Research Article

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Random attractors for stochastic plate equations with memory in unbounded domains

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Abstract: In this paper, we investigate the dynamics of stochastic plate equations with memory in unbounded domains. More specifically, we obtain the uniform time estimates for solutions of the problem. Based on the estimates above, we prove the existence and uniqueness of random attractors in unbounded domains.

Keywords: memory, pullback attractors, plate equation, unbounded domains, multiplicative noise

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the standard probability space, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathcal{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. There is a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathcal{P})$, which is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega, \ t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is an ergodic parametric dynamical system.

Considering the following non-autonomous stochastic plate equation with memory and multiplicative noise in unbounded domain $\mathbb{R}^n$:

$$u_{tt} + \Delta^2 u + h(u_t) + \int_0^\infty \mu(s)\Delta^2(u(t) - u(t-s))\mathrm{d}s + \lambda u + f(x, u) = g(x, t) + \varepsilon u \circ \frac{\mathrm{d}w}{\mathrm{d}t} \tag{1.1}$$

with the initial value conditions

$$u(x, \tau) = u_0(x), \quad u_t(x, \tau) = u_1(x), \tag{1.2}$$

where $x \in \mathbb{R}^n$, $t > \tau$ with $\tau \in \mathbb{R}$, $\lambda > 0$ and ε are constants, μ is the memory kernel, $h(u_t)$ is a nonlinear damping term, f is a given interaction term, g is a given function satisfying $g \in L^2_{\mathrm{loc}}(\mathbb{R}, H^1(\mathbb{R}^n))$, and w is a two-sided real-valued Wiener process on a probability space. The stochastic equation (1.1) is understood in the sense of Stratonovich's integration.

Many studies have been carried out regarding the dynamics of a variety of systems related to equation (1.1). For example, if the random term is vanished, $\mu = 0$ and $g(x, t) = g(x)$, then (1.1) changes into a deterministic autonomous plate equation. The existence and uniqueness of the global attractor of the corresponding

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dynamical system were investigated in [1–11]; besides, the uniform attractor of the dynamical system generated by the non-autonomous plate equation was obtained in [12].

For the stochastic case, if $\mu = 0$ and the forcing term $g(x, t) = g(x)$, then the existence of a random attractor of (1.1)–(1.2) in bounded domain has been established in [13–16]; if $\mu \neq 0$, the existence of random attractors for plate equations with memory and additive noise in bounded domain was considered in [17,18]. Recently, in the unbounded domain, the authors investigated the asymptotic behavior for stochastic plate equation driven by different noises (see [19–22] for details).

However, studies on the stochastic plate equation with memory still lack. Motivated by the literature above, we investigate the asymptotic behaviors for stochastic plate equation driven by multiplicative noise in unbounded domains in this paper. More precisely, compared to [19–22], we have the memory effects. The main features of our work are summarized as follows.

(i) Note that Sobolev embeddings are no longer compact in unbounded domains. It leads to a major difficulty for us to prove the asymptotic compactness of solutions by standard method. To overcome this difficulty, we refer to [23,24] which provide uniform estimates on the far-field values of solutions.

(ii) There is no applicable compact embedding property in the “history” space. In this case, we solve it with the help of a useful result in [25]. For our purpose, we introduce a new variable and an extended Hilbert space. Moreover, we still need uniform estimates of solutions in higher regular space due to the memory term.

(iii) The influence of multiplicative noise and additive noise on the solutions of plate equations is quite different. When dealing with random attractors of a stochastic equation, we often transform the stochastic equation into a deterministic one with random parameters. If the equation is driven by additive noise, then the transformation does not change the structure of the original equation. Therefore, one can obtain all necessary uniform estimates of solutions, and then get the existence of random attractors for additive noise with any intensity (see, e.g., [20]). However, if the plate equation is driven by multiplicative noise, then there are several additional terms appearing after the equation is transformed (see $(3.12)_2$ in Section 3). These additional terms involve the unknown variable u and have great effect on the way to derive uniform estimates of solutions. This is the reason why, in this paper, we only study the existence of random attractors for the stochastic equation (1.1) when the intensity ε of the multiplicative noise is sufficiently small.

(iv) In this manuscript, $\varphi_0 \in E$, so we cannot obtain the higher order estimate by using the classical energy method. To this end, we split the system into a linear system and a zero initial data nonlinear system. The energy of the linear system decays to 0, while the energy of nonlinear system is bounded in higher regular space. Then we can use this property to deduce the compactness.

In the next section, we recall some notations and results regarding random attractors for non-autonomous stochastic equations. We then define a continuous cocycle for equation (1.1) and derive necessary uniform estimates in Sections 3 and 4, respectively. Finally, we prove the existence of random attractors in Section 5.

Throughout the paper, the letters c and c_i $(i = 1, 2, \dots)$ are generic positive constants which do not depend on ε.

2 Preliminaries

In this section, we present some definitions and known results regarding pullback attractors of non-autonomous random dynamical systems from [26,27].

Definition 2.1. Let $\theta : \mathbb{R} \times \Omega \to \Omega$ be a $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable mapping. We say $(\Omega, \mathcal{F}, \mathcal{P}, \theta)$ is a parametric dynamical system if $\theta(0, \cdot)$ is the identity on Ω, $\theta(s + t, \cdot) = \theta(t, \cdot) \circ \theta(s, \cdot)$ for all $t, s \in \mathbb{R}$, and $P\theta(t, \cdot) = P$ for all $t \in \mathbb{R}$.

Definition 2.2. Let $K : \mathbb{R} \times \Omega \to 2^X$ be a set-valued mapping with closed nonempty images. We say K is measurable with respect to $\mathcal{F}$ in Ω if the mapping $\omega \in \Omega \to \mathrm{d}(x, K(\tau, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$.

Definition 2.3. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $t, s \in \mathbb{R}^+$, the following conditions (1)–(4) are satisfied:
(1) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;
(2) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;
(3) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;
(4) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Hereafter, we assume Φ is a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$, and $\mathcal{D}$ is the collection of some families of nonempty bounded subsets of X parameterized by $\tau \in \mathbb{R}$ and $\omega \in \Omega$:

$$\mathcal{D} = \{D = \{D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \varnothing, \tau \in \mathbb{R}, \omega \in \Omega\}\}.$$

Definition 2.4. Let $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of nonempty subsets of X. For every $\tau \in \mathbb{R}$, $\omega \in \Omega$, let

$$\Omega(B, \tau, \omega) = \bigcap_{r\geq 0} \overline{\bigcup_{t\geq r} \Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega))}.$$

Then the family $\{\Omega(B, \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is called the Ω-limit set of B and is denoted by $\Omega(B)$.

Definition 2.5. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then K is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and for every $B \in \mathcal{D}$, there exists $T = T(B, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \quad \text{for all } t \geq T.$$

If, in addition, $K(\tau, \omega)$ is closed in X and is measurable in ω with respect to $\mathcal{F}$, then K is called a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 2.6. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\left\{\Phi\left(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n\right)\right\}_{n=1}^{\infty} \quad \text{has a convergent subsequence in } X$$

whenever $t_n \to \infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n}\omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.7. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions (1)–(3) are fulfilled: for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,
(1) $\mathcal{A}(\tau, \omega)$ is compact in X and is measurable in ω with respect to $\mathcal{F}$;
(2) $\mathcal{A}$ is invariant, that is,

$$\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t\omega);$$

(3) For every $B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t\to\infty} d(\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau, \omega)) = 0.$$

Proposition 2.8. *Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty subsets of X, and Φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$. If Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$, then Φ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ which is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\mathcal{A}(\tau, \omega) = \Omega(K, \tau, \omega) = \bigcup_{D\in\mathcal{D}} \Omega(B, \tau, \omega).$$

3 Cocycles for stochastic plate equation

In this section, we outline some basic settings about (1.1)–(1.2) and show that it generates a continuous cocycle in $E = H^2 \times L^2 \times \mathfrak{R}_{\mu,2}$.

Let $-\Delta$ denote the Laplace operator in $\mathbb{R}^n$, $A = \Delta^2$ and $D(A) = H^4(\mathbb{R}^n)$. We can define the powers A^r of A for $r \in \mathbb{R}$. The space $V_r = D\left(A^{\frac{r}{4}}\right)$ is a Hilbert space with the following inner product and norm:

$$(u, v)_r = \left(A^{\frac{r}{4}}u, A^{\frac{r}{4}}v\right), \quad \|\cdot\|_r = \|A^{\frac{r}{4}}u\|, \quad \forall u, v \in V_r.$$

In particular, $V_0 = L^2(\mathbb{R}^n)$, $V_1 = H^1(\mathbb{R}^n)$, $V_2 = H^2(\mathbb{R}^n)$.

For brevity, the notation $(\cdot,\cdot)$ for L^2-inner product will also be used for the notation of duality pairing between dual spaces.

Following Dafermos [28], we introduce a Hilbert "history" space $\mathfrak{R}_{\mu,2} = L^2_\mu(\mathbb{R}^+, V_2)$ with the inner product

$$(\eta_1, \eta_2)_{\mu,2} = \int_0^\infty \mu(s)(\Delta\eta_1(s), \Delta\eta_2(s))\mathrm{d}s, \quad \forall \eta_1, \eta_2 \in \mathfrak{R}_{\mu,2},$$

and new variables

$$\eta(x, t, s) = u(x, t) - u(x, t - s), \quad (x, s) \in \mathbb{R}^n \times \mathbb{R}^+, \ t \geq 0.$$

By differentiation we have

$$\eta_t(x, t, s) = -\eta_s(x, t, s) + u_t(x, t), \quad (x, s) \in \mathbb{R}^n \times \mathbb{R}^+, \ t \geq 0.$$

Denote $E = H^2 \times L^2 \times \mathfrak{R}_{\mu,2}$, with the Sobolev norm

$$\|y\|_{H^2\times L^2\times\mathfrak{R}_{\mu,2}} = (\|v\|^2 + \|u\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,2})^{\frac{1}{2}}, \quad \text{for } y = (u, v, \eta)^\top \in E. \tag{3.1}$$

Let $\xi = u_t + \delta u$, where δ is a small positive constant whose value will be determined later. Substituting $u_t = \xi - \delta u$ into (1.1) we find

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + \delta u = \xi, \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}t} - \delta\xi + (\lambda + \delta^2 + A)u + h(\xi - \delta u) + \displaystyle\int_0^\infty \mu(s)A\eta(s)\mathrm{d}s + f(x, u) = g(x, t) + \varepsilon u \circ \dfrac{\mathrm{d}w}{\mathrm{d}t}, \\ \eta_t + \eta_s = u_t, \end{cases} \tag{3.2}$$

with the initial value conditions

$$u(x, \tau) = u_0(x), \quad \xi(x, \tau) = \xi_0(x), \quad \eta(x, \tau, s) = \eta_0(x, s) = u(x, \tau) - u(x, \tau - s), \tag{3.3}$$

where $\xi_0(x) = u_1(x) + \delta u_0(x)$, $x \in \mathbb{R}^n$, $s \in \mathbb{R}^+$.

Assumption I. Assume that the memory kernel function $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, nonlinear functions $h \in C^1(\mathbb{R})$ and $f \in C^1(\mathbb{R})$ satisfy the following conditions:

(1) $\forall s \in \mathbb{R}^+$ and some $\varpi > 0$.

$$\mu(s) \geq 0, \quad \mu'(s) + \varpi\mu \leq 0, \tag{3.4}$$

note that (3.4) implies $m_0 \overset{\text{def}}{=} \|\mu\|_{L^1(\mathbb{R}^+)} = \int_0^\infty \mu(s)\mathrm{d}s > 0$.

(2) Let $F(x, u) = \int_0^u f(x, s)\mathrm{d}s$ for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$, there exist positive constants c_i $(i = 1, 2, 3, 4)$, such that

$$|f(x, u)| \leq c_1|u|^p + \phi_1(x), \quad \phi_1 \in L^2(\mathbb{R}^n), \tag{3.5}$$

$$f(x, u)u - c_2F(x, u) \geq \phi_2(x), \quad \phi_2 \in L^1(\mathbb{R}^n), \tag{3.6}$$

$$F(x,u) \geq c_3|u|^{p+1} - \phi_3(x), \quad \phi_3 \in L^1(\mathbb{R}^n), \tag{3.7}$$

$$\left|\frac{\partial f}{\partial u}(x,u)\right| \leq \beta, \quad \left|\frac{\partial f}{\partial x}(x,u)\right| \leq \phi_4(x), \quad \phi_4 \in L^2(\mathbb{R}^n), \tag{3.8}$$

where $\beta > 0$, $1 \leq p \leq \frac{n+4}{n-4}$. Note that (3.5) and (3.6) imply

$$F(x,u) \leq c(|u|^2 + |u|^{p+1} + \phi_1^2 + \phi_2). \tag{3.9}$$

(3) There exist two constants β_1, β_2 such that

$$h(0) = 0, \quad 0 < \beta_1 \leq h'(v) \leq \beta_2 < \infty. \tag{3.10}$$

By (3.4), the space $\mathfrak{R}_{\mu,r} = L^2_\mu(\mathbb{R}^+, V_r)$ $(r \in \mathbb{R})$ is a Hilbert space of V_r-valued functions on $\mathbb{R}^+$ with the inner product and norm:

$$(\eta_1,\eta_2)_{\mu,r} = \int_0^\infty \mu(s)\Big(A^{\frac{r}{4}}\eta_1(s), A^{\frac{r}{4}}\eta_2(s)\Big)\mathrm{d}s, \qquad \forall \eta, \eta_1, \eta_2 \in V_r,$$
$$\|\eta\|^2_{\mu,r} = \int_0^\infty \mu(s)\Big(A^{\frac{r}{4}}\eta(s), A^{\frac{r}{4}}\eta(s)\Big)\mathrm{d}s,$$

and on $\mathfrak{R}_{\mu,r}$, the linear operator $-\partial_s$ has domain

$$D(-\partial_s) = \{\eta \in H^1_\mu(\mathbb{R}^+, V_r) : \eta(0) = 0\}, \quad \text{where} \quad H^1_\mu(\mathbb{R}^+, V_r) = \{\eta : \eta(s), \partial_s\eta \in L^2_\mu(\mathbb{R}^+, V_r)\}.$$

To study the dynamical behavior of problem (3.2), we need to convert problem (3.2) into a deterministic system with a random parameter. We now introduce an Ornstein-Uhlenbeck process given by the Brownian motion. Put

$$z(\theta_t\omega) = z(t,\omega) = -\delta \int_{-\infty}^0 e^{\delta s}(\theta_t\omega)(s)\mathrm{d}s, \tag{3.11}$$

which is called the Ornstein-Uhlenbeck process and solves the Itô equation $\mathrm{d}z + \delta z \mathrm{d}t = \mathrm{d}\omega$, $z(-\infty) = 0$.

From [30], it is known that the random variable $|z(\omega)|$ is a stationary, ergodic, and tempered stochastic process, and there is a θ_t-invariant set $\widetilde{\Omega} \subset \Omega$ of full $\mathcal{P}$ measure such that $z(\theta_t\omega)$ is continuous in t for every $\omega \in \widetilde{\Omega}$. For convenience, we shall simply write $\widetilde{\Omega}$ as Ω.

To show that problem (3.2) generates a cocycle, we let

$$v(x,t) = \xi(x,t) - \varepsilon u(x,t) z(\theta_t\omega),$$

then (3.2) can be rewritten as the equivalent system with random coefficients but without multiplicative noise

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + \delta u - v = \varepsilon u z(\theta_t\omega), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} - \delta v + (\lambda + \delta^2 + A)u + \displaystyle\int_0^\infty \mu(s) A\eta(s)\mathrm{d}s + f(x,u) \\ \qquad = g(x,t) - h(v + \varepsilon u z(\theta_t\omega) - \delta u) - \varepsilon(v - 3\delta u + \varepsilon u z(\theta_t\omega)) z(\theta_t\omega), \\ \eta_t + \eta_s = u_t, \end{cases} \tag{3.12}$$

with the initial value conditions

$$u(x,\tau) = u_0(x), \quad v(x,\tau) = v_0(x), \quad \eta(x,\tau,s) = \eta_0(x,s) = u(x,\tau) - u(x,\tau - s), \tag{3.13}$$

where $v_0(x) = \xi_0(x) - \varepsilon u_0(x) z(\theta_t\omega)$, $x \in \mathbb{R}^n$, $s \in \mathbb{R}^+$.

The well-posedness of the deterministic problems (3.12)–(3.13) in $H^2 \times L^2 \times \mathfrak{R}_{\mu,2}$ can be established by standard methods as in [29–31], more precisely, under conditions (3.4)–(3.8) and (3.10), for every $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $(u_0, v_0, \eta_0) \in E$, we can obtain the following lemma:

Lemma 3.1. *Put* $\varphi(t+\tau, \tau, \theta_{-\tau}\omega, \varphi_0) = (u(t+\tau, \tau, \theta_{-\tau}\omega, u_0), v(t+\tau, \tau, \theta_{-\tau}\omega, v_0), \eta(t+\tau, \tau, \theta_{-\tau}\omega, \eta_0, s))^\top$, *where* $\varphi_0 = (u_0, v_0, \eta_0)^\top$, *and let conditions* (3.4)–(3.8) *and* (3.10) *hold. Then for every* $\omega \in \Omega$, $\tau \in \mathbb{R}$ *and* $\varphi_0 \in E(\mathbb{R}^n)$, *problem* (3.12)–(3.13) *has a unique* $(\mathcal{F}, \mathcal{B}(H^2(\mathbb{R}^n)) \times \mathcal{B}(L^2(\mathbb{R}^n)) \times \mathcal{B}(\mathfrak{R}_{\mu,2}))$*-measurable solution* $\varphi(\cdot, \tau, \omega, \varphi_0) \in C([\tau, \infty), E(\mathbb{R}^n))$ *with* $\varphi(\tau, \tau, \omega, \varphi_0) = \varphi_0$, $\varphi(t, \tau, \omega, \varphi_0) \in E(\mathbb{R}^n)$ *being continuous in* φ_0 *with respect to the usual norm of* $E(\mathbb{R}^n)$ *for each* $t > \tau$. *Moreover, for every* $(t, \tau, \omega, \varphi_0) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E(\mathbb{R}^n)$, *the mapping*

$$\Phi(t, \tau, \omega, \varphi_0) = \varphi(t+\tau, \tau, \theta_{-\tau}\omega, \varphi_0) \tag{3.14}$$

generates a continuous cocycle from $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times E(\mathbb{R}^n)$ *to* $E(\mathbb{R}^n)$ *over* $\mathbb{R}$ *and* $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Introducing the homeomorphism $P(\theta_t\omega)(u, v, \eta)^\top = (u, v + z(\theta_t\omega), \eta)^\top$, $(u, v, \eta)^\top \in E(\mathbb{R}^n)$ with an inverse homeomorphism $P^{-1}(\theta_t\omega)(u, v, \eta)^\top = (u, v - z(\theta_t\omega), \eta)^\top$. Then, the transformation

$$\widetilde{\Phi}(t, \tau, \omega, (u_0, \xi_0, \eta_0)) = P(\theta_t\omega)\Phi(t, \tau, \omega, (u_0, v_0, \eta_0))P^{-1}(\theta_t\omega) \tag{3.15}$$

generates a continuous cocycle with (3.2)–(3.3) over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Note that these two continuous cocycles are equivalent. By (3.15), it is easy to check that $\widetilde{\Phi}$ has a random attractor provided Φ possesses a random attractor. Then, we only need to consider the continuous cocycle Φ.

Assumption II. We assume that σ, δ, ε, and $g(x,t)$ satisfy the following conditions:

$$\sigma = \frac{1}{2}\min\left\{\delta, \delta\left(1 - \frac{2m_0\delta}{\varpi}\right), \frac{\varpi}{4}, \delta c_2\right\}, \tag{3.16}$$

$$\delta > 0 \quad \text{satisfies} \quad \lambda + \delta^2 - \beta_2\delta > 0, \quad \beta_1 > 5\delta + \frac{\beta^2}{\delta(\lambda + \delta^2 - \beta_2\delta)}, \tag{3.17}$$

$$|\varepsilon| < \min\left\{\frac{-4\sqrt{\delta}(\gamma_2\gamma_3 + \gamma_1)\varpi + 2\sqrt{4\delta(\gamma_2\gamma_3 + \gamma_1)^2\varpi^2 + \pi\delta\varpi(2\sigma\gamma_2\varpi + 8\sigma\gamma_2 m_0)}}{(2\varpi + 8m_0)\gamma_2\sqrt{\pi}},\right.$$
$$\left.\frac{-4\sqrt{\delta}(\gamma_2\gamma_3 + 1)\varpi + 2\sqrt{4\delta(\gamma_2\gamma_3 + 1)^2\varpi^2 + \pi\delta\varpi(2\sigma\gamma_2\varpi + 8\sigma\gamma_2 m_0)}}{(2\varpi + 8m_0)\gamma_2\sqrt{\pi}}\right\}, \tag{3.18}$$

where $\gamma_1 = \max\left\{1, \frac{c_1 c_3^{-1}}{2}\right\}$, $\gamma_2 = 1 + \frac{1}{\lambda + \delta^2 - \beta_2\delta}$, $\gamma_3 = \frac{3}{2}\delta + \frac{1}{2}\beta_2 + (\beta_2 - \beta_1)\delta + 1$. Moreover,

$$\int_{-\infty}^{0} e^{\sigma s}\|g(\cdot, \tau + s)\|_1^2 \mathrm{d}s < \infty, \quad \forall \tau \in \mathbb{R}, \tag{3.19}$$

and

$$\lim_{k\to\infty}\int_{-\infty}^{0} e^{\sigma s}\int_{|x|\geq k}|g(x, \tau + s)|^2 \mathrm{d}x\mathrm{d}s = 0, \quad \forall \tau \in \mathbb{R}, \tag{3.20}$$

where $|\cdot|$ denotes the absolute value of real number in $\mathbb{R}$.

Given a bounded nonempty subset B of E, we write $\|B\| = \sup_{\phi\in B}\|\phi\|_E$. Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of E such that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$,

$$\lim_{s\to-\infty} e^{\sigma s}\|D(\tau + s, \theta_s\omega)\|_E^2 = 0. \tag{3.21}$$

Let $\mathcal{D}$ be the collection of all such families, that is,

$$\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies (3.21)}\}. \tag{3.22}$$

4 Uniform estimates of solutions

In this section, we derive a series of uniform estimates for solutions of problems (3.12)–(3.13).

We define a new norm $\|\cdot\|_E$ by

$$\|Y\|_E = (\|v\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,2})^{\frac{1}{2}}, \quad \text{for } Y = (u, v, \eta) \in E. \tag{4.1}$$

It is easy to check that $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H^2\times L^2\times\mathfrak{R}_{\mu,2}}$ in (3.1).

First we show that the cocycle Φ has a pullback $\mathcal{D}$-absorbing set in $\mathcal{D}$.

Lemma 4.1. *Under Assumptions I and II, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$ the solution of problem* (3.12)–(3.13) *satisfies*

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|_E^2 \leq R(\tau, \omega),$$

and $R(\tau, \omega)$ is given by

$$R(\tau, \omega) = M \int_{-\infty}^{0} e^{2\int_0^s \left[\sigma - \gamma_1|\varepsilon||z(\theta_r\omega)| - \gamma_2\left(\left(\frac{1}{2}\varepsilon^2 + \frac{2m_0\varepsilon^2}{\varpi}\right)|z(\theta_r\omega)|^2 + \gamma_3|\varepsilon||z(\theta_t\omega)|\right)\right]\mathrm{d}r} \cdot (\|g(\cdot, s + \tau)\|^2 + |\varepsilon||z(\theta_s\omega)|)\mathrm{d}s, \tag{4.2}$$

where M is a positive constant independent of τ, ω, D, and ε.

Proof. Taking the inner product of $(3.12)_2$ with v in $L^2(\mathbb{R}^n)$, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 - (\delta - \varepsilon z(\theta_t\omega))\|v\|^2 + (\lambda + \delta^2)(u, v) + (Au, v) + \int_0^\infty \mu(s)(A\eta(s), v)\mathrm{d}s + (f(x, u), v)\\ &\quad = \varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))(u, v) - (h(v + \varepsilon uz(\theta_t\omega) - \delta u), v) + (g(x, t), v).\end{aligned} \tag{4.3}$$

By $(3.12)_1$, we have

$$v = u_t - \varepsilon uz(\theta_t\omega) + \delta u. \tag{4.4}$$

It follows from (3.10) and Lagrange's mean value theorem that

$$\begin{aligned}-(h(v + \varepsilon uz(\theta_t\omega) - \delta u), v) &= -(h(v + \varepsilon uz(\theta_t\omega) - \delta u) - h(0), v)\\ &= -(h'(\vartheta)(v + \varepsilon uz(\theta_t\omega) - \delta u), v)\\ &\leq -\beta_1\|v\|^2 - (h'(\vartheta)(\varepsilon uz(\theta_t\omega) - \delta u), v)\\ &\leq -\beta_1\|v\|^2 + \beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\| + h'(\vartheta)\delta(u, v),\end{aligned} \tag{4.5}$$

where ϑ is between 0 and $v + \varepsilon uz(\theta_t\omega) - \delta u$.

By (3.10) and (4.4), we know

$$h'(\vartheta)\delta(u, v) = h'(\vartheta)\delta(u, u_t - \varepsilon uz(\theta_t\omega) + \delta u) \leq \beta_2\delta \cdot \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \beta_2\delta^2\|u\|^2 - \beta_1\delta|\varepsilon||z(\theta_t\omega)|\|u\|^2. \tag{4.6}$$

Substituting (4.4) into the third and fourth terms on the left-hand side of (4.3), we find that

$$(u, v) = (u, u_t - \varepsilon uz(\theta_t\omega) + \delta u) \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \delta\|u\|^2 - |\varepsilon||z(\theta_t\omega)|\|u\|^2 \tag{4.7}$$

and

$$(Au, v) = (\Delta u, \Delta v) = (\Delta u, \Delta u_t - \varepsilon z(\theta_t\omega)\Delta u + \delta \Delta u) \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + \delta\|\Delta u\|^2 - |\varepsilon||z(\theta_t\omega)|\|\Delta u\|^2. \tag{4.8}$$

Using the Cauchy-Schwarz inequality and Young's inequality, we have

$$\begin{aligned}
&\varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))(u, v) + \beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\| \\
&= (3\delta\varepsilon z(\theta_t\omega) - \varepsilon^2 z^2(\theta_t\omega))(u, v) + \beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\| \\
&\leq (3\delta|\varepsilon||z(\theta_t\omega)| + \varepsilon^2|z(\theta_t\omega)|^2)\|u\|\|v\| + \beta_2|\varepsilon||z(\theta_t\omega)|\|u\|\|v\| \\
&= ((3\delta + \beta_2)|\varepsilon||z(\theta_t\omega)| + \varepsilon^2|z(\theta_t\omega)|^2)\|u\|\|v\| \\
&\leq \left(\frac{1}{2}(3\delta + \beta_2)|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)(\|u\|^2 + \|v\|^2),
\end{aligned} \tag{4.9}$$

$$(g, v) \leq \|g\|\|v\| \leq \frac{\|g\|^2}{2(\beta_1 - \delta)} + \frac{\beta_1 - \delta}{2}\|v\|^2. \tag{4.10}$$

Let $\widetilde{F}(x, u) = \int_{\mathbb{R}^n} F(x, u)\mathrm{d}x$. Then for the last term on the left-hand side of (4.3) we have

$$(f(x, u), v) = (f(x, u), u_t - \varepsilon z(\theta_t\omega)u + \delta u) = \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{F}(x, u) + \delta(f(x, u), u) - \varepsilon z(\theta_t\omega)(f(x, u), u). \tag{4.11}$$

From (3.5) and (3.7), we obtain

$$\begin{aligned}
\varepsilon z(\theta_t\omega)(f(x, u), u) &\leq c_1|\varepsilon||z(\theta_t\omega)| \int_{\mathbb{R}^n} |u|^{\gamma+1}\mathrm{d}x + |\varepsilon||z(\theta_t\omega)|\|\phi_1\|^2 + |\varepsilon||z(\theta_t\omega)|\|u\|^2 \\
&\leq c_1 c_3^{-1}|\varepsilon||z(\theta_t\omega)| \int_{\mathbb{R}^n} (F(x, u) + \phi_3)\mathrm{d}x + |\varepsilon||z(\theta_t\omega)|\|\phi_1\|^2 + |\varepsilon||z(\theta_t\omega)|\|u\|^2 \\
&\leq c_1 c_3^{-1}|\varepsilon||z(\theta_t\omega)|\widetilde{F}(x, u) + c|\varepsilon||z(\theta_t\omega)| + |\varepsilon||z(\theta_t\omega)|\|u\|^2.
\end{aligned} \tag{4.12}$$

A direct calculation deduces that

$$\begin{aligned}
\int_0^\infty \mu(s)(A\eta(s), v)\mathrm{d}s &= \int_0^\infty \mu(s)(\Delta^2\eta(s), v)\mathrm{d}s \\
&= \int_0^\infty \mu(s)(\Delta\eta(s), \Delta(u_t - \varepsilon u z(\theta_t\omega) + \delta u))\mathrm{d}s \\
&= \int_0^\infty \mu(s)(\Delta\eta(s), \Delta u_t)\mathrm{d}s - \varepsilon z(\theta_t\omega)\int_0^\infty \mu(s)(\Delta\eta(s), \Delta u)\mathrm{d}s + \delta\int_0^\infty \mu(s)(\Delta\eta(s), \Delta u)\mathrm{d}s.
\end{aligned} \tag{4.13}$$

Using $(3.12)_3$, then integrating by parts with respect to s, we get

$$\int_0^\infty \mu(s)(\Delta\eta(s), \Delta u_t)\mathrm{d}s \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|^2_{\mu,2} + \frac{\varpi}{2}\|\eta\|^2_{\mu,2}. \tag{4.14}$$

Using Young's inequality, we have

$$-\varepsilon z(\theta_t\omega)\int_0^\infty \mu(s)(\Delta\eta(s), \Delta u)\mathrm{d}s \geq -\frac{\varpi}{8}\|\eta\|^2_{\mu,2} - \frac{2m_0\varepsilon^2}{\varpi}|z(\theta_t\omega)|^2\|\Delta u\|^2 \tag{4.15}$$

and

$$\delta\int_0^\infty \mu(s)(\Delta\eta(s), \Delta u)\mathrm{d}s \geq -\frac{\varpi}{8}\|\eta\|^2_{\mu,2} - \frac{2m_0\delta^2}{\varpi}\|\Delta u\|^2. \tag{4.16}$$

Combining with (4.14)–(4.16) and (4.13), we get

$$\int_0^\infty \mu(s)(A\eta(s), v)\mathrm{d}s \ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{\mu,2}^2 + \frac{\varpi}{4}\|\eta\|_{\mu,2}^2 - \frac{2m_0\varepsilon^2}{\varpi}|z(\theta_t\omega)|^2\|\Delta u\|^2 - \frac{2m_0\delta^2}{\varpi}\|\Delta u\|^2. \tag{4.17}$$

Substitute (4.5)–(4.17) into (4.3) and together with (3.6) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,2}^2 + 2\widetilde{F}(x, u)) \\
&\quad + \delta(\|v\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u\|^2) + \delta\left(1 - \frac{2m_0\delta}{\varpi}\right)\|\Delta u\|^2 + \frac{\varpi}{4}\|\eta\|_{\mu,2}^2 + \delta c_2\widetilde{F}(x, u) \\
&\le \left(\frac{1}{2}(3\delta + \beta_2)|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)(\|u\|^2 + \|v\|^2) + \frac{2m_0\varepsilon^2}{\varpi}|z(\theta_t\omega)|^2\|\Delta u\|^2 \\
&\quad + |\varepsilon||z(\theta_t\omega)|(\|v\|^2 + (\lambda + \delta^2 - \beta_1\delta)\|u\|^2 + \|\Delta u\|^2) + |\varepsilon||z(\theta_t\omega)|\|u\|^2 \\
&\quad + \frac{3\delta - \beta_1}{2}\|v\|^2 + \frac{\|g\|^2}{2(\beta_1 - \delta)} + c_1c_3^{-1}|\varepsilon||z(\theta_t\omega)|\widetilde{F}(x, u) + c|\varepsilon||z(\theta_t\omega)| \\
&\le \left(\frac{1}{2}(3\delta + \beta_2)|\varepsilon||z(\theta_t\omega)| + \left(\frac{1}{2}\varepsilon^2 + \frac{2m_0\varepsilon^2}{\varpi}\right)|z(\theta_t\omega)|^2\right)(\|u\|^2 + \|v\|^2 + \|\Delta u\|^2) \\
&\quad + \gamma_1|\varepsilon||z(\theta_t\omega)|(\|v\|^2 + (\lambda + \delta^2 - \beta_1\delta)\|u\|^2 + \|\Delta u\|^2 + 2\widetilde{F}(x, u)) \\
&\quad + |\varepsilon||z(\theta_t\omega)|\|u\|^2 + c(\|g\|^2 + |\varepsilon||z(\theta_t\omega)|),
\end{aligned} \tag{4.18}$$

where $\gamma_1 = \max\left\{1, \frac{c_1c_3^{-1}}{2}\right\}$.

Choosing δ small enough such that $1 - \frac{2m_0\delta}{\varpi} > 0$, then let $\sigma = \frac{1}{2}\min\left\{\delta, \delta\left(1 - \frac{2m_0\delta}{\varpi}\right), \frac{\varpi}{4}, \delta c_2\right\}$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,2}^2 + 2\widetilde{F}(x, u)) \\
&\le -\left[\sigma - \gamma_1|\varepsilon||z(\theta_t\omega)| - \gamma_2\left(\left(\frac{1}{2}\varepsilon^2 + \frac{2m_0\varepsilon^2}{\varpi}\right)|z(\theta_t\omega)|^2 + \gamma_3|\varepsilon||z(\theta_t\omega)|\right)\right] \\
&\quad \times (\|v\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,2}^2 + 2\widetilde{F}(x, u)) + c(\|g\|^2 + |\varepsilon||z(\theta_t\omega)|),
\end{aligned} \tag{4.19}$$

where $\gamma_2 = 1 + \frac{1}{\lambda + \delta^2 - \beta_2\delta}$, $\gamma_3 = \frac{3}{2}\delta + \frac{1}{2}\beta_2 + (\beta_2 - \beta_1)\delta + 1$.

Denote

$$\varrho(\tau, \omega) = \sigma - \gamma_1|\varepsilon||z(\theta_t\omega)| - \gamma_2\left(\left(\frac{1}{2}\varepsilon^2 + \frac{2m_0\varepsilon^2}{\varpi}\right)|z(\theta_t\omega)|^2 + \gamma_3|\varepsilon||z(\theta_t\omega)|\right). \tag{4.20}$$

Using the Gronwall inequality to integrate (4.19) over $(\tau - t, \tau)$ with $t \ge 0$, we get

$$\begin{aligned}
&\|v(\tau, \tau - t, \omega, v_0)\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u(\tau, \tau - t, \omega, u_0)\|^2 \\
&\quad + \|\Delta u(\tau, \tau - t, \omega, u_0)\|^2 + \|\eta(\tau, \tau - t, \omega, \eta_0, s)\|_{\mu,2}^2 + 2\widetilde{F}(x, u(\tau, \tau - t, \omega, u_0)) \\
&\le (\|v_0\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|_{\mu,2}^2 + 2\widetilde{F}(x, u_0))e^{2\int_\tau^{\tau - t}\varrho(s,\omega)\mathrm{d}s} \\
&\quad + c\int_{\tau - t}^{\tau} e^{2\int_\tau^s \varrho(r,\omega)\mathrm{d}r}(\|g(\cdot, s)\|^2 + |\varepsilon||z(\theta_s\omega)|)\mathrm{d}s.
\end{aligned} \tag{4.21}$$

Replacing ω by $\theta_{-\tau}\omega$ in the above we obtain, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$,

$$\begin{aligned}
&\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 \\
&\quad + \|\Delta u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 + \|\eta(\tau, \tau - t, \theta_{-\tau}\omega, \eta_0, s)\|_{\mu,2}^2 + 2\widetilde{F}(x, u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)) \\
&\le (\|v_0\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|_{\mu,2}^2 + 2\widetilde{F}(x, u_0))e^{2\int_\tau^{\tau - t}\varrho(s-\tau,\omega)\mathrm{d}s} \\
&\quad + c\int_{\tau - t}^{\tau} e^{2\int_\tau^s \varrho(r-\tau,\omega)\mathrm{d}r}(\|g(\cdot, s)\|^2 + |\varepsilon||z(\theta_{s-\tau}\omega)|)\mathrm{d}s,
\end{aligned} \tag{4.22}$$

then

$$
\begin{aligned}
&\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_0)\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 \\
&\quad + \|\Delta u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 + \|\eta(\tau, \tau - t, \theta_{-\tau}\omega, \eta_0, s)\|^2_{\mu,2} + 2\widetilde{F}(x, u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)) \\
&\leq (\|v_0\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|^2_{\mu,2} + 2\widetilde{F}(x, u_0))e^{2\int_0^{-t} \varrho(s,\omega)\mathrm{d}s} \\
&\quad + c\int_{-t}^{0} e^{2\int_0^{s} \varrho(r,\omega)\mathrm{d}r}(\|g(\cdot, s + \tau)\|^2 + |\varepsilon||z(\theta_s\omega)|)\mathrm{d}s.
\end{aligned}
\tag{4.23}
$$

Since $|z(\theta_t\omega)|$ is stationary and ergodic, from (3.11) and the ergodic theorem we can get

$$
\lim_{t\to\infty} \frac{1}{t}\int_{-t}^{0} |z(\theta_r\omega)|\mathrm{d}r = \mathbf{E}(|z(\theta_r\omega)|) = \frac{1}{\sqrt{\pi\delta}}, \tag{4.24}
$$

$$
\lim_{t\to\infty} \frac{1}{t}\int_{-t}^{0} |z(\theta_r\omega)|^2\mathrm{d}r = \mathbf{E}(|z(\theta_r\omega)|^2) = \frac{1}{2\delta}. \tag{4.25}
$$

By (4.24)–(4.25), there exists $T_1(\omega) > 0$ such that for all $t \geq T_1(\omega)$,

$$
\int_{-t}^{0} |z(\theta_r\omega)|\mathrm{d}r < \frac{2}{\sqrt{\pi\delta}}t, \qquad \int_{-t}^{0} |z(\theta_r\omega)|^2\mathrm{d}r < \frac{1}{\delta}t. \tag{4.26}
$$

Next we show that for any $s \leq -T_1$

$$
e^{2\int_0^s \varrho(r,\omega)\mathrm{d}r} \leq e^{\sigma s}. \tag{4.27}
$$

By using the two inequalities in (4.26), we have

$$
\begin{aligned}
&\int_0^s \left[\sigma - \gamma_1|\varepsilon||z(\theta_r\omega)| - \gamma_2\left(\left(\frac{1}{2}\varepsilon^2 + \frac{2m_0\varepsilon^2}{\varpi}\right)|z(\theta_r\omega)|^2 + \gamma_3|\varepsilon||z(\theta_r\omega)|\right)\right]\mathrm{d}r \\
&\quad > \sigma s - |\varepsilon|\frac{2\gamma_1}{\sqrt{\pi\delta}}s - \gamma_2\left[\left(\frac{1}{2}\varepsilon^2 + \frac{2m_0\delta^2}{\varpi}\right)\frac{1}{\delta} + \gamma_3|\varepsilon|\frac{2}{\sqrt{\pi\delta}}\right]s \\
&\quad = -\frac{\gamma_2}{\delta}\left(\frac{1}{2}\varepsilon^2 + \frac{2m_0\varepsilon^2}{\varpi}\right)s - \frac{2}{\sqrt{\pi\delta}}[\gamma_3\gamma_2 + \gamma_1]|\varepsilon|s + \sigma s.
\end{aligned}
\tag{4.28}
$$

In order to have the inequality in (4.27) valid, we need

$$
\int_0^s \left[\sigma - \gamma_1|\varepsilon||z(\theta_r\omega)| - \gamma_2\left(\left(\frac{1}{2}\varepsilon^2 + \frac{2m_0\varepsilon^2}{\varpi}\right)|z(\theta_r\omega)|^2 + \gamma_3|\varepsilon||z(\theta_r\omega)|\right)\right]\mathrm{d}r \leq \frac{\sigma}{2}s.
$$

Since $s \leq -T_1$, then it requires that

$$
\frac{\gamma_2}{\delta}\left(\frac{1}{2}\varepsilon^2 + \frac{2m_0\varepsilon^2}{\varpi}\right) + \frac{2}{\sqrt{\pi\delta}}[\gamma_3\gamma_2 + \gamma_1]|\varepsilon| - \frac{\sigma}{2} < 0.
$$

Solving this quadratic inequality, ε needs to satisfy (3.18) as we have assumed in Assumption II.

Since $|z(\theta_t\omega)|$ is tempered, by (3.19) and (4.27), we see that the following integral is convergent:

$$
R_1^2(\tau, \omega) = c\int_{-\infty}^{0} e^{2\int_0^s \varrho(r,\omega)\mathrm{d}r}(\|g(\cdot, s + \tau)\|^2 + |\varepsilon||z(\theta_s\omega)|)\mathrm{d}s. \tag{4.29}
$$

Note that (3.9) implies

$$\int_{\mathbb{R}^n} F(x, u_0)\mathrm{d}x \le c(1 + \|u_0\|^2 + \|u_0\|_{H^2}^{p+1}). \tag{4.30}$$

Since $D \in \mathcal{D}$ and $(u_0, v_0, \eta_0) \in D(\tau - t, \theta_{-t}\omega)$, for all $t \ge T_1$, we get from (4.29) and (4.30) that

$$\begin{aligned}
&(\|v_0\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|_{\mu,2}^2 + 2\widetilde{F}(x, u_0))e^{2\int_0^{-t} \varrho(s,\omega)\mathrm{d}s} \\
&\le\ ce^{-\sigma t}(1 + \|v_0\|^2 + \|u_0\|_{H^2}^2 + \|u_0\|_{H^2}^{p+1} + \|\eta_0\|_{\mu,2}^2) \\
&\le\ ce^{-\sigma t}(1 + \|D(\tau - t, \theta_{-t}\omega)\|^2 + \|D(\tau - t, \theta_{-t}\omega)\|^{p+1}) \to 0, \quad \text{as } t \to +\infty.
\end{aligned} \tag{4.31}$$

From (4.1), (4.23), (4.29), and (4.31), there exists $T_2 = T_2(\tau, \omega, D) \ge T_1$ such that for all $t \ge T_2$,

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|_E^2 \le cR_1^2(\tau, \omega),$$

thus the proof is completed. □

The following lemma will be used to show the uniform estimates of solutions as well as to establish pullback asymptotic compactness.

Lemma 4.2. *Under Assumptions I and II, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \ge T$, $s \in [-t, 0]$, the solution of problem* (3.12)–(3.13) *satisfies*

$$\|\varphi(\tau + s, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|_E^2 \le R(\tau, \omega)e^{2\int_s^0 \varrho(r,\omega)\mathrm{d}r},$$

where $(u_0, v_0, \eta_0)^{\top} \in D(\tau - t, \theta_{-t}\omega)$, M is a positive constant independent of τ, ω, D, and ε, and $R(\tau, \omega)$ is a specific random variable.

Proof. Similar to (4.23), integrating (4.19) over $(\tau - t, \tau + s)$ with $t \ge 0$ and $s \in [-t, 0]$, we get

$$\begin{aligned}
&\|v(\tau + s, \tau - t, \omega, v_0)\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u(\tau + s, \tau - t, \omega, u_0)\|^2 \\
&\quad + \|\Delta u(\tau + s, \tau - t, \omega, u_0)\|^2 + \|\eta(\tau + s, \tau - t, \omega, \eta_0, s)\|_{\mu,2}^2 + 2\widetilde{F}(x, u(\tau + s, \tau - t, \omega, u_0)) \\
&\le (\|v_0\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|_{\mu,2}^2 + 2\widetilde{F}(x, u_0))e^{2\int_{\tau+s}^{\tau-t} \varrho(r-t,\omega)\mathrm{d}r} \\
&\quad + c\int_{\tau-t}^{\tau+s} e^{2\int_{\tau+s}^{\zeta} \varrho(r-\tau,\omega)\mathrm{d}r}(\|g(\cdot, \zeta)\|^2 + |\varepsilon||z(\theta_{\zeta-\tau}\omega)|)\mathrm{d}\zeta \\
&\le (\|v_0\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|_{\mu,2}^2 + 2\widetilde{F}(x, u_0))e^{2\int_s^{-t} \varrho(r,\omega)\mathrm{d}r} \\
&\quad + c\int_{-t}^{s} e^{2\int_s^{\zeta} \varrho(r,\omega)\mathrm{d}r}(\|g(\cdot, \zeta + \tau)\|^2 + |\varepsilon||z(\theta_\zeta\omega)|)\mathrm{d}\zeta.
\end{aligned} \tag{4.32}$$

For the last integral term on the right-hand side of (4.32), we have

$$\begin{aligned}
&c\int_{-t}^{s} e^{2\int_s^{\zeta} \varrho(r-t,\omega)\mathrm{d}r}(1 + \|g(\cdot, \zeta + \tau)\|^2 + |\varepsilon||z(\theta_\zeta\omega)|)\mathrm{d}\zeta \\
&\quad = c\left[\int_{-t}^{-T_1} e^{2\int_s^{\zeta} \varrho(r,\omega)\mathrm{d}r} + \int_{-T_1}^{s} e^{2\int_s^{\zeta} \varrho(r,\omega)\mathrm{d}r}\right](\|g(\cdot, \zeta + \tau)\|^2 + |\varepsilon||z(\theta_\zeta\omega)|)\mathrm{d}\zeta \\
&\quad \le\ ce^{2\int_s^{0} \varrho(r,\omega)\mathrm{d}r}\int_{-t}^{-T_1} e^{2\int_0^{\zeta} \varrho(r,\omega)\mathrm{d}r}(\|g(\cdot, \zeta + \tau)\|^2 + |\varepsilon||z(\theta_\zeta\omega)|)\mathrm{d}\zeta
\end{aligned} \tag{4.33}$$

$$
\begin{aligned}
&+ ce^{2\int_s^0 \varrho(r,\omega)\mathrm{d}r} \int_{-T_1}^{0} e^{2\int_0^\zeta \varrho(r,\omega)\mathrm{d}r} (\|g(\cdot,\zeta+\tau)\|^2 + |\varepsilon||z(\theta_\zeta\omega)|)\mathrm{d}\zeta \\
&\le ce^{2\int_s^0 \varrho(r,\omega)\mathrm{d}r} \int_{-t}^{-T_1} e^{\sigma\zeta} (\|g(\cdot,\zeta+\tau)\|^2 + |\varepsilon||z(\theta_\zeta\omega)|)\mathrm{d}\zeta \\
&+ ce^{2\int_s^0 \varrho(r,\omega)\mathrm{d}r} \int_{-T_1}^{0} e^{2\int_0^\zeta \varrho(r,\omega)\mathrm{d}r} (\|g(\cdot,\zeta+\tau)\|^2 + |\varepsilon||z(\theta_\zeta\omega)|)\mathrm{d}\zeta \\
&\le e^{2\int_s^0 \varrho(r,\omega)\mathrm{d}r} R_2(\varepsilon,\tau,\omega),
\end{aligned}
$$

where

$$
R_2(\tau,\omega) = c\int_{-\infty}^{0} e^{\sigma\zeta}(\|g(\cdot,\zeta+\tau)\|^2 + |\varepsilon||z(\theta_\zeta\omega)|)\mathrm{d}\zeta + c\int_{-T_1}^{0} e^{2\int_0^\zeta \varrho(r,\omega)\mathrm{d}r}(\|g(\cdot,\zeta+\tau)\|^2 + |\varepsilon||z(\theta_\zeta\omega)|)\mathrm{d}\zeta.
$$

As in (4.31), we find that there exists $T_3 = T_3(\tau,\omega,D) \ge T_1$ such that for all $t \ge T_3$,

$$
\begin{aligned}
&(\|v_0\|^2 + (\lambda+\delta^2-\beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|^2_{\mu,2} + 2\widetilde{F}(x,u_0))e^{2\int_s^{-t} \varrho(r,\omega)\mathrm{d}r} \\
&\le\ ce^{2\int_s^0 \varrho(r,\omega)\mathrm{d}r} e^{2\int_0^{-t} \varrho(r,\omega)\mathrm{d}r} (\|v_0\|^2 + (\lambda+\delta^2-\beta_2\delta)\|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|^2_{\mu,2} + 2\widetilde{F}(x,u_0)) \\
&\le\ e^{2\int_s^0 \varrho(r,\omega)\mathrm{d}r} R_2(\tau,\omega).
\end{aligned}
\tag{4.34}
$$

It follows from (4.32)–(4.34) and (4.30) that, for all $t \ge T_3$, $s \in [-t, 0]$, and ε satisfying (3.18),

$$
\begin{aligned}
&\|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2 + (\lambda+\delta^2-\beta_2\delta)\|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2 \\
&+ \|\Delta u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2 + \|\eta(\tau+s,\tau-t,\omega,\eta_0,s)\|^2_{\mu,2} \le 2e^{2\int_s^0 \varrho(r,\omega)\mathrm{d}r} R_2(\tau,\omega).
\end{aligned}
\tag{4.35}
$$

The proof is completed. □

Next, we will give higher order estimates for $\varphi^{(\varepsilon)}$.

Lemma 4.3. *Under Assumptions I and II, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D) > 0$ such that for all $t \ge T$ the solution of problem* (3.12)–(3.13) *satisfies*

$$
\|A^{\frac14}\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|^2_E \le R(\tau,\omega),
$$

and $R(\tau,\omega)$ is given by

$$
R(\tau,\omega) = R_3^2(\varepsilon,\tau,\omega) + ce^{-\sigma t}\left(\left\|A^{\frac14}v_0\right\|^2 + \left\|A^{\frac14}u_0\right\|^2 + \left\|A^{\frac34}u_0\right\|^2\right),
\tag{4.36}
$$

where $(u_0, v_0, \eta_0)^\top \in D(\tau-t,\theta_{-t}\omega)$, c is a positive constant independent of τ,ω,D, and ε, and $R_3(\tau,\omega)$ is a specific random variable.

Proof. Taking the inner product of $(3.12)_2$ with $A^{\frac12}v$ in $L^2(\mathbb{R}^n)$, we find that

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\left\|A^{\frac14}v\right\|^2 - (\delta-\varepsilon z(\theta_t\omega))\left\|A^{\frac14}v\right\|^2 + (\lambda+\delta^2)\left(u, A^{\frac12}v\right) + \left(Au, A^{\frac12}v\right) \\
&\quad+ \int_0^\infty \mu(s)\left(A\eta(s), A^{\frac12}v\right)\mathrm{d}s + (f(x,u), A^{\frac12}v) \\
&\quad= \varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))\left(u, A^{\frac12}v\right) - (h(v+\varepsilon uz(\theta_t\omega)-\delta u), A^{\frac12}v) + \left(g(x,t), A^{\frac12}v\right).
\end{aligned}
\tag{4.37}
$$

Similar to the proof of Lemma 4.1, we have the following estimates:

$$\begin{aligned}
-\Big(h(v+\varepsilon uz(\theta_t\omega)-\delta u),A^{\frac12}v\Big) &= -\Big(h(v+\varepsilon uz(\theta_t\omega)-\delta u)-h(0),A^{\frac12}v\Big)\\
&= -\Big(h'(\vartheta)(v+\varepsilon uz(\theta_t\omega)-\delta u),A^{\frac12}v\Big)\\
&\le -\beta_1\left\|A^{\frac14}v\right\|^2-\Big(h'(\vartheta)(\varepsilon uz(\theta_t\omega)-\delta u),A^{\frac12}v\Big)\\
&\le -\beta_1\left\|A^{\frac14}v\right\|^2+\beta_2|\varepsilon||z(\theta_t\omega)|\left\|A^{\frac14}u\right\|\left\|A^{\frac14}v\right\|+h'(\vartheta)\delta\Big(u,A^{\frac12}v\Big),
\end{aligned}\tag{4.38}$$

$$\begin{aligned}
h'(\vartheta)\delta\Big(u,A^{\frac12}v\Big) &= h'(\vartheta)\delta\Big(u,A^{\frac12}u_t-\varepsilon z(\theta_t\omega)A^{\frac12}u\Big)+\delta A^{\frac12}u\Big)\\
&\le \beta_2\delta\cdot\frac12\frac{\mathrm d}{\mathrm dt}\left\|A^{\frac14}u\right\|^2+\beta_2\delta^2\left\|A^{\frac14}u\right\|^2-\beta_1\delta|\varepsilon||z(\theta_t\omega)|\left\|A^{\frac14}u\right\|^2,
\end{aligned}\tag{4.39}$$

$$\begin{aligned}
\Big(u,A^{\frac12}v\Big) &= \Big(u,A^{\frac12}u_t-\varepsilon z(\theta_t\omega)A^{\frac12}u+\delta A^{\frac12}u\Big)\\
&\ge \frac12\frac{\mathrm d}{\mathrm dt}\left\|A^{\frac14}u\right\|^2+\delta\left\|A^{\frac14}u\right\|^2-|\varepsilon||z(\theta_t\omega)|\left\|A^{\frac14}u\right\|^2,
\end{aligned}\tag{4.40}$$

$$\begin{aligned}
(Au,A^{\frac12}v) &= \Big(Au,A^{\frac12}u_t-\varepsilon z(\theta_t\omega)A^{\frac12}u+\delta A^{\frac12}u\Big)\\
&\ge \frac12\frac{\mathrm d}{\mathrm dt}\left\|A^{\frac34}u\right\|^2+\delta\left\|A^{\frac34}u\right\|^2-|\varepsilon||z(\theta_t\omega)|\left\|A^{\frac34}u\right\|^2,
\end{aligned}\tag{4.41}$$

$$\begin{aligned}
&\varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))\Big(u,A^{\frac12}v\Big)+\beta_2|\varepsilon||z(\theta_t\omega)|\left\|A^{\frac14}u\right\|\left\|A^{\frac14}v\right\|\\
&= (3\delta\varepsilon z(\theta_t\omega)-\varepsilon^2z^2(\theta_t\omega))\Big(u,A^{\frac12}v\Big)+\beta_2|\varepsilon||z(\theta_t\omega)|\left\|A^{\frac14}u\right\|\left\|A^{\frac14}v\right\|\\
&\le (3\delta|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2)\left\|A^{\frac14}u\right\|\left\|A^{\frac14}v\right\|+\beta_2|\varepsilon||z(\theta_t\omega)|\left\|A^{\frac14}u\right\|\left\|A^{\frac14}v\right\|\\
&= ((3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\varepsilon^2|z(\theta_t\omega)|^2)\left\|A^{\frac14}u\right\|\left\|A^{\frac14}v\right\|\\
&\le \left(\frac12(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac12\varepsilon^2|z(\theta_t\omega)|^2\right)\left(\left\|A^{\frac14}u\right\|^2+\left\|A^{\frac14}v\right\|^2\right),
\end{aligned}\tag{4.42}$$

$$\Big(g,A^{\frac12}v\Big)\le\|g\|_1\left\|A^{\frac14}v\right\|\le\frac{\|g\|_1^2}{2(\beta_1-\delta)}+\frac{\beta_1-\delta}{2}\left\|A^{\frac14}v\right\|^2,\tag{4.43}$$

$$\int_0^\infty\mu(s)\Big(A\eta(s),A^{\frac12}v\Big)\mathrm ds\ge\frac12\frac{\mathrm d}{\mathrm dt}\left\|A^{\frac14}\eta\right\|_{\mu,2}^2+\frac{\varpi}{4}\left\|A^{\frac14}\eta\right\|_{\mu,2}^2-\frac{2m_0\varepsilon^2}{\varpi}|z(\theta_t\omega)|^2\left\|A^{\frac34}u\right\|^2-\frac{2m_0\delta^2}{\varpi}\left\|A^{\frac34}u\right\|^2.\tag{4.44}$$

For the last term on the left-hand side of (4.37), by (3.8), we have

$$\begin{aligned}
-\Big(f(x,u),A^{\frac12}v\Big) &= -\int_{\mathbb R^n}\frac{\partial}{\partial x}f(x,u)\cdot A^{\frac14}v\,\mathrm dx-\int_{\mathbb R^n}\frac{\partial}{\partial u}f(x,u)\cdot A^{\frac14}u\cdot A^{\frac14}v\,\mathrm dx\\
&\le \int_{\mathbb R^n}\left|\frac{\partial}{\partial x}f(x,u)\right|\cdot\left|A^{\frac14}v\right|\mathrm dx+\beta\int_{\mathbb R^n}\left|A^{\frac14}u\right|\cdot\left|A^{\frac14}v\right|\mathrm dx\\
&\le \int_{\mathbb R^n}|\eta_4|\cdot\left|A^{\frac14}v\right|\mathrm dx+\beta\int_{\mathbb R^n}\left|A^{\frac14}u\right|\cdot\left|A^{\frac14}v\right|\mathrm dx\\
&\le \|\eta_4\|\left\|A^{\frac14}v\right\|+\beta\left\|A^{\frac14}u\right\|\left\|A^{\frac14}v\right\|\\
&\le c+\left(\delta+\frac{\beta^2}{2\delta(\lambda+\delta^2-\beta_2\delta)}\right)\left\|A^{\frac14}v\right\|^2+\frac12\delta(\lambda+\delta^2-\beta_2\delta)\left\|A^{\frac14}u\right\|^2.
\end{aligned}\tag{4.45}$$

Substitute (4.38)–(4.45) into (4.37) and together with (3.17) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\Big\|A^{\frac{1}{4}}v\Big\|^2+(\lambda+\delta^2-\beta_2\delta)\Big\|A^{\frac{1}{4}}u\Big\|^2+\Big\|A^{\frac{3}{4}}u\Big\|^2+\Big\|A^{\frac{1}{4}}\eta\Big\|^2_{\mu,2}\bigg)\\&\quad+\sigma\bigg(\Big\|A^{\frac{1}{4}}v\Big\|^2+(\lambda+\delta^2-\beta_2\delta)\Big\|A^{\frac{1}{4}}u\Big\|^2+\Big\|A^{\frac{3}{4}}u\Big\|^2+\Big\|A^{\frac{1}{4}}\eta\Big\|^2_{\mu,2}\bigg)\\&\le\bigg(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\Big(\frac{1}{2}\varepsilon^2+\frac{2m_0\varepsilon^2}{\varpi}\Big)|z(\theta_t\omega)|^2\bigg)\bigg(\Big\|A^{\frac{1}{4}}u\Big\|^2+\Big\|A^{\frac{1}{4}}v\Big\|^2+\Big\|A^{\frac{3}{4}}u\Big\|^2\bigg)\\&\quad+|\varepsilon||z(\theta_t\omega)|\bigg(\Big\|A^{\frac{1}{4}}v\Big\|^2+(\lambda+\delta^2-\beta_1\delta)\Big\|A^{\frac{1}{4}}u\Big\|^2+\Big\|A^{\frac{3}{4}}u\Big\|^2+\Big\|A^{\frac{1}{4}}\eta\Big\|^2_{\mu,2}\bigg)+\frac{\|g\|_1^2}{2(\beta_1-\delta)}.\end{aligned}\tag{4.46}$$

Then

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\Big\|A^{\frac{1}{4}}v\Big\|^2+(\lambda+\delta^2-\beta_2\delta)\Big\|A^{\frac{1}{4}}u\Big\|^2+\Big\|A^{\frac{3}{4}}u\Big\|^2+\Big\|A^{\frac{1}{4}}\eta\Big\|^2_{\mu,2}\bigg)\\&\le-\bigg[\sigma-|\varepsilon||z(\theta_t\omega)|-\gamma_2\bigg(\Big(\frac{1}{2}\varepsilon^2+\frac{2m_0\varepsilon^2}{\varpi}\Big)|z(\theta_t\omega)|^2+\gamma_3|\varepsilon||z(\theta_t\omega)|\bigg)\bigg]\\&\quad\times\bigg(\Big\|A^{\frac{1}{4}}v\Big\|^2+(\lambda+\delta^2-\beta_2\delta)\Big\|A^{\frac{1}{4}}u\Big\|^2+\Big\|A^{\frac{3}{4}}u\Big\|^2+\Big\|A^{\frac{1}{4}}\eta\Big\|^2_{\mu,2}\bigg)+\frac{\|g\|_1^2}{2(\beta_1-\delta)}.\end{aligned}\tag{4.47}$$

Let us denote

$$\varrho_1(\tau,\omega)=\sigma-|\varepsilon||z(\theta_t\omega)|-\gamma_2\bigg(\Big(\frac{1}{2}\varepsilon^2+\frac{2m_0\varepsilon^2}{\varpi}\Big)|z(\theta_t\omega)|^2+\gamma_3|\varepsilon||z(\theta_t\omega)|\bigg).\tag{4.48}$$

Using the Gronwall inequality to integrate (4.47) over $(\tau-t,\tau)$ with $t\ge 0$, we get

$$\begin{aligned}&\Big\|A^{\frac{1}{4}}v(\tau,\tau-t,\omega,v_0)\Big\|^2+(\lambda+\delta^2-\beta_2\delta)\Big\|A^{\frac{1}{4}}u(\tau,\tau-t,\omega,u_0)\Big\|^2\\&\quad+\Big\|A^{\frac{3}{4}}u(\tau,\tau-t,\omega,u_0)\Big\|^2+\Big\|A^{\frac{1}{4}}\eta(\tau,\tau-t,\omega,\eta_0,s)\Big\|^2_{\mu,2}\\&\le\bigg(\Big\|A^{\frac{1}{4}}v_0\Big\|^2+(\lambda+\delta^2-\beta_2\delta)\Big\|A^{\frac{1}{4}}u_0\Big\|^2+\Big\|A^{\frac{3}{4}}u_0\Big\|^2+\Big\|A^{\frac{1}{4}}\eta_0\Big\|^2_{\mu,2}\bigg)e^{2\int_\tau^{\tau-t}\varrho_1(s,\omega)\mathrm{d}s}\\&\quad+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho_1(r,\omega)\mathrm{d}r}\|g(\cdot,s)\|_1^2\mathrm{d}s.\end{aligned}\tag{4.49}$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.49), for every $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$, and $\omega\in\Omega$,

$$\begin{aligned}&\Big\|A^{\frac{1}{4}}v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\Big\|^2+(\lambda+\delta^2-\beta_2\delta)\Big\|A^{\frac{1}{4}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\Big\|^2\\&\quad+\Big\|A^{\frac{3}{4}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\Big\|^2+\Big\|A^{\frac{1}{4}}\eta(\tau,\tau-t,\theta_{-\tau}\omega,\eta_0,s)\Big\|^2_{\mu,2}\\&\le\bigg(\Big\|A^{\frac{1}{4}}v_0\Big\|^2+(\lambda+\delta^2-\beta_2\delta)\Big\|A^{\frac{1}{4}}u_0\Big\|^2+\Big\|A^{\frac{3}{4}}u_0\Big\|^2+\Big\|A^{\frac{1}{4}}\eta_0\Big\|^2_{\mu,2}\bigg)e^{2\int_\tau^{\tau-t}\varrho_1(s-\tau,\omega)\mathrm{d}s}\\&\quad+c\int_{\tau-t}^{\tau}e^{2\int_\tau^s\varrho_1(r-\tau,\omega)\mathrm{d}r}\|g(\cdot,s)\|_1^2\mathrm{d}s,\end{aligned}\tag{4.50}$$

then

$$\begin{aligned}
&\left\|A^{\frac{1}{4}}v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)\right\|^2+(\lambda+\delta^2-\beta_2\delta)\left\|A^{\frac{1}{4}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2\\
&\quad+\left\|A^{\frac{3}{4}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\right\|^2+\left\|A^{\frac{1}{4}}\eta(\tau,\tau-t,\theta_{-\tau}\omega,\eta_0,s)\right\|_{\mu,2}^2\\
&\le\left(\left\|A^{\frac{1}{4}}v_0\right\|^2+(\lambda+\delta^2-\beta_2\delta)\left\|A^{\frac{1}{4}}u_0\right\|^2+\left\|A^{\frac{3}{4}}u_0\right\|^2+\left\|A^{\frac{1}{4}}\eta_0\right\|_{\mu,2}^2\right)e^{2\int_0^{-t}\varrho_1(s,\omega)\mathrm{d}s}\\
&\quad+c\int_{-t}^{0}e^{2\int_0^{s}\varrho_1(r,\omega)\mathrm{d}r}\|g(\cdot,s+\tau)\|_1^2\mathrm{d}s.
\end{aligned}\tag{4.51}$$

Next we show that for any $s\le -T_1$

$$e^{2\int_0^{s}\varrho_1(r,\omega)\mathrm{d}r}\le e^{\sigma s}.\tag{4.52}$$

In fact, using the two inequalities in (4.26), we have

$$\begin{aligned}
&\int_0^s\left[\sigma-|\varepsilon||z(\theta_r\omega)|-\gamma_2\left(\left(\frac{1}{2}\varepsilon^2+\frac{2m_0\varepsilon^2}{\varpi}\right)|z(\theta_r\omega)|^2+\gamma_3|\varepsilon||z(\theta_r\omega)|\right)\right]\mathrm{d}r\\
&\quad>\ \sigma s-|\varepsilon|\frac{2}{\sqrt{\pi\delta}}s-\gamma_2\left[\left(\frac{1}{2}\varepsilon^2+\frac{2m_0\varepsilon^2}{\varpi}\right)\frac{1}{\delta}+\gamma_3|\varepsilon|\frac{2}{\sqrt{\pi\delta}}\right]s\\
&\quad=\ -\frac{\gamma_2}{\delta}\left(\frac{1}{2}\varepsilon^2+\frac{2m_0\varepsilon^2}{\varpi}\right)s-\frac{2}{\sqrt{\pi\delta}}[\gamma_3\gamma_2+1]|\varepsilon|s+\sigma s.
\end{aligned}$$

In order to have the inequality in (4.52) valid, we need

$$\int_0^s\left[\sigma-|\varepsilon||z(\theta_r\omega)|-\gamma_2\left(\left(\frac{1}{2}\varepsilon^2+\frac{2m_0\varepsilon^2}{\varpi}\right)|z(\theta_r\omega)|^2+\gamma_3|\varepsilon||z(\theta_r\omega)|\right)\right]\mathrm{d}r\le\frac{\sigma}{2}s.$$

Since $s\le -T_1$, then it requires that

$$\frac{\gamma_2}{\delta}\left(\frac{1}{2}\varepsilon^2+\frac{2m_0\varepsilon^2}{\varpi}\right)+\frac{2}{\sqrt{\pi\delta}}[\gamma_3\gamma_2+1]|\varepsilon|-\frac{\sigma}{2}<0.$$

Solving this quadratic inequality, ε needs to satisfy (3.18).

By (3.19) and (4.52), we see that the following integral is convergent:

$$R_3^2(\tau,\omega)=c\int_{-\infty}^{0}e^{2\int_0^{s}\varrho_1(r,\omega)\mathrm{d}r}\|g(\cdot,s+\tau)\|_1^2\mathrm{d}s.\tag{4.53}$$

For all $t\ge T_1$, we get from (4.52) that

$$\begin{aligned}
&\left(\left\|A^{\frac{1}{4}}v_0\right\|^2+(\lambda+\delta^2-\beta_2\delta)\left\|A^{\frac{1}{4}}u_0\right\|^2+\left\|A^{\frac{3}{4}}u_0\right\|^2+\left\|A^{\frac{1}{4}}\eta_0\right\|_{\mu,2}^2\right)e^{2\int_0^{-t}\Gamma_1(s,\omega)\mathrm{d}s}\\
&\quad\le\ ce^{-\sigma t}\left(\left\|A^{\frac{1}{4}}v_0\right\|^2+\left\|A^{\frac{1}{4}}u_0\right\|^2+\left\|A^{\frac{3}{4}}u_0\right\|^2+\left\|A^{\frac{1}{4}}\eta_0\right\|_{\mu,2}^2\right).
\end{aligned}\tag{4.54}$$

From (4.1), (4.51), (4.53), and (4.54), there exists $T_4=T_4(\tau,\omega,D)\ge T_1$ such that for all $t\ge T_4$,

$$\left\|A^{\frac{1}{4}}\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\right\|_E^2\le R_3^2(\tau,\omega)+ce^{-\sigma t}\left(\left\|A^{\frac{1}{4}}v_0\right\|^2+\left\|A^{\frac{1}{4}}u_0\right\|^2+\left\|A^{\frac{3}{4}}u_0\right\|^2+\left\|A^{\frac{1}{4}}\eta_0\right\|_{\mu,2}^2\right).\tag{4.55}$$

Thus, the proof is completed. □

In what follows, we derive uniform estimates on the tails of solutions when x and t approach infinity. These estimates will be used to overcome the difficulty caused by non-compactness in unbounded domains and are crucial for proving the pullback asymptotic compactness of the cocycle.

Lemma 4.4. *Under Assumptions I and II, for every $\eta > 0, \tau \in \mathbb{R}, \omega \in \Omega, D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \eta) > 0$, $K = K(\tau, \omega, \eta) \geq 1$ such that for all $t \geq T$, $k \geq K$, the solutions of problems (3.12)–(3.13) satisfy*

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|^2_{E(\mathbb{R}^n \setminus \mathbb{B}_k)} \leq \eta, \tag{4.56}$$

where for $k \geq 1$, $\mathbb{B}_k = \{x \in \mathbb{R}^n : |x| \leq k\}$ and $\mathbb{R}^n \setminus \mathbb{B}_k$ is the complement of $\mathbb{B}_k$.

Proof. Take a smooth function ρ, such that $0 \leq \rho \leq 1$ for $s \in \mathbb{R}$, and

$$\rho(s) = \begin{cases} 0, & \text{if } 0 \leq |s| \leq 1, \\ 1, & \text{if } |s| \geq 2, \end{cases} \tag{4.57}$$

and there exist constants $\mu_1, \mu_2, \mu_3, \mu_4$ such that $|\rho'(s)| \leq \mu_1$, $|\rho''(s)| \leq \mu_2$, $|\rho'''(s)| \leq \mu_3$, $|\rho''''(s)| \leq \mu_4$ for $s \in \mathbb{R}$. Taking the inner product of $(3.12)_2$ with $\rho\left(\frac{|x|^2}{k^2}\right)v$ in $L^2(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\mathrm{d}x - (\delta - \varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\mathrm{d}x + \int_{\mathbb{R}^n}\int_0^\infty \mu(s)A\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)v\mathrm{d}s\mathrm{d}x \\
&\quad + (\lambda + \delta^2)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uv\mathrm{d}x + \int_{\mathbb{R}^n}(Au)\rho\left(\frac{|x|^2}{k^2}\right)v\mathrm{d}x + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x, u)v\mathrm{d}x \\
&= \varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uv\mathrm{d}x - \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(h(v + \varepsilon uz(\theta_t\omega) - \delta u))v\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)g(x, t)v\mathrm{d}x.
\end{aligned} \tag{4.58}$$

First, by (3.10), similar to (4.5), we have

$$\begin{aligned}
&-\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(h(v + \varepsilon uz(\theta_t\omega) - \delta u))v\mathrm{d}x \\
&\quad = -\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(h(v + \varepsilon uz(\theta_t\omega) - \delta u) - h(0))v\mathrm{d}x \\
&\quad \leq -\beta_1\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\mathrm{d}x + h'(\vartheta)\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uv\mathrm{d}x \\
&\qquad + \beta_2|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u||v|\mathrm{d}x.
\end{aligned} \tag{4.59}$$

Taking (4.59) into (4.58), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\mathrm{d}x - (\delta - \varepsilon z(\theta_t\omega) - \beta_1)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\mathrm{d}x + \int_{\mathbb{R}^n}\int_0^\infty \mu(s)A\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)v\mathrm{d}s\mathrm{d}x \\
&\quad + (\lambda + \delta^2 - h'(\vartheta)\delta)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uv\mathrm{d}x + \int_{\mathbb{R}^n}(Au)\rho\left(\frac{|x|^2}{k^2}\right)v\mathrm{d}x + \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x, u)v\mathrm{d}x
\end{aligned} \tag{4.60}$$

$$\le \varepsilon z(\theta_t\omega)(3\delta - \varepsilon z(\theta_t\omega)) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) uv\,\mathrm{d}x + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) g(x,t)v\,\mathrm{d}x$$
$$+ \beta_2|\varepsilon||z(\theta_t\omega)| \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|u||v|\,\mathrm{d}x.$$

For the fourth term on the left-hand side of (4.60), we have

$$\begin{aligned}
&(\lambda + \delta^2 - h'(\vartheta)\delta) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) uv\,\mathrm{d}x \\
&= (\lambda + \delta^2 - h'(\vartheta)\delta) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) u\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \delta u - \varepsilon u z(\theta_t\omega)\right)\mathrm{d}x \\
&= (\lambda + \delta^2 - h'(\vartheta)\delta) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}u^2 + (\delta - \varepsilon z(\theta_t\omega))u^2\right)\mathrm{d}x \\
&\ge (\lambda + \delta^2 - \beta_2\delta)\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|u|^2\mathrm{d}x + \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|u|^2\mathrm{d}x\right) \\
&\quad - (\lambda + \delta^2 + \beta_2\delta)|\varepsilon||z(\theta_t\omega)| \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|u|^2\mathrm{d}x.
\end{aligned} \tag{4.61}$$

For the fifth term on the left-hand side of (4.60), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} (Au)\rho\left(\frac{|x|^2}{k^2}\right) v\,\mathrm{d}x &= \int_{\mathbb{R}^n} (Au)\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \delta u - \varepsilon u z(\theta_t\omega)\right)\mathrm{d}x \\
&= \int_{\mathbb{R}^n} (\Delta^2 u)\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \delta u - \varepsilon z(\theta_t\omega)u\right)\mathrm{d}x \\
&= \int_{\mathbb{R}^n} (\Delta u)\Delta\left(\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \delta u - \varepsilon z(\theta_t\omega)u\right)\right)\mathrm{d}x \\
&= \int_{\mathbb{R}^n} (\Delta u)\left(\left(\frac{2}{k^2}\rho'\left(\frac{|x|^2}{k^2}\right) + \frac{4x^2}{k^4}\rho''\left(\frac{|x|^2}{k^2}\right)\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \delta u - \varepsilon z(\theta_t\omega)u\right)\right. \\
&\quad + 2\cdot\frac{2|x|}{k^2}\rho'\left(\frac{|x|^2}{k^2}\right)\nabla\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \delta u - \varepsilon z(\theta_t\omega)u\right) + \rho\left(\frac{|x|^2}{k^2}\right)\Delta\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \delta u \right. \\
&\quad \left.\left. - \varepsilon z(\theta_t\omega)u\right)\right)\mathrm{d}x \\
&\ge - \int_{k<x<\sqrt{2}k} \left(\frac{2\mu_1}{k^2} + \frac{4\mu_2 x^2}{k^4}\right)|(\Delta u)v|\mathrm{d}x - \int_{k<x<\sqrt{2}k} \frac{4\mu_1 x}{k^2}|(\Delta u)(\nabla v)|\mathrm{d}x \\
&\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x + \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x - \varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x \\
&\ge - \int_{\mathbb{R}^n} \left(\frac{2\mu_1 + 8\mu_2}{k^2}\right)|(\Delta u)v|\mathrm{d}x - \int_{\mathbb{R}^n} \frac{4\sqrt{2}\mu_1}{k}|(\Delta u)(\nabla v)|\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x \\
&\quad + \delta \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x - \varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x
\end{aligned} \tag{4.62}$$

$$\geq -\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2)-\frac{4\sqrt{2}\mu_1}{k}\|\Delta u\|\|\nabla v\|+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x$$
$$+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x-\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x$$
$$\geq -\frac{\mu_1+4\mu_2}{k^2}(\|\Delta u\|^2+\|v\|^2)-\frac{2\sqrt{2}\mu_1}{k}(\|\Delta u\|^2+\|\nabla v\|^2)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x$$
$$-(|\varepsilon||z(\theta_t\omega)|-\delta)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x.$$

For the sixth term on the left-hand side of (4.60), we have

$$\begin{aligned}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)v\mathrm{d}x&=\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)\left(\frac{\mathrm{d}u}{\mathrm{d}t}+\delta u-\varepsilon z(\theta_t\omega)u\right)\mathrm{d}x\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(x,u)\mathrm{d}x+\delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)u\mathrm{d}x\\&\quad-\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)u\mathrm{d}x.\end{aligned}\tag{4.63}$$

By (3.6), we see that

$$\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)u\mathrm{d}x\geq c_2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(x,u)\mathrm{d}x+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\phi_2(x)\mathrm{d}x.\tag{4.64}$$

On the other hand, by (3.5) and (3.7),

$$\begin{aligned}\varepsilon z(\theta_t\omega)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)f(x,u)u\mathrm{d}x\leq\ &c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(x,u)\mathrm{d}x+c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^2\mathrm{d}x\\&+c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|\phi_1|^2+|\phi_3|)\mathrm{d}x.\end{aligned}\tag{4.65}$$

Similar to (4.9) and (4.10) in Lemma 4.1, we get

$$\begin{aligned}&\varepsilon z(\theta_t\omega)(3\delta-\varepsilon z(\theta_t\omega))\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)uv\mathrm{d}x+\beta_2|\varepsilon||z(\theta_t\omega)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u||v|\mathrm{d}x\\&\leq\left(\frac{1}{2}(3\delta+\beta_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|u|^2+|v|^2)\mathrm{d}x.\end{aligned}\tag{4.66}$$

$$\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)g(x,t)v\mathrm{d}x\leq\frac{1}{2(\beta_1-\delta)}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|g(x,t)|^2\mathrm{d}x+\frac{\beta_1-\delta}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|v|^2\mathrm{d}x.\tag{4.67}$$

For the third term on the left-hand side of (4.60), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_0^\infty \mu(s)A\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)v\mathrm{d}s\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^n}\int_0^\infty \mu(s)A\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t}+\delta u-\varepsilon uz(\theta_t\omega)\right)\mathrm{d}s\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)\Delta\eta(s)\right)\Delta\left(\rho\left(\frac{|x|^2}{k^2}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t}+\delta u-\varepsilon uz(\theta_t\omega)\right)\right)\mathrm{d}s\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^n}\left(\int_0^\infty \mu(s)\Delta\eta(s)\right)\left(\frac{2}{k^2}\rho'\left(\frac{|x|^2}{k^2}\right)+\frac{4x^2}{k^4}\rho''\left(\frac{|x|^2}{k^2}\right)\right)\left(\frac{\mathrm{d}u}{\mathrm{d}t}+\delta u-\varepsilon uz(\theta_t\omega)\right)\\
&\qquad+2\cdot\frac{2|x|}{k^2}\rho'\left(\frac{|x|^2}{k^2}\right)\nabla\left(\frac{\mathrm{d}u}{\mathrm{d}t}+\delta u-\varepsilon uz(\theta_t\omega)\right)+\rho\left(\frac{|x|^2}{k^2}\right)\Delta\left(\frac{\mathrm{d}u}{\mathrm{d}t}+\delta u-\varepsilon uz(\theta_t\omega)\right)\mathrm{d}s\mathrm{d}x\\
&\quad\geq -\int_{k<x<\sqrt{2}k}\left(\frac{2\mu_1}{k^2}+\frac{4\mu_2x^2}{k^4}\right)\int_0^\infty \mu(s)|\Delta\eta(s)v|\mathrm{d}s\mathrm{d}x \qquad (4.68)\\
&\qquad-\int_{k<x<\sqrt{2}k}\frac{4\mu_1x}{k^2}\int_0^\infty \mu(s)|\Delta\eta(s)\nabla v|\mathrm{d}s\mathrm{d}x+\int_{\mathbb{R}^n}\int_0^\infty \mu(s)\Delta\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)\Delta u_t\mathrm{d}s\mathrm{d}x\\
&\qquad+\delta\int_{\mathbb{R}^n}\int_0^\infty \mu(s)\Delta\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)\Delta u\mathrm{d}s\mathrm{d}x-|\varepsilon|\int_{\mathbb{R}^n}\int_0^\infty \mu(s)|\Delta\eta(s)|\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u||z(\theta_t\omega)|\mathrm{d}s\mathrm{d}x\\
&\quad\geq -\frac{2\mu_1+8\mu_2}{k^2}\int_{\mathbb{R}^n}\int_0^\infty \mu(s)|\Delta\eta(s)v|\mathrm{d}s\mathrm{d}x-\frac{4\sqrt{2}\mu_1}{k}\int_{\mathbb{R}^n}\int_0^\infty \mu(s)|\Delta\eta(s)\nabla v|\mathrm{d}s\mathrm{d}x\\
&\qquad+\int_{\mathbb{R}^n}\int_0^\infty \mu(s)\Delta\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)\Delta u_t\mathrm{d}s\mathrm{d}x+\delta\int_{\mathbb{R}^n}\int_0^\infty \mu(s)\Delta\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)\Delta u\mathrm{d}s\mathrm{d}x\\
&\qquad-|\varepsilon|\int_{\mathbb{R}^n}\int_0^\infty \mu(s)|\Delta\eta(s)|\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u||z(\theta_t\omega)|\mathrm{d}s\mathrm{d}x.
\end{aligned}
$$

Using Young's inequality, we get

$$
-\frac{2\mu_1+8\mu_2}{k^2}\int_{\mathbb{R}^n}\int_0^\infty \mu(s)|\Delta\eta(s)v|\mathrm{d}s\mathrm{d}x\geq -\frac{\mu_1+4\mu_2}{k^2}(\|\eta\|^2_{\mu,2}+m_0\|v\|^2), \qquad (4.69)
$$

and

$$
-\frac{4\sqrt{2}\mu_1}{k}\int_{\mathbb{R}^n}\int_0^\infty \mu(s)|\Delta\eta(s)\nabla v|\mathrm{d}s\mathrm{d}x\geq -\frac{2\sqrt{2}\mu_1}{k}(\|\eta\|^2_{\mu,2}+m_0\|\nabla v\|^2). \qquad (4.70)
$$

Integrating by parts with respect to s and using (3.4), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_0^\infty \mu(s)\Delta\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)\Delta u_t\mathrm{d}s\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^n}\int_0^\infty \mu(s)\Delta\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)\Delta(\eta_t+\eta_s)\mathrm{d}s\mathrm{d}x \qquad (4.71)\\
&\quad\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,2}\mathrm{d}x+\frac{\varpi}{2}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,2}\mathrm{d}x,
\end{aligned}
$$

$$\delta \int_{\mathbb{R}^n} \int_0^\infty \mu(s)\Delta\eta(s)\rho\left(\frac{|x|^2}{k^2}\right)\Delta u \mathrm{d}s\mathrm{d}x \geq -\frac{\varpi}{8}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,2}\mathrm{d}x - \frac{2m_0\delta^2}{\varpi}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x, \tag{4.72}$$

$$\begin{aligned} -|\varepsilon| \int_{\mathbb{R}^n} \int_0^\infty \mu(s)|\Delta\eta(s)|\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u||z(\theta_t\omega)|\mathrm{d}s\mathrm{d}x \geq &-\frac{\varpi}{8}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,2}\mathrm{d}x \\ &- \frac{2m_0\varepsilon^2}{\varpi}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2|z(\theta_t\omega)|^2\mathrm{d}x. \end{aligned} \tag{4.73}$$

Then it follows from (4.60)–(4.73) that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2 + |\Delta u|^2 + |\eta(s)|^2_{\mu,2} + 2F(x,u))\mathrm{d}x \\ &\quad + \delta\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2)\mathrm{d}x + \delta\left(1 - \frac{2m_0\delta}{\varpi}\right)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\Delta u|^2\mathrm{d}x \\ &\quad + \frac{\varpi}{4}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|\eta(s)|^2_{\mu,2}\mathrm{d}x + \delta c_2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(x,u)\mathrm{d}x \\ &\leq c(\|\Delta u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \|\eta(s)\|^2_{\mu,2}) + \left(\frac{1}{2}(3\delta + \beta_2)|\varepsilon||z(\theta_t\omega)| + \left(\frac{1}{2}\varepsilon^2 + \frac{2m_0\varepsilon^2}{\varpi}\right)|z(\theta_t\omega)|^2\right) \\ &\quad \times \int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|u|^2 + |v|^2 + |\Delta u|^2)\mathrm{d}x + c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|g(x,t)|^2\mathrm{d}x \\ &\quad + c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)F(x,u)\mathrm{d}x + c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|u|^2\mathrm{d}x \\ &\quad + c|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|\phi_1|^2 + |\phi_3|)\mathrm{d}x + c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\phi_2(x)\mathrm{d}x \\ &\quad + |\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|v|^2 + (\lambda + \delta^2 - \beta_1\delta)|u|^2 + |\Delta u|^2)\mathrm{d}x. \end{aligned} \tag{4.74}$$

Since that $\phi_1 \in L^2(\mathbb{R}^n)$, $\phi_2, \phi_3 \in L^1(\mathbb{R}^n)$, for given $\eta > 0$, there exists $K_0 = K_0(\eta) \geq 1$ such that for all $k \geq K_0$,

$$\begin{aligned} c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|\phi_1|^2 + |\phi_2| + |\phi_3|)\mathrm{d}x &= c\int_{|x|\geq k}\rho\left(\frac{|x|^2}{k^2}\right)(|\phi_1|^2 + |\phi_2| + |\phi_3|)\mathrm{d}x \\ &\leq c\int_{|x|\geq k}(|\phi_1|^2 + |\phi_2| + |\phi_3|)\mathrm{d}x \leq \eta. \end{aligned} \tag{4.75}$$

Using the expression (4.20), we conclude from (4.74) that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2 + |\Delta u|^2 + |\eta(s)|^2_{\mu,2} + 2F(x,u))\mathrm{d}x \\ &\leq -\varrho(t,\omega)\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|v|^2 + (\lambda + \delta^2 - \beta_2\delta)|u|^2 + |\Delta u|^2 + |\eta(s)|^2_{\mu,2} + 2F(x,u))\mathrm{d}x \\ &\quad + c(\|\Delta u\|^2 + \|v\|^2 + \|\nabla v\|^2 + \|\eta(s)\|^2_{\mu,2}) + c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|g(x,t)|^2\mathrm{d}x + \eta(1 + |\varepsilon||z(\theta_t\omega)|). \end{aligned} \tag{4.76}$$

Integrating (4.76) over $(\tau - t, \tau)$ for $t \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|v(\tau,\tau-t,\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\omega,u_0)|^2)\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|\Delta u(\tau,\tau-t,\omega,u_0)|^2+|\eta(\tau,\tau-t,\omega,\eta_0,s)|^2_{\mu,2}+2F(x,u(\tau,\tau-t,\omega,u_0)))\mathrm{d}x\\
&\le e^{2\int_\tau^{\tau-t}\varrho(\mu,\omega)\mathrm{d}\mu}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|v_0(x)|^2+(\lambda+\delta^2-\beta_2\delta)|u_0(x)|^2)\mathrm{d}x\\
&\quad+e^{2\int_\tau^{\tau-t}\varrho(\mu,\omega)\mathrm{d}\mu}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|\Delta u_0(x)|^2+|\eta_0(s)|^2_{\mu,2}+2F(x,u_0(x)))\mathrm{d}x\\
&\quad+c\int_{\tau-t}^{\tau}e^{2\int_\tau^{s}\varrho(\mu,\omega)\mathrm{d}\mu}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|g(x,s)|^2\mathrm{d}s\mathrm{d}x+\eta\int_{\tau-t}^{\tau}e^{2\int_\tau^{s}\varrho(\mu,\omega)\mathrm{d}\mu}(1+|\varepsilon||z(\theta_s\omega)|)\mathrm{d}s\\
&\quad+c\int_{\tau-t}^{\tau}e^{2\int_\tau^{s}\varrho(\mu,\omega)\mathrm{d}\mu}(\|\Delta u(s,\tau-t,\omega,u_0)\|^2+\|v(s,\tau-t,\omega,v_0)\|^2)\mathrm{d}s\\
&\quad+c\int_{\tau-t}^{\tau}e^{2\int_\tau^{s}\varrho(\mu,\omega)\mathrm{d}\mu}(\|\nabla v(s,\tau-t,\omega,u_0)\|^2+\|\eta(s,\tau-t,\omega,\eta_0,s)\|^2_{\mu,2})\mathrm{d}s.
\end{aligned}\tag{4.77}$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.77), for every $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$, and $\omega\in\Omega$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\eta(\tau,\tau-t,\theta_{-\tau}\omega,\eta_0,s)|^2_{\mu,2}\\
&\quad+2F(x,u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)))\mathrm{d}x\\
&\le e^{2\int_\tau^{\tau-t}\varrho(\mu-\tau,\omega)\mathrm{d}\mu}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|v_0(x)|^2+(\lambda+\delta^2-\beta_2\delta)|u_0(x)|^2)\mathrm{d}x\\
&\quad+e^{2\int_\tau^{\tau-t}\varrho(\mu-\tau,\omega)\mathrm{d}\mu}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|\Delta u_0(x)|^2+|\eta_0(s)|^2_{\mu,2}+2F(x,u_0(x)))\mathrm{d}x\\
&\quad+c\int_{\tau-t}^{\tau}e^{2\int_\tau^{s}\varrho(\mu-\tau,\omega)\mathrm{d}\mu}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|g(x,s)|^2\mathrm{d}s\mathrm{d}x+\eta\int_{\tau-t}^{\tau}e^{2\int_\tau^{s}\varrho(\mu-\tau,\omega)\mathrm{d}\mu}(1+|\varepsilon||z(\theta_{s-\tau}\omega)|)\mathrm{d}s\\
&\quad+c\int_{\tau-t}^{\tau}e^{2\int_\tau^{s}\varrho(\mu-\tau,\omega)\mathrm{d}\mu}(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)\mathrm{d}s\\
&\quad+c\int_{\tau-t}^{\tau}e^{2\int_\tau^{s}\varrho(\mu-\tau,\omega)\mathrm{d}\mu}(\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\eta(s,\tau-t,\theta_{-\tau}\omega,\eta_0,s)\|^2_{\mu,2})\mathrm{d}s\\
&\le e^{2\int_0^{-t}\varrho(\mu,\omega)\mathrm{d}\mu}(\|v_0(x)\|^2+(\lambda+\delta^2-\beta_2\delta)\|u_0(x)\|^2+\|\Delta u_0(x)\|^2+\|\eta_0(s)\|^2_{\mu,2}+2\widetilde{F}(x,u_0(x)))\mathrm{d}x\\
&\quad+c\int_{-t}^{0}e^{2\int_0^{s}\varrho(\mu,\omega)\mathrm{d}\mu}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)|g(x,s+\tau)|^2\mathrm{d}s\mathrm{d}x+\eta\int_{-t}^{0}e^{2\int_0^{s}\varrho(\mu,\omega)\mathrm{d}\mu}(1+|\varepsilon||z(\theta_s\omega)|)\mathrm{d}s\\
&\quad+c\int_{\tau-t}^{\tau}e^{2\int_\tau^{s}\varrho(\mu-\tau,\omega)\mathrm{d}\mu}(\|\Delta u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|v(s,\tau-t,\theta_{-\tau}\omega,v_0)\|^2)\mathrm{d}s\\
&\quad+c\int_{\tau-t}^{\tau}e^{2\int_\tau^{s}\varrho(\mu-\tau,\omega)\mathrm{d}\mu}(\|\nabla v(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|\eta(s,\tau-t,\theta_{-\tau}\omega,\eta_0,s)\|^2_{\mu,2})\mathrm{d}s.
\end{aligned}\tag{4.78}$$

It is similar to (4.31), for an arbitrarily given $\eta > 0$, there exists $T = T(\tau, \omega, D, \eta)$ such that for all $t \geq T$,

$$e^{2\int_0^{-t} \varrho(\mu,\omega)\mathrm{d}\mu}(\|v_0(x)\|^2 + (\lambda + \delta^2 - \beta_2\delta)\|u_0(x)\|^2 + \|\Delta u_0(x)\|^2 + \|\eta_0(s)\|^2_{\mu,2} + 2\widetilde{F}(x, u_0(x)))\mathrm{d}x \leq \eta. \tag{4.79}$$

For the fourth and fifth terms on the right-hand side of (4.78), by Lemmas 4.1 and 4.3, for all $t \geq \max\{T_2, T_4\}$,

$$\begin{aligned} &c\int_{\tau-t}^{\tau} e^{2\int_\tau^s \varrho(\mu-\tau,\omega)\mathrm{d}\mu}(\|\Delta u(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 + \|v(s, \tau - t, \theta_{-\tau}\omega, v_0)\|^2)\mathrm{d}s \\ &\quad + c\int_{\tau-t}^{\tau} e^{2\int_\tau^s \varrho(\mu-\tau,\omega)\mathrm{d}\mu}(\|\nabla v(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^2 + \|\eta(s, \tau - t, \theta_{-\tau}\omega, \eta_0, s)\|^2_{\mu,2})\mathrm{d}s \\ &\leq \eta(R_1^2(\varepsilon, \tau, \omega) + R_3^2(\varepsilon, \tau, \omega)). \end{aligned} \tag{4.80}$$

For the second term on the right-hand side of (4.78), there exists $K_1 = K_1(\tau, \omega) \geq 1$ such that for all $k \geq K_1$, by (4.27), we get

$$\begin{aligned} &\int_{-\infty}^{0} e^{2\int_0^s \varrho(\mu,\omega)\mathrm{d}\mu} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|g(x, s + \tau)|^2\mathrm{d}s\mathrm{d}x \\ &\leq \int_{-\infty}^{-T_1} e^{2\int_0^s \varrho(\mu,\omega)\mathrm{d}\mu} \int_{|x|\geq k} |g(x, s + \tau)|^2\mathrm{d}s\mathrm{d}x + \int_{-T_1}^{0} e^{2\int_0^s \varrho(\mu,\omega)\mathrm{d}\mu} \int_{|x|\geq k} |g(x, s + \tau)|^2\mathrm{d}s\mathrm{d}x \\ &\leq \int_{-\infty}^{-T_1} e^{\sigma s} \int_{|x|\geq k} |g(x, s + \tau)|^2\mathrm{d}s\mathrm{d}x + e^{c^*}\int_{-T_1}^{0} e^{\sigma s} \int_{|x|\geq k} |g(x, s + \tau)|^2\mathrm{d}s\mathrm{d}x, \end{aligned} \tag{4.81}$$

where $c^* > 0$ is a random variable independent of $\tau \in \mathbb{R}$ and $D \in \mathcal{D}$.

Therefore, by (3.20) there exists $K_2(\tau, \omega) \geq K_1$ such that for all $k \geq K_2$, we obtain

$$c\int_{-\infty}^{0} e^{2\int_0^s \varrho(\mu,\omega)\mathrm{d}\mu} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)|g(x, s + \tau)|^2\mathrm{d}s\mathrm{d}x \leq e^c\int_{-\infty}^{0} e^{\sigma s} \int_{|x|\geq k} |g(x, s + \tau)|^2\mathrm{d}s\mathrm{d}x \leq \eta. \tag{4.82}$$

Let

$$R_4^2(\tau, \omega) = \int_{-\infty}^{0} e^{2\int_0^s \varrho(\mu,\omega)\mathrm{d}\mu}(1 + |\varepsilon||z(\theta_s\omega)|)\mathrm{d}s, \tag{4.83}$$

by (4.27), we know that the integral of (4.83) is convergent.

Together with (4.78)–(4.82), we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)(|v(\tau, \tau - t, \theta_{-\tau}\omega, v_0)|^2 + (\lambda + \delta^2 - \beta_2\delta)|u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)|^2)\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right)(|\Delta u(\tau, \tau - t, \theta_{-\tau}\omega, u_0)|^2 + |\eta(\tau, \tau - t, \theta_{-\tau}\omega, \eta_0, s)|^2_{\mu,2}) \\ &\quad + 2F(x, u(\tau, \tau - t, \theta_{-\tau}\omega, u_0))\mathrm{d}x \\ &\leq 2\eta(1 + R_1^2(\tau, \omega) + R_3^2(\tau, \omega) + R_4^2(\tau, \omega)). \end{aligned} \tag{4.84}$$

It follows from (4.30) and (4.84) that there exists $K_3 = K_3(\tau, \omega) \geq K_2$, such that for all $k \geq K_3$, $t \geq \max\{T_2, T_4\}$,

$$\begin{aligned}&\int\limits_{|x|\geq\sqrt{2}k}\rho\Big(\frac{|x|^2}{k^2}\Big)(|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)|^2+(\lambda+\delta^2-\beta_2\delta)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2)\mathrm{d}x\\&\quad+\int\limits_{\mathbb{R}^n}\rho\Big(\frac{|x|^2}{k^2}\Big)(|\Delta u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)|^2+|\eta(\tau,\tau-t,\theta_{-\tau}\omega,\eta_0,s)|^2_{\mu,2})\mathrm{d}x\\&\leq 3\eta(1+R_1^2(\tau,\omega)+R_3^2(\tau,\omega)+R_4^2(\tau,\omega)),\end{aligned}$$

which implies (4.56). □

In order to obtain the precompactness of the solutions for (3.12)–(3.13) in bounded domain $\mathbb{B}_{2k}$ later, we decompose $\varphi=(u,v,\eta)^\top$ of (3.12)–(3.13) into $\varphi=\varphi_L+\varphi_N$, where $\varphi_L=(u_L,\xi_L,\eta_L)^T$ and $\varphi_N=(u_N,v_N,\eta_N)^T$ solve, respectively,

$$\begin{cases}\dfrac{\mathrm{d}u_L}{\mathrm{d}t}+\delta u_L=\xi_L,\\ \dfrac{\mathrm{d}\xi_L}{\mathrm{d}t}-\delta\xi_L+h(u_t)-h(u_{N,t})+(\lambda+\delta^2+A)u_L+\displaystyle\int_0^\infty\mu(s)A\eta_L(s)\mathrm{d}s=0,\\ \eta_{L,t}+\eta_{L,s}=u_{L,t},\\ u_L(x,\tau)=u_0(x),\quad \xi_L(x,\tau)=\xi_0(x),\\ \eta_L(x,\tau,s)=\eta_0(x,s)=u_L(x,\tau)-u_L(x,\tau-s)\end{cases}\tag{4.85}$$

and

$$\begin{cases}\dfrac{\mathrm{d}u_N}{\mathrm{d}t}+\delta u_N=v_N+\varepsilon uz(\theta_t\omega),\\ \dfrac{\mathrm{d}v_N}{\mathrm{d}t}-\delta v_N+h(u_{N,t})+(\lambda+\varepsilon^2+A)u_N+\displaystyle\int_0^\infty\mu(s)A\eta_N(s)\mathrm{d}s+f(x,u)\\ \qquad\qquad=g(x,t)+\varepsilon(v-3\delta u+\varepsilon uz(\theta_t\omega)),\\ \eta_{N,t}+\eta_{N,s}=u_{N,t},\\ u_N(x,\tau)=0,\quad v_N(x,\tau)=0,\\ \eta_N(x,\tau,s)=0.\end{cases}\tag{4.86}$$

For the solutions of equations (4.85) and (4.86), by Lemmas 4.1 and 4.3, we can easily get the following estimates and regularity results, respectively.

Lemma 4.5. *Assume that* (3.4) *and* (3.10) *hold. Then for any* $(u_L,\xi_L,\eta_L)^T$ *of the solution of* (4.85) *satisfies*

$$\|\varphi_L(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{L,0})\|_E^2\to 0,\quad \text{when } t\to\infty.\tag{4.87}$$

Lemma 4.6. *Under Assumptions I and II, for every* $\tau\in\mathbb{R}$, $\omega\in\Omega$, $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, *there exists* $T=T(\tau,\omega,D)>0$ *such that for all* $t\geq T$ *the solution of problem* (4.86) *satisfies*

$$\Big\|A^{\frac{1}{4}}\varphi_N(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{N,0})\Big\|_E^2\leq R_3(\tau,\omega),$$

where $R_3(\tau,\omega)$ *is a random variable.*

5 Random attractors

In this section, we prove existence and uniqueness of $\mathcal{D}$-pullback attractors for the stochastic system (3.12)–(3.13). First we also need the following results to prove the asymptotic compactness about memory term as well as the existence of random attractors.

Lemma 5.1. [25] *Let X_0, X, X_1 be three Banach spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$, the first injection being compact. Let $Y \subset L^2_\mu(\mathbb{R}^+, X)$ satisfy the following hypotheses:*
(*i*) *Y is bounded in $L^2_\mu(\mathbb{R}^+, X_0) \cap H^1_\mu(\mathbb{R}^+, X_1)$;*
(*ii*) $\sup_{\eta\in Y}\|\eta(s)\|^2_X \le K_0$, $\forall s \in \mathbb{R}^+$ *for some* $K_0 > 0$.

Then Y is relatively compact in $L^2_\mu(\mathbb{R}^+, X)$.

Note that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t_m \ge 0$,

$$\begin{aligned}&\eta_N(\tau, \tau - t_m, \theta_{-\tau}\omega, \eta_{N,0}\big(\theta_{-t_m}\omega\big), s)\\ &\quad= \begin{cases} u_N\Big(\tau, \tau - t_m, \theta_{-\tau}\omega, u_{N,0}\big(\theta_{-t_m}\omega\big)\Big) - u_N\Big(\tau - s, \tau - t_m, \theta_{-\tau+s}\omega, u_{N,0}\big(\theta_{-t_m+s}\omega\big)\Big), & s \le t,\\ u_N\Big(\tau, \tau - t_m, \theta_{-\tau}\omega, u_{N,0}\big(\theta_{-t_m}\omega\big)\Big), & s \ge t\end{cases}\end{aligned} \tag{5.1}$$

and

$$\eta_{N,s}\Big(\tau, \tau - t_m, \theta_{-\tau}\omega, \eta_{N,0}\big(\theta_{-t_m}\omega\big)\Big) = \begin{cases} u_{N,t}\Big(\tau - s, \tau - t_m, \theta_{-\tau+s}\omega, u_{N,0}\big(\theta_{-t_m+s}\omega\big)\Big), & s \le t,\\ 0, & s \ge t.\end{cases} \tag{5.2}$$

Then from Lemma 4.6 and (5.2), it follows that

$$\begin{aligned}&\max\Big\{\|\eta_{N,s}\big(\tau, \tau - t_m, \theta_{-\tau}\omega, \eta_{N,0}\big(\theta_{-t_m}\omega\big), s\big)\|^2_{\mu,1}, \|\eta_N\big(\tau, \tau - t_m, \theta_{-\tau}\omega, \eta_{N,0}\big(\theta_{-t_m}\omega\big), s\big)\|^2_{\mu,3}\Big\}\\ &\le 2R_3(\tau, \omega), \quad \forall s \ge 0,\end{aligned} \tag{5.3}$$

which implies that $\{\eta_N(\tau, \tau - t_m, \theta_{-\tau}\omega, \eta_{N,0}(\theta_{-t_m}\omega), s)\}_{m=1}^\infty$ is bounded in $L^2_\mu(\mathbb{R}^+, V_3) \cap H^1_\mu(\mathbb{R}^+, V_1)$. For brevity, we denote $\hat{B}(\tau, \omega) = \{\eta_N(\tau, \tau - t_m, \theta_{-\tau}\omega, \eta_{N,0}(\theta_{-t_m}\omega), s)\}_{m=1}^\infty$. Again, by Lemmas 4.1, Lemma 4.6, and (5.1), we have

$$\sup_{\eta\in\hat{B}(\tau,\omega), s\ge 0}\|\Delta\eta(s)\|^2 = \sup_{t_m\ge 0}\ \sup_{\eta_0\left(\theta_{-t_m}\omega\right)\in D\left(\tau - t_m, \theta_{-t_m}\omega\right)}\|\Delta\eta_N\big(\tau, \tau - t_m, \theta_{-\tau}\omega, \eta_{N,0}\big(\theta_{-t_m}\omega\big), s\big)\|^2 \le 2R_1(\tau, \omega). \tag{5.4}$$

Thus, by (3.4) and (5.4), it follows that for any $\eta \in \hat{B}(\tau, \omega)$,

$$\|\eta(s)\|^2_{\mu,2} = \int_0^{+\infty}\mu(s)\|\Delta\eta(s)\|^2 ds \le 2R_1(\tau, \omega)\int_0^{+\infty} e^{-\delta s} ds \le \frac{2R_1(\tau, \omega)}{\delta}, \tag{5.5}$$

which shows that $\{\eta_N(\tau, \tau - t_m, \theta_{-\tau}\omega, \eta_{N,0}(\theta_{-t_m}\omega), s)\}_{m=1}^\infty \subset L^2_\mu(\mathbb{R}^+, H^2(\mathbb{B}_{2k_1}))$ is a bounded subset. By Lemma 4.5 and 5.1, we know that the sequence $\{\eta(\tau, \tau - t_m, \theta_{-\tau}\omega, \eta_0(\theta_{-t_m}\omega), s)\}_{m=1}^\infty$ is compact in $L^2_\mu(\mathbb{R}^+, H^2(\mathbb{B}_{2k_1}))$.

Then we apply the lemmas shown in Section 4 to prove the asymptotic compactness of solutions of (3.12)–(3.13) in E.

Lemma 5.2. *Under Assumptions I and II, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, the sequence of solutions of* (3.12)–(3.13), $\{\varphi(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,n})\}_{m=1}^\infty$ *has a convergent subsequence in E whenever $t_m \to \infty$ and $\varphi_{0,n} \in D(\tau - t_m, \theta_{-t_m}\omega)$ with $D \in \mathcal{D}$.*

Proof. By Lemma 4.4, for every $\zeta > 0$, there exist $k_0 = k_0(\tau, \omega, \zeta) \ge 1$ and $m_2 = m_2(\tau, \omega, D, \zeta) \ge m_1$ such that for all $m \ge m_1$,

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{0,n})\|^2_{E\left(\mathbb{R}^n\setminus\mathbb{B}_{k_0}\right)} \le \zeta, \tag{5.6}$$

By Lemma 4.6, there exist $k_1 = k_1(\tau, \omega) \ge k_0$, such that

$$\Big\|A^{\frac14}u_N(\tau, \tau - t_m, \theta_{-\tau}\omega, u_{N,0}\big(\theta_{-t_m}\omega\big)\Big\|^2_{H^2\left(\mathbb{B}_{2k_1}\right)} + \Big\|A^{\frac14}v_N(\tau, \tau - t_m, \theta_{-\tau}\omega, v_{N,0}\big(\theta_{-t_m}\omega\big)\Big\|^2_{L^2\left(\mathbb{B}_{2k_1}\right)} \le R_3(\tau, \omega),$$

which along with the compact embedding $H^3(\mathbb{B}_{2k_1}) \times H^1(\mathbb{B}_{2k_1}) \hookrightarrow H^2(\mathbb{B}_{2k_1}) \times L^2(\mathbb{B}_{2k_1})$, we know the sequences $\{(u_N(\tau, \tau - t_m, \theta_{-\tau}\omega, u_{N,0}(\theta_{-t_m}\omega)), v_N(\tau, \tau - t_m, \theta_{-\tau}\omega, v_{N,0}(\theta_{-t_m}\omega)))\}_{m=1}^{\infty}$ are precompact in $H^2(\mathbb{B}_{2k_1}) \times L^2(\mathbb{B}_{2k_1})$. By Lemma 4.5, we deduce that the sequences $\{(u(\tau, \tau - t_m, \theta_{-\tau}\omega, u_0(\theta_{-t_m}\omega)), v(\tau, \tau - t_m, \theta_{-\tau}\omega, v_0(\theta_{-t_m}\omega)))\}_{m=1}^{\infty}$ are precompact in $H^2(\mathbb{B}_{2k_1}) \times L^2(\mathbb{B}_{2k_1})$. In addition, the sequences $\{\eta(\tau, \tau - t_m, \theta_{-\tau}\omega, \eta_0(\theta_{-t_m}\omega), s)\}_{m=1}^{\infty}$ are precompact in $L^2_\mu(\mathbb{R}^+, H^2(\mathbb{B}_{2k_1}))$. Thus, $\{\varphi(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,n})\}_{m=1}^{\infty}$ is precompact in $E(\mathbb{B}_{2k_1})$, this together with (5.6) implies $\{\varphi(\tau, \tau - t_m, \theta_{-\tau}\omega, \varphi_{0,n})\}_{m=1}^{\infty}$ has a convergent subsequence in $E(\mathbb{R}^n)$. □

Since Lemma 4.1 implies a pullback $\mathcal{D}$-absorbing set for Φ, and Φ_ε is pullback $\mathcal{D}$-asymptotically compact in E from Lemma 5.2, we immediately get the following existence theorem by Proposition 2.1.

Theorem 5.3. *Under Assumptions I and II, the cocycle Φ generated by the stochastic plate equation* (3.12)–(3.13) *has a unique pullback $\mathcal{D}$-attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in the space E.*

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