is based on the characteristic function whose zeros are eigenvalues, which applies to the arbitrary boundary condition, so the results in this article are not restricted to self-adjoint problems, but include non-self-adjoint problems. We defined the partition of the interval, the values of p, q, w, and determined that $\Delta(\lambda)$ is the polynomial about λ, which shows that the number of eigenvalues is finite. Moreover, we analyzed the number of eigenvalues.

Remark 2. Condition (2.5) does not restrict the sign of any of the coefficients r, q, w. Also, each of r, q, w is allowed to be identically zero on subintervals of J. If r is identically zero on a subinterval J_1, then there exists a solution y, which is identically zero on J_1, but its quasi-derivative $v = py'$ is a nonzero constant on J_1. Such an interval of zero is counted as a single zero, which is given in [2].

Definition 1. A solution y is called a trivial solution of equation (2.1) on some interval, if y is identically zero and its quasi-derivative $v = py'$ is also zero on this interval.

Lemma 1. *Let* (2.5) *hold and let* $\Phi(x, \lambda) = [\varphi_{st}(x, \lambda)]$ *denote the fundamental matrix of system* (2.6) *determined by the initial condition* $\Phi(a, \lambda) = I$, *I is the identity matrix. Then, a complex number λ is an eigenvalue of the Sturm-Liouville problem with n transmission conditions* (2.1)–(2.3) *if and only if*

$$\Delta(\lambda) = \det[A_\lambda + B_\lambda \Phi(b, \lambda)] = 0. \tag{2.7}$$

Moreover, it can be written as follows:

$$\Delta(\lambda) = \det(A_\lambda) + \det(B_\lambda) + h_{11}(\lambda)\varphi_{11}(b, \lambda) + h_{12}(\lambda)\varphi_{12}(b, \lambda) + h_{21}(\lambda)\varphi_{21}(b, \lambda) + h_{22}(\lambda)\varphi_{22}(b, \lambda), \tag{2.8}$$

where $H(\lambda) = \begin{pmatrix} h_{11}(\lambda) & h_{12}(\lambda) \\ h_{21}(\lambda) & h_{22}(\lambda) \end{pmatrix}$, *with*

$$h_{11}(\lambda) = (\alpha_2'\lambda^2 + \beta_2'\lambda + \gamma_2')(\alpha_3\lambda^2 + \beta_3\lambda + \gamma_3) - (\alpha_1'\lambda^2 + \beta_1'\lambda + \gamma_1')(\alpha_4\lambda^2 + \beta_4\lambda + \gamma_4),$$
$$h_{12}(\lambda) = (\alpha_1\lambda^2 + \beta_1\lambda + \gamma_1)(\alpha_4\lambda^2 + \beta_4\lambda + \gamma_4) - (\alpha_2\lambda^2 + \beta_2\lambda + \gamma_2)(\alpha_3\lambda^2 + \beta_3\lambda + \gamma_3),$$
$$h_{21}(\lambda) = (\alpha_2'\lambda^2 + \beta_2'\lambda + \gamma_2')(\alpha_3'\lambda^2 + \beta_3'\lambda + \gamma_3') - (\alpha_1'\lambda^2 + \beta_1'\lambda + \gamma_1')(\alpha_4'\lambda^2 + \beta_4'\lambda + \gamma_4'),$$
$$h_{22}(\lambda) = (\alpha_1\lambda^2 + \beta_1\lambda + \gamma_1)(\alpha_4'\lambda^2 + \beta_4'\lambda + \gamma_4') - (\alpha_2\lambda^2 + \beta_2\lambda + \gamma_2)(\alpha_3'\lambda^2 + \beta_3'\lambda + \gamma_3').$$

Proof. The proof of the first part of this lemma is similar to the one in [8], and the second part comes from a straightforward computation. □

Definition 2. The Sturm-Liouville problem with n transmission conditions and boundary conditions depending quadratically on the eigenparameter (2.1)–(2.3), or equivalently (2.2), (2.3), and (2.6), is said to be degenerate if in (2.8) either $\Delta(\lambda) \equiv 0$ for all $\lambda \in \mathbb{C}$ or $\Delta(\lambda) \neq 0$ for any $\lambda \in \mathbb{C}$.

3 The main conclusion

In this section, we assume (2.4) holds and there exists a partition of interval J

$$
\begin{aligned}
a = a_0^1 < a_1^1 < a_2^1 < \cdots < a_{2m_1}^1 < a_{2m_1+1}^1 = c_1, \quad & \text{on } [a, c_1), \\
c_1 = a_0^2 < a_1^2 < a_2^2 < \cdots < a_{2m_2}^2 < a_{2m_2+1}^2 = c_2, \quad & \text{on } (c_1, c_2), \\
\cdots\cdots\cdots\cdots\cdots & \\
c_n = a_0^{n+1} < a_1^{n+1} < a_2^{n+1} < \cdots < a_{2m_{n+1}}^{n+1} < a_{2m_{n+1}+1}^{n+1} = b, \quad & \text{on } (c_n, b],
\end{aligned}
\tag{3.1}
$$

for n and some integers m_i, $i = 1, 2, \ldots, n + 1$, such that

$$
r = 0 \text{ on } (a_{2k}^i, a_{2k+1}^i), \quad \int_{a_{2k}^i}^{a_{2k+1}^i} w \neq 0, \quad k = 0, 1, \ldots, m_i, \quad i = 1, 2, \ldots, n + 1
\tag{3.2}
$$

and

$$
q = w = 0 \text{ on } (a_{2k+1}^i, a_{2k+2}^i), \quad \int_{a_{2k+1}^i}^{a_{2k+2}^i} r \neq 0, \quad k = 0, 1, \ldots, m_i - 1, \quad i = 1, 2, \ldots, n + 1.
\tag{3.3}
$$

Next, we let

$$
\begin{aligned}
r_k^i &= \int_{a_{2k+1}^i}^{a_{2k+2}^i} r(x)\mathrm{d}x, \quad k = 0, 1, \ldots, m_i - 1, \quad i = 1, 2, \ldots, n + 1; \\
q_k^i &= \int_{a_{2k}^i}^{a_{2k+1}^i} q(x)\mathrm{d}x, \quad k = 0, 1, \ldots, m_i, \quad i = 1, 2, \ldots, n + 1; \\
w_k^i &= \int_{a_{2k}^i}^{a_{2k+1}^i} w(x)\mathrm{d}x, \quad k = 0, 1, \ldots, m_i, \quad i = 1, 2, \ldots, n + 1.
\end{aligned}
\tag{3.4}
$$

Lemma 2. *Let* (2.5) *and* (3.1)–(3.3) *hold. Let* $\Phi(x, \lambda) = [\varphi_{ij}(x, \lambda)]$ *be the fundamental matrix of the system* (2.6) *determined by the initial condition* $\Phi(a, \lambda) = I$ *for each* $\lambda \in \mathbb{C}$. *Then, we have that*

$$
\Phi(a_1^1, \lambda) = \begin{pmatrix} 1 & 0 \\ q_0^1 - \lambda w_0^1 & 1 \end{pmatrix} \Phi(a, \lambda),
\tag{3.5}
$$

$$
\Phi(a_3^1, \lambda) = \begin{pmatrix} 1 + (q_0^1 - \lambda w_0^1)r_0^1 & r_0^1 \\ \varphi_{21}^1(a_3^1, \lambda) & 1 + (q_1^1 - \lambda w_1^1)r_0^1 \end{pmatrix},
\tag{3.6}
$$

where

$$
\varphi_{21}^1(a_3^1, \lambda) = (q_0^1 - \lambda w_0^1) + (q_1^1 - \lambda w_1^1) + (q_0^1 - \lambda w_0^1)(q_1^1 - \lambda w_1^1)r_0^1.
$$

In general, for $1 \le k \le m_1$,

$$
\Phi(a_{2k+1}^1, \lambda) = \begin{pmatrix} 1 & r_{k-1}^1 \\ q_k^1 - \lambda w_k^1 & 1 + (q_k^1 - \lambda w_k^1)r_{k-1}^1 \end{pmatrix} \Phi(a_{2k-1}^1, \lambda).
\tag{3.7}
$$

Proof. Observe from (2.6) that u is constant on each subinterval, where r is identically zero and v is constant on each subinterval, where both q and w are identically zero. The result follows from repeated applications of (2.6). $\qquad\square$

Lemma 3. *Let* (2.5) *and* (3.1)–(3.3) *hold. Let* $\Phi_i(x, \lambda) = [\varphi_{st}^i(x, \lambda)]$ $(x \in (c_i, c_{i+1}), c_{n+1} = b = a_{2m_{n+1}+1}^{n+1})$ *be the fundamental matrix solution of system* (2.6) *determined by the initial condition* $\Phi_i(c_i+, \lambda) = I$ (*here* $\Phi_i(c_i+, \lambda) = \Phi_i(a_0^{i+1}, \lambda) = \Phi(c_i+, \lambda), i = 1, 2, \ldots, n$) *denote the right limit at point* c_i *for each* $\lambda \in \mathbb{C}$. *Then, we have that*

$$\Phi_i(a_1^{i+1}, \lambda) = \begin{pmatrix} 1 & 0 \\ q_0^{i+1} - \lambda w_0^{i+1} & 1 \end{pmatrix} \Phi_i(a_0^{i+1}, \lambda), \tag{3.8}$$

$$\Phi_i(a_3^{i+1}, \lambda) = \begin{pmatrix} 1 + (q_0^{i+1} - \lambda w_0^{i+1})r_0^{i+1} & r_0^{i+1} \\ \varphi_{21}^{i+1}(a_3^{i+1}, \lambda) & 1 + (q_1^{i+1} - \lambda w_1^{i+1})r_0^{i+1} \end{pmatrix}, \tag{3.9}$$

where

$$\varphi_{21}^{i+1}(a_3^{i+1}, \lambda) = (q_0^{i+1} - \lambda w_0^{i+1}) + (q_1^{i+1} - \lambda w_1^{i+1}) + (q_0^{i+1} - \lambda w_0^{i+1})(q_1^{i+1} - \lambda w_1^{i+1})r_0^{i+1}.$$

In general, for $1 \le k \le m_{i+1}$,

$$\Phi_i(a_{2k+1}^{i+1}, \lambda) = \begin{pmatrix} 1 & r_{k-1}^{i+1} \\ q_k^{i+1} - \lambda w_k^{i+1} & 1 + (q_k^{i+1} - \lambda w_k^{i+1})r_{k-1}^{i+1} \end{pmatrix} \Phi_i(a_{2k-1}^{i+1}, \lambda). \tag{3.10}$$

Proof. The proof is similar to the one as in Lemma 2. □

Lemma 4. *Let* (2.5) *and* (3.1)–(3.3) *hold. Let* $\Phi(x, \lambda) = [\varphi_{st}(x, \lambda)]$ *be the fundamental matrix solution of the system* (2.6) *determined by the initial condition* $\Phi(a, \lambda) = I$ *for each* $\lambda \in \mathbb{C}$, *and* $\Phi_i(x, \lambda) = [\varphi_{st}^i(x, \lambda)]$ *be given as in* Lemma 3. *Then, we have that*

$$\Phi(b, \lambda) = \Phi_n(b, \lambda)G_n\Phi_{n-1}(c_n, \lambda)G_{n-1}\Phi_{n-2}(c_{n-1}, \lambda)\cdots G_1\Phi(c_1, \lambda), \tag{3.11}$$

where

$$G_i = (g_{st})_{2\times2} = -D_i^{-1}C_i$$

and

$$\Phi(c_1, \lambda) = \Phi(c_1-, \lambda) = \Phi\left(a_{2m_1+1}^1, \lambda\right),$$
$$\Phi_i(c_{i+1}, \lambda) = \Phi_i(c_{i+1}-, \lambda) = \Phi_i\left(a_{2m_{i+1}+1}^{i+1}, \lambda\right) = \Phi(c_{i+1}, \lambda), \quad (i = 1, 2, \ldots, n-1),$$
$$\Phi_n(b, \lambda) = \Phi_n\left(a_{2m_{n+1}+1}^{n+1}, \lambda\right)$$

denote the left limit at point c_i $(i = 1, 2, \ldots, n)$.

Proof. From the transmission conditions (2.3), we know that $C_i\Phi(c_i-, \lambda) + D_i\Phi(c_i+, \lambda) = 0$; thus, $\Phi(c_i+, \lambda) = -D_i^{-1}C_i\Phi(c_i-, \lambda)$. By using Lemma 4 in [16], when $i = 1$, in $(a, c_1) \cup (c_1, c_2)$, we have that $\Phi(c_2, \lambda) = \Phi_1(c_2, \lambda)G_1\Phi(c_1, \lambda)$, when $i = 2$, we have that in $(a, c_1) \cup (c_1, c_2) \cup (c_2, c_3)$, $\Phi(c_3, \lambda) = \Phi_2(c_3, \lambda)G_2\Phi_1(c_2, \lambda)G_1\Phi(c_1, \lambda)$. By repeated application of Lemmas 2–4 in [16], it can be concluded that (3.11) follows. □

Note that $c_i = a_{2m_i+1}^i = a_0^{i+1}$, $b = a_{2m_{n+1}+1}^{n+1}$ and (3.11). Then, the structure of the fundamental matrix solution $\Phi(b, \lambda)$ is given in Lemmas 2 and 3 and mathematical induction yields the following.

Corollary 1. *If* $g_{12}^i \ne 0$, $i = 1, 2, \ldots, n$, *then for the fundamental matrix* $\Phi(b, \lambda)$, *we have that*

$$\varphi_{11}(b, \lambda) = G_{0n} \cdot \prod_{i=1}^{n-1} g_{12}^i \cdot \prod_{i=1}^{n+1} R_i \cdot \prod_{i=1}^{m_{n+1}-1} \cdot \prod_{i=0}^{m_n-1} \prod_{i=0}^{m_{n-1}} \cdots \prod_{i=0}^{m_2} \prod_{i=0}^{m_1} (q_0^1 - \lambda\omega_0^1) + \tilde\varphi_{11}(\lambda), \tag{3.12}$$

$$\varphi_{12}(b, \lambda) = G_{0n} \cdot \prod_{i=1}^{n-1} g_{12}^i \cdot \prod_{i=1}^{n+1} R_i \cdot \prod_{i=1}^{m_{n+1}-1} \cdot \prod_{i=0}^{m_n-1} \prod_{i=0}^{m_{n-1}} \cdots \prod_{i=0}^{m_2} \prod_{i=1}^{m_1} (q_1^1 - \lambda\omega_1^1) + \tilde\varphi_{12}(\lambda), \tag{3.13}$$

$$\varphi_{21}(b, \lambda) = G_{0n} \cdot \prod_{i=1}^{n-1} g_{12}^i \cdot \prod_{i=1}^{n+1} R_i \cdot \prod_{i=1}^{m_{n+1}} \cdot \prod_{i=0}^{m_n-1} \cdot \prod_{i=0}^{m_{n-1}} \cdots \prod_{i=0}^{m_2} \cdot \prod_{i=0}^{m_1} (q_0^1 - \lambda\omega_0^1) + \tilde{\varphi}_{21}(\lambda),$$
(3.14)

$$\varphi_{22}(b, \lambda) = G_{0n} \cdot \prod_{i=1}^{n-1} g_{12}^i \cdot \prod_{i=1}^{n+1} R_i \cdot \prod_{i=1}^{m_{n+1}} \cdot \prod_{i=0}^{m_n-1} \cdot \prod_{i=0}^{m_{n-1}} \cdots \prod_{i=0}^{m_2} \cdot \prod_{i=1}^{m_1} (q_1^1 - \lambda\omega_1^1) + \tilde{\varphi}_{22}(\lambda),$$
(3.15)

where

$$G_{0n} = \left[g_{12}^n \left(q_{m_n}^n - \lambda\omega_{m_n}^n \right)(q_0^{n+1} - \lambda\omega_0^{n+1}) + g_{11}^n (q_0^{n+1} - \lambda\omega_0^{n+1}) + g_{22}^n \left(q_{m_n}^n - \lambda\omega_{m_n}^n \right) + g_{21}^n \right],$$

$$R_i = \prod_{k=0}^{m_i-1} r_k^i, \; i = 1, 2, \ldots, n+1, \qquad \prod_{i=1}^{m_{n+1}-1} = \prod_{i=1}^{m_{n+1}-1} (q_i^{n+1} - \lambda\omega_i^{n+1}),$$

$$\prod_{i=0}^{m_n-1} = \prod_{i=0}^{m_n-1} (q_i^n - \lambda\omega_i^n), \qquad \prod_{i=0}^{m_k} = \prod_{i=0}^{m_k} (q_i^k - \lambda\omega_i^k), \; k = 2, \ldots, n-1,$$

$\tilde{\varphi}_{st}(\lambda) = o(\prod_{i=1}^{n+1} R_i), \, s, t = 1, 2$ *as* $\min\{r_k^i\} \to \infty$ *for fixed* $q, \omega,$ *and* λ.

From Corollary 1, we can see that each of the entries in Φ (i.e., $\varphi_{st}, s, t = 1, 2$) is a polynomial of λ.

In the next, we construct regular Sturm-Liouville problems with n transmission conditions (2.3) with general self-adjoint and non-self-adjoint boundary conditions (2.2), which have exactly m eigenvalues for each $m \in \mathbb{N}$.

Theorem 1. *Let* $m_i \in \mathbb{N}$ $(i = 1, 2, \ldots, n+1)$, $g_{12}^i \neq 0$, $i = 1, 2, \ldots, n$, *and let* (2.5) *and* (3.1)–(3.3) *hold. Let* $H = (h_{st})_{2\times2}$ *be defined as in* Lemma 1. *Then:*
1. *If* $\alpha_2'\alpha_3' - \alpha_1'\alpha_4' \neq 0$ *in* $h_{21}(\lambda)$, *then the Sturm-Liouville problem with n transmission conditions* (2.1)–(2.3) *has at most* $\sum_{i=1}^{n+1} m_i + n + 5$ *eigenvalues.*
2. *If* $\alpha_2'\alpha_3' - \alpha_1'\alpha_4' = 0$ *and* $\alpha_2'\beta_3' + \alpha_3'\beta_2' - \alpha_1'\beta_4' - \alpha_4'\beta_1' \neq 0$ *in* $h_{21}(\lambda)$, *or* $\alpha_2'\alpha_3 - \alpha_1'\alpha_4 \neq 0$ *in* $h_{11}(\lambda)$, *or* $\alpha_4'\alpha_1 - \alpha_3'\alpha_2 \neq 0$ *in* $h_{22}(\lambda)$, *then the Sturm-Liouville problem with n transmission conditions* (2.1)–(2.3) *has at most* $\sum_{i=1}^{n+1} m_i + n + 4$ *eigenvalues.*
3. *If* $\alpha_2'\alpha_3' - \alpha_1'\alpha_4' = 0$, $\alpha_2'\beta_3' + \alpha_3'\beta_2' - \alpha_1'\beta_4' - \alpha_4'\beta_1' = 0$ *and* $\alpha_2'\gamma_3' + \beta_2'\beta_3' + \alpha_3'\gamma_2' - \alpha_1'\gamma_4' - \beta_1'\beta_4' - \alpha_4'\gamma_1' \neq 0$ *in* $h_{21}(\lambda)$, *or* $\alpha_2'\alpha_3 - \alpha_1'\alpha_4 = 0$ *and* $\alpha_2'\beta_3 + \alpha_3\beta_2' - \alpha_1'\beta_4 - \alpha_4\beta_1' \neq 0$ *in* $h_{11}(\lambda)$, *or* $\alpha_4'\alpha_1 - \alpha_3'\alpha_2 = 0$ *and* $\alpha_1\beta_4' + \alpha_4'\beta_1 - \alpha_2\beta_3' - \alpha_3'\beta_2 \neq 0$ *in* $h_{22}(\lambda)$, *or* $\alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0$ *in* $h_{12}(\lambda)$, *then the Sturm-Liouville problem with n transmission conditions* (2.1)–(2.3) *has at most* $\sum_{i=1}^{n+1} m_i + n + 3$ *eigenvalues.*
4. *If none of the aforementioned conditions holds, then the Sturm-Liouville problem with n transmission conditions* (2.1)–(2.3) *either has l eigenvalues for* $l \in \left\{ 1, 2, \ldots, \sum_{i=1}^{n+1} m_i + n + 2 \right\}$ *or is degenerate.*

Proof. We prove case 1, and other cases can be proved in the same way. We note from (3.2) that the degrees of $\varphi_{11}(b, \lambda), \varphi_{12}(b, \lambda), \varphi_{21}(b, \lambda), \varphi_{22}(b, \lambda)$ in λ are $\sum_{i=1}^{n+1} m_i + n$, $\sum_{i=1}^{n+1} m_i + n - 1$, $\sum_{i=1}^{n+1} m_i + n + 1$, $\sum_{i=1}^{n+1} m_i + n$, respectively. Thus, from Lemma 1 and Corollary 1, we get that maximum degree of $\Delta(\lambda)$ can be $\sum_{i=1}^{n+1} m_i + n + 5$ if $\alpha_2'\alpha_3' - \alpha_1'\alpha_4' \neq 0$ in $h_{21}(\lambda)$. Here, from fundamental theorem of algebra, one gets that $\Delta(\lambda)$ has at most $\sum_{i=1}^{n+1} m_i + n + 5$ roots, i.e., the Sturm-Liouville problem (2.1)–(2.3) has at most $\sum_{i=1}^{n+1} m_i + n + 5$ eigenvalues. Then, case 1 of this theorem is proved, and other cases can be proved in the similar way. □

Remark 3. If the rank of the matrix in (2.2) is one, then it turns into the ordinary constants, and it is the result of [22].

Theorem 2. *Let* $m_i \in \mathbb{N}$ $(i = 1, 2, \ldots, n+1)$, $g_{12}^n = 0$, *but* $g_{11}^n\omega_0^{n+1} + g_{22}^n\omega_{m_n}^n \neq 0$, $g_{12}^i \neq 0, i = 1, 2, \ldots, n-1$, *and let* (2.4) *and* (3.1)–(3.3) *hold. Let* $H = (h_{st})_{2\times2}$ *be defined as in* Lemma 1. *Then:*
1. *If* $\alpha_2'\alpha_3' - \alpha_1'\alpha_4' \neq 0$ *in* $h_{21}(\lambda)$, *then the Sturm-Liouville problem with n transmission conditions* (2.1)–(2.3) *has at most* $\sum_{i=1}^{n+1} m_i + n + 4$ *eigenvalues.*

2. *If $\alpha_2'\alpha_3' - \alpha_1'\alpha_4' = 0$ and $\alpha_2'\beta_3' + \alpha_3'\beta_2' - \alpha_1'\beta_4' - \alpha_4'\beta_1' \neq 0$ in $h_{21}(\lambda)$, or $\alpha_2'\alpha_3 - \alpha_1'\alpha_4 \neq 0$ in $h_{11}(\lambda)$, or $\alpha_4'\alpha_1 - \alpha_3'\alpha_2$*
   $\neq 0$ in $h_{22}(\lambda)$, then the Sturm-Liouville problem with n transmission conditions (2.1)–(2.3) has at most
   $\sum_{i=1}^{n+1} m_i + n + 3$ eigenvalues.
3. *If $\alpha_2'\alpha_3' - \alpha_1'\alpha_4' = 0$, $\alpha_2'\beta_3' + \alpha_3'\beta_2' - \alpha_1'\beta_4' - \alpha_4'\beta_1' = 0$ and $\alpha_2'\gamma_3' + \beta_2'\beta_3' + \alpha_3'\gamma_2' - \alpha_1'\gamma_4' - \beta_1'\beta_4' - \alpha_4'\gamma_1' \neq 0$ in $h_{21}(\lambda)$,*
   or $\alpha_2'\alpha_3 - \alpha_1'\alpha_4 = 0$ and $\alpha_2'\beta_3 + \alpha_3\beta_2' - \alpha_1'\beta_4 - \alpha_4\beta_1' \neq 0$ in $h_{11}(\lambda)$, or $\alpha_4'\alpha_1 - \alpha_3'\alpha_2 = 0$ and $\alpha_1\beta_4' + \alpha_4'\beta_1 - \alpha_2\beta_3' -$
   $\alpha_3'\beta_2 \neq 0$ in $h_{22}(\lambda)$, or $\alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0$ in $h_{12}(\lambda)$, then the Sturm-Liouville problem with n transmission con-
   ditions (2.1)–(2.3) has at most $\sum_{i=1}^{n+1} m_i + n + 2$ eigenvalues.
4. *If none of the aforementioned conditions holds, then the Sturm-Liouville problem with n transmission*
   conditions (2.1)–(2.3) either has l eigenvalues for $l \in \left\{1, 2, \ldots, \sum_{i=1}^{n+1} m_i + n + 1\right\}$ or is degenerate.

Proof. The proof is similar to Theorem 1 only by noting that $g_{12}^n \neq 0$, but $g_{11}^n \omega_0^{n+1} + g_{22}^n \omega_{m_n}^n \neq 0$, and the degree of λ will decrease by one and hence is omitted here. ☐

Remark 4. If $\alpha_i = \beta_i = \alpha_i' = \beta_i' = 0$ ($i = 1, \ldots, 4$), the Sturm-Liouville problems (2.1)–(2.3) will reduce to the finite spectrum of Sturm-Liouville problems with n transmission conditions (see [22]).

Remark 5. If $\alpha_i = \alpha_i' = 0$ ($i = 1, \ldots, 4$) and $\beta_2'\beta_3' - \beta_1'\beta_4' \neq 0$, the Sturm-Liouville problems (2.1)–(2.3) will reduce to the Sturm-Liouville problems with n transmission conditions and spectral parameter boundary conditions of first-degree polynomials. Similar to the method of [18], we can prove that the problems have at most $\sum_{i=1}^{n+1} m_i + n + 3$ eigenvalues.

Finally, we give examples to illustrate our main results.

Example 1. Let $n = 2$ and consider the Sturm-Liouville problem with two transmission conditions

$$\begin{cases} -(py')' + qy = \lambda wy, & \text{on } J \in (-5, -2) \cup (-2, 1) \cup (1, 6), \\ A_\lambda Y(-5) + B_\lambda Y(6) = 0, \\ -2(py')(-2-) + y(-2+) = 0, & y(-2-) + (py')(-2+) = 0, \\ 2y(1-) - (py')(1+) = 0, & (py')(1-) + y(1+) = 0, \end{cases} \tag{3.16}$$

where

$$A_\lambda = \begin{pmatrix} 1 & 2\lambda^2 \\ 0 & \lambda^2 + 2\lambda \end{pmatrix}, \quad B_\lambda = \begin{pmatrix} 1 & \lambda^2 + \lambda \\ 0 & \lambda^2 \end{pmatrix}.$$

Choose $m_1 = 1$, $m_2 = 1$, $m_3 = 2$, and p, q, w are piecewise constant functions, which are defined as follows:

$$p(x) = \begin{cases} \infty, & (-5, -4), \\ 1, & (-4, -3), \\ \infty, & (-3, -2), \\ \infty, & (-2, -1), \\ \frac{1}{2}, & (-1, 0), \\ \infty, & (0, 1), \\ \infty, & (1, 2), \\ 1, & (2, 3), \\ \infty, & (3, 4), \\ \frac{1}{2}, & (4, 5), \\ \infty, & (5, 6), \end{cases} \quad q(x) = \begin{cases} 0, & (-5, -4), \\ 0, & (-4, -3), \\ 1, & (-3, -2), \\ 2, & (-2, -1), \\ 0, & (-1, 0), \\ -1, & (0, 1), \\ 1, & (1, 2), \\ 0, & (2, 3), \\ 3, & (3, 4), \\ 0, & (4, 5), \\ 1, & (5, 6), \end{cases} \quad w(x) = \begin{cases} 1, & (-5, -4), \\ 0, & (-4, -3), \\ 1, & (-3, -2), \\ 1, & (-2, -1), \\ 0, & (-1, 0), \\ 1, & (0, 1), \\ 1, & (1, 2), \\ 0, & (2, 3), \\ 1, & (3, 4), \\ 0, & (4, 5), \\ 1, & (5, 6). \end{cases} \tag{3.17}$$

From (3.16), we have

$$C_1 = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\det(C_1) = 2 > 0, \quad \det(D_1) = 1 > 0, \quad \det(C_2) = 2 > 0, \quad \det(D_2) = 1 > 0,$$

$$G_1 = -D_1^{-1}C_1 = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}, \quad g_{12}^1 = 2 \neq 0, \quad G_2 = -D_2^{-1}C_2 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \quad g_{12}^2 = 1 \neq 0.$$

By deduction, it can be obtained that the characteristic function

$$\Delta(\lambda) = \lambda^2 + 2\lambda + \lambda^2 + (\lambda^2 + 2\lambda)\varphi_{11}(b, \lambda) + (-\lambda^4 + 3\lambda^3 + 2\lambda^2)\varphi_{21}(b, \lambda) + \lambda^2\varphi_{22}(b, \lambda)$$
$$= 8\lambda^{11} - 128\lambda^{10} + 780\lambda^9 - 2134\lambda^8 + 1792\lambda^7 + 3107\lambda^6 - 6303\lambda^5 + 636\lambda^4 + 3171\lambda^3 + 109\lambda^2 - 446\lambda.$$

Hence, the Sturm-Liouville problem (3.16), (3.17) has exactly $m_1 + m_2 + m_3 + n + 5 = 11$ eigenvalues:

$$\lambda_0 = 0, \ \lambda_1 = 4.9561, \ \lambda_2 = 1.8806, \ \lambda_3 = 0.3886, \ \lambda_4 = 2.072, \ \lambda_5 = -1.2265, \ \lambda_6 = 2.4901,$$
$$\lambda_7 = -0.461 - 0.1783i, \ \lambda_8 = 3.2205 - 0.1718i, \ \lambda_9 = -0.461 + 0.1783i, \ \lambda_{10} = 3.2205 + 0.1718i.$$

Example 2. Let $n = 2$ and consider the Sturm-Liouville problem with two transmission conditions:

$$\begin{cases} -(py')' + qy = \lambda wy, \ \text{on} \ J \in (-5, -2) \cup (-2, 1) \cup (1, 6), \\ A_\lambda Y(-5) + B_\lambda Y(6) = 0, \\ -2(py')(-2-) + y(-2+) = 0, \quad y(-2-) + (py')(-2+) = 0, \\ 2y(1-) - (py')(1+) = 0, \quad (py')(1-) + y(1+) = 0, \end{cases} \tag{3.18}$$

where

$$A_\lambda = \begin{pmatrix} 1 & 2\lambda^2 \\ 0 & \lambda^2 \end{pmatrix}, \quad B_\lambda = \begin{pmatrix} \lambda & 2\lambda^2 \\ 0 & \lambda^2 \end{pmatrix}.$$

Choose $m_1 = 1$, $m_2 = 1$, $m_3 = 2$, and p, q, w are defined as in Example 1. Then, we get the characteristic function:

$$\Delta(\lambda) = \lambda^2 + \lambda^3 + \lambda^3\varphi_{11}(b, \lambda) + \lambda^2\varphi_{22}(b, \lambda) = 8\lambda^9 - 84\lambda^8 + 252\lambda^7 - 2\lambda^6 - 947\lambda^5 + 713\lambda^4 + 878\lambda^3 - 720\lambda^2.$$

Hence, the Sturm-Liouville problem (3.17), (3.18) has exactly $m_1 + m_2 + m_3 + n + 3 = 9$ eigenvalues:

$$\lambda_0 = 0, \ \lambda_1 = 0, \ \lambda_2 = 4.907, \ \lambda_3 = 0.7737, \ \lambda_4 = -1.227, \ \lambda_5 = 2.2893, \ \lambda_6 = 1.9257, \ \lambda_7 = 3.2005, \ \lambda_8 = -1.3692.$$
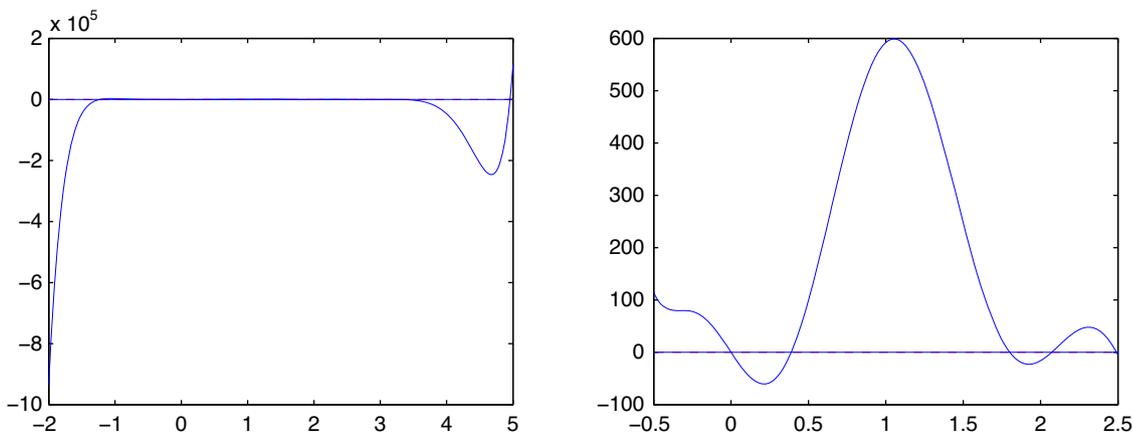

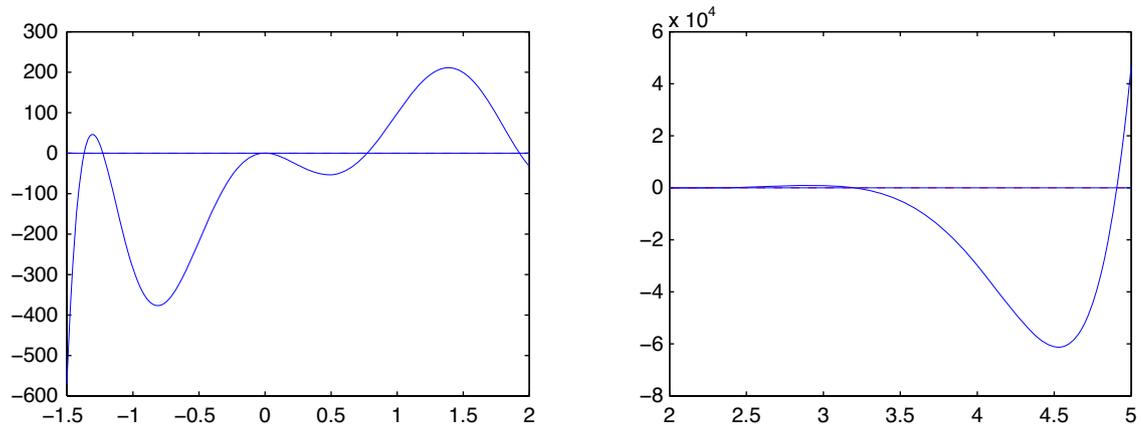**Figure 1:** Characteristic function in Example 1.

Figure 2: Characteristic function in Example 2.

Figures 1 and 2 show the graph of the $\Delta(\lambda)$ in Examples 1 and 2, respectively (the graph of complex eigenvalues is not shown here).

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