Research Article

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Universal inequalities of the poly-drifting Laplacian on smooth metric measure spaces

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Abstract: In this paper, we study the eigenvalue problem of poly-drifting Laplacian on complete smooth metric measure space $(M, \langle , \rangle, e^{-\phi}\mathrm{d}v)$, with nonnegative weighted Ricci curvature $\mathrm{Ric}^{\phi} \geq 0$ for some $\phi \in C^2(M)$, which is uniformly bounded from above, and successfully obtain several universal inequalities of this eigenvalue problem.

Keywords: eigenvalues, universal inequalities, poly-drifting Laplacian, smooth metric measure space, weighted Ricci curvature

MSC 2020: 35P15, 53C20, 53C42

1 Introduction

A smooth metric measure space (SMMS) is actually a Riemannian manifold equipped with some measure which is conformal to the usual Riemannian measure. More precisely, for a given complete n-dimensional Riemannian manifold $(M, \langle,\rangle)$ with metric $\langle,\rangle$, the triple $(M, \langle,\rangle, e^{-\phi}\mathrm{d}v)$ is called a SMMS, where ϕ is a *smooth real-valued* function on M and $\mathrm{d}v$ is the Riemannian volume element related to $\langle,\rangle$ (sometimes, we also call $\mathrm{d}v$ the volume density). On a SMMS $(M, \langle,\rangle, e^{-\phi}\mathrm{d}v)$, we can define the so-called ∞-*Bakry-Émery Ricci tensor* Ric^{ϕ} by

$$\mathrm{Ric}^{\phi} = \mathrm{Ric} + \mathrm{Hess}\phi,$$

which sometimes is also called the *weighted Ricci curvature*. Here Ric, Hess are the Ricci curvature and the Hessian operator on M, respectively.

The SMMS is not a trivial generalization of the corresponding Riemannian manifold, and actually, some classical results in the Riemannian geometry cannot be extended trivially to the case of SMMSs. For example, when Ric^{ϕ} is bounded from below, Myers's theorem, Bishop-Gromov's volume comparison, Cheeger-Gromoll's splitting theorem and Abresch-Gromoll excess estimate cannot hold as the Riemannian case. On the other hand, the equation $\mathrm{Ric}^{\phi} = \kappa\langle,\rangle$ for some constant κ is called the gradient Ricci soliton equation, which plays an important role in the study of the Ricci flow. If $\kappa = 0$, $\kappa > 0$ or $\kappa < 0$, the gradient Ricci soliton $(M, \langle,\rangle, e^{-\phi}\mathrm{d}v, \kappa)$ is said to be steady, shrinking or expanding, respectively – for more details

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about Ricci solitons, we refer to [1]. One can define the so-called *drifting Laplacian* (or the *weighted Laplacian*) $\mathbb{L}_\phi$ as follows:

$$\mathbb{L}_\phi := \Delta - \langle \nabla\phi, \nabla\cdot\rangle,$$

where ∇, Δ are the gradient and the Laplace operators, respectively. When $\phi = \frac{1}{4}|x|^2$, then the drifting Laplacian $\mathbb{L}_\phi$ is the operator $\mathbb{L} := \Delta - \frac{1}{2}\langle x, \nabla(\cdot)\rangle$, which was introduced by Colding and Minicozzi [2] to study self-shrinker hypersurfaces. Some interesting results for eigenvalues of the drifting Laplacian can be found in [3–7] and references therein.

Let Ω be a bounded domain in an n-dimensional complete SMMS $(M, \langle,\rangle, e^{-\phi}\mathrm{d}v)$. In this paper, we consider the following eigenvalue problem:

$$\begin{cases} (-\mathbb{L}_\phi)^m u = \Lambda u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν denotes the outward unit normal vector field of $\partial\Omega$, and m is an arbitrary positive integer. We know that $\mathbb{L}_\phi^m$ is self-adjoint on the space of functions

$$\mathbb{F} = \left\{ f \in C^{m+2}(\Omega) \cap C^{m+1}(\partial\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial \nu}\bigg|_{\partial\Omega} = \cdots = \frac{\partial^{m-1} f}{\partial \nu^{m-1}}\bigg|_{\partial\Omega} = 0 \right\}$$

with respect to the inner product

$$\langle\langle f, g\rangle\rangle = \int_\Omega fg\,\mathrm{d}\mu,$$

so the eigenvalue problem (1.1) has a discrete spectrum whose elements (i.e., eigenvalues) can be listed increasingly as follows:

$$0 < \Lambda_1 \le \Lambda_2 \le \cdots \le \Lambda_k \le \cdots,$$

where each eigenvalue is repeated according to its multiplicity. Especially, when ϕ is a constant function, problem (1.1) becomes the eigenvalue problem of the poly-harmonic operator on Riemannian manifolds. In recent years, "*Universal inequalities*" for eigenvalues (i.e., not depending on the domain) of some self-adjoint elliptic operators is a hot research topic in Geometric Analysis. For eigenvalues of the poly-harmonic operator on Riemannian manifolds, some universal inequalities have been obtained in [8–12] and references therein.

A natural question is:

- *Could we generalize those universal inequalities for the Laplacian on Riemannian manifolds to the case of drifting Laplacian on SMMSs*?

In fact, when $m = 1$, Xia and Xu [13] investigated eigenvalues of the Dirichlet problem of the drifting Laplacian on compact manifolds and got some universal inequalities; when $m = 2$, Du et al. [14] obtained some universal inequalities of Yang type for eigenvalues of the bi-drifting Laplacian either on a compact Riemannian manifold with boundary (possibly empty) immersed in a Euclidean space, a unit sphere or a projective space, or on bounded domains of complete manifolds supporting some special functions. Moreover, some new results for the eigenvalues of bi-drifting Laplacian can be found in [15,16]. Recently, when m is an arbitrary integer no less than 2, Pereira et al. [17] and Du et al. [18] gave some universal inequalities for eigenvalues of the poly-drifting Laplacian. More results on universal inequalities for eigenvalues of the drifting Laplacian can be seen, e.g., in [19–24].

Based on the existing results mentioned above, in this paper, we will give some universal inequalities for eigenvalues of problem (1.1) when m is an arbitrary integer no less than 2.

First, we have:

Theorem 1.1. *Let* $(M, \langle,\rangle, e^{-\phi}dv)$ *be a complete SMMS with nonnegative weighted Ricci curvature* $\mathrm{Ric}^{\phi} \geq 0$ *for some* $\phi \in C^2(M)$, *which is bounded above uniformly on* M. *Denote by* Λ_i *the ith eigenvalue of problem* (1.1). *Then, for* $m \geq 2$, *we have*

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \left(\frac{4m(2m-1)}{l}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\Lambda_i^{\frac{m-1}{m}}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Lambda_i^{\frac{1}{m}}\right)^{\frac{1}{2}}, \tag{1.2}$$

where l *is the dimension of the Euclidean* l*-space* $\mathbb{R}^l$ *in the splitting of* M *given in* Lemma 2.5.

Remark 1.2. We prefer to give two explanations to show further and clearly our motivation of considering universal inequalities for eigenvalues of weighted operators:

(1) From the work [25] of Wei and Wylie, we know that a SMMS is not necessarily compact when $\mathrm{Ric}^{\phi} \geq \lambda > 0$, unlike in the case of Riemannian manifolds where such a complete one is compact if its Ricci curvature is bounded from below uniformly by some positive constant. So, it is meaningful to study the geometry of SMMSs.

(2) For what kind of $(M, \langle,\rangle, e^{-\phi}dv)$, does there exist universal inequalities for the eigenvalues of problem (1.1) for $m \geq 2$? Actually, Pereira et al. [17] have given some universal inequalities on bounded domains in a Euclidean space or a unit sphere. Du et al. [18] gave some universal inequalities for eigenvalues of the poly-drifting Laplacian on bounded domains in the Gaussian and cylinder shrinking solitons. Comparing with the results in [17,18], here we have given some universal inequalities for the eigenvalues of problem (1.1) in some new SMMSs.

From the inequality (1.2), we can get universal bounds on the $(k+1)$th eigenvalue in terms of the first k eigenvalues.

Corollary 1.3. *Under the assumption of* Theorem 1.1, *we have*

$$\Lambda_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\Lambda_i + \frac{2m(2m-1)}{k^2 l}\left(\sum_{i=1}^{k}\Lambda_i^{\frac{l-1}{l}}\right)\left(\sum_{i=1}^{k}\Lambda_i^{\frac{1}{l}}\right) + \left\{\frac{4m^2(2m-1)^2}{k^4 l^2}\left(\sum_{i=1}^{k}\Lambda_i^{\frac{l-1}{l}}\right)^2\left(\sum_{i=1}^{k}\Lambda_i^{\frac{1}{l}}\right)^2 - \frac{1}{k}\sum_{i=1}^{k}\left(\Lambda_i - \frac{1}{k}\sum_{j=1}^{k}\Lambda_j\right)^2\right\}^{\frac{1}{2}} \tag{1.3}$$

and

$$\Lambda_{k+1} \leq \frac{(l+2m(2m-1))}{kl}\sum_{i=1}^{l}\Lambda_i + \left\{\frac{(l+2m(2m-1))^2}{k^2 l^2}\left(\sum_{i=1}^{l}\Lambda_i\right)^2 - \frac{(l+4m(2m-1))}{kl}\sum_{i=1}^{l}\left(\Lambda_i - \frac{1}{k}\sum_{j=1}^{l}\Lambda_j\right)^2\right\}^{\frac{1}{2}}. \tag{1.4}$$

Our last result is the universal inequality for lower order eigenvalues of the problem (1.1).

Theorem 1.4. *Under the assumption of* Theorem 1.1, *we have*

$$\sum_{i=1}^{l}(\Lambda_{i+1}-\Lambda_1)^{\frac{1}{2}} \leq 2\sqrt{(2m-1)l}\,\Lambda_1^{\frac{1}{2}}. \tag{1.5}$$

2 Some useful facts

In this section, we will give some facts which will play an important role in the proofs of our results listed in Section 1. First, we will state the following general inequality for eigenvalues of the poly-drifting Laplacian which were given in [17,18].

Lemma 2.1. ([17, Theorem 2.1], [18, Lemma 2.1]) *Let $(M, \langle,\rangle, e^{-\phi}\mathrm{d}v)$ be an n-dimensional compact SMMS with boundary (possibly empty), and let Λ_i be the ith eigenvalue of the following eigenvalue problem:*

$$\begin{cases} (-\mathbb{L}_{\phi})^m u = \Lambda u & \text{in } M, \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0 & \text{on } \partial M, \end{cases}$$

and u_i be the orthonormal eigenfunction corresponding to Λ_i, that is,

$$\begin{cases} (-\mathbb{L}_{\phi})^m u_i = \Lambda_i u_i & \text{in } M, \\ u_i = \dfrac{\partial u_i}{\partial \nu} = \cdots = \dfrac{\partial^{m-1} u_i}{\partial \nu^{m-1}} = 0 & \text{on } \partial M, \\ \displaystyle\int_M u_i u_j \mathrm{d}\mu = \delta_{ij} & \forall i,\ j = 1, 2, \ldots, \end{cases}$$

where ν is the outward unit normal vector field of ∂M and $\mathrm{d}\mu = e^{-\phi}dv$. Then:

(*I*) *For any function $h \in C^{m+2}(M) \cap C^{m+1}(\partial M)$ and any positive integer k, we have*

$$\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \|u_i \nabla h\|^2 \le \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \int_M h u_i ((-\mathbb{L}_{\phi})^m (h u_i) - \Lambda_i h u_i) + \sum_{i=1}^{k} \frac{(\Lambda_{k+1} - \Lambda_i)}{\delta} \left\| \langle \nabla h, \nabla u_i \rangle + \frac{u_i \mathbb{L}_{\phi} h}{2} \right\|^2, \tag{2.1}$$

where δ is any positive constant and $\|f\|^2 = \int_M f^2 \mathrm{d}\mu$.

(*II*) *If $h_i \in C^{m+2}(M) \cap C^{m+1}(\partial M)$, $i \ge 2$, satisfies $\int_{\Omega} h_i u_1 u_{j+1} = 0$ for $1 \le j < i$, then for any positive integer i, we have*

$$(\Lambda_{i+1} - \Lambda_1)^{\frac{1}{2}} \int_M |u_1 \nabla h_i|^2 \mathrm{d}\mu \le \delta \int_M h_i u_1 ((-\mathbb{L}_{\phi})^m (h_i u_1) - \Lambda_1 h_i u_1) \mathrm{d}\mu + \frac{1}{\delta} \left\| \langle \nabla h_i, \nabla u_1 \rangle + \frac{1}{2} u_1 \mathbb{L}_{\phi} h_i \right\|^2, \tag{2.2}$$

where δ is any positive constant and $\|f\|^2 = \int_M f^2 \mathrm{d}\mu$.

Lemma 2.2. [17, Lemma 2.2] *Let Λ_i and u_i, $i = 1, 2, \ldots$, be as in* Lemma 2.1, *then*

$$0 \le \int_M u_i (-\mathbb{L}_{\phi})^k u_i \le (\Lambda_i)^{\frac{k}{m}}, \quad k = 1, 2, \ldots, m-1. \tag{2.3}$$

Lemma 2.3. (Reverse Chebyshev inequality [26]) *Suppose $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ are two real sequences with $\{a_i\}$ increasing and $\{b_i\}$ decreasing, then we have*

$$\sum_{i=1}^{k} a_i b_i \le \frac{1}{k} \left(\sum_{i=1}^{k} a_i \right) \left(\sum_{i=1}^{k} b_i \right). \tag{2.4}$$

Lemma 2.4. (Weighted Chebyshev inequality [26]) *Let $\{a_i\}_{i=1}^k$, $\{b_i\}_{i=1}^k$ and $\{c_i\}_{i=1}^k$ be three sequences of non-negative real numbers with $\{a_i\}_{i=1}^k$ decreasing, $\{b_i\}_{i=1}^k$ and $\{c_i\}_{i=1}^k$ increasing. Then the following inequality*

$$\left(\sum_{i=1}^{k} a_i^2 b_i \right) \left(\sum_{i=1}^{k} a_i c_i \right) \le \left(\sum_{i=1}^{k} a_i^2 \right) \left(\sum_{i=1}^{k} a_i b_i c_i \right) \tag{2.5}$$

holds.

Lemma 2.5. [27, Theorem 1.1] *Let (M, g) be a complete connected Riemannian manifold with weighted Ricci curvature* $\mathrm{Ric}^{\phi} \ge 0$ *for some $\phi \in C^2(M)$, which is bounded above uniformly on M. Then it splits isometrically as $N \times \mathbb{R}^l$, where N is some complete Riemannian manifold without lines and $\mathbb{R}^l$ is the l-Euclidean space. Furthermore, the function ϕ is constant on each $\mathbb{R}^l$ in this splitting.*

3 Proofs of our results

In this section, we will give the detailed proofs of the main results.

Proof of Theorem 1.1. By Lemma 2.5, we know that $(M, \langle,\rangle, d\mu)$ splits isometrically as $N \times \mathbb{R}^l$. Let $\overline{x} = (t, x)$ be the standard coordinate functions of $N \times \mathbb{R}^l$, where $t \in N$ and $x = (x_1, \dots, x_l) \in \mathbb{R}^l$. Since ϕ is constant on each $\mathbb{R}^l$ in this splitting, for $\alpha = 1, 2, \dots, l$ and for any vector field $X \in \Xi(M)$, we have

$$\mathbb{L}_\phi x_\alpha = \Delta x_\alpha - \langle \nabla\phi, \nabla x_\alpha\rangle = 0, \quad |\nabla x_\alpha| = 1, \quad \mathrm{Ric}^\phi(\nabla x_\alpha, X) = 0, \tag{3.1}$$

where $\Xi(M)$ denotes the set of smooth vector fields on M. By the Bochner formula (see [5,25]) and using (3.1), for any functions $g \in C^3(\Omega)$, where Ω is a bounded domain of M, we have

$$\mathbb{L}_\phi\langle\nabla x_\alpha, \nabla g\rangle = 2\langle\nabla^2 x_\alpha, \nabla^2 g\rangle + \langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi g)\rangle + \langle\nabla g, \nabla(\mathbb{L}_\phi x_\alpha)\rangle + 2\mathrm{Ric}^\phi(\nabla x_\alpha, \nabla g) = \langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi g)\rangle. \tag{3.2}$$

By $\mathbb{L}_\phi x_\alpha = 0$ and (3.2), we can get

$$(-\mathbb{L}_\phi)^m(x_\alpha u_i) = x_\alpha(-\mathbb{L}_\phi)^m(u_i) + 2m(-1)^m\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-1}u_i)\rangle = \Lambda_i x_\alpha u_i + 2m(-1)^m\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-1}u_i)\rangle. \tag{3.3}$$

Taking $h = x_\alpha$ into (2.1) and summing α from 1 to l, then we infer from

$$|\nabla x_\alpha| = 1, \quad \mathbb{L}_\phi x_\alpha = 0, \quad \sum_{\alpha=1}^{l}\langle\nabla x_\alpha, \nabla u_i\rangle^2 \le |\nabla u_i|^2$$

that

$$l\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 \le \sum_{i=1}^{k}\sum_{\alpha=1}^{l}\delta(\Lambda_{k+1} - \Lambda_i)^2\int_\Omega 2m(-1)^m x_\alpha u_i\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-1}u_i)\rangle d\mu + \sum_{i=1}^{k}\frac{(\Lambda_{k+1} - \Lambda_i)}{\delta}\int_\Omega |\nabla u_i|^2 d\mu. \tag{3.4}$$

Since

$$(-\mathbb{L}_\phi)^{m-1}(x_\alpha u_i) = x_\alpha(-\mathbb{L}_\phi)^{m-1}(u_i) + 2(m-1)(-1)^{m-1}\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-2}u_i)\rangle,$$

we have

$$\begin{aligned}\int_\Omega x_\alpha u_i\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-1}u_i)\rangle d\mu &= \int_\Omega x_\alpha u_i \mathbb{L}_\phi^{m-1}\langle\nabla x_\alpha, \nabla u_i\rangle d\mu \\ &= \int_\Omega \mathbb{L}_\phi^{m-1}(x_\alpha u_i)\langle\nabla x_\alpha, \nabla u_i\rangle d\mu \\ &= \int_\Omega (x_\alpha\mathbb{L}_\phi^{m-1}u_i + 2(m-1)\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-2}u_i)\rangle)\langle\nabla x_\alpha, \nabla u_i\rangle d\mu.\end{aligned} \tag{3.5}$$

On the other hand, we follow from the divergence theorem that

$$\int_\Omega x_\alpha u_i\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-1}u_i)\rangle d\mu = -\int_\Omega \mathbb{L}_\phi^{m-1}u_i(u_i|\nabla x_\alpha|^2 + x_\alpha\langle\nabla x_\alpha, \nabla u_i\rangle)d\mu. \tag{3.6}$$

Combining (3.5) and (3.6), one has

$$\int_\Omega x_\alpha u_i\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-1}u_i)\rangle d\mu = \int_\Omega\left((m-1)\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-2}u_i)\rangle\langle\nabla x_\alpha, \nabla u_i\rangle + \frac{1}{2}u_i\mathbb{L}_\phi^{m-1}u_i|\nabla x_\alpha|^2\right)d\mu. \tag{3.7}$$

When $m = 2p$, $p \in \mathbb{Z}_+$, let $g = x_\alpha$, $\alpha = 1, 2, \dots, l$, we have

$$\begin{aligned}&\int_\Omega \langle\nabla g, \nabla\langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p-1}u_i)\rangle\rangle^2 d\mu \\ &\le \int_\Omega |\nabla g|^2|\nabla\langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p-1}u_i)\rangle|^2 d\mu\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \langle \nabla g, \nabla((-\mathbb{L}_\phi)^p u_i)\rangle\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle \mathrm{d}\mu \\
&= \int_\Omega -(-\mathbb{L}_\phi)^p u_i \langle \nabla g, \nabla\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle\rangle \mathrm{d}\mu \\
&\le \left\{\int_\Omega ((-\mathbb{L}_\phi)^p u_i)^2 \mathrm{d}\mu\right\}^{\frac{1}{2}} \left\{\int_\Omega \langle \nabla g, \nabla\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle\rangle^2 \mathrm{d}\mu\right\}^{\frac{1}{2}} \\
&\le \left\{\int_\Omega u_i(-\mathbb{L}_\phi)^{2p} u_i \mathrm{d}\mu\right\}^{\frac{1}{2}} \left\{\int_\Omega |\nabla g|^2 |\nabla\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle|^2 \mathrm{d}\mu\right\}^{\frac{1}{2}} \\
&= \Lambda_i^{\frac{1}{2}} \left\{\int_\Omega \langle \nabla g, \nabla\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle\rangle^2 \mathrm{d}\mu\right\}^{\frac{1}{2}},
\end{aligned}
$$

which implies that

$$
\int_\Omega \langle \nabla g, \nabla\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle\rangle^2 \mathrm{d}\mu \le \Lambda_i. \tag{3.8}
$$

By (3.8) and Lemma 2.2, we have

$$
\begin{aligned}
&\int_\Omega \langle \nabla g, \nabla((-\mathbb{L}_\phi)^{m-2} u_i)\rangle\langle \nabla g, \nabla u_i\rangle \mathrm{d}\mu \\
&\quad= \int_\Omega \langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle \mathrm{d}\mu \\
&\quad= \int_\Omega -(-\mathbb{L}_\phi)^{p-1} u_i \langle \nabla g, \langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle\rangle \mathrm{d}\mu \\
&\quad\le \left\{\int_\Omega ((-\mathbb{L}_\phi)^{p-1} u_i)^2 \mathrm{d}\mu\right\}^{\frac{1}{2}} \left\{\int_\Omega \langle \nabla g, \nabla\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle\rangle^2 \mathrm{d}\mu\right\}^{\frac{1}{2}} \\
&\quad\le \Lambda_i^{\frac{m-2}{2m}} \Lambda_i^{\frac{1}{2}} \\
&\quad\le \Lambda_i^{\frac{m-1}{m}}.
\end{aligned} \tag{3.9}
$$

When $m = 2p + 1$, $p \in \mathbb{Z}_+$,

$$
\begin{aligned}
&\int_\Omega \langle \nabla g, \nabla\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle\rangle^2 \mathrm{d}\mu \\
&\quad\le \int_\Omega |\nabla g|^2 |\nabla\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle|^2 \mathrm{d}\mu \\
&\quad= \int_\Omega \langle \nabla g, \nabla((-\mathbb{L}_\phi)^p u_i)\rangle\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle \mathrm{d}\mu \\
&\quad= \int_\Omega -(-\mathbb{L}_\phi)^p u_i \langle \nabla g, \nabla\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle\rangle \mathrm{d}\mu \\
&\quad\le \left\{\int_\Omega ((-\mathbb{L}_\phi)^p u_i)^2 \mathrm{d}\mu\right\}^{\frac{1}{2}} \left\{\int_\Omega \langle \nabla g, \nabla\langle \nabla g, \nabla((-\mathbb{L}_\phi)^{p-1} u_i)\rangle\rangle^2 \mathrm{d}\mu\right\}^{\frac{1}{2}}
\end{aligned}
$$

$$\le \left\{\int_\Omega u_i(-\mathbb{L}_\phi)^{2p}u_i\,\mathrm{d}\mu\right\}^{\frac{1}{2}}\left\{\int_\Omega |\nabla g|^2|\nabla\langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p-1}u_i)\rangle|^2\mathrm{d}\mu\right\}^{\frac{1}{2}}$$
$$\le \Lambda_i^{\frac{m-1}{2m}}\left\{\int_\Omega \langle\nabla g, \nabla\langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p-1}u_i)\rangle\rangle^2\mathrm{d}\mu\right\}^{\frac{1}{2}},$$

which implies that

$$\int_\Omega \langle\nabla g, \nabla\langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p-1}u_i)\rangle\rangle^2 \le \Lambda_i^{\frac{m-1}{m}}. \tag{3.10}$$

By (3.10) and Lemma 2.2, we have

$$\begin{aligned}
&\int_\Omega \langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p-2}u_i)\rangle\langle\nabla g, \nabla u_i\rangle\mathrm{d}\mu \\
&= \int_\Omega \langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p}u_i)\rangle\langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p-1}u_i)\rangle\mathrm{d}\mu \\
&= \int_\Omega -(-\mathbb{L}_\phi)^{p}u_i\langle\nabla g, \langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p-1}u_i)\rangle\rangle\mathrm{d}\mu \\
&\le \left\{\int_\Omega ((-\mathbb{L}_\phi)^{p}u_i)^2\mathrm{d}\mu\right\}^{\frac{1}{2}}\left\{\int_\Omega \langle\nabla g, \nabla\langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p-1}u_i)\rangle\rangle^2\mathrm{d}\mu\right\}^{\frac{1}{2}} \\
&\le \Lambda_i^{\frac{m-1}{2m}}\left\{\int_\Omega |\nabla g|^2|\nabla\langle\nabla g, \nabla((-\mathbb{L}_\phi)^{p}u_i)\rangle|^2\mathrm{d}\mu\right\}^{\frac{1}{2}} \\
&\le \Lambda_i^{\frac{m-1}{m}}.
\end{aligned} \tag{3.11}$$

It follows from (3.9) and (3.11) that

$$\int_\Omega \langle\nabla g, \nabla((-\mathbb{L}_\phi)^{m-2}u_i)\rangle\langle\nabla g, \nabla u_i\rangle\mathrm{d}\mu \le \Lambda_i^{\frac{m-1}{m}}.$$

By the above inequality, we have

$$\sum_{\alpha=1}^{l}\int_\Omega \langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-2}u_i)\rangle\langle\nabla x_\alpha, \nabla u_i\rangle\mathrm{d}\mu \le l\Lambda_i^{\frac{m-1}{m}}, \tag{3.12}$$

we infer from (3.7), (3.12) and Lemma 2.2 that

$$\begin{aligned}
&\sum_{\alpha=1}^{l}\int_\Omega (-1)^m x_\alpha u_i\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-1}u_i)\rangle\mathrm{d}\mu \\
&= \sum_{\alpha=1}^{l}\int_\Omega (-1)^m\left((m-1)\langle\nabla x_\alpha, \nabla(\mathbb{L}_\phi^{m-2}u_i)\rangle\langle\nabla x_\alpha, \nabla u_i\rangle + \frac{1}{2}u_i\mathbb{L}_\phi^{m-1}u_i|\nabla x_\alpha|^2\right)\mathrm{d}\mu \\
&\le (m-1)l\Lambda_i^{\frac{m-1}{m}} + \int_\Omega \frac{l}{2}u_i(-\mathbb{L}_\phi)^{m-1}u_i\mathrm{d}\mu \\
&= \left(m-\frac{1}{2}\right)l\Lambda_i^{\frac{m-1}{m}}.
\end{aligned} \tag{3.13}$$

Also by Lemma 2.2, one has

$$\int_\Omega |\nabla u_i|^2 d\mu = \int_\Omega u_i(-\mathbb{L}_\phi u_i) d\mu \le \Lambda_i^{\frac{1}{m}}. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.4), we have

$$l\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \le \sum_{i=1}^k \delta m(2m-1)l(\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{\frac{m-1}{m}} + \sum_{i=1}^k \frac{(\Lambda_{k+1} - \Lambda_i)}{\delta} \Lambda_i^{\frac{1}{m}}. \tag{3.15}$$

Taking

$$\delta = \left\{ \frac{\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)\Lambda_i^{\frac{1}{m}}}{\sum_{i=1}^k m(2m-1)l(\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{\frac{m-1}{m}}} \right\}$$

in (3.15), we have

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \le \left(\frac{4m(2m-1)}{l} \right)^{\frac{1}{2}} \left(\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{\frac{m-1}{m}} \right)^{\frac{1}{2}} \left(\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{\frac{1}{m}} \right)^{\frac{1}{2}}.$$

This completes the proof of Theorem 1.1. □

Proof of Corollary 1.3. By Lemma 2.3, with $\{(\Lambda_{k+1} - \Lambda_i)\}_{i=1}^k$ decreasing, $\left\{\Lambda_i^{\frac{l-1}{l}}\right\}_{i=1}^k$ and $\left\{\Lambda_i^{\frac{1}{l}}\right\}_{i=1}^k$ increasing, we have

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{\frac{1}{l}} \le \frac{1}{k} \left(\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \right) \left(\sum_{i=1}^k \Lambda_i^{\frac{1}{l}} \right) \tag{3.16}$$

and

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{\frac{l-1}{l}} \le \frac{1}{k} \left(\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \right)^2 \left(\sum_{i=1}^k \Lambda_i^{\frac{l-1}{l}} \right). \tag{3.17}$$

Putting (3.16) and (3.17) into (1.2), we have

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \le \frac{4m(2m-1)}{k^2 l} \left(\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \right) \left(\sum_{i=1}^k \Lambda_i^{\frac{l-1}{l}} \right) \left(\sum_{i=1}^k \Lambda_i^{\frac{1}{l}} \right).$$

Solving this quadratic polynomial about Λ_{k+1}, we have (1.3) directly.

By Lemma 2.4, we have

$$\left(\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{\frac{l-1}{l}} \right) \left(\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{\frac{1}{l}} \right) \le \left(\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \right) \left(\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i \right).$$

Together with (1.2), one has

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \le \frac{4m(2m-1)}{l} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i.$$

Solving this quadratic polynomial about Λ_{k+1}, we have (1.4) directly. This finishes the proof of Corollary 1.3. □

Proof of Theorem 1.4. By Lemma 2.5, we know that M can split into product manifold $N \times \mathbb{R}^l$. Let $x_1, x_2, \ldots, x_l$ be the standard coordinate functions of $\mathbb{R}^l$. Define an $(l \times l)$-matrix B by

$$B := (b_{ij}),$$

where $b_{ij} = \int_\Omega x_i u_1 u_{j+1}$. Using the orthogonalization of Gram and Schmidt, one knows that there exist an upper triangle matrix $R = (R_{ij})$ and an orthogonal matrix $Q = (q_{ij})$ such that $R = QB$, i.e.,

$$R_{ij} = \sum_{k=1}^{l} q_{ik} b_{kj} = \int_\Omega \sum_{k=1}^{l} q_{ik} x_k u_1 u_{j+1} = 0, \quad \text{for } 1 \le j < i \le n.$$

Defining $y_i = \sum_{k=1}^{l} q_{ik} x_k$, we have $\int_\Omega y_i u_1 u_{j+1} = 0$, where $1 \le j < i \le l$. On the other hand, since $Q = (q_{ij})$ is an orthogonal matrix, we know that $y_1, \ldots, y_l$ are also the standard coordinate functions of $\mathbb{R}^l$ and $|\nabla y_i| = 1$, $\mathbb{L}_\phi y_i = 0$. Then taking $h_i = y_i$ in (2.2) and summing over i from 1 to l, we can obtain

$$\sum_{i=1}^{l} (\Lambda_{i+1} - \Lambda_1)^{\frac{1}{2}} \le \delta \sum_{i=1}^{l} \int_M y_i u_1((-\mathbb{L}_\phi)^m (y_i u_1) - \Lambda_1 y_i u_1) d\mu + \frac{1}{\delta} \sum_{i=1}^{l} \| \langle \nabla y_i, \nabla u_1 \rangle \|^2. \tag{3.18}$$

Using a similar calculation as in the proof of Theorem 1.1, we have

$$\begin{aligned} &\sum_{i=1}^{l} \int_\Omega y_i u_1((-\mathbb{L}_\phi)^m (y_i u_1) - \Lambda_1 y_i u_1) d\mu \\ &= \sum_{i=1}^{l} \int_\Omega \left((m-1) \langle \nabla y_i, \nabla(\mathbb{L}_\phi^{m-2} u_1) \rangle \langle \nabla y_i, \nabla u_1 \rangle + \frac{1}{2} u_1 \mathbb{L}_\phi^{m-1} u_1 |\nabla y_i|^2 \right) d\mu \\ &\le (m-1) l \Lambda_1^{\frac{m-1}{m}} + \int_\Omega \frac{l}{2} u_1 (-\mathbb{L}_\phi)^{m-1} u_1 d\mu \\ &= \left(m - \frac{1}{2} \right) l \Lambda_1^{\frac{m-1}{m}} \end{aligned} \tag{3.19}$$

and

$$\sum_{i=1}^{l} \| \langle \nabla y_i, \nabla u_1 \rangle \|^2 \le \int_\Omega |\nabla u_1|^2 d\mu = \int_\Omega u_1 - \mathbb{L}_\phi u_1 d\mu \le \Lambda_1^{\frac{1}{m}}. \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.18), one can obtain

$$\sum_{i=1}^{l} (\Lambda_{i+1} - \Lambda_1)^{\frac{1}{2}} \le \delta (2m-1) l \Lambda_1^{\frac{m-1}{m}} + \frac{1}{\delta} \Lambda_1^{\frac{1}{m}}.$$

Taking $\delta = \left(\Lambda_1^{\frac{1}{m}} / (2m-1) l \Lambda_1^{\frac{m-1}{m}} \right)^{1/2}$ in the above inequality, we have

$$\sum_{i=1}^{l} (\Lambda_{i+1} - \Lambda_1)^{\frac{1}{2}} \le 2 \left\{ (2m-1) l \Lambda_1^{\frac{m-1}{m}} \Lambda_1^{\frac{1}{m}} \right\}^{\frac{1}{2}} = 2 \sqrt{(2m-1) l}\, \Lambda_1^{\frac{1}{2}},$$

which is (1.5) exactly. This completes the proof of Theorem 1.4. □

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