Research Article

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Existence of random attractors and the upper semicontinuity for small random perturbations of 2D Navier-Stokes equations with linear damping

Abstract: The incompressible 2D stochastic Navier-Stokes equations with linear damping are considered in this paper. Based on some new calculation estimates, we obtain the existence of random attractor and the upper semicontinuity of the random attractors as $\varepsilon \to 0^+$ on the two-dimensional space.

1 Introduction

This paper considers the following stochastic Navier-Stokes equations with linear damping in a two-dimensional domain $D \subset \mathbb{R}^2$,

$$
\begin{cases}
\dfrac{\partial u_\varepsilon}{\partial t} - \nu \Delta u_\varepsilon + \gamma u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p = f(x) + \varepsilon \phi \dfrac{\mathrm{d}W(t)}{\mathrm{d}t}, & x \in D, \quad t > 0, \\
\nabla \cdot u_\varepsilon = 0, & x \in D, \quad t > 0, \\
u_\varepsilon(x, t)|_{\partial D} = 0, & x \in \partial D, \quad t > 0, \\
u_\varepsilon(x, 0) = u_0(x), & x \in D,
\end{cases}
\tag{1.1}
$$

where D is the abounded domain with boundary ∂D, $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})^T$ is the velocity, which depends on the disturbance parameter ε. p is the pressure, $\nu > 0$ is the kinematic viscosity, γu_ε is the linear damping which parallels the velocity u_ε, and the constant γ is positive. $u_0(x)$ is the initial velocity. The symbol $W(t)$ is a real valued two-sided Wiener process.

Equations (1.1) describe the movement of incompressible fluids in geophysical dynamics. The constant γ is Rayleigh's friction coefficient or Ekman suction/dissipation constant. The linear damping γu is a simulation of the bottom friction in a two-dimensional ocean model or a line in a two-dimensional atmospheric model. Especially, $\gamma = 0$, equations (1.1) are the classical stochastic 2D Navier-Stokes equation. For $\varepsilon = 0$, equation (1.1) become non-stochastic system. In the past 20 years, many extensive and in-depth studies emerge. Ilyin et al. [1] studied the limit of small viscosity coefficient $\nu \to 0^+$ and derived that the linear damping term γu_ε plays an important role in reducing the number of degrees of freedom in the two-dimensional model. The estimates for the number of determining modes and nodes are comparable to the

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sharp estimates for the fractal dimension of the global attractor. For details, we can refer to the literature [2]. Constantin and Ramos [3] derived that in $\mathbb{R}^2$ space the rate of dissipation of enstrophy vanishes. The stationary statistical solutions of the damped and driven Navier-Stokes equations converge to renormalized stationary statistical solutions of the damped and driven Euler equations, and the solutions obey the enstrophy balance [3]. On arbitrary open sets, Rosa [4] deduced the existence of the global attractor. Under the condition $f(x) \in (L^2(\mathbb{R}^2))$, Zhao and Zheng [5] proved the existence of global attractor and studied the deformations of the Navier-Stokes equation by limit behavior. Li [6] established the existence of uniform random attractor for stochastic Navier-Stokes equations in the space H.

The theory on the stochastic dynamical system is investigated in [7–11]. Our investigation of the Navier-Stokes equations with linear damping is inspired by [6,12,13]. We focus on the random attractor and its upper semicontinuity. By calculations, we derives the existence of random attractors and the upper semicontinuity for small random perturbations of Navier-Stokes equations with linear damping on the two-dimensional space, which enriches the theoretical results of the model.

This paper is arranged as follows. In Section 2, we recall some fundamental concepts and some lemmas which are used in the sequel. In Section 3, we conduce the existence of random attractors. In Section 4, we derive the upper semicontinuity for random attractors.

2 Preliminaries

This section introduces some basic related concepts for the random attractors, which were developed by Crauel and Flandoli [8,14].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ a family of measures which preserves transformations. For all $s, t \in \mathbb{R}$, the mapping $(t, \omega) \to \theta_t \omega$ is measurable, $\theta_0 = id$, $\theta_{t+s} = \theta_t \theta_s$. The θ_t with the probability space $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a measurable dynamical system. For the integrity of knowledge, it introduces the following concepts.

Definition 2.1. For any $\omega \in \Omega$, if the function $\varphi : \mathbb{R}^+ \times \Omega \times X \to X$ satisfies the following conditions:
 (i) $\varphi(0, \omega, \cdot)$ is the identity of X,
(ii) $\varphi(t + s, \omega, x) = \varphi(t, \omega, x)\varphi(s, \omega, x)$ for all $t, s \in \mathbb{R}^+$, $x \in X$ and P-almost every (a.e.) $\omega \in \Omega$,
(iii) $\varphi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}^+$,
the function $\varphi : \mathbb{R}^+ \times \Omega \times X \to X$ is called a continuous random dynamical system (RDS) on a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$.

Next is the concept of a random absorption set [14].

Definition 2.2. The symbol $\mathcal{D}$ is a collection of families of random subsets $\{B(\omega)\}_{\omega\in\Omega}$ of space X. If for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, φ on RDS in $\mathcal{D}$, and P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) \geq 0$, for any $t \geq t_B(\omega)$ such that

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \tag{2.2}$$

we called the random set $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ a random absorbing set for φ on RDS in $\mathcal{D}$.

The following three theorems were given and proved in [12].

Lemma 2.3. *Let $K(\omega)$ be a random compact set which absorbs every bounded non-random set $B \subset X$, the set*

$$\mathcal{A}(\omega) = \bigcup_{B \subset X} \Lambda(B, \omega) \tag{2.3}$$

is a random attractor for φ, where the union is taken over all $B \subset X$ bounded, and $\Lambda(B, \omega)$ is the omega-limit set of B and is given by

$$\Lambda(B, \omega) = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \varphi(t, \theta_{-t}\omega)B}. \tag{2.4}$$

When we add the random element which depends on a parameter $\varepsilon \in (0, 1]$, the random term $\varepsilon\phi\frac{dW(t)}{dt}$ perturbs the unperturbation system. According to the conventional theory, it derives an RDS which depends on the parameter $\varepsilon \in (0, 1]$

$$\varphi_\varepsilon : \mathbb{R}^+ \times \Omega \times X$$

such that for P-a.e. $\omega \in \Omega$ and all $t \in \mathbb{R}^+$,

$$\varphi_\varepsilon(t, \theta_{-t}\omega, x) \to S(t)x, \qquad \text{as } \varepsilon \to 0^+; \tag{C1}$$

uniformly on bounded sets of X.

Lemma 2.4. *For all $\varepsilon \in (0, 1]$, let $\mathcal{A}_\varepsilon(\omega)$ be a random attractor of the system* (1.1). *Assume that there exists a compact set K such that, P-a.s.*

$$\lim_{\varepsilon \to 0^+} \text{dist}(\mathcal{A}_\varepsilon(\omega), K) = 0. \tag{C2}$$

Then,

$$\lim_{\varepsilon \to 0^+} \text{dist}(\mathcal{A}_\varepsilon(\omega), \mathcal{A}) = 0,$$

with probability one.

Conditions ($C1$) and ($C2$) are necessary and sufficient for the upper semicontinuity property. Condition ($C2$) is a similar property for the random absorbing sets, which are used to derive the random attractors (see [12] for more details). Here, we just list the result as follows.

Lemma 2.5. *Let $K_\varepsilon(\omega)$ be a family of random compact absorbing sets which are uniformly in disturbance parameter ε, that is, for P-a.e. $\omega \in \Omega$ and all $B \subset X$, there exists $t_B(\omega)$ which is independent in disturbance parameter ε such that, for any $t > t_B(\omega)$ and $\varepsilon \in (0, 1]$,*

$$\varphi_\varepsilon(t, \theta_{-t}\omega)B \subset K_\varepsilon(\omega),$$

and there exists a compact set K such that P-a.s.

$$\lim_{\varepsilon \to 0^+} dist(K_\varepsilon(\omega), K) = 0. \tag{C2$'$}$$

Then, for each $\varepsilon \in (0, 1]$, there exists a random attractor $\mathcal{A}_\varepsilon(\omega)$, and ($C2$) holds.

For convenience, it introduces the function spaces and inequalities. Denote by

$$(u, v) = \int_D (u_1v_1 + u_2v_2)dx, \quad u = (u_1, u_2) \in (L^2(D)), \quad v = (v_1, v_2) \in (L^2(D))$$

and

$$((u, v)) = \int_D (\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2)dx, \quad u, v \in (H_0^1(D))^2$$

and the associated norms $|u| = (u, u)^{\frac{1}{2}}$, $\|u\| = ((u, u))^{\frac{1}{2}}$.

Let λ_1 be the first eigenvalue of the operator $A = -\mathcal{P}\triangle$, where $\mathcal{P}$ is the orthogonal projection of $(L^2(D))^2$. For all $u, v, w, z \in V$, according to Sobolev's relevant knowledge, it has the following inequalities:

$$\lambda_1|u|^2 \le \|u\|^2, \tag{2.5}$$

$$|(B(u, v), w)| = |b(u, v, w)| \le c_0 |u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}|v|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|w\|, \tag{2.6}$$

and

$$(B(u, z), z) = 0. \tag{2.7}$$

3 Existence of random attractors

We will derive computational inequalities and use Lemma 2.5 to study the random attractors of system (1.1) in this section. Assume $f \in H$, $\phi \in D(A)$ and set

$$v_\varepsilon(t, w) = u_\varepsilon(t, w) - \varepsilon\phi z(t, w),$$

where

$$z(t) = \int_{-\infty}^{t} \exp(-\alpha(t - \tau))\mathrm{d}W(\tau).$$

The constant in the above inequality $\alpha > 0$ is large enough and fixed.

The $z = (z_1(t, \omega), z_2(t, \omega), \ldots, z_n(t, \omega))$ mentioned in the above formula is the Ornstein-Uhlenbeck process

$$z = \sum_{j=1}^{m} \phi_j z_j,$$

with $z_j = \int_{-\infty}^{t} e^{-\alpha(t-\tau)}\mathrm{d}w_j(\tau)$ and the real valued two-sided Wiener process $W(t) = (w_1(t, \omega), w_2(t, \omega), \ldots, w_n(t, \omega))$. It derives that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \exp(-\alpha(t - t))\frac{\mathrm{d}W(t)}{\mathrm{d}t} - \alpha \int_{-\infty}^{t} \exp(-\alpha(t - \tau))\mathrm{d}W(\tau) = \frac{\mathrm{d}W(t)}{\mathrm{d}t} - \alpha z. \tag{3.8}$$

As we know, $z(t)$ is a stationary process and its trajectories are P-a.s. continuous. Introducing the projection operator $\mathcal{P}$ and the linear operator A, the first formula of equation (1.1) becomes the following form:

$$\frac{\mathrm{d}v_\varepsilon}{\mathrm{d}t} + \nu Av_\varepsilon + \gamma v_\varepsilon + B(v_\varepsilon + \varepsilon\phi z, v_\varepsilon + \varepsilon\phi z) = f + (\alpha - \gamma)\varepsilon\phi z - \varepsilon\nu A(\phi z). \tag{3.9}$$

The existence and uniqueness of solutions for (3.9) are derived by using the method which is similar to [11,10]. We omit it here. Define an RDS $(\varphi(t, \tau; \omega))_{t \geq \tau, \omega \in \Omega}$ associated with (3.9).

$$\varphi(t, \tau; \omega)u_\tau = u_\varepsilon(t, \omega) = v_\varepsilon(t, \omega) + \varepsilon\phi z(t, \omega).$$

Taking the scalar product in (3.9), it derives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_\varepsilon| + \nu\|v_\varepsilon\|^2 + (b(v_\varepsilon + \varepsilon\phi z, v_\varepsilon + \varepsilon\phi z), v_\varepsilon) = (f, v_\varepsilon) + (\alpha - \gamma)(\varepsilon\phi z, v_\varepsilon) - ((\varepsilon\nu\phi z, v_\varepsilon)), \tag{3.10}$$

$$|(b(v_\varepsilon + \varepsilon\phi z, v_\varepsilon + \varepsilon\phi z), v_\varepsilon)| = |(b(v_\varepsilon + \varepsilon\phi z, \varepsilon\phi z), v_\varepsilon + \varepsilon\phi z)|$$

$$= \left|\left(b\left(v_\varepsilon + \varepsilon\phi z, \varepsilon\phi \sum_{j=1}^{m} \phi_j z_j\right), v_\varepsilon + \varepsilon\phi z\right)\right|$$

$$\leq c_1\varepsilon\left(\sum_{j=1}^{m} |z_j|\right)|v_\varepsilon + \varepsilon\phi z|^2 \tag{3.11}$$

$$\leq c_1\varepsilon\left(\sum_{j=1}^{m} |z_j|\right)(|v_\varepsilon|^2 + \varepsilon^2 |z|^2)$$

$$\leq 2c_1\left(\sum_{j=1}^{m} |z_j|\right)(|v_\varepsilon|^2 + |z|^2).$$

Using the Young inequality with ε, it follows

$$(f, v_\varepsilon) \le \frac{\nu\lambda_1}{8}|v_\varepsilon|^2 + \frac{2}{\nu\lambda_1}|f|^2, \tag{3.12}$$

$$(\alpha - \gamma)(\varepsilon\phi z, v_\varepsilon) \le \frac{\nu\lambda_1}{8}|v_\varepsilon|^2 + \frac{2(\alpha - \gamma)^2\varepsilon^2}{\nu\lambda_1}|z|^2, \tag{3.13}$$

$$\nu((\varepsilon\phi z, v_\varepsilon)) \le \frac{\nu}{2}\|v_\varepsilon\|^2 + \frac{\nu}{2}\|z\|^2. \tag{3.14}$$

Substitute (3.11)–(3.14) into (3.10) and deduce

$$\frac{1}{2}\frac{d}{dt}|v_\varepsilon|^2 + \gamma|v_\varepsilon|^2 + \nu\|v_\varepsilon\|^2 \le 2c_1\left(\sum_{j=1}^{m}|v_\varepsilon|^2\right) + \frac{\nu\lambda_1}{4}|v_\varepsilon|^2 + \frac{\nu}{2}\|v_\varepsilon\|^2 + g, \tag{3.15}$$

where

$$g = 2c_1\left(\sum_{j=1}^{m}|z_j|\right)|z|^2 + \frac{2}{\nu\lambda_1}|f|^2 + \frac{2(\alpha - \gamma)^2}{\nu\lambda_1} + \frac{\nu}{2}\|z\|^2 \tag{3.16}$$

and λ_1 is the first eigenvalue of the operator A.

For $\forall v_\varepsilon \in V$, it satisfies

$$\|v_\varepsilon\| \ge \lambda_1|v_\varepsilon|.$$

Noting

$$\nu\|v_\varepsilon\|^2 \ge \frac{\nu}{4}\|v_\varepsilon\|^2 + \frac{3\nu}{4}\|v_\varepsilon\|^2 \ge \frac{\nu}{4}\|v_\varepsilon\|^2 + \frac{3\lambda_1\nu}{4}|v_\varepsilon|^2$$

and substituting the above inequality into (3.15), it derives

$$\frac{d}{dt}|v_\varepsilon|^2 + \frac{\nu}{4}\|v_\varepsilon\|^2 + \left(2\gamma + \frac{\nu\lambda_1}{4} - 4c_1\sum_{j=1}^{m}|z_j|\right)|v_\varepsilon|^2 \le 2g. \tag{3.17}$$

By Gronwall's lemma, for $s \le -1$, and $t \in [-1, 0]$,

$$
\begin{aligned}
|v_\varepsilon(t)|^2 &\le |v_\varepsilon(s)|^2 \exp\left(-\int_s^t\left(2\gamma + \frac{\nu\lambda_1}{4} - 4c_1\sum_{j=1}^{m}|z_j(\sigma)|\right)d\sigma\right) \\
&\quad + 2\int_s^t g(\sigma)\exp\left(-\int_s^t\left(2\gamma + \frac{\nu\lambda_1}{4} - 4c_1\sum_{j=1}^{m}|z_j(\tau)|\right)d\tau\right)d\sigma \\
&\le c_2\,|v_\varepsilon(s)|^2\exp\left(s\left(2\gamma + \frac{\nu\lambda_1}{4} + \frac{4c_1}{s}\int_s^0\sum_{j=1}^{m}|z_j(\sigma)|\right)d\sigma\right) \\
&\quad + 2c_2\int_s^0 g(\sigma)\exp\left(-\int_\sigma^0\left(2\gamma + \frac{\nu\lambda_1}{4} - 4c_1\sum_{j=1}^{m}|z_j(\tau)|\right)d\tau\right)d\sigma,
\end{aligned}
\tag{3.18}
$$

where

$$c_2 = \exp\left(2\gamma + \frac{\nu\lambda_1}{4}\right).$$

By the ergodic theorem, the stationary and ergodic process $\sum_{j=1}^{m}|z_j|$ satisfies

$$-\frac{1}{s}\int_{s}^{0}\sum_{j=1}^{m}|z_j(\sigma)|\mathrm{d}\sigma \to E\left(\sum_{j=1}^{m}|z_j(0)|\right), \tag{3.19}$$

when $s \to -\infty$. Then, there exists $s_0(\omega)$ such that for any $s \le s_0(\omega)$,

$$-\frac{1}{s}\int_{s}^{0}\sum_{j=1}^{m}|z_j(\sigma)|\mathrm{d}\sigma \le 2E\left(\sum_{j=1}^{m}|z_j(0)|\right) \tag{3.20}$$

and

$$\exp\left(s\left(2\gamma + \frac{\nu\lambda_1}{4} + \frac{4c_1}{s}\int_{s}^{0}\sum_{j=1}^{m}|z_j(\sigma)|\mathrm{d}\sigma\right)\right) \le \exp\left(s\left(2\gamma + \frac{\nu\lambda_1}{4} + 8c_1E\left(\sum_{j=1}^{m}|z_j(0)|\right)\right)\right). \tag{3.21}$$

Note that

$$E\left(\sum_{j=1}^{m}|z_j(0)|\right) \le \sum_{j=1}^{m}E(|z_j(0)|^2)^{\frac{1}{2}} = \frac{m}{(2\alpha)^{\frac{1}{2}}}. \tag{3.22}$$

Taking the constant α large enough, it derives

$$\sum_{j=1}^{m}E(|z_j(0)|^2)^{\frac{1}{2}} \le \frac{\nu\lambda_1}{64c_1}. \tag{3.23}$$

Considering $s \to -\infty$, the last part of the index in the first term which comes from (3.18) decays to 0. Now, we estimate the second term of (3.18).

$$z_j = z_j(0) - \alpha\int_{t}^{0}z_j(s)\mathrm{d}s + w_j(t)$$

which shows that $\frac{|z_j(t)|}{t}$ is bounded at $-\infty$. The term $g(t)$ grows at most polynomially. Since $g(t)$ is multiplied by function which decays exponentially by (3.19) and (3.23) the integral converges. Thus for $s < s_0(\omega)$ and $t \in [-1, 0]$,

$$\begin{aligned}
|v_\varepsilon(t)|^2 &\le c_2|v_\varepsilon(s)|^2\exp\left(s\left(2\gamma + \frac{\nu\lambda_1}{8}\right)\right) \\
&\quad + 2c_2\int_{-\infty}^{0}g(\sigma)\exp\left(\sigma\left(2\gamma + \frac{\nu\lambda_1}{4} + \frac{2c_1}{\sigma}\int_{0}^{\sigma}\sum_{j=1}^{m}|z_j(\tau)|\mathrm{d}\tau\right)\right)\mathrm{d}\sigma \\
&\le 2c_2|u_\varepsilon(s)|^2\exp\left(s\left(2\gamma + \frac{\nu\lambda_1}{8}\right)\right) + 2c_2|z(s)|^2\exp\left(s\left(2\gamma + \frac{\nu\lambda_1}{8}\right)\right) \\
&\quad + 2c_2\int_{-\infty}^{0}g(\sigma)\exp\left(\sigma\left(2\gamma + \frac{\nu\lambda_1}{4} + \frac{2c_1}{\sigma}\int_{0}^{\sigma}\sum_{j=1}^{m}|z_j(\tau)|\mathrm{d}\tau\right)\right)\mathrm{d}\sigma,
\end{aligned} \tag{3.24}$$

so, there exist $s_1(\omega, B)$ which depends only on B and ω such that for $s < s_1(\omega, B)$, $t \in [-1, 0]$ and deduce

$$\begin{aligned}
|v_\varepsilon(t)|^2 &\le r_0(\omega) \\
&= 2c_2\int_{-\infty}^{0}g(\sigma)\exp\left(\sigma\left(2\gamma + \frac{\nu\lambda_1}{4} + \frac{2c_1}{\sigma}\int_{0}^{\sigma}\sum_{j=1}^{m}|z_j(\tau)|\mathrm{d}\tau\right)\right)\mathrm{d}\sigma \\
&\quad + 2c_2\sup_{s\in(-\infty,-1]}\left(|z(s)|^2\exp\left(s\frac{\nu\lambda_1}{8}\right)\right) + 1.
\end{aligned} \tag{3.25}$$

Moreover, we can integrate (3.10) from -1 to 0 and deduce

$$\int_{-1}^{0} \|v_\varepsilon(s)\|^2 \le r_1(\omega) = \frac{8}{\nu} \int_{-1}^{0} g(\sigma) d\sigma + \frac{8c_1}{\nu} \left(\int_{-1}^{0} \sum_{j=1}^{m} |z_j(\sigma)| d\sigma \right) r_0(\omega). \tag{3.26}$$

Now, we estimate the $\|v_\varepsilon\|^2$. Taking the scalar product of (3.9) by v_ε in V and using the following inequality:

$$|b(u, u)| \le c_3 |u|^{\frac{1}{2}} |Au|^{\frac{1}{2}} \|u\|,$$

it has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} &\|v_\varepsilon\|^2 + \gamma \|v_\varepsilon\|^2 + \nu |Av_\varepsilon|^2 \\ &= ((f, v_\varepsilon)) + (\alpha - \gamma)\varepsilon((\phi z, v_\varepsilon)) - \varepsilon\nu(A\phi z, Av_\varepsilon) - (b(v_\varepsilon + z, v_\varepsilon + z), Av_\varepsilon) \\ &\le \frac{4}{\nu} |f|^2 + \frac{4(\alpha - \gamma)^2 \varepsilon^2}{\nu} |z|^2 + 4\nu\varepsilon |A\phi z|^2 + \frac{4c_3^2}{\nu} \varepsilon |v_\varepsilon + \varepsilon\phi z| |A\phi z| \|v_\varepsilon + \varepsilon\phi z\|^2 \\ &\quad + \frac{32c_3^4}{\nu} \varepsilon |v_\varepsilon + \varepsilon\phi z|^2 \|\phi z\|^4 + \frac{32c_3^4}{\nu^3} \varepsilon |v_\varepsilon + \varepsilon\phi z|^2 \|v_\varepsilon\|^2. \end{aligned} \tag{3.27}$$

For any $t \in [-1, 0]$, it deduces that

$$\|v_\varepsilon(0)\|^2 \le \|v_\varepsilon(s)\|^2 e^{\int_t^0 N(\sigma) d\sigma} + \int_t^0 M(\sigma) e^{\int_\sigma^0 N(\tau) d\tau d\sigma} \le (\|v_\varepsilon(s)\|^2 + \int_{-1}^0 M(\sigma) d\sigma) e^{\int_{-1}^0 N(\sigma) d\sigma}, \tag{3.28}$$

with

$$M(t) = \frac{4}{\nu} |f|^2 + \frac{4(\alpha - \gamma)^2 \varepsilon^2}{\nu} |z|^2 + 4\nu\varepsilon |A\phi z|^2 + \frac{4c_3^2}{\nu} \varepsilon |v_\varepsilon + \varepsilon\phi z| |A\phi z| \|v_\varepsilon + \varepsilon\phi z\|^2 + \frac{32c_3^4}{\nu} \varepsilon |v_\varepsilon + \varepsilon\phi z|^2 \|\phi z\|^4,$$

$$N(t) = \frac{32c_3^4}{\nu^3} \varepsilon |v_\varepsilon + \varepsilon\phi z|^2 \|v_\varepsilon\|^2.$$

Integration (3.28) with respect to t on $[-1, 0]$,

$$\|v_\varepsilon(0)\|^2 \le \left(\int_{-1}^0 \|v(t)\|^2 dt + \int_{-1}^0 M(\sigma) d\sigma \right) e^{\int_{-1}^0 N(\sigma) d\sigma}. \tag{3.29}$$

Finally, it can deduce that there exists $r_\varepsilon(\omega)$ such that, if $t_0 < t(\omega)$,

$$\|v_\varepsilon(0)\|^2 \le r_\varepsilon^2. \tag{3.30}$$

Taking $K_\varepsilon(\omega)$ as the ball in space V of radius $r_\varepsilon(\omega) + \varepsilon\|\phi z(0)\|$, it has a compact absorbing set which is uniformly in ε in H for φ. It is clear that

$$\lim_{\varepsilon \to 0^+} (r_\varepsilon(\omega) + \varepsilon\|\phi z(0)\|) \le r_d. \tag{3.31}$$

r_d is independent of $\omega \in \Omega$. So, the assumption $(C2')$ of Lemma 2.5 holds. Then, for ε, there is a random attractor $\mathcal{A}_\varepsilon(\omega)$. According to Lemma 2.5, the equation $(C2)$ is also guaranteed.

From the discussion above, the following theorem on the existence of random attractors holds.

Theorem 3.1. *Assume $f \in H$, $\phi \in D(A)$, $\varepsilon \in (0, 1]$ the system* (1.1) *has random attractors.*

4 Upper semicontinuity of attractors

Section 3 derives that ($C2$) which comes from Lemma 2.4 holds. In order to apply Lemma 2.4 to derive the upper semicontinuity of the random attractor, we just need to prove that $C1$ is established. By calculations, it has the following Theorem 4.1.

Theorem 4.1. *Let $u_\varepsilon(0, \omega; -t_0, u_0)$ be the solution of system* (1.1) *and $u(t_0; u_0)$ be the solution of the unperturbed problem ($\varepsilon = 0$). When the perturbed parameter $\varepsilon \to 0$, $u_\varepsilon(0, \omega; -t_0, u_0)$ converges in space H P-a.s. to $u(t_0; u_0)$, uniformly on bounded sets of initial conditions, that is, for P-a.e. $\omega \in \Omega$, $t_0 \in R^+$ and $\forall\, G \subset H$ bounded*

$$\lim_{\varepsilon \to 0^+} |u_\varepsilon(0, \omega; -t_0, u_0) - u(t_0; u_0)| = 0, \quad u_0 \in G. \tag{4.32}$$

Proof. The proof of the theorem is similar to Section 3.3 in [12]. In order to the completeness of the description, we describe as follows.

Set $v_\varepsilon(t, \omega) = u_\varepsilon(t, \omega) - u(t)$ as the difference between the solutions of the perturbed and the unperturbed equation with the same initial condition u_0 at $-t_0$. It is clear that v_ε satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} v_\varepsilon + \gamma v_\varepsilon + A v_\varepsilon + B(v_\varepsilon + u, v_\varepsilon + u) - B(u, u) = \varepsilon \phi \frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad v_\varepsilon(-t_0) = 0.$$

Considering the operator B is bilinear, it derives

$$\frac{\mathrm{d}}{\mathrm{d}t} v_\varepsilon + \gamma v_\varepsilon + A v_\varepsilon + B(v_\varepsilon, v_\varepsilon) + B(v_\varepsilon, u) + B(u, v_\varepsilon) = \varepsilon \phi \frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad v_\varepsilon(-t_0) = 0.$$

Let

$$z_\varepsilon = v_\varepsilon - \varepsilon \phi W(t).$$

It obtains

$$\frac{\mathrm{d}z_\varepsilon}{\mathrm{d}t} + \gamma z_\varepsilon + \varepsilon \gamma \phi W(t) + \varepsilon v A(\phi W(t)) + B(z_\varepsilon + \varepsilon \phi W(t), z_\varepsilon + \varepsilon \phi W(t))$$
$$+ B(z_\varepsilon + \varepsilon \phi W(t), u) + B(u, z_\varepsilon + \varepsilon \phi W(t)) = 0.$$

Multiplying (4.33) by z_ε, and using the bilinearity of operater B, it derives

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z_\varepsilon|^2 &+ \gamma\,|z_\varepsilon|^2 + \gamma\varepsilon(\phi W(t), z_\varepsilon) + v(A z_\varepsilon, z_\varepsilon) + v(\varepsilon W(t) A\phi, z_\varepsilon) \\
&+ (B(z_\varepsilon, \varepsilon\phi W(t)), z_\varepsilon) + (B(\varepsilon\phi W(t), \varepsilon\phi W(t)), z_\varepsilon) + (B(z_\varepsilon, u), z_\varepsilon) \\
&+ (B(\varepsilon\phi W(t), u), z_\varepsilon) + (B(u, \varepsilon\phi W(t)), z_\varepsilon) = 0.
\end{aligned} \tag{4.33}$$

By Young's inequality, we have the following estimates:

$$\varepsilon\gamma(\phi W(t), z_\varepsilon) \le \frac{\varepsilon^2}{4}|\phi|^2 |W(t)|^2 + \gamma^2 |z_\varepsilon|^2, \tag{4.34}$$

$$(\varepsilon v W(t) A\phi, z_\varepsilon) \le \varepsilon |W(t)| |A\phi| |z_\varepsilon| \le \frac{\varepsilon^2 v^2 W(t)^2 |A\phi|^2}{2} + \frac{|z_\varepsilon|^2}{2}, \tag{4.35}$$

$$(B(z, \varepsilon\phi W(t)), z_\varepsilon) \le \varepsilon|W(t)| |(B(z_\varepsilon, \phi), z_\varepsilon)| \le c_1\varepsilon|W(t)| |z_\varepsilon|^2, \tag{4.36}$$

$$\begin{aligned}
(B(\varepsilon\phi W(t), \varepsilon\phi W(t)), z_\varepsilon) &\le \varepsilon|W(t)| |(B(z_\varepsilon, \phi), z_\varepsilon)| \\
&\le c_3\varepsilon^2 W(t)^2 |\phi|^{\frac{1}{2}}\|\phi\||A\phi|^{\frac{1}{2}}|z_\varepsilon| \\
&\le \frac{c_3^2}{2}\varepsilon^4 W(t)^4 |\phi|\|\phi\|^2|A\phi| + \frac{|z_\varepsilon|^2}{2},
\end{aligned} \tag{4.37}$$

$$(B(z_\varepsilon, u), z_\varepsilon) \le c_4 |z_\varepsilon| \|u\| \|z_\varepsilon\| \le \nu \|z_\varepsilon\|^2 + c_5 \|u\|^2 |z_\varepsilon|^2, \tag{4.38}$$

$$\begin{aligned}
(B(\varepsilon \phi W(t), u), z_\varepsilon) &\le \varepsilon |W(t)| |(B(\phi, u), z_\varepsilon)| \\
&\le \varepsilon |W(t)| |\phi|^{\frac{1}{2}} |A\phi|^{\frac{1}{2}} \|u\| |z_\varepsilon| \\
&\le \frac{1}{2} \varepsilon^2 W(t)^2 |\phi| |A\phi| \|u\|^2 + \frac{1}{2} |z_\varepsilon|^2,
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
(B(u, \varepsilon \phi W(t)), z_\varepsilon) &\le \varepsilon |W(t)| |(B(u, \phi), z_\varepsilon)| \\
&\le \varepsilon |W(t)| |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} |\phi|^{\frac{1}{2}} |A\phi|^{\frac{1}{2}} |z_\varepsilon| \\
&\le \varepsilon |W(t)|^2 |u| \|u\| \|\phi\| |A\phi| + \frac{1}{2} |z_\varepsilon|^2.
\end{aligned} \tag{4.40}$$

Taking the estimates (4.34)–(4.40) into (4.33), it has

$$\frac{\mathrm{d}}{\mathrm{d}t} |z_\varepsilon|^2 \le g(t) + h(t) |z_\varepsilon|^2, \tag{4.41}$$

where

$$g(t) = K\varepsilon (W(t)^2) |A\phi|^2 + W(t)^4 |\phi| \|\phi\|^2 |A\phi| + W(t)^2 |\phi| |A\phi| \|u\|^2 + W(t)^2 |u| \|u\| \|\phi\| |A\phi| + W(t)^2 |\phi|^2 \tag{4.42}$$

and

$$h(t) = 2 + \gamma^2 + c_1 \varepsilon |W(t)| + c_5 \|u\|^2. \tag{4.43}$$

Using Gronwall's lemma,

$$|z_\varepsilon|^2 \le g(t) + \int_{-t_0}^{t} g(s) h(s) \exp\left(\int_{s}^{t} k(\tau) \mathrm{d}\tau \right) \mathrm{d}s, \tag{4.44}$$

and so $|z_\varepsilon|^2 \to$ as $\varepsilon \to 0^+$ for all $t \ge -t_0$. Then

$$\lim_{\varepsilon \to 0^+} |v_\varepsilon|^2 \le \lim_{\varepsilon \to 0^+} 2(|z_\varepsilon|^2 + \varepsilon^2 |\phi|^2 W(t)^2) = 0. \tag{4.45}$$

Taking $t = 0$, it completes the proof. □

From the above analysis, we obtain the upper semicontinuity of the random attractors.

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