

Research Article

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The G -sequence shadowing property and G -equicontinuity of the inverse limit spaces under group action

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Abstract: First, we give the concepts of G -sequence shadowing property, G -equicontinuity and G -regularly recurrent point. Second, we study their dynamical properties in the inverse limit space under group action. The following results are obtained. (1) The self-mapping f has the G -sequence shadowing property if and only if the shift mapping σ has the \bar{G} -sequence shadowing property; (2) The self-mapping f is G -equicontinuous if and only if the shift mapping σ is \bar{G} -equicontinuous; (3) $RR_{\bar{G}}(\sigma) = \varprojlim (RR_G(f), f)$. These conclusions make up for the lack of theory in the inverse limit space under group action.

Keywords: inverse limit spaces, G -sequence shadowing property, G -equicontinuity, G -regularly recurrent point

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1 Introduction

The inverse limit space is a kind of very important space, which has always been the focus of research. However, the theory of inverse limit space has been very perfect. Scholar put forward the concept of the inverse limit space under group action and proved that the shift mapping and the self-mapping are equivariant to each other in G -shadowing property and G -strong shadowing property, see [1]. In addition, the shadowing property and equicontinuity are very important properties in the dynamical systems. Many scholars have studied their dynamical properties and obtained many meaningful results, see [2–15]. Zhong and Wang [2] gave a sufficient and necessary condition for a point to be an equicontinuous point of dynamical system. In [3] it is shown that every ergodic invariant measure of a mean equicontinuous system has discrete spectrum; Ji, Chen and Zhang [4] proved that the shift map has the Lipschitz shadowing property if and only if the self-map has the Lipschitz shadowing property in the inverse limit space. In this paper, first, we give the concepts of G -sequence shadowing property, G -equicontinuity and G -regularly recurrent point. Second, we study their dynamical properties in the inverse limit space under group action and will get the following theorem.

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Theorem A. Let $(X_f, \bar{G}, \bar{d}, \sigma)$ be the inverse limit space of (X, G, d, f) under group action. If the map $f : X \rightarrow X$ is equivariant and surjective, we have that the self-mapping f has the G -sequence shadowing property if and only if the shift mapping σ has the \bar{G} -sequence shadowing property.

Theorem B. Let $(X_f, \bar{G}, \bar{d}, \sigma)$ be the inverse limit space of (X, G, d, f) under group action. If the map $f : X \rightarrow X$ is equivariant and surjective, we have that the self-mapping f is G -equicontinuous if and only if the shift mapping σ is \bar{G} -equicontinuous.

Theorem C. Let $(X_f, \bar{G}, \bar{d}, \sigma)$ be the inverse limit space of (X, G, d, f) under group action. If for any $i \geq 0$ the map $\pi_i : X_f \rightarrow X$ is open, we have $RR_{\bar{G}}(\sigma) = \varprojlim (RR_G(f), f)$.

We will prove Theorems A, B, C in Sections 2, 3, 4, respectively.

2 G -sequence shadowing property

For the convenience of the reader, we will give the concepts used in this section. Now we start with the following definitions.

Definition 2.1. [16] Let (X, d) be a metric space, G be a topological group and $\theta : G \times X \rightarrow X$ be a continuous map. The triple (X, G, θ) is called to be a metric G -space if the following conditions are satisfied:

- (1) $\theta(e, x) = x$, where for all $x \in X$ and e is the identity of G ;
- (2) $\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)$ for all $x \in X$ and $g_1, g_2 \in G$.

If (X, d) is compact, then (X, G, θ) is also said to be compact metric G -space. For the convenience of writing, $\theta(g, x)$ is usually abbreviated as gx .

Definition 2.2. [17] Let (X, d) be a metric G -space and f be a continuous map from X to X . The map f is said to be a equivariant map if we have $f(px) = pf(x)$ for all $x \in X$ and $p \in G$.

Definition 2.3. [1] Let (X, d) be a metric G -space and f be a continuous map from X to X . $\varprojlim(X, f)$ is said to be the inverse limit space if we write $\varprojlim(X, f) = \{(x_0, x_1, x_2 \dots) : f(x_{i+1}) = x_i, i \geq 0\}$, where $\varprojlim(X, f)$ is denoted by X_f .

The metric \bar{d} in X_f is defined by $\bar{d}(\bar{x}, \bar{y}) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}$, where $\bar{x} = (x_0, x_1, x_2 \dots)$ and $\bar{y} = (y_0, y_1, y_2 \dots)$. The shift mapping $\sigma : X_f \rightarrow X_f$ is defined by $\sigma(\bar{x}) = (f(x_0), x_0, x_1 \dots)$. Thus, (X_f, \bar{d}) is compact metric space and the shift mapping σ is homeomorphism.

Definition 2.4. [1] Let (X, d) be a metric G -space and f be an equivariant map from X to X . Write $\bar{G} = \{(g, g, g \dots) : g \in G\}$ and $G_{\infty} = \prod_{i=0}^{\infty} G_i$ where $G_i = G$. The map $\theta : \bar{G} \times X_f \rightarrow X_f$ is defined by $\theta(\bar{g}, \bar{x}) = \bar{g}\bar{x} = (gx_0, gx_1, gx_2 \dots)$, where $\bar{g} = (g, g, g \dots) \in \bar{G}$ and $\bar{x} = (x_0, x_1, x_2 \dots) \in X_f$. Then (X_f, \bar{G}, θ) is a metric \bar{G} -space.

Let $(X_f, \bar{G}, \bar{d}, \sigma)$ and (X, G, d, f) be shown as above. The space $(X_f, \bar{G}, \bar{d}, \sigma)$ is called to be the inverse limit spaces of (X, G, d, f) under group action.

Definition 2.5. [18] Let (X, d) be a metric G -space and f be a continuous map from X to X . The sequence $\{x_i\}_{i \geq 0}$ is called to be (G, δ) -pseudo orbit of f if for any $i \geq 0$ there exists $t_i \in G$ such that $d(t_i f(x_i), x_{i+1}) < \delta$.

Definition 2.6. [18] Let (X, d) be a metric G -space and f be a continuous map from X to X . The sequence $\{x_i\}_{i \geq 0}$ is said to be (G, δ) -shadowed by a point $y \in X$ if for any $i \geq 0$ there exists $t_i \in G$ such that $d(f^i(y), t_i x_i) < \delta$.

Remark 2.1. By Definitions 2.5 and 2.6, we will give the concept of G -sequence shadowing property.

Definition 2.7. Let (X, d) be a metric G -space and f be a continuous map from X to X . The map f has G -sequence shadowing property if each $\varepsilon > 0$ there exists $\delta > 0$ such that for any (G, δ) -pseudo orbit $\{x_i\}_{i \geq 0}$ of f , there exists a point $y \in X$ and nonnegative integer sequence $\{n_i\}_{i \geq 0}$ such that the sequence $\{x_{n_i}\}_{i \geq 0}$ is (G, ε) -shadowed by the point y .

Now, we start to prove Theorem A.

Theorem A. Let $(X_f, \bar{G}, \bar{d}, \sigma)$ be the inverse limit space of (X, G, d, f) under group action. If the map $f : X \rightarrow X$ is equivalent and surjective, we have that the self-mapping f has the G -sequence shadowing property if and only if the shift mapping σ has the \bar{G} -sequence shadowing property.

Proof. \Rightarrow Suppose that the map f has the G -sequence shadowing property. Since X is compact metric space, it is bounded. Write $M = \text{diam}(X)$. Then for any $\varepsilon > 0$, there exists $m > 0$ such that

$$\frac{M}{2^m} < \frac{\varepsilon}{2}.$$

Since the map f is uniformly continuous, it follows that for any $0 \leq i \leq m$, there exists $0 < \delta_1 < \frac{\varepsilon}{4}$ such that $d(x, y) < \delta_1$ implies

$$d(f^i(x), f^i(y)) < \frac{\varepsilon}{4}. \tag{1}$$

Note that the map f has the G -sequence shadowing property, it follows that there exists $0 < \delta_2 < \delta_1$ such that any (G, δ_2) -pseudo orbit $\{x_i\}_{i \geq 0}$ of f , there exists a point $y \in X$ and nonnegative integer sequence $\{n_i\}_{i \geq 0}$ such that the sequence $\{x_{n_i}\}_{i \geq 0}$ is (G, δ_1) -shadowed by the point y . Let $\{\bar{y}_k\}_{k \geq 0}$ be $(\bar{G}, \frac{\delta_2}{2^m})$ -pseudo orbit, where $\bar{y}_k = (y_k^0, y_k^1, y_k^2 \dots) \in X_f$. Hence for any $k \geq 0$ there exists $\bar{g}_k = (g_k, g_k, g_k \dots) \in \bar{G}$ such that

$$\bar{d}(\bar{g}_k \sigma(\bar{y}_k), \bar{y}_{k+1}) < \frac{\delta_2}{2^m}.$$

That is, for any $k \geq 0$, we have

$$d(g_k f(y_k^m), y_{k+1}^m) < \delta_2.$$

Thus, $\{y_k^m\}_{k \geq 0}$ is (G, δ_2) -pseudo orbit in X . Hence, there exists $x_0 \in X$, $t_k \in G$ and nonnegative integer sequence $\{n_k\}_{k \geq 0}$ such that

$$d(f^{n_k}(x_0), t_k y_{n_k}^m) < \delta_1.$$

By (1) and the map f is equivalent, for any $k \geq 0$ and $0 \leq i \leq m$, we have

$$d(f^{k+i}(x_0), t_k y_{n_k}^{m-i}) < \frac{\varepsilon}{4}. \tag{2}$$

Since the map f is surjective, we can choose $\bar{s} = (f^m(x_0), f^{m-1}(x_0), f^{m-2}(x_0) \dots x_0 \dots) \in X_f$ and $\bar{t}_k = (t_k, t_k, t_k \dots) \in \bar{G}$.

By (2), for any $k \geq 0$, it follows that

$$\bar{d}(\sigma^k(\bar{s}), \bar{t}_k \bar{y}_{n_k}) < \sum_{i=0}^m \frac{\varepsilon}{2^{i+2}} + \frac{M}{2^m} < \varepsilon.$$

Hence, the shift mapping σ has the \bar{G} -sequence shadowing property.

⇐ Suppose the shift mapping σ has the \bar{G} -sequence shadowing property. Let $m_0 > 0$. For each $\eta > 0$ there exists $\delta_3 > 0$ such that for any (\bar{G}, δ_3) -pseudo orbit $\{\bar{z}_k\}_{k \geq 0}$ of σ , there exists a point $z \in X$ and non-negative integer sequence $\{n_k\}_{k \geq 0}$ such that the sequence $\{\bar{z}_{n_k}\}_{k \geq 0}$ is $(\bar{G}, \frac{\eta}{2^{m_0}})$ -shadowed by the point z and

$$\frac{M}{2^{m_0}} < \frac{\delta_3}{2}. \tag{3}$$

Since the map f is uniformly continuous, it follows that for any $0 \leq i \leq m_0$ there exists $0 < \delta_4 < \frac{\delta_3}{4}$ such that $d(x, y) < \delta_4$ implies

$$d(f^i(x), f^i(y)) < \frac{\delta_3}{4}. \tag{4}$$

Suppose that $\{x_k\}_{k \geq 0}$ is (G, δ_4) -pseudo orbit in X . Thus, for any $k > 0$ there exists $l_k \in G$ such that

$$d(l_k f(x_k), x_{k+1}) < \delta_4.$$

By (4) and the map f is equivalent, for any $k > 0$ and $0 \leq i \leq m_0$, we have

$$d(l_k f^{i+1}(x), f^i(x_{k+1})) < \frac{\delta_3}{4}. \tag{5}$$

Since the map f is surjective, for each $k > 0$ we can choose $\bar{z}_k = (f^{m_0}(x_k), f^{m_0-1}(x_k), \dots, x_k, \dots) \in X_f$ and $\bar{l}_k = (l_k, l_k, l_k, \dots) \in \bar{G}$. According to (3) and (5) for any $k > 0$, it follows that

$$\bar{d}(\bar{l}_k \sigma(\bar{z}_k, \bar{z}_{k+1})) < \sum_{i=0}^{m_0} \frac{\delta_3}{2^{i+2}} + \frac{M}{2^{m_0}} < \delta_3.$$

Hence, $\{\bar{z}_k\}_{k \geq 0}$ is (\bar{G}, δ_3) -pseudo orbit in X_f . Thus, there exists $\bar{z} = (z_0, z_1, z_2, \dots) \in X_f$, $\bar{p}_k = (p_k, p_k, p_k, \dots) \in \bar{G}$ and nonnegative integer sequence $\{n_k\}_{k \geq 0}$ such that

$$\bar{d}(\sigma^k(\bar{z}), \bar{p}_k \bar{z}_{n_k}) < \frac{\eta}{2^{m_0}}.$$

So, for any $k > 0$, we have

$$d(f^k(z_{m_0}), p_k x_k) < \eta.$$

Hence, the map f has the G -sequence shadowing property. Thus, we end the proof. □

Next, we give an example satisfying G -sequence shadowing property.

Example 2.1. Let $X = \{0, -1, -\frac{1}{n}, -1 + \frac{1}{n}\}$. The metric d in X is defined by $d(x, y) = |x - y|$ where $x, y \in X$. Let $G = Z_2 = \{0, 1\}$. Defined by $0 \cdot x = x, 1 \cdot x = -1 - x$ for every $x \in X$. The map $f : X \rightarrow X$ is defined by

$$f(0) = 0, \quad f\left(-\frac{1}{2}\right) = -\frac{1}{2}, \quad f(-1) = -1, \\ f\left(-\frac{1}{n+1}\right) = -\frac{1}{n}, \quad f\left(-1 + \frac{1}{n+1}\right) = -1 + \frac{1}{n}, \quad n > 2.$$

Now, we start to prove that the map f has the G -sequence shadowing property.

Proof. It is very easy to know that (X, d) is a compact metric G -space and the map f is equivalent. For any $\eta > 0$, there exists $m > 0$ such that $\frac{1}{m} < \eta$. Write $\delta = \frac{1}{4m(m+1)}$. Let $\{x_i\}_{i \geq 0}$ be (G, δ) -pseudo orbit of the map f . Hence for any $i \geq 0$ there exists $g_i \in G$ such that

$$d(g_i f(x_i), x_{i+1}) < \delta.$$

Obviously, the distance between any two different points is greater than δ in $\left[-1 + \frac{1}{m+1}, -\frac{1}{m+1}\right] \cap X$. Hence, we have two cases.

Case 1: There exists $k \in N$ such that

$$x_k \in \left(-1 + \frac{1}{m}, -\frac{1}{m}\right) \cap X.$$

According to the inequality $d(g_k f(x_k), x_{k+1}) < \delta$, we have that

$$x_{k+1} = g_k f(x_k)$$

and

$$x_{k+1} \in \left(-1 + \frac{1}{m}, -\frac{1}{m}\right) \cap X.$$

Keep going, we can get $x_{i+1} = g_i f(x_i)$ and $x_i \in \left(-1 + \frac{1}{m}, -\frac{1}{m}\right) \cap X$ when $i \geq k$. If $k = 0$, according to that the map f is equivalent, we have that

$$x_i = g_i g_{i-1} \cdots g_0 f^i(x_0).$$

If $k \geq 1$, according to the inequality $d(g_{k-1} f(x_{k-1}), x_k) < \delta$, we have that

$$x_k = g_{k-1} f(x_{k-1})$$

and

$$x_{k-1} \in \left(-1 + \frac{1}{m}, -\frac{1}{m}\right) \cap X.$$

Keep going, we can get $x_i = g_{i-1} f(x_{i-1})$ and $x_i \in \left(\frac{1}{m}, 1 - \frac{1}{m}\right) \cap X$ when $i \leq k$. Hence, when $i \geq 1$, we have that

$$x_i = g_{i-1} f(x_{i-1}).$$

According to that the map f is equivalent, we have that

$$x_i = g_i g_{i-1} \cdots g_0 f^i(x_0).$$

Hence, we have $d(g_i g_{i-1} \cdots g_0 f^i(x_0), x_i) = 0 < \eta$. Thus, the map f has the G -sequence shadowing property.

Case 2: For any $i \in N$, we have that

$$x_i \in \left\{ \left[-1, -1 + \frac{1}{m}\right] \cup \left[-\frac{1}{m}, 0\right] \right\} \cap X.$$

When $x_i \in \left[-1, -1 + \frac{1}{m}\right] \cap X$, we write $g_i = 1$. When $x_i \in \left[-\frac{1}{m}, 0\right] \cap X$, we write $g_i = 0$. Thus, we can get that

$$d(g_i f^i(0), x_i) < \frac{1}{m} < \eta.$$

Thus, the map f has the G -sequence shadowing property. □

3 G -equicontinuous

Let N_+ be the set of positive integers in this paper.

Definition 3.1. Let (X, d) be a metric space and f be a continuous map from X to X . The map f is said to be equicontinuous if for any $\varepsilon > 0$ and $n \in N_+$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f^n(x), f^n(y)) < \varepsilon$.

Remark 3.1. According to the definition of equicontinuity, we will give the concept of G -equicontinuity.

Definition 3.2. Let (X, d) be a metric G -space and f be a continuous map from X to X . The map f is said to be G -equicontinuous if each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $n \in \mathbb{N}_+$ there exists $g_n, p_n \in G$ such that $d(x, y) < \delta$ implies $d(f^n(g_n x), f^n(p_n y)) < \varepsilon$.

Remark 3.2. Let Z_+ be the set of nonnegative positive integers. If $G = Z_+$, then (X, Z_+, φ) is a semi discrete dynamical system. According to [19], there exists a continuous map f from X to X such that for any $x \in X$ and $m \in Z_+$, we have $\varphi(m, x) = f^m(x)$. In this case, the map f has G -equicontinuous means that each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $n \in \mathbb{N}_+$ there exists $m, k \in Z_+$ such that $d(x, y) < \delta$ implies $d(f^{n+m}(x), f^{n+k}(y)) < \varepsilon$. Hence, Definition 3.2 is broader than Definition 3.1 even for $G = Z_+$.

Now, we start to prove Theorem B.

Theorem B. Let $(X_f, \bar{G}, \bar{d}, \sigma)$ be the inverse limit space of (X, G, d, f) under group action. If the map $f : X \rightarrow X$ is equivariant and surjective, we have that the self-mapping f is G -equicontinuous if and only if the shift mapping σ is \bar{G} -equicontinuous.

Proof. \Rightarrow Suppose the map f is G -equicontinuous. Hence, for any $\varepsilon > 0$ there exists $0 < \delta < \frac{\varepsilon}{4}$ such that for any $n \geq 0$ there exists $g_n, p_n \in G$ such that $d(x, y) < \delta$ implies

$$d(f^n(g_n x), f^n(p_n y)) < \frac{\varepsilon}{4}. \tag{6}$$

Let $\delta_0 < \delta$ and $\bar{d}(\bar{x}, \bar{y}) < \delta_0$, where $\bar{x} = (x_0, x_1, x_2 \dots) \in X_f$ and $\bar{y} = (y_0, y_1, y_2 \dots) \in X_f$. Thus, we have

$$d(x_0, y_0) < \delta_0 < \delta.$$

By (6), for any $n \geq 0$ there exists $g_n, p_n \in G$ such that

$$d(f^n(g_n x_0), f^n(p_n y_0)) < \frac{\varepsilon}{4}.$$

Let $\bar{g}_n = (g_n, g_{n-1} \dots g_1, g_0, e, e, e \dots) \in \bar{G}$ and $\bar{k}_n = (k_n, k_{n-1} \dots k_1, k_0, e, e, e \dots) \in \bar{G}$. According to the map σ is an equivalent map, it follows that

$$\begin{aligned} \bar{d}(\sigma^n(\bar{g}_n \cdot \bar{x}), \sigma^n(\bar{k}_n \cdot \bar{y})) &= \bar{d}(\bar{g}_n \sigma^n(\bar{x}), \bar{k}_n \sigma^n(\bar{y})) \\ &= \sum_{i=0}^n \frac{d(g_{n-i} f^{n-i}(x_0), k_{n-i} f^{n-i}(y_0))}{2^i} + \sum_{i=n+1}^{\infty} \frac{d(f^i x_i, f^i y_i)}{2^i} \\ &= \sum_{i=0}^n \frac{d(f^{n-i}(g_{n-i} x_0), f^{n-i}(k_{n-i} y_0))}{2^i} + \frac{1}{2^n} \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i} \\ &< \frac{\varepsilon}{2} + \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i} \\ &< \frac{\varepsilon}{2} + \bar{d}(\bar{x}, \bar{y}) < \varepsilon. \end{aligned}$$

So the shift mapping σ is G -equicontinuous.

\Leftarrow Suppose the shift mapping σ is \bar{G} -equicontinuous. For any $\varepsilon > 0$, there exists $0 < \delta_1 < \frac{\varepsilon}{4}$ such that for any $n \geq 0$, there exists $\bar{g}_n, \bar{k}_n \in \bar{G}$ such that $\bar{d}(\bar{x}, \bar{y}) < \delta_1$ implies

$$d(\sigma^n(\bar{g}_n \bar{x}), \sigma^n(\bar{k}_n \bar{y})) < \varepsilon, \tag{7}$$

where $\bar{g}_n = (g_n^0, g_n^1, g_n^2 \dots) \in \bar{G}$ and $\bar{k}_n = (k_n^0, k_n^1, k_n^2 \dots) \in \bar{G}$. Since X is compact metric space, it is bounded. Write $M = \text{diam}(X)$. Let $m > 0$ such that

$$\sum_{i=m+1}^{\infty} \frac{M}{2^i} < \frac{\delta_1}{2}.$$

Since the map f is uniformly continuous, it follows that for any $0 \leq i \leq m$ there exists $0 < \delta_2 < \frac{\delta_1}{4}$ such that $d(x, y) < \delta_2$ implies

$$d(f^i(x), f^i(y)) < \frac{\delta_1}{4}. \quad (8)$$

Let $x_0, y_0 \in X$ such that $d(x_0, y_0) < \delta_2$. By (8), we have

$$d(f^i(x_0), f^i(y_0)) < \varepsilon.$$

Since the map f is surjective, we can choose that

$$\bar{x} = (f^m(x_0), f^{m-1}(x_0), \dots, f(x_0), x_0, x_1, x_2, \dots) \in X_f,$$

$$\bar{y} = (f^m(y_0), f^{m-1}(y_0), \dots, f(y_0), y_0, y_1, y_2, \dots) \in X_f.$$

Hence, we have that

$$\bar{d}(\bar{x}, \bar{y}) < \sum_{i=0}^m \frac{d(f^{m-i}(x_0), f^{m-i}(y_0))}{2^i} + \sum_{i=m+1}^{\infty} \frac{M}{2^i} < \frac{\delta_1}{2} + \frac{\delta_1}{2} < \delta_1.$$

By (7), for any $n \geq 0$, we have that

$$\bar{d}(\sigma^n(\bar{g}_n \bar{x}), \sigma^n(\bar{k}_n \bar{y})) < \varepsilon,$$

where $\bar{g}_n = (g_n^0, g_n^1, g_n^2, \dots) \in \bar{G}$ and $\bar{k}_n = (k_n^0, k_n^1, k_n^2, \dots) \in \bar{G}$. Thus, it follows that

$$d(f^n(g_n^0 f^m(x_0)), f^n(k_n^0 f^m(y_0))) < \varepsilon.$$

By the map f is equivalent, we get

$$d(f^{n+m}(g_n^0 x_0), f^{n+m}(k_n^0 y_0)) < \varepsilon.$$

So the map f is G -equicontinuous. This completes the proof. \square

Now, we give an example satisfying G -equicontinuous.

Example 3.1. Let $X = [0, 1]$. The metric d in X is defined by $d(x, y) = |x - y|$ where $x, y \in X$. The map $f : X \rightarrow X$ is defined by $f(x) = x$. Let $G = \{0, 1\}$ act on X by $0 \cdot x = x$, $1 \cdot x = 1 - x$ for every $x \in X$. It is very easy to know that (X, d) is a metric G -space. For any $\varepsilon > 0$ and $n \in \mathbb{N}_+$, let $0 < \delta < \varepsilon$ and $g_n = p_n = 0$. If $d(x, y) < \delta$, then we have that

$$d(f^n(g_n x), f^n(p_n y)) = d(x, y) < \delta < \varepsilon.$$

So the map f is G -equicontinuous.

4 G -regularly recurrent point

Definition 4.1. Let (X, d) be a metric space and f be a continuous map from X to X . A point $x \in X$ is called to be regularly recurrent point if for each open set U containing the point x , there exists $m > 0$ such that for any $k > 0$, we have $f^{km}(x) \in U$. Denoted by $RR(f)$ the regularly recurrent point set of the map f .

Remark 4.1. According to the definition of regularly recurrent point, we will give the concept of G -regularly recurrent point.

Definition 4.2. Let (X, d) be a metric G -space and f be a continuous map from X to X . A point $x \in X$ is called to be G -regularly recurrent point if for each open set U containing the point x , there exists $m > 0$ such that for any $k > 0$ there exists $g_k \in G$ such that $g_k f^{km}(x) \in U$. Denoted by $RR_G(f)$ the G -regularly recurrent point set of the map f .

Now, we start to prove Theorem C.

Theorem C. Let $(X_f, \bar{G}, \bar{d}, \sigma)$ be the inverse limit space of (X, G, d, f) under group action. If for any $i \geq 0$ the map $\pi_i : X_f \rightarrow X$ is open, we have $RR_{\bar{G}}(\sigma) = \varprojlim (RR_G(f), f)$.

Proof. \Rightarrow Suppose $\bar{x} \in RR_{\bar{G}}(\sigma)$ where $\bar{x} = (x_0, x_1, x_2 \dots)$. For any $i \geq 0$, let U_i be an any open set containing the point x_i . Thus, $\pi_i^{-1}(U_i)$ is an open set containing the point \bar{x} . Hence, there exists $m > 0$ such that for any $k > 0$ there exists $\bar{g}_k = (g_k, g_k, g_k \dots) \in \bar{G}$ such that $\bar{g}_k \sigma^{km}(\bar{x}) \in \pi_i^{-1}(U_i)$. Thus, $\pi_i(\bar{g}_k \sigma^{km}(\bar{x})) \in U_i$. That is $g_k f^{km}(x_i) \in U_i$. So $x_i \in RR_G(f)$. Hence, $RR_{\bar{G}}(\sigma) \subset \varprojlim (RR_G(f), f)$.

\Leftarrow Suppose $\bar{y} \in \varprojlim (RR_G(f), f)$ where $\bar{y} = (y_0, y_1, y_2 \dots)$. Then for any $i \geq 0$ we have $y_i \in RR_G(f)$. Let V be an open set containing the point y_i . Then $\pi_i(V)$ is an open set containing the point y_i . There exists $n > 0$ such that for any $p > 0$, there exists $t_p \in G$ such that $t_p f^{pn}(y_i) \in \pi_i(V)$. Let $\bar{t}_p = (t_p, t_p, t_p \dots) \in \bar{G}$. Then we have $\bar{t}_p \sigma^{pn}(\bar{y}) \in V$. Thus, $\bar{y} \in RR_{\bar{G}}(\sigma)$. Hence, $\varprojlim (RR_G(f), f) \subset RR_{\bar{G}}(\sigma)$. This completes the proof. \square

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