

Research Article

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On multi-step methods for singular fractional q -integro-differential equations

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Abstract: The objective of this paper is to investigate, by applying the standard Caputo fractional q -derivative of order α , the existence of solutions for the singular fractional q -integro-differential equation $\mathcal{D}_q^\alpha[k](t) = \Omega(t, k_1, k_2, k_3, k_4)$, under some boundary conditions where Ω is singular at some point $0 \leq t \leq 1$, on a time scale $\mathbb{T}_{t_0} = \{t : t = t_0 q^n\} \cup \{0\}$, for $n \in \mathbb{N}$ where $t_0 \in \mathbb{R}$ and $q \in (0, 1)$. We consider the compact map and avail the Lebesgue dominated theorem for finding solutions of the addressed problem. Besides, we prove the main results in context of completely continuous functions. Our attention is concentrated on fractional multi-step methods of both implicit and explicit type, for which sufficient existence conditions are investigated. Finally, we present some examples involving graphs, tables and algorithms to illustrate the validity of our theoretical findings.

Keywords: singularity, multi-step methods, q -integro-differential equation

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1 Introduction

The field of fractional calculus plays a fundamental role in mathematical analysis. It provides efficient techniques to solve fractional differential equations and inclusions [1–10]. On the other hand, one of the most interesting topics is q -difference equations which were introduced by Jackson in [11]. Later, many researchers studied and presented their significant applications [12–22].

In 2007, Atici and Eloe studied discrete fractional calculus and considered a family of finite fractional linear difference equations. They developed the theory of linear finite fractional difference equations analogously to the theory of finite difference equations. In [23], the fractional problem

$$\mathcal{D}^\sigma[k](r) + w(r, k(r), \mathcal{D}^\xi[k](r)) = 0,$$

with boundary conditions $k(0) = k(1) = 0$ was investigated, where $0 < r < 1$, $1 < \sigma < 2$, $0 < \xi \leq \sigma - 1$, \mathcal{D}^σ is the standard Riemann-Liouville fractional derivative, w satisfies the Carathéodory conditions on

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$[0, 1] \times (0, \infty) \times \mathbb{R}$, w is positive and $w(t, k, l)$ is singular at $t = 0$. In [24,25], the fractional differential equation $\mathcal{D}^\sigma[k](r) + w(r, k(r)) = 0$ with boundary conditions $k(0) = k''(0) = 0$ and $k(1) = \lambda \int_0^1 k(s)ds$ was studied, where $0 < r < 1$, $2 < \sigma < 3$, $0 < \lambda < 2$, \mathcal{D}^σ is the Caputo fractional derivative and $w : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function. In [26], the singular fractional problem

$${}^c\mathcal{D}^\alpha[k](r) + w(r, k(r)), {}^c\mathcal{D}^\sigma[k](r) = 0,$$

with boundary conditions $k(0) = k'(0) = 0$ and $k'(1) = {}^c\mathcal{D}^\sigma[k](1)$ was considered, where $0 < r < 1$, $2 < \alpha < 3$, $0 < \sigma < 1$, $w : (0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous with $w(t, k, l)$ may be singular at $t = 0$ and ${}^c\mathcal{D}^\alpha$ is the Caputo derivative.

In 2015, Zhang et al. and through the spectral analysis and fixed point index theorem obtained the existence of positive solutions of the singular nonlinear fractional differential equation $\mathcal{D}^\alpha[u](t) = w(t, [u](t), \mathcal{D}^\beta[u](t))$ for $0 < t < 1$, with integral boundary value conditions $\mathcal{D}^\beta[u](0) = 0$ and $\mathcal{D}^\beta[u](1) = \int_0^1 \mathcal{D}^\beta[u](r)dN(r)$, where $\alpha \in (1, 2]$, $\beta \in (0, 1]$, $w(t, u, v)$ may be singular at both $t = 0, 1$ and $u = v = 0$, $\int_0^1 u(r)dN(r)$ denotes the Riemann-Stieltjes integral with a signed measure, in which $N : [0, 1] \rightarrow \mathbb{R}$ is a function of bounded variation [27]. Ahmad et al. investigated the existence of solutions for a q -antiperiodic boundary value problem of fractional q -difference inclusions given by

$${}^c\mathcal{D}_q^\alpha[k](t) \in F(t, k(t), \mathcal{D}_q[k](t), \mathcal{D}_q^2[k](t)),$$

for $t \in [0, 1]$, $q \in (0, 1)$, $2 < \alpha \leq 3$, $0 < \beta \leq 3$ with conditions $k(0) + k(1) = 0$,

$$\mathcal{D}_q[k](0) + \mathcal{D}_q[k](1) = 0, \quad \mathcal{D}_q^2[k](0) + \mathcal{D}_q^2[k](1) = 0,$$

where ${}^c\mathcal{D}_q^\alpha$ denotes Caputo fractional q -derivative of order α and $F : [0, 1] \times \mathbb{R}^3 \rightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map with $\mathcal{P}(\mathbb{R})$ a class of all subsets of \mathbb{R} [24]. In 2019, Ntouyas et al. in [20], by applying definition of the fractional q -derivative of the Caputo-type and the fractional q -integral of the Riemann-Liouville-type, studied the existence and uniqueness of solutions for a multi-term nonlinear fractional q -integro-differential equations under some boundary conditions

$${}^c\mathcal{D}_q^\sigma[k](r) = \Omega(r, k(r), (\varphi_1 k)(r), (\varphi_2 k)(r), {}^c\mathcal{D}_q^{\beta_1}[k](r), {}^c\mathcal{D}_q^{\beta_2}[k](r), \dots, {}^c\mathcal{D}_q^{\beta_n}[k](r)).$$

In [21], Liang et al. investigated the existence of solutions for a nonlinear problem regular and singular fractional q -differential equation

$${}^c\mathcal{D}_q^\sigma[k](t) = \Omega(r, k(r), k'(r), {}^c\mathcal{D}_q^\beta[k](r)),$$

with conditions $k(0) = c_1 k(1)$, $k'(0) = c_2 {}^c\mathcal{D}_q^\beta[k](1)$ and $k^{(m)}(0) = 0$ for $2 \leq m \leq n - 1$, here $n - 1 < \sigma < n$ with $n \geq 3$, $\beta, q, c_1 \in (0, 1)$, $c_2 \in (0, \Gamma_q(2 - \beta))$, function Ω is a L^k -Carathéodory, $\Omega(r, k_1, k_2, k_3)$ may be singular and ${}^c\mathcal{D}_q^\sigma$ the fractional Caputo-type q -derivative. Furthermore, they discussed the existence of solutions for the fractional q -derivative inclusions

$${}^c\mathcal{D}_q^\sigma[k](r) \in \mathcal{F}(r, k(r), k'(r), {}^c\mathcal{D}_q^\beta[k](r)),$$

under conditions

$$\begin{cases} k(0) + k'(0) + {}^c\mathcal{D}_q^\beta[k](0) = \int_0^{\eta_1} k(s)ds, \\ k(1) + k'(1) + {}^c\mathcal{D}_q^\beta[k](1) = \int_0^{\eta_2} k(s)ds, \end{cases}$$

for any $t \in I$ and $q, \eta_1, \eta_2, \beta \in (0, 1)$, where \mathcal{F} maps $I \times \mathbb{R}^3$ into $2^{\mathbb{R}}$ is a compact valued multifunction and ${}^c\mathcal{D}_q^\sigma$ is the fractional Caputo-type q -derivative operator of order $\sigma \in (1, 2]$, and

$$\Gamma_q(2 - \beta)(\eta_1^2 \eta_2 - \eta_1 \eta_2^2 - \eta_1^2 + \eta_2^2 + 4\eta_1 - 2\eta_2 - 2) + 2(1 - \eta_1) \neq 0,$$

such that $\sigma - \beta > 1$ [16]. Relevant results have been presented in other studies, for example [27–31].

In this paper and motivated by the aforementioned achievements, we investigate the singular fractional q -integro-differential equation of the form

$$\mathcal{D}_q^\sigma[k](t) = \Omega \left(t, k(t), k'(t), \mathcal{D}_q^\zeta[k](t), \int_0^t f(r)k(r)dr \right), \quad (1)$$

for $0 < t < 1$ under boundary conditions $k(0) = 0$ and $k(1) = k'(1) = \dots = k^{(n)}(1) = \mathcal{D}_q^\eta[k](\tau)$, where $k \in C^1(J)$, $n = [\eta] + 1$, $\sigma \geq 2$, $\zeta, \eta, \tau \in (0, 1)$, $f \in L^1(J)$ is nonnegative with $\|f\|_1 = m$, $\Omega(t, k_1, k_2, k_3, k_4)$ is singular at some points of $t \in J := (0, 1)$ and \mathcal{D}_q^σ is the Caputo fractional q -derivative of order σ . Existence of solutions is studied via multi-step methods. We prove the main results in context of completely continuous functions and by the help of the Lebesgue dominated theorem. Examples are presented and MATLAB routines [32] are implemented to demonstrate the validity of the proposed results.

The rest of the paper is organized as follows: Section 2 recalls some preliminary concepts and fundamental results of q -calculus. Sections 3 and 4 are devoted to the main results and examples illustrating the obtained results and some algorithms for the addressed problem, respectively.

2 Essential preliminaries

This section is devoted to starting some notations and essential preliminaries that are acting as necessary prerequisites for the results of the subsequent sections.

2.1 q -Fractional derivative and integral

Throughout this article, we shall apply the time scale calculus notations [17]. In fact, we consider the fractional q -calculus on the specific time scale

$$\mathbb{T}_{t_0} = \{0\} \cup \{t : t = t_0 q^n\},$$

for $n \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and $q \in (0, 1)$. If there is no confusion concerning t_0 we shall denote \mathbb{T}_{t_0} by \mathbb{T} . Let $a \in \mathbb{R}$. Define $[a]_q = (1 - q^a)/(1 - q)$ [11]. The q -factorial function $(x - y)_q^{(n)}$ with $n \in \mathbb{N}_0$ is defined by

$$(x - y)_q^{(n)} = \prod_{k=0}^{n-1} (x - yq^k), \quad (2)$$

and $(x - y)_q^{(0)} = 1$, where x and y are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [12].

Algorithm 1. MATLAB lines for calculation of q -factorial function $(x - y)_q^{(n)}$

```

1  function p = qfunction(x, y, q, n)
2  if n==0
3      s=1;
4  else
5      s=1;
6      for k=0:n-1
7          s = s*(x-y*q^k);
8      end;
9      p=s;
10 end;
11 end

```

Also, for $\sigma \in \mathbb{R}$ and $a \neq 0$, we have

$$(x - y)_q^{(\sigma)} = x^\sigma \prod_{k=0}^{\infty} \frac{x - yq^k}{x - yq^{\sigma+k}}. \quad (3)$$

Algorithms 1 and 2 simplify q -factorial functions $(x - y)_q^{(n)}$ and $(x - y)_q^{(\sigma)}$, respectively. In the previous study [34], the authors proved $(x - y)_q^{(\sigma+\nu)} = (x - y)_q^{(\sigma)}(x - q^\sigma y)_q^{(\nu)}$ and $(ax - ay)_q^{(\sigma)} = a^\sigma(x - y)_q^{(\sigma)}$.

Algorithm 2. MATLAB lines for calculation of q -factorial function $(x - y)_q^{(\sigma)}$

```

1  function p = qfunctionreal(x,y,q,sigma,n)
2  if n==0
3      p=1;
4  else
5      s=1;
6      for k=0:n-1
7          s = s*(x-y*q^k)/(x-y*q^(sigma+k));
8      end;
9      p=s*x^sigma;
10 end;
11 end

```

If $y = 0$, then it is clear that $x^{(\sigma)} = x^\sigma$. The q -Gamma function is given by

$$\Gamma_q(z) = (1 - q)^{1-z} (1 - q)_q^{(z-1)},$$

where $z \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ [11]. In fact, by using (3), we have

$$\Gamma_q(z) = (1 - q)^{1-z} \prod_{k=0}^{\infty} \frac{1 - q^{k+1}}{1 - q^{z+k-1}}. \quad (4)$$

Algorithm 3. MATLAB lines for calculation of $\Gamma_q(x)$

```

1  function p = qGamma(q,x,n)
2  s=1;
3  for k=0:n
4      s=s*(1-q^(k+1))/(1-q^(x+k-1));
5  end;
6  p = s*(1-q)^(1-x);
7  end

```

Algorithm 3 shows the MATLAB lines for calculation of $\Gamma_q(x)$ which we tend n to infinity in it. Note that, $\Gamma_q(z + 1) = [z]_q \Gamma_q(z)$ [34, Lemma 1]. For a function $w : \mathbb{T} \rightarrow \mathbb{R}$, the q -derivative of w , is

$$\mathcal{D}_q[w](x) = \left(\frac{d}{dx} \right)_q w(x) = \frac{w(x) - w(qx)}{(1 - q)x}, \quad (5)$$

for all $t \in \mathbb{T} \setminus \{0\}$, and $\mathcal{D}_q[w](0) = \lim_{x \rightarrow 0} \mathcal{D}_q[w](x)$ [12]. Also, the higher order q -derivative of the function w is defined by $\mathcal{D}_q^n[w](x) = \mathcal{D}_q[\mathcal{D}_q^{n-1}[w]](x)$, for all $n \geq 1$, where $\mathcal{D}_q^0[w](x) = w(x)$ [12]. In fact

$$\mathcal{D}_q^n[w](x) = \frac{1}{x^n(1-q)^n} \sum_{k=0}^n \frac{(1-q^{-n})_q^{(k)}}{(1-q)_q^{(k)}} q^k w(xq^k), \quad (6)$$

for $x \in \mathbb{T} \setminus \{0\}$ [33].

Remark 2.1. By using equation (2), we can change equation (6) as follows:

$$\mathcal{D}_q^n[w](x) = \frac{1}{x^n(1-q)^n} \sum_{k=0}^n \prod_{i=0}^{k-1} \frac{(1-q^{i-n})}{(1-q^{i+1})} q^k w(xq^k). \quad (7)$$

Algorithms 4 and 5 show the MATLAB codes for calculation of equations (5) and (7), respectively.

Algorithm 4. MATLAB lines for calculation of $\mathcal{D}_q[w](x)$

```

1 function p = Dq(q, x, fun)
2 if x==0
3     p=limit((subs(fun,x)-subs(fun,q*x))/((1-q)*x), x, 0);
4 else
5     p=(eval(subs(fun,x))-eval(subs(fun,q*x)))/((1-q)*x);
6 end;
7 end

```

Algorithm 5. MATLAB lines for calculation of $\mathcal{D}_q^n[w](x)$

```

1 function g = Dqnatural(q, x, n, fun)
2 s=0;
3 for k=0:n
4     p=1;
5     for i=0:k-1
6         p=p*(1-q^(i-n))/(1-q^(i+1));
7     end;
8     p=p*q^k*eval(subs(fun,x*q^k));
9     s=s+p;
10 end;
11 g=s/(x^n*(1-q)^n);
12 end

```

The q -integral of the function w is defined by

$$\mathcal{I}_q[w](x) = \int_0^x w(s) d_q s = x(1-q) \sum_{k=0}^{\infty} q^k w(xq^k), \quad (8)$$

for $0 \leq x \leq b$, provided the series is absolutely converged [12].

Algorithm 6. MATLAB lines for calculation of $I_q[w](t)$

```

1 function p = Iq(q,x,n,fun)
2 s=1;
3 for k=0:n
4     s=s+q^k*eval(subs(fun,x*q^k));
5 end;
6 p=x*(1-q)*s;
7 end

```

By using Algorithm 6, we can obtain the numerical results of $I_q[w](x)$ when $n \rightarrow \infty$. If a in $[0, b]$, then

$$\int_a^b w(s)d_q s = I_q[w](b) - I_q[w](a) = (1 - q) \sum_{k=0}^{\infty} q^k [bw(bq^k) - aw(aq^k)], \tag{9}$$

whenever the series exists. The operator I_q^n is given by $I_q^0[w](x) = w(x)$ and

$$I_q^n[w](x) = I_q[I_q^{n-1}[w]](x),$$

for $n \geq 1$ and $g \in C([0, b])$ [12]. It has been proved that $\mathcal{D}_q[I_q[w]](x) = w(x)$, and $I_q[\mathcal{D}_q[w]](x) = w(x) - w(0)$, whenever the function w is continuous at $x = 0$ [12].

The fractional Riemann-Liouville-type q -integral of the function w is defined by

$$I_q^\sigma[w](t) = \frac{1}{\Gamma_q(\sigma)} \int_0^t (t - s)_{q^{\sigma-1}} w(s) d_q s, \quad I_q^0[w](t) = w(t), \tag{10}$$

for $t \in [0, 1]$ and $\sigma > 0$ [14,33].

Remark 2.2. By using equations (3), (4) and (8), we obtain

$$\frac{1}{\Gamma_q(\sigma)} \int_0^t (t - s)_{q^{\sigma-1}} w(s) d_q s = \frac{1}{\Gamma_q(\sigma)} \int_0^t t^{\sigma-1} \prod_{i=0}^{\infty} \frac{t - sq^i}{t - sq^{\sigma+i-1}} w(s) d_q s = t^\sigma (1 - q)^\sigma \prod_{i=0}^{\infty} \frac{1 - q^{\sigma+i-1}}{1 - q^{i+1}} \sum_{k=0}^{\infty} q^k \prod_{i=0}^{\infty} \frac{1 - q^{k+i}}{1 - q^{\sigma+k+i-1}} w(tq^k).$$

Therefore,

$$I_q^\sigma[w](t) = t^\sigma (1 - q)^\sigma \lim_{n \rightarrow \infty} \sum_{k=0}^n q^k \prod_{i=0}^n \frac{(1 - q^{\sigma+i-1})(1 - q^{k+i})}{(1 - q^{i+1})(1 - q^{\sigma+k+i-1})} w(tq^k), \tag{11}$$

Algorithm 7 shows the MATLAB codes of numerical technique.

Algorithm 7. MATLAB lines for calculation of $I_q^\sigma[w](x)$

```

1 function g = Iq_sigma(q,sigma,t,n,fun)
2 p=0;
3 for k=0:n
4     s=1;
5     for i=0:n
6         s=s*(1-q^(sigma+i-1))*(1-q^(k+i))/((1-q^(i+1))*(1-q^(sigma+k+i-1)));
7     end
8     p=p+q^k*s*eval(subs(fun,t*q^k));
9 end;
10 g=round(p*(t^sigma)*(1-q)^sigma,6);
11 end

```

The Caputo fractional q -derivative of the function w is defined by

$${}^C\mathcal{D}_q^\sigma[w](t) = I_q^{[\sigma]-\sigma}[\mathcal{D}_q^{[\sigma]}[w]](t) = \frac{1}{\Gamma_q([\sigma] - \sigma)} \int_0^t (t - s)_q^{([\sigma]-\sigma-1)} \mathcal{D}_q^{[\sigma]}[w](s) d_qs \tag{12}$$

for $t \in [0, 1]$ and $\sigma > 0$ [14,36]. It has been proved that $I_q^\nu[I_q^\sigma[w]](t) = I_q^{\sigma+\nu}[w](t)$, and ${}^C\mathcal{D}_q^\sigma[I_q^\sigma[w]](t) = w(t)$, where $\sigma, \nu \geq 0$ [14]. Also,

$$I_q^\sigma[\mathcal{D}_q^n[w]](t) = \mathcal{D}_q^n[I_q^\sigma[w]](t) - \sum_{k=0}^{n-1} \frac{t^{\sigma+k-n}}{\Gamma_q(\sigma + k - n + 1)} \mathcal{D}_q^k[w](0),$$

where $\sigma > 0$ and $n \geq 1$ [14].

Remark 2.3. From equation (4), Remark 2.1 and equation (11) in Remark 2.2, we obtain

$$\begin{aligned} & \frac{1}{\Gamma_q([\sigma] - \sigma)} \int_0^t (t - s)_q^{([\sigma]-\sigma-1)} \mathcal{D}_q^{[\sigma]}[w](s) d_qs \\ &= \frac{1}{\Gamma_q([\sigma] - \sigma)} \int_0^t t^{[\sigma]-\sigma-1} \left[\prod_{i=0}^{\infty} \frac{t - sq^i}{t - sq^{i([\sigma]-\sigma-1+i)}} \right] \left(\frac{1}{t^{[\sigma](1-q)^{[\sigma]}}} \sum_{k=0}^{[\sigma]} \left[\prod_{i=0}^{k-1} \frac{(1 - q^{i-[\sigma]})}{(1 - q^{i+1})} \right] q^k w(sq^k) \right) d_qs \\ &= \frac{1}{t^\sigma(1-q)^{\sigma-[\sigma]}} \sum_{k=0}^{\infty} \left(\left[\prod_{i=0}^{\infty} \frac{(1 - q^{[\sigma]-\sigma+i-1})(1 - q^{k+i})}{(1 - q^{i+1})(1 - q^{[\sigma]-\sigma-1+k+i})} \right] \left(\sum_{m=0}^{[\sigma]} \left[\prod_{i=0}^{m-1} \frac{(1 - q^{i-[\sigma]})}{(1 - q^{i+1})} \right] q^m w(tq^{k+m}) \right) \right). \end{aligned}$$

Thus, we have

$${}^C\mathcal{D}_q^\sigma[w](t) = \frac{1}{t^\sigma(1-q)^{\sigma-[\sigma]}} \lim_{n \rightarrow \infty} \sum_{k=0}^n \left(\left[\prod_{i=0}^n \frac{(1 - q^{[\sigma]-\sigma+i-1})(1 - q^{k+i})}{(1 - q^{i+1})(1 - q^{[\sigma]-\sigma-1+k+i})} \right] \left(\sum_{m=0}^{[\sigma]} \left[\prod_{i=0}^{m-1} \frac{(1 - q^{i-[\sigma]})}{(1 - q^{i+1})} \right] q^m w(tq^{k+m}) \right) \right). \tag{13}$$

Algorithm 8 shows the MATLAB codes of numerical technique.

Algorithm 8. MATLAB lines for calculation of ${}^C\mathcal{D}_q^\sigma[w](t)$

```

1 function g = IqCaputo_sigma(q, sigma, t, n, fun)
2 S=0;
3 for k=0:n
4     p1=1;
5     for i=0:n
6         p1=p1*(1-q^(floor(sigma)-sigma+i-1))*(1-q^(k+i))/((1-q^(i+1)) ...
7             *(1-q^(floor(sigma)-sigma+k+i-1)));
8     end;
9     s2=0;
10    for m=0:floor(sigma)
11        p2=1;
12        for i=0:m-1
13            p2=p2*(1-q^(i-floor(sigma)))/(1-q^(i+1));
14        end;
15        p2=p2*q^m*eval(subs(fun, t*q^(k+m)));
16        s2=s2+p2;
17    end;
18    S=S+p1*s2;
19 end;
20 g=round(S/( t^sigma*(1-q)^(sigma-floor(sigma))),6);
21 end

```

Throughout this article, we consider

$$\|k\|_1 := \int_0^1 |k(t)| dt, \quad \|k\| := \sup\{|k(t)| : t \in J\}, \quad \|k\|_* := \max\{\|k\|, \|k'\|\},$$

as the norm of $\mathcal{L} = L^1(J)$, $\mathcal{A} = C(J)$ and $\mathcal{B} = C^1(J)$, respectively.

The following lemmas are used in the subsequent sections.

Lemma 2.1. [37] *Suppose that $0 < n - 1 \leq \sigma < n$ and $k \in \mathcal{A} \cap \mathcal{L}$. Then*

$$I_q^{\sigma[C]} \mathcal{D}_q^{\sigma}[k](t) = k(t) + \sum_{i=0}^{n-1} c_i t^i,$$

for some constants $c_0, \dots, c_{n-1} \in \mathbb{R}$.

Lemma 2.2. [38] *If C is a closed, bounded and convex subset of a Banach space X and $\Phi : C \rightarrow C$ is completely continuous, then Φ has a fixed point in C .*

Lemma 2.3. [39] *Let X be a Banach space, C a closed and convex subset of X , O a relatively open subset of C with $0 \in O$ and $\Omega : O \rightarrow C$ a continuous and compact map. Then either Ω has a fixed point in \bar{O} or there exist $a \in \partial O$ and $\lambda \in (0, 1)$ such that $a = \lambda\Omega(a)$.*

2.2 Linear multi-step methods

As in the case of ordinary differential equations, linear multi-step methods for fractional differential equations makes use of approximations of values of $k_1(t), k_2(t), k_3(t), k_4(t)$ and $\Omega(t, k_1(t), k_2(t), k_3(t), k_4(t))$ on some points of a partition $s_0 < s_1 < \dots < s_n$ [32,35]. We can therefore write linear multi-step methods for the solution of (1) in the form

$$\sum_{j=0}^n {}_1\alpha_j ({}_{n-j}k_1, {}_{n-j}k_2, {}_{n-j}k_3, {}_{n-j}k_4) = h^\tau \sum_{j=0}^n {}_2\alpha_j \Omega({}_{n-j}k_1, {}_{n-j}k_2, {}_{n-j}k_3, {}_{n-j}k_4), \tag{14}$$

where ${}_1\alpha_j$ and ${}_2\alpha_j$ are real parameters and we will indicate with ${}_1\rho_n(\xi)$ and ${}_2\rho_n(\xi)$ the generating polynomials ${}_k\rho_n(\xi) = \sum_{j=0}^n \alpha_j \xi^{n-j}$. Numerical methods (14) are requested to be consistent with the original problem (1), in the sense that, as $h \rightarrow 0$, the discretized problem is expected to tend asymptotically to the continuous one [32]. In order to formally introduce the consistency concept and study order conditions, it is usually to introduce, associated with (14), the linear difference operator

$$\begin{aligned} \mathcal{L}_h[(z_1(t), z_2(t), z_3(t), z_4(t)), t, \tau] &= \sum_{j=0}^n {}_1\alpha_j ({}_{n-j}z_1(t-hj), {}_{n-j}z_2(t-hj), {}_{n-j}z_3(t-hj), {}_{n-j}z_4(t-hj)) \\ &\quad - h^\tau \sum_{j=0}^n {}_2\alpha_j {}^C\mathcal{D}_q^{\tau}[{}_{n-j}z_1, {}_{n-j}z_2, {}_{n-j}z_3, {}_{n-j}z_4](t-hj), \end{aligned}$$

where $(z_1(t), z_2(t), z_3(t), z_4(t))$ is a sufficiently smooth function [32]. The linear multi-step method (14) is said to be consistent if, for any initial value problem (1), with exact solution $(k_1(t), k_2(t), k_3(t), k_4(t))$, it holds

$$\lim_{h \rightarrow 0} \frac{1}{h^\tau} \mathcal{L}_h[({}_{n-j}k_1(t), {}_{n-j}k_2(t), {}_{n-j}k_3(t), {}_{n-j}k_4(t)), t, \tau] = (0, 0, 0, 0),$$

with h and n related by $t = s_0 + hn$. Moreover, the method is said to be of order ℓ if

$$\frac{1}{h^\tau} \mathcal{L}_h[({}_{n-j}k_1(t), {}_{n-j}k_2(t), {}_{n-j}k_3(t), {}_{n-j}k_4(t)), t, \tau] = O(h^\ell),$$

as h tends to zero. Under the assumption that $(k_1(t), k_2(t), k_3(t), k_4(t))$ is $(m + 1)$ -times differentiable, $t = s_n$, we can expand the true solution

$$\begin{aligned} &(k_1(t - jh), k_2(t - jh), k_3(t - jh), k_4(t - jh)) \\ &= (k_1(s_0 + (n - j)h), k_2(s_0 + (n - j)h), k_3(s_0 + (n - j)h), k_4(s_0 + (n - j)h)), \end{aligned}$$

of (1) as

$$\begin{aligned} &(k_1(t - jh), k_2(t - jh), k_3(t - jh), k_4(t - jh)) \\ &= (k_1(s_0), k_2(s_0), k_3(s_0), k_4(s_0)) + \sum_{d=1}^m \frac{(n - j)^d h^d}{d!} (k_1^d(s_0), k_2^d(s_0), k_3^d(s_0), k_4^d(s_0)) \\ &\quad + \frac{h^{m+1}}{d!} \int_0^{n-j} (n - j - \xi)_q^{(m)} (k_1^{m+1}(s_0 + h\xi), k_2^{m+1}(s_0 + h\xi), k_3^{m+1}(s_0 + h\xi), k_4^{m+1}(s_0 + h\xi)) d_q \xi, \end{aligned}$$

and its τ -fractional q -derivative as

$$\begin{aligned} &{}^C \mathcal{D}_q^\tau [{}_{n-j}z_1, {}_{n-j}z_2, {}_{n-j}z_3, {}_{n-j}z_4](t - hj) \\ &= \sum_{d=1}^m \frac{h^{d-\tau} (n - j)^{d-\tau}}{\Gamma_q(d + 1 - \tau)} (k_1^d(s_0), k_2^d(s_0), k_3^d(s_0), k_4^d(s_0)) \\ &\quad + \frac{h^{m+1-\tau}}{\Gamma_q(m + 1 - \tau)} \int_0^{n-j} (n - j - \xi)_q^{(m-\tau)} (k_1^{m+1}(s_0 + h\xi), k_2^{m+1}(s_0 + h\xi), k_3^{m+1}(s_0 + h\xi), k_4^{m+1}(s_0 + h\xi)) d_q \xi. \end{aligned}$$

In this way, we can write the difference operator

$$\mathcal{L}_h[(k_1(t), k_2(t), z_3(t), z_4(t)), t, \tau],$$

as

$$\mathcal{L}_h[(k_1(t), k_2(t), z_3(t), z_4(t)), t, \tau] = C_0(n, \tau) + \sum_{d=1}^m h^d C_d(n, \tau) (k_1^d(s_0), k_2^d(s_0), z_3^d(s_0), z_4^d(s_0)) + h^{m+1} R_{m+1},$$

where the remainder R_{m+1} is obtained from Taylor's expansions and

$$\begin{aligned} C_0(n, \tau) &= \sum_{j=0}^n {}_1\alpha_j, \\ C_d(n, \tau) &= \frac{1}{d!} \sum_{j=0}^n (n - j)_q^d {}_1\alpha_j - \frac{1}{\Gamma_q(d + 1 - \tau)} \sum_{j=0}^n {}_2\alpha_j (n - j)^{d-\tau}, \end{aligned}$$

for $d = 1, 2, \dots, m$.

3 Main results

We employ the multi-step methods to prove the main results in this section. First, we adopt the following lemma.

Lemma 3.1. *Let $z \in \mathcal{L}$. The unique solution of problem*

$$\mathcal{D}_q^\sigma[k](t) + z(t) = 0,$$

with boundary conditions $k(0) = 0$ and $k(1) = k'(1) = \dots = k^{(n)}(t) = \mathcal{D}_q^\eta[k](\tau)$, ($n = [\eta] + 1$), is $k_0(t) = \int_0^1 G_q(t, s)z(s)d_qs$, where

$$G_q(t, s) = \begin{cases} \frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)}, & t \leq s, \tau \leq s, \\ \frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} - \frac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)}, & \tau \leq s \leq t, \\ \frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} - \frac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\lambda\Gamma_q(\sigma-\eta)}, & t \leq s \leq \tau, \\ \frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} - \frac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\lambda\Gamma_q(\sigma-\eta)} - \frac{\lambda(t-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)}, & s \leq t, s \leq \tau, \end{cases} \tag{15}$$

for $t, s \in J$, $\sigma \geq 2$, $\eta, \tau \in (0, 1)$ where

$$\lambda := 1 - \frac{\tau^{1-\eta}}{\Gamma_q(2-\eta)} \neq 0. \tag{16}$$

Proof. Assume that k be a solution for the problem. By applying Lemma 2.1, we get

$$k(t) = -I_q^\sigma[z](t) + d_{n-1}t^{n-1} + \dots + d_1t + d_0,$$

where $n - 1 \leq \sigma < n$. By utilizing the boundary conditions, we conclude $d_0 = 0$. Hence,

$$\mathcal{D}_q^\eta[k](\tau) = -I_q^{\sigma-\eta}[z](\tau) + d_1 \frac{\tau^{1-\eta}}{\Gamma_q(2-\eta)}$$

and $k(1) = -I_q^\sigma[z](1) + d_1$. Since $k(1) = \mathcal{D}_q^\eta[k](\tau)$, we conclude that

$$d_1 \left(1 - \frac{\tau^{1-\eta}}{\Gamma_q(2-\eta)} \right) = I_q^\sigma[z](1) - I_q^{\sigma-\eta}[z](\tau)$$

and so $d_1 = \frac{1}{\lambda} [I_q^\sigma[z](1) - I_q^{\sigma-\eta}[z](\tau)]$. Thus, we have

$$k(t) = -I_q^\sigma[k](t) + \frac{t}{\lambda} [I_q^\sigma[z](1) - I_q^{\sigma-\eta}[z](\tau)].$$

Therefore, we have two cases.

(1) If $t \leq \tau$, then we can see that

$$\begin{aligned} k(t) &= -I_q^\sigma[z](t) + \frac{t}{\lambda} I_q^\sigma[z](t) + \frac{t}{\lambda\Gamma_q(\sigma)} \left[\int_t^\tau (1-s)_q^{(\sigma-1)} z(s) d_qs + \int_\tau^1 (1-s)_q^{\sigma-1} z(s) d_qs \right] \\ &\quad - \frac{t}{\lambda} \left[I_q^{\sigma-\eta}[z](t) + \frac{1}{\Gamma_q(\sigma-\eta)} \int_t^\tau (\tau-s)_q^{(\sigma-\eta-1)} z(s) d_qs \right] \\ &= \int_0^t \left[\frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} - \frac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} - \frac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\Gamma_q(\sigma-\eta)} \right] z(s) d_qs \\ &\quad + \int_t^\tau \left[\frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} - \frac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\lambda\Gamma_q(\sigma-\eta)} \right] z(s) d_qs + \int_\tau^1 \frac{t(1-s)_q^{(\sigma-1)}}{\lambda\Gamma_q(\sigma)} z(s) d_qs. \end{aligned}$$

(2) If $t \geq \tau$, then we can see that

$$\begin{aligned} k(t) &= -\mathcal{I}_q^\sigma[z](\tau) - \int_\tau^t \frac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} z(s) d_qs + \frac{t}{\lambda} \mathcal{I}_q^\sigma[z](\tau) \\ &+ \int_\tau^t \frac{t(1-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)} z(s) d_qs + \int_t^1 \frac{t(1-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)} z(s) d_qs - \frac{t}{\lambda} \mathcal{I}_q^{\sigma-\eta}[z](\tau) \\ &= \int_0^\tau \left[\frac{t(1-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)} - \frac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} - \frac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\lambda \Gamma_q(\sigma-\eta)} \right] z(s) d_qs \\ &+ \int_\tau^t \left[\frac{t(1-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)} - \frac{\lambda(t-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)} \right] z(s) d_qs + \int_t^1 \frac{t(1-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)} z(s) d_qs. \end{aligned}$$

This implies that, $k(t) = \int_0^1 G_q(t, s)z(s) d_qs = k_0(t)$ for each t . □

Remark 3.1. If $k \in \mathcal{B}$, then

$$\mathcal{D}_q^\beta[k](t) = \frac{1}{\Gamma_q(1-\beta)} \int_0^t (t-s)_q^{-\beta} k'(s) d_qs$$

and so

$$|\mathcal{D}_q^\beta[k](t)| \leq \frac{\|k'\|}{\Gamma(1-\beta)} \int_0^t (t-qs)^{-\beta} d_qs = \frac{\|k'\|}{\Gamma(2-\beta)} t^{1-\beta}.$$

Thus, $\mathcal{D}_q^\beta[k] \in \mathcal{A}$ and

$$|\mathcal{D}_q^\beta[k]| \leq \frac{\|k'\|}{\Gamma_q(2-\beta)}.$$

Since $\int_0^1 f(r) dr = m \in (0, \infty)$,

$$\left| \int_0^t f(r)k(r) dr \right| \leq \|k\| \int_0^t f(r) dr \leq m\|k\|.$$

Now, we give our main result.

Theorem 3.2. *The singular problem (1) has a solution whenever the following assumptions hold.*

(1) *There exist the maps $f_i : J \rightarrow \mathbb{R}$ with $\int_0^1 f_i(r) dr < \infty$ for all $i = 1, 2, 3, 4$ such that*

$$|\Omega(t, k_1, k_2, k_3, k_4) - \Omega(t, l_1, l_2, l_3, l_4)| \leq \sum_{i=1}^4 f_i(t) |k_i(t) - l_i(t)|,$$

for all $(k_1, k_2, k_3, k_4), (l_1, l_2, l_3, l_4) \in \mathbb{R}^4$ and $t \in J$.

(2) *There exist $g \in \mathcal{L}$ and $\Theta \in \mathcal{A}^4$ such that*

$$|\Omega(t, k_1, k_2, k_3, k_4)| \leq g(t)\Theta(k_1, k_2, k_3, k_4),$$

for each $(k_1, k_2, k_3, k_4) \in \mathbb{R}^4$, almost all $t \in J$. Also

$$\|\Theta\|_{\mathcal{A}} = \sup\{|\Theta(k_1, k_2, k_3, k_4)| : (k_1, k_2, k_3, k_4) \in \mathbb{R}^4\} < \infty. \tag{17}$$

Proof. We first define a map $T : \mathcal{B} \rightarrow \mathcal{B}$ by

$$T_k(t) = \int_0^1 G_q(t, s) \tilde{\Omega}(k, s) d_qs = -I_q^\sigma[\tilde{\Omega}](k, t) + \frac{t}{\lambda} [I_q^\sigma[\tilde{\Omega}](k, 1) - I_q^{\sigma-\eta}[\tilde{\Omega}](k, \tau)], \tag{18}$$

for each $k \in \mathcal{B}$ and $t \in J$ where

$$\tilde{\Omega}(z, \bar{t}) = \Omega \left(\bar{t}, z(\bar{t}), z'(\bar{t}), \mathcal{D}_q^\zeta[z](\bar{t}), \int_0^{\bar{t}} f(r)z(r)dr \right).$$

Suppose that $k_1, k_2 \in \mathcal{B}$. Then we have

$$\begin{aligned} |T_{k_1}(t) - T_{k_2}(t)| &\leq I_q^\sigma[|\tilde{\Omega}(k_1, s) - \tilde{\Omega}(k_2, s)|] + \frac{t}{\lambda} I_q^\sigma[|\tilde{\Omega}(k_1, 1) - \tilde{\Omega}(k_2, 1)|] + \frac{t}{\lambda} I_q^{\sigma-\eta}[|\tilde{\Omega}(k_1, \tau) - \tilde{\Omega}(k_2, \tau)|] \\ &\leq I_q^\sigma \left[|f_1(t)| |k_1(t) - k_2(t)| + |f_2(t)| |k_1'(t) - k_2'(t)| + |f_3(t)| |\mathcal{D}_q^\zeta[k_1](t) - \mathcal{D}_q^\zeta[k_2](t)| \right. \\ &\quad \left. + |f_4(t)| \left| \int_0^t f(r)(k_1(r) - k_2(r))dr \right| \right] \\ &\quad + \frac{t}{\lambda} \left[I_q^\sigma \left[|f_1(1)| |k_1(1) - k_2(1)| + |f_2(1)| |k_1'(1) - k_2'(1)| + |f_3(1)| |\mathcal{D}_q^\zeta[k_1](1) - \mathcal{D}_q^\zeta[k_2](1)| \right. \right. \\ &\quad \left. \left. + |f_4(1)| \left| \int_0^1 f(r)(k_1(r) - k_2(r))dr \right| \right] \right] \\ &\quad + I_q^{\sigma-\eta} \left[|f_1(\tau)| |k_1(\tau) - k_2(\tau)| + |f_2(\tau)| |k_1'(\tau) - k_2'(\tau)| + |f_3(\tau)| |\mathcal{D}_q^\zeta[k_1](\tau) - \mathcal{D}_q^\zeta[k_2](\tau)| \right. \\ &\quad \left. + |f_4(\tau)| \left| \int_0^1 f(r)(k_1(r) - k_2(r))dr \right| \right] \Bigg] \\ &\leq \|k_1 - k_2\| I_q^\sigma(f_1(t) + mf_4(t)) + \|k_1' - k_2'\| I_q^\sigma \left[f_2(t) + \frac{f_3(t)}{\Gamma_q(2-\zeta)} \right] \\ &\quad + \|k_1 - k_2\| \frac{t}{\lambda} I_q^\sigma[f_1(1) + mf_4(1)] + \|k_1' - k_2'\| \frac{t}{\lambda} I_q^\sigma \left[f_2(1) + \frac{f_3(1)}{\Gamma_q(2-\zeta)} \right] \\ &\quad + \|k_1 - k_2\| \frac{t}{\lambda} I_q^{\sigma-\eta}[f_1 + mf_4](\tau) + \|k_1' - k_2'\| I_q^{\sigma-\eta} \left[f_2(\tau) + \frac{f_3(\tau)}{\Gamma_q(2-\zeta)} \right] \\ &\leq \|k_1 - k_2\| \int_0^1 (1-s)_q^{(\sigma-\eta-1)} \left[\frac{2f_1(s) + 2mf_4(s)}{\Gamma_q(\sigma)} + \frac{f_1(s) + mf_4(s)}{\lambda\Gamma_q(\sigma-\eta)} \right] d_qs \\ &\quad + \|k_1' - k_2'\| \int_0^1 (1-s)_q^{(\sigma-\eta-1)} \left[\frac{2f_2(s)}{\Gamma_q(\sigma)} + \frac{f_2(s)}{\lambda\Gamma_q(\sigma-\eta)} + \frac{2f_3(s)}{\Gamma_q(\sigma)\Gamma_q(2-\zeta)} + \frac{f_3(s)}{\lambda\Gamma_q(\sigma-\eta)\Gamma_q(2-\zeta)} \right] d_qs \\ &\leq \Lambda_1(\|k_1 - k_2\| + \|k_1' - k_2'\|) = \Lambda_1\|k_1 - k_2\|_*, \end{aligned}$$

where

$$\Lambda_1 = \max \left\{ \int_0^1 (1-s)_q^{\sigma-\eta-1} \left(\frac{2f_1(s) + 2mf_4(s)}{\Gamma_q(\sigma)} + \frac{f_1(s) + mf_4(s)}{\lambda\Gamma_q(\sigma-\eta)} \right) d_q s, \right. \\ \left. \int_0^1 (1-s)_q^{(\sigma-\eta-1)} \left(\frac{2f_2(s)}{\Gamma_q(\sigma)} + \frac{f_2(s)}{\lambda\Gamma_q(\sigma-\eta)} + \frac{2f_3(s)}{\Gamma_q(\sigma)\Gamma_q(2-\zeta)} + \frac{f_3(s)}{\lambda\Gamma_q(\sigma-\eta)\Gamma_q(2-\zeta)} \right) d_q s \right\} < \infty.$$

On the other hand, we get

$$|T'_{k_1}(t) - T'_{k_2}(t)| \leq \int_0^1 \frac{\partial G_q(t, s)}{\partial t} |\tilde{\Omega}(k_1, s) - \tilde{\Omega}(k_2, s)| d_q s \\ \leq \|k_1 - k_2\| \int_0^1 (1-s)_q^{(\sigma-2)} \left(\frac{f_1(s)}{\Gamma_q(\sigma-1)} + \frac{f_1(s)}{\Gamma_q(\sigma)} + \frac{f_1(s)}{\lambda\Gamma_q(\sigma-\eta)} + \frac{mf_4(s)}{\Gamma_q(\sigma-1)} + \frac{mf_4(s)}{\Gamma_q(\sigma)} + \frac{mf_4(s)}{\lambda\Gamma_q(\sigma-\eta)} \right) d_q s \\ + \|k'_1 - k'_2\| \int_0^1 (1-s)_q^{(\sigma-2)} \left(\frac{f_2(s)}{\Gamma_q(\sigma-1)} + \frac{f_2(s)}{\Gamma_q(\sigma)} + \frac{f_2(s)}{\lambda\Gamma_q(\sigma-\eta)} + \frac{f_3(s)}{\Gamma_q(\sigma-1)\Gamma_q(2-\zeta)} \right. \\ \left. + \frac{f_3(s)}{\Gamma_q(\sigma)\Gamma_q(2-\zeta)} + \frac{f_3(s)}{\lambda\Gamma_q(\sigma-\eta)\Gamma_q(2-\zeta)} \right) d_q s \\ \leq \Lambda_2 (\|k_1 - k_2\| + \|k'_1 - k'_2\|) = \Lambda_2 \|k_1 - k_2\|_*,$$

where

$$\Lambda_2 = \max \left\{ \int_0^1 (1-s)_q^{(\sigma-2)} \left(\frac{f_1(s)}{\Gamma_q(\sigma-1)} + \frac{f_1(s)}{\Gamma_q(\sigma)} + \frac{f_1(s)}{\lambda\Gamma_q(\sigma-\eta)} + \frac{mf_4(s)}{\Gamma_q(\sigma-1)} + \frac{mf_4(s)}{\Gamma_q(\sigma)} + \frac{mf_4(s)}{\lambda\Gamma_q(\sigma-\eta)} \right) d_q s, \right. \\ \left. \int_0^1 (1-s)_q^{(\sigma-2)} \left(\frac{f_2(s)}{\Gamma_q(\sigma-1)} + \frac{f_2(s)}{\Gamma_q(\sigma)} + \frac{f_2(s)}{\lambda\Gamma_q(\sigma-\eta)} + \frac{f_3(s)}{\Gamma_q(\sigma-1)\Gamma_q(2-\zeta)} + \frac{f_3(s)}{\Gamma_q(\sigma)\Gamma_q(2-\beta)} \right. \right. \\ \left. \left. + \frac{f_3(s)}{\lambda\Gamma_q(\sigma-\eta)\Gamma_q(2-\zeta)} \right) d_q s \right\} < \infty.$$

Put

$$M_1 = \frac{1}{\Gamma_q(\sigma)} + \frac{1}{\lambda\Gamma_q(\sigma)} + \frac{1}{\lambda\Gamma_q(\sigma-\eta)}, \tag{19} \\ M_2 = \frac{1}{\Gamma_q(\sigma-1)} + \frac{1}{\lambda\Gamma_q(\sigma)} + \frac{1}{\lambda\Gamma_q(\sigma-\eta)},$$

$$m_0 = \int_0^1 (1-s)_q^{(\sigma-\eta-1)} g(s) d_q s \\ = (1-q) \sum_{k=0}^{\infty} q^k (1-q^k)_q^{(\sigma-\eta-1)} g(q^k) \tag{20} \\ = (1-q) \sum_{k=0}^{\infty} \left(q^k g(q^k) \left[\prod_{i=0}^{\infty} \frac{1-q^{k+i}}{1-q^{k+\sigma-\eta-1+i}} \right] \right),$$

and $r_0 = m_0 \|\Theta\|_{\mathcal{L}} \max\{M_1, M_2\}$, $\Lambda_0 = \max\{\Lambda_1, \Lambda_2\}$. Since $g \in \mathcal{L}$, $m_0 < \infty$. Then we have

$$\|T_{k_1}(t) - T_{k_2}(t)\|_* \leq \Lambda_0 \|k_1 - k_2\|_*$$

and so $\|T_{k_1}(t) - T_{k_2}(t)\|_* \rightarrow 0$ as $\|k_1 - k_2\|_* \rightarrow 0$. Consider $k \in \mathcal{B}$ and

$$B_{r_0} = \{k \in \mathcal{B} : \|k\|_* \leq r_0\}.$$

Then, we have

$$\begin{aligned} |T_k(t)| &\leq I_q^\sigma \left[g(t) \Theta \left(t, k(t), k'(t), \mathcal{D}_q^\beta[k](t), \int_0^t f(r)k(r)dr \right) \right] \\ &\quad + \frac{1}{\lambda} \left[I_q^\sigma \left[g(1) \Theta \left(1, k(1), k'(1), \mathcal{D}_q^\zeta[k](1), \int_0^1 f(r)k(r)dr \right) \right] \right] \\ &\quad + \left[I_q^{\sigma-\eta} \left[g(\tau) \Theta \left(\tau, k(\tau), k'(\tau), \mathcal{D}_q^\zeta[u](\tau), \int_0^\tau f(r)k(r)dr \right) \right] \right] \\ &\leq \|\Theta\|_{\mathcal{A}} \left(\frac{1}{\Gamma_q(\sigma)} + \frac{1}{\lambda \Gamma_q(\sigma)} + \frac{1}{\lambda \Gamma_q(\sigma - \eta)} \right) \int_0^1 (1-s)_q^{(\sigma-\eta-1)} g(s) d_qs \\ &= m_0 \|\Theta\|_{\mathcal{A}} M_1, \end{aligned}$$

for each $t \in J$. Note that, $\int_0^1 (1 - qs)^{(\sigma-1)} g(s) d_qs \leq m_0$. Also, we can conclude that

$$\begin{aligned} T'_k(t) &= \int_0^1 \frac{\partial G_q(t, s)}{\partial t} \Omega \left(s, k(s), k'(s), \mathcal{D}_q^\zeta[k](s), \int_0^s f(r)k(r)dr \right) d_qs \\ &= -I_q^{\sigma-1} \Omega \left(t, k(t), k'(t), \mathcal{D}_q^\zeta k(t), \int_0^t f(r)k(r)dr \right) \\ &\quad + \frac{1}{\lambda} \left[I_q^\sigma \Omega \left(1, k(1), k'(1), \mathcal{D}_q^\zeta k(1), \int_0^1 f(r)k(r)dr \right) \right] \\ &\quad + I_q^{\sigma-\eta} \Omega \left(\tau, k(\tau), k'(\tau), \mathcal{D}_q^\zeta k(\tau), \int_0^\tau f(r)k(\tau)dr \right) \Big], \end{aligned}$$

and so

$$|T'_k(t)| \leq \|\Theta\|_{\mathcal{A}} \left(\frac{1}{\Gamma_q(\sigma-1)} + \frac{1}{\lambda \Gamma_q(\sigma)} + \frac{1}{\lambda \Gamma_q(\sigma - \eta)} \right) \int_0^1 (1-s)_q^{(\sigma-\eta-1)} g(s) d_qs = m_0 \|\Theta\|_{\mathcal{A}} M_2.$$

Hence, $\|T_k\|_* = \max\{\|T_k\|, \|T'_k\|\} \leq r_0$. Therefore, T maps B_{r_0} into B_{r_0} . Similarly, one can check that T maps bounded sets into bounded sets. Let $t_1, t_2 \in J$ with $t_1 \leq t_2$. Then, we have

$$\begin{aligned} \|T_k(t_1) - T_k(t_2)\| &\leq \frac{1}{\Gamma_q(\sigma)} \int_0^{t_1} [(t_2 - s)_q^{(\sigma-1)} - (t_1 - s)_q^{(\sigma-1)}] |\tilde{\Omega}(k, s)| d_qs + \frac{1}{\Gamma_q(\sigma)} \int_{t_1}^{t_2} (t_1 - s)_q^{(\sigma-1)} |\tilde{\Omega}(k, s)| d_qs \\ &\quad + \frac{|t_2 - t_1|}{\Delta \Gamma_q(\sigma)} \int_0^1 (1-s)_q^{(\sigma-1)} |\tilde{\Omega}(k, s)| d_qs + \frac{|t_2 - t_1|}{\lambda \Gamma_q(\sigma - \eta)} \int_0^\mu (\mu - s)_q^{(\sigma-\eta-1)} |\tilde{\Omega}(k, s)| d_qs \\ &\leq \|\Theta\|_{\mathcal{A}} \left[\frac{1}{\Gamma_q(\sigma)} \int_0^{t_1} [(t_2 - s)_q^{(\sigma-1)} - (t_1 - s)_q^{(\sigma-1)}] g(s) d_qs + \frac{1}{\Gamma_q(\sigma)} \int_{t_1}^{t_2} (t_1 - s)_q^{\sigma-1} g(s) d_qs \right. \\ &\quad \left. + |t_2 - t_1| \left(\frac{1}{\lambda_q \Gamma_q(\sigma)} + \frac{1}{\lambda \Gamma_q(\sigma - \eta)} \right) \int_0^1 (1-s)_q^{(\sigma-\eta-1)} g(s) d_qs \right]. \end{aligned}$$

Since $g \in \mathcal{L}$, $\int_0^1 (1-s)_q^{(\sigma-\eta-1)} g(s) d_qs < \infty$. Also, we have

$$\sup_{t_1, t_2 \in J} \left(\int_0^{t_1} [(t_2-s)_q^{(\sigma-1)} - (t_1-s)_q^{(\sigma-1)}] g(s) d_qs \right) \leq \int_0^1 (1-s)_q^{(\sigma-1)} g(s) d_qs < \infty.$$

Since $(t_2-s)_q^{(\sigma-1)} - (t_1-s)_q^{(\sigma-1)} \rightarrow 0$, as $t_2 \rightarrow t_1$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|t_2 - t_1| < \delta$ implies

$$(t_2-s)_q^{(\sigma-1)} - (t_1-s)_q^{(\sigma-1)} < \varepsilon.$$

If $0 < \delta < \varepsilon$ and $|t_2 - t_1| < \delta$, then

$$\int_0^{t_1} [(t_2-s)_q^{(\sigma-1)} - (t_1-s)_q^{(\sigma-1)}] g(s) d_qs \leq \varepsilon \int_0^1 g(s) d_qs,$$

and so

$$\int_0^{t_1} [(t_2-s)_q^{(\sigma-1)} - (t_1-s)_q^{(\sigma-1)}] g(s) d_qs \rightarrow 0,$$

as $t_2 \rightarrow t_1$. Similarly, we conclude that

$$\int_{t_1}^{t_2} (t_1-s)_q^{(\sigma-1)} g(s) d_qs$$

and

$$\int_0^1 (1-s)_q^{(\sigma-\eta-1)} g(s) d_qs$$

tend to 0 as $t_2 \rightarrow t_1$. Thus, $|T_k(t_2) - T_k(t_1)| \rightarrow 0$ as $t_2 \rightarrow t_1$. Note that

$$|T'_k(t_2) - T'_k(t_1)| \leq \frac{\|\Theta\|_{\mathcal{A}}}{\Gamma_q(\sigma-1)} \left[\int_0^{t_1} [(t_2-s)_q^{(\sigma-2)} - (t_1-s)_q^{(\sigma-2)}] g(s) d_qs + \int_{t_1}^{t_2} (t_1-s)_q^{(\sigma-2)} g(s) d_qs \right].$$

By using a similar way, we conclude that $|T'_k(t_2) - T'_k(t_1)| \rightarrow 0$ as $t_2 \rightarrow t_1$. Hence,

$$\|T_k(t_2) - T_k(t_1)\|_* \rightarrow 0,$$

as $t_2 \rightarrow t_1$ and so T is equi-continuous on B_{r_0} . Hence, $T : B_{r_0} \rightarrow B_{r_0}$ is completely continuous. At present, Lemma 2.2 implies that T has a fixed point on B_{r_0} which is the solution of the problem (1). The proof is completed. \square

Note that in Theorem 3.2, the map $\Omega(t, \cdot, \cdot, \cdot, \cdot)$ could be discontinuous at points of a subset of J of measure zero. One can obtain solutions of the problem (1) under some different conditions. For example in next result, the map $\Omega(t, \cdot, \cdot, \cdot, \cdot)$ could be discontinuous at $t = 0$.

Theorem 3.3. *Let $\Omega : J \times \mathcal{B}^4 \rightarrow \mathbb{R}$ be a map. Then the problem (1) has a solution, whenever the following assumptions hold for all $(k_1, k_2, k_3, k_4) \in \mathcal{B}^4$ and almost all $t \in J$.*

- (1) $\Omega(t, \cdot, \cdot, \cdot, \cdot) : J \rightarrow \mathbb{R}$ is continuous and $\Omega(t, k_1, k_2, k_3, k_4) \geq 0$.
- (2) There exist $g \in \mathcal{L}$ and $\Theta_1, \Theta_2 : \mathbb{R}^4 \rightarrow [0, \infty)$ such that Θ_1 and Θ_2 are nondecreasing in all components,

$$\lim_{k \rightarrow \infty} \Theta_1(k, k, k, k)/k = 0, \quad \lim_{l \rightarrow \infty} \Theta_2(l, l, l, l) = \ell < \infty$$

and

$$\Omega(t, k_1, k_2, k_3, k_4) \leq g(t)\Theta_1(k_1, k_2, k_3, k_4) + \Theta_2(k_1, k_2, k_3, k_4).$$

Proof. For each $k \in \mathcal{B}$ and $i \geq 1$ define

$$(k)_i(t) = \min \left\{ \frac{-1}{i}, k(t) \right\},$$

whenever $k(t) < 0$ and $(k)_i(t) = \max \left\{ \frac{1}{i}, k(t) \right\}$ whenever $k(t) \geq 0$. Put

$$\Omega_i(t, k_1, k_2, k_3, k_4) = \Omega(t, (k_1)_i, (k_2)_i, (k_3)_i, (k_4)_i),$$

for all i, t and k_1, k_2, k_3, k_4 . By simple method, we conclude that $(k)_i(t) \rightarrow k(t)$ and each Ω_i is a regular function on J . A regular function at a point a is a function that is regular in some neighborhood of a . For each i , consider the regular fractional q -integro-differential equation

$$\mathcal{D}_q^\sigma[k](t) + \Omega_i \left(t, k(t), k'(t), \mathcal{D}_q^\zeta[k](t), \int_0^t f(r)k(r)dr \right) = 0, \tag{21}$$

under the boundary condition of the problem (1). Suppose that $\|g\|_1 = m > 0$ and $\varepsilon_0 > 0$ be given. Choose $r_1 > 0$ and $r_2 > 0$ such that $\left| \frac{\ell}{k} \right| < \frac{1}{2}\varepsilon_0$ for each $|k| > r_1$ and

$$\frac{\Theta_1(k, k, k, k)}{k} < \frac{1}{2\|g\|_1}\varepsilon_0,$$

for each $|k| > r_2$, respectively. Take $r_0 := \max\{r_1, r_2\}$, then for all $|k| > r_0$, we obtain

$$\frac{\ell + \|g\|_1\Theta_1(k, k, k, k)}{k} < \varepsilon_0.$$

Put $\Lambda_0 := \max\{M_1, M_2\}$, here M_1 and M_2 are defined in equation (19), and $\varepsilon_0 = \frac{1}{\Lambda_0}$. If

$$r > r_0 \max \left\{ 1, \frac{1}{\Gamma_q(2 - \zeta)}, m \right\},$$

then

$$\frac{1}{r} \left[\ell + \|g\|_1 \Theta \left(r, r, \frac{r}{\Gamma_q(2 - \zeta)}, mr \right) \right] < \frac{1}{\Lambda_0}. \tag{22}$$

At present, consider the set

$$B_r = \{k \in \mathcal{B} : \|k\|_* < r\}.$$

For each $i \geq 1$, define $T_i : \bar{B}_r \rightarrow \mathcal{B}$ as (18) in which we replaced Ω by Ω_i . If $\{k_i\}$ is a convergent sequence in \bar{B}_r , then $k_i \rightarrow k$ and $k_i' \rightarrow k'$ uniformly on J . Since

$$\|\mathcal{D}_q^\zeta[k_i](t) - \mathcal{D}_q^\zeta[k](t)\| \leq \frac{\|k_i - k\|}{\Gamma_q(2 - \zeta)}$$

and $\mathcal{D}_q^\zeta[k_i](t) \rightarrow \mathcal{D}_q^\zeta[k](t)$. Also, we have

$$\left| \int_0^t f(r)k_i(r)dr - \int_0^t f(r)k(r)dr \right| \leq \int_0^t f(r)|k_i(r) - k(r)|dr \leq m\|k_i - k\|$$

and so

$$\lim_{i \rightarrow \infty} \int_0^t f(r)k_i(r)dr = \int_0^t f(r)k(r)dr.$$

Thus, $\lim_{i \rightarrow \infty} \tilde{\Omega}_i(k_i, t) = \tilde{\Omega}_i(k, t)$. Note that

$$\begin{aligned} |T_n[k_i](t) - T_n[k](t)| &\leq \int_0^1 \left[\frac{(t-s)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} + \frac{t(t-s)_q^{(\sigma-1)}}{\lambda \Gamma_q(\sigma)} + \frac{t(\tau-s)_q^{(\sigma-\eta-1)}}{\lambda \Gamma_q(\sigma-\eta)} \right] |\tilde{\Omega}_n(k_i, s) - \tilde{\Omega}_n(k, s)| d_q s \\ &\leq M_1 \int_0^1 |\tilde{\Omega}_n(k_i, s) - \tilde{\Omega}_n(k, s)| d_q s. \end{aligned}$$

By using a similar method, we have

$$|T'_n[k_i](t) - T'_n[k](t)| \leq M_2 \int_0^1 |\tilde{\Omega}_n(k_i, s) - \tilde{\Omega}_n(k, s)| d_q s.$$

Thus, $\|T_n[k_i](t) - T_n[k](t)\|_* \rightarrow 0$ as $k_i \rightarrow k$. Hence, $\{T_n[k_i]\}_{i=1}^{\infty}$ is relatively compact in \bar{B}_r and so T_i is a completely continuous operator on \bar{B}_r for all i . Suppose that $i \geq 1$ be given and there exist $z \in \partial B_r$ and $0 < c < 1$ such that $z = cT_i[z]$. Since $\|z\|_* = r$, $\|z\| \leq r$, $\|z'\| \leq r$,

$$\|\mathcal{D}_q^\zeta[z]\| \leq \frac{\|z'\|}{\Gamma_q(2-\zeta)} \leq \frac{r}{\Gamma_q(2-\zeta)}$$

and $\left\| \int f(r)z(r)dr \right\| \leq mr$. By using the assumption, we have

$$\begin{aligned} |z(t)| = |cT_n[z](t)| &= \left| c \int_0^1 G_q(t, qs) \tilde{\Omega}_i(z, s) d_q s \right| \\ &< M_1 \int_0^1 \Theta_2 \left(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr \right) d_q s + \int_0^1 f(s) \Theta_1 \left(z(s), z'(s), \mathcal{D}_q^\sigma[z](s), \int_0^s f(r)z(r)dr \right) d_q s \\ &\leq M_1 \left(\ell + \|g\|_1 \Theta_1 \left(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr \right) \right) \end{aligned}$$

and

$$|z'(t)| = |cT'_n[z](t)| = \left| c \int_0^1 \frac{\partial G_q(t, s)}{\partial t} \tilde{\Omega}_i(z, s) d_q s \right| < M_2 \left(\ell + \|g\|_1 \Theta_1 \left(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr \right) \right).$$

Hence,

$$\|z\|_* < \max\{M_1, M_2\} \left(\ell + \|g\|_1 \Theta_1 \left(r, r, \frac{r}{\Gamma_q(2-\beta)}, mr \right) \right),$$

and so

$$r < \Lambda_0 \left(\ell + \|g\|_1 \Theta_1 \left(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr \right) \right).$$

Thus,

$$\ell + \|g\|_1 \Theta_1 \left(r, r, \frac{r}{\Gamma_q(2-\zeta)}, mr \right) > \frac{r}{\Lambda_0},$$

which is a contradiction to (22). This implies that $z \notin \partial B_r$. By employing Lemma 2.3, T_i has a fixed point $k_i \in \bar{B}_r$ for each i , that is the problem (21) has a solution. Let $(k)_i$ be the solution of the problem (21). As we proved, $\{(k)_i\}$ is relatively compact and $(k)_i \rightarrow k$ for some $k \in \mathcal{B}$. Thus, $k \in \bar{B}_r$. Similar to last result, we can show that $\lim_{i \rightarrow \infty} \mathcal{D}_q^\beta[k_i](t) = \mathcal{D}_q^\beta[k](t)$, $\lim_{i \rightarrow \infty} k_i'(t) = k'(t)$ and

$$\lim_{i \rightarrow \infty} \int_0^t f(r)k_i(r)dr = \int_0^t f(r)k(r)dr,$$

for each $t \in J$. Consequently, we get $\lim_{i \rightarrow \infty} \tilde{\Omega}_i(k, t) = \tilde{\Omega}(k, t)$ and

$$|G_q(t, s)\tilde{\Omega}_i(k, t) - \tilde{\Omega}(k, t)| \leq M_1 \left[g(s)\Theta_2 \left(r, r, \frac{r}{\Gamma_q(2 - \zeta)}, mr \right) \right] < \infty.$$

By applying the Lebesgue dominated theorem, we obtain

$$k(t) = \int_0^1 G_q(t, s)\tilde{\Omega}(k, s)d_qs,$$

for all $t \in J$. This completes the proof. □

4 Illustrative examples via computational results

In this section, we present two illustrative examples. For problems for which the analytical solution is not known, we will use, as reference solution, the numerical approximation obtained with a tiny step h by the implicit trapezoidal PI rule, which, as we will see, usually shows an excellent accuracy [32]. For this purpose, we need to present a simplified analysis that is able to execute the values of the q -Gamma function. We provided a pseudo-code description of the method for calculation of the q -Gamma function of order n in Algorithms 3, 4, 6 and 7; for more details see <https://www.dm.uniba.it/members/garrappa/software>.

All the experiments are carried out in MATLAB Ver. 8.5.0.197613 (R2015a) on a computer equipped with a CPU AMD Athlon(tm) II X2 245 at 2.90 GHz running under the operating system Windows 7.

Example 4.1. Consider the fractional q -integro-differential problem

$$\mathcal{D}_q^{\frac{25}{9}}[k](t) + g(t) \left[\frac{|k(t)|}{3 + |k(t)|} + \frac{|k'(t)|}{3 + |k'(t)|} + \frac{|\mathcal{D}_q^{\frac{9}{14}}[k](t)|}{3 + |\mathcal{D}_q^{\frac{9}{14}}[k](t)|} + \frac{|z_k(t)|}{3 + |z_k(t)|} \right] = 0, \tag{23}$$

for $t \in J$, $k \in C^1(J)$ and for each $q \in (0, 1)$, under conditions $k(0) = 0$ and

$$k(1) = \mathcal{D}_q^{\frac{5}{7}}[k] \left(\frac{8}{9} \right),$$

where

$$z_k(t) = \int_0^t f(r)k(r)dr,$$

$f(t) = \frac{3}{\sqrt{t+1}} \in \mathcal{L}$ and $m = \|f\|_1 = 3$. Clearly in the problem $\sigma = \frac{25}{9} \geq 2$, $\zeta = \frac{9}{14} \in (0, 1)$, $\tau = \frac{8}{9} \in (0, 1)$, $\eta = \frac{5}{7} \in (0, 1)$. We define $g(t)$ by

$$g(t) = \begin{cases} \frac{1}{t^{p_1}}, & t \in (0, \gamma_1], \\ \frac{1}{(t - \gamma_1)^{p_2}}, & t \in (\gamma_1, \gamma_2], \\ \vdots & \vdots \\ \frac{1}{(t - \gamma_k)^{p_{N_0+1}}}, & t \in (\gamma_k, 1), \end{cases}$$

where $p_1, \dots, p_{N_0+1} \in (0, 1)$ ($k \geq 1$), and $\gamma_1, \gamma_2, \dots, \gamma_{N_0}$ be real numbers such that

$$0 < \gamma_1 < \gamma_2 < \dots < \gamma_{N_0} < 1.$$

For $N_0 = 4$, we take

$$g(t) = \begin{cases} \frac{1}{t^{\frac{1}{8}}}, & t \in \left(0, \frac{1}{4}\right], \\ \frac{1}{\left(t - \frac{1}{4}\right)^{\frac{3}{7}}}, & t \in \left(\frac{1}{4}, \frac{1}{2}\right], \\ \frac{1}{\left(t - \frac{1}{2}\right)^{\frac{5}{7}}}, & t \in \left(\frac{1}{2}, \frac{3}{4}\right], \\ \frac{1}{\left(t - \frac{3}{4}\right)^{\frac{9}{10}}}, & t \in \left(\frac{3}{4}, 1\right). \end{cases} \quad (24)$$

Now, define

$$\Theta(k_1, k_2, k_3, k_4) = \sum_{i=1}^4 \frac{|k_i|}{3 + |k_i|},$$

for $(k_1, k_2, k_3, k_4) \in \mathbb{R}^4$. One can see that Θ satisfies in equation (17). Then we have

$$\begin{aligned} & |\Omega(t, k_1, k_2, k_3, k_4) - \Omega(t, l_1, l_2, l_3, l_4)| \\ &= \left| g(t) \left[\frac{|k_1(t)|}{3 + |k_1(t)|} + \frac{|k_2'(t)|}{3 + |k_2'(t)|} + \frac{|\mathcal{D}_q^{\frac{9}{14}}[k_3](t)|}{3 + |\mathcal{D}_q^{\frac{9}{14}}[k_3](t)|} + \frac{|z_{k_4}(t)|}{3 + |z_{k_4}(t)|} \right] \right. \\ & \quad \left. - g(t) \left[\frac{|l_1(t)|}{3 + |l_1(t)|} + \frac{|l_2'(t)|}{3 + |l_2'(t)|} + \frac{|\mathcal{D}_q^{\frac{9}{14}}[l_3](t)|}{3 + |\mathcal{D}_q^{\frac{9}{14}}[l_3](t)|} + \frac{|z_{l_4}(t)|}{3 + |z_{l_4}(t)|} \right] \right| \\ &\leq |g(t)| \left[\frac{1}{3} |k_1(t) - l_1(t)| + \frac{1}{3} |k_2(t) - l_2(t)| + \frac{1}{3} |k_3(t) - l_3(t)| + \frac{1}{3} |k_4(t) - l_4(t)| \right] \\ &\leq |g(t)| \sum_{i=1}^4 |k_i(t) - l_i(t)|. \end{aligned}$$

Thus, by applying the definition of $g(t)$, we get

$$|\Omega(t, k_1, k_2, k_3, k_4) - \Omega(t, l_1, l_2, l_3, l_4)| \leq \begin{cases} \frac{1}{t^{\frac{1}{8}}} \sum_{i=1}^4 |k_i(t) - l_i(t)|, & t \in \left(0, \frac{1}{4}\right], \\ \frac{1}{\left(t - \frac{1}{4}\right)^{\frac{3}{7}}} \sum_{i=1}^4 |k_i(t) - l_i(t)|, & t \in \left(\frac{1}{4}, \frac{1}{2}\right], \\ \frac{1}{\left(t - \frac{1}{2}\right)^{\frac{5}{7}}} \sum_{i=1}^4 |k_i(t) - l_i(t)|, & t \in \left(\frac{1}{2}, \frac{3}{4}\right], \\ \frac{1}{\left(t - \frac{3}{4}\right)^{\frac{9}{10}}} \sum_{i=1}^4 |k_i(t) - l_i(t)|, & t \in \left(\frac{3}{4}, 1\right). \end{cases}$$

Therefore,

$$f_i(t) = \frac{1}{\sqrt[8]{t}}, \quad \frac{1}{\sqrt[7]{\left(t - 0.25\right)^3}}, \quad \frac{1}{\sqrt[7]{\left(t - 0.5\right)^5}}, \quad \frac{1}{\sqrt[10]{\left(t - 0.75\right)^9}},$$

for $t \in \left(0, \frac{1}{4}\right], \left(\frac{1}{4}, \frac{1}{2}\right], \left(\frac{1}{2}, \frac{3}{4}\right], \left(\frac{3}{4}, 1\right)$, respectively, for $i = 1, 2, 3, 4$. In addition by using equations (16) and (19), we obtain

$$\lambda = 1 - \frac{\tau^{1-\eta}}{\Gamma_q(2-\eta)} = 1 - \frac{\left(\frac{8}{9}\right)^{\left(\frac{2}{7}\right)}}{\Gamma_q\left(\frac{9}{7}\right)} \cong \begin{cases} 0.0062, & q = \frac{1}{10}, \\ -0.0412, & q = \frac{1}{2}, \\ -0.0664, & q = \frac{6}{7}. \end{cases}$$

On the other hand,

$$M_1 = \frac{1}{\Gamma_q(\sigma)} + \frac{1}{\lambda \Gamma_q(\sigma)} + \frac{1}{\lambda \Gamma_q(\sigma - \eta)} = \frac{1}{\Gamma_q\left(\frac{25}{9}\right)} + \frac{1}{\lambda \Gamma_q\left(\frac{25}{9}\right)} + \frac{1}{\lambda \Gamma_q\left(\frac{130}{63}\right)},$$

$$M_2 = \frac{1}{\Gamma_q(\sigma - 1)} + \frac{1}{\lambda \Gamma_q(\sigma)} + \frac{1}{\lambda \Gamma_q(\sigma - \eta)} = \frac{1}{\Gamma_q\left(\frac{16}{9}\right)} + \frac{1}{\lambda \Gamma_q\left(\frac{25}{9}\right)} + \frac{1}{\lambda \Gamma_q\left(\frac{130}{63}\right)}$$

and from equation (20),

$$m_0 = \int_0^1 (1-s)_q^{(\sigma-\eta-1)} g(s) d_q s = \int_0^1 (1-s)_q^{\left(\frac{25}{9}-\frac{5}{7}-1\right)} g(s) d_q s = \int_0^1 (1-s)_q^{\left(\frac{67}{63}\right)} g(s) d_q s$$

$$= (1-q) \sum_{k=0}^{\infty} \left(q^k g(q^k) \left[\prod_{i=0}^{\infty} \frac{1 - q^{k+i}}{1 - q^{k+\sigma-\eta-1+i}} \right] \right) = (1-q) \sum_{k=0}^{\infty} \left(q^k g(q^k) \left[\prod_{i=0}^{\infty} \frac{1 - q^{k+i}}{1 - q^{k+i+\frac{67}{63}}} \right] \right).$$

Thus, we have $M_1 \cong 313.0401, -41.1026, -23.6644, M_2 \cong 313.1262, -40.7920, -23.2307, m_0 \cong 0.1372, 0.6360, 1.5717, r_0 = \|\Theta\|_{\mathcal{A}} \times (42.9762), \|\Theta\|_{\mathcal{A}} \times (-25.9439), \|\Theta\|_{\mathcal{A}} \times (-36.5123)$ for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$, respectively. These results are obtained by Algorithms 9, 10 and 11. Now, for showing the numerical results, we consider the problem (23) as follows:

$$\mathcal{D}_q^{\frac{25}{9}} [k](t) + g(t) \left[\frac{|k(t)|}{3 + |k(t)|} + \frac{|k'(t)|}{3 + |k'(t)|} + \frac{|\mathcal{D}_q^{\frac{9}{14}} [k](t)|}{3 + |\mathcal{D}_q^{\frac{9}{14}} [k](t)|} + \frac{|z_k(t)|}{3 + |z_k(t)|} \right]$$

$$\leq \mathcal{D}_q^{\frac{25}{9}} [k](t) + g(t) \left[|k(t)| + |k'(t)| + |\mathcal{D}_q^{\frac{9}{14}} [k](t)| + |z_k(t)| \right] = 0. \tag{25}$$

Table 1: Some numerical results of $\lambda, M_1, M_2, m_0, r_0$ in Example 4.1 for $t \in \bar{J}$ and $q = \frac{1}{10}$

$(q = \frac{1}{10})n$	λ	M_1	M_2	m_0	r_0
1	0.0067	287.9505	288.0367	0.1189	34.2533
2	0.0062	310.3480	310.4341	0.1348	41.8468
3	0.0062	312.7840	312.8701	0.1369	42.8388
4	0.0062	313.0398	313.1259	0.1372	42.9627
5	0.0062	313.0401	313.1262	0.1372	42.9746
6	0.0062	313.0401	313.1262	0.1372	42.9762
7	0.0062	313.0401	313.1262	0.1372	42.9762

Table 2: Some numerical results of $\lambda, M_1, M_2, m_0, r_0$ in Example 4.1 for $t \in \bar{J}$ and $q = \frac{1}{2}$

$(q = \frac{1}{2})n$	λ	M_1	M_2	m_0	r_0
1	0.0083	178.5039	178.8292	0.3858	68.9912
2	-0.0172	-92.2014	-91.8824	0.4974	-45.7011
3	-0.0294	-55.8206	-55.5059	0.559	-31.0280
⋮	⋮	⋮	⋮	⋮	⋮
9	-0.0410	-41.2640	-40.9543	0.6338	-25.9558
10	-0.0411	-41.1831	-40.8735	0.6348	-25.9454
11	-0.0412	-41.1423	-40.8327	0.6353	-25.9421
12	-0.0412	-41.1219	-40.8123	0.6356	-25.9416
13	-0.0412	-41.1122	-40.8027	0.6358	-25.9423
14	-0.0412	-41.1069	-40.7973	0.6359	-25.9427
15	-0.0412	-41.1042	-40.7947	0.6359	-25.9430
16	-0.0412	-41.1034	-40.7939	0.6360	-25.9437
17	-0.0412	-41.1025	-40.7929	0.6360	-25.9437
18	-0.0412	-41.1025	-40.7930	0.6360	-25.9441
19	-0.0412	-41.1026	-40.7930	0.6360	-25.9443
20	-0.0412	-41.1016	-40.7920	0.6360	-25.9437
21	-0.0412	-41.1016	-40.7920	0.6360	-25.9438
22	-0.0412	-41.1016	-40.7920	0.6360	-25.9438
23	-0.0412	-41.1016	-40.7920	0.6360	-25.9439
24	-0.0412	-41.1016	-40.7920	0.6360	-25.9439

Let $t_1 = \frac{1}{8}, t_2 = \frac{4}{11}, t_3 = \frac{5}{9}$ and $t_4 = \frac{16}{19}$. Then from definition of $g(t)$ in equation (24), we have $g(t_1) = 1.2668, g(t_2) = 2.5396, g(t_3) = 7.8817$ and $g(t_4) = 8.5535$, which, upon substitution in equation (25), leads to

$$\begin{aligned}
 \mathcal{D}_q^{\frac{25}{9}}[k](t) + \frac{1}{\sqrt[8]{8}} \left[|k(t)| + |k'(t)| + |\mathcal{D}_q^{\frac{9}{14}}[k](t)| \right] &= -\frac{1}{\sqrt[8]{8}} \left| \int_0^t f_1(r)k(r)dr \right|, \\
 \mathcal{D}_q^{\frac{25}{9}}[k](t) + \frac{1}{\sqrt[7]{\left(\frac{5}{44}\right)^3}} \left[|k(t)| + |k'(t)| + |\mathcal{D}_q^{\frac{9}{14}}[k](t)| \right] &= -\frac{1}{\sqrt[7]{\left(\frac{5}{44}\right)^3}} \left| \int_0^t f_2(r)k(r)dr \right|, \\
 \mathcal{D}_q^{\frac{25}{9}}[k](t) + \frac{1}{\sqrt[7]{\left(\frac{1}{18}\right)^5}} \left[|k(t)| + |k'(t)| + |\mathcal{D}_q^{\frac{9}{14}}[k](t)| \right] &= -\frac{1}{\sqrt[7]{\left(\frac{1}{18}\right)^5}} \left| \int_0^t f_3(r)k(r)dr \right|, \\
 \mathcal{D}_q^{\frac{25}{9}}[k](t) + \frac{1}{\sqrt[10]{\left(\frac{7}{76}\right)^9}} \left[|k(t)| + |k'(t)| + |\mathcal{D}_q^{\frac{9}{14}}[k](t)| \right] &= -\frac{1}{\sqrt[10]{\left(\frac{7}{76}\right)^9}} \left| \int_0^t f_4(r)k(r)dr \right|.
 \end{aligned}
 \tag{26}$$

Table 3: Some numerical results of $\lambda, M_1, M_2, m_0, r_0$ in Example 4.1 for $t \in \bar{J}$ and $q = \frac{6}{7}$

$(q = \frac{6}{7})n$	λ	M_1	M_2	m_0	r_0
1	0.2188	2.2717	2.6290	0.6443	1.6939
2	0.1616	3.9488	4.3474	0.7476	3.2501
3	0.1192	6.4012	6.8259	0.9636	6.5775
⋮	⋮	⋮	⋮	⋮	⋮
52	-0.0663	-23.6855	-23.2518	1.5712	-36.5341
53	-0.0664	-23.6825	-23.2489	1.5713	-36.5307
54	-0.0664	-23.6798	-23.2461	1.5713	-36.5275
⋮	⋮	⋮	⋮	⋮	⋮
67	-0.0664	-23.6666	-23.2329	1.5716	-36.5139
68	-0.0664	-23.6663	-23.2327	1.5717	-36.5137
69	-0.0664	-23.666	-23.2324	1.5717	-36.5135
⋮	⋮	⋮	⋮	⋮	⋮
84	-0.0664	-23.6647	-23.231	1.5717	-36.5126
85	-0.0664	-23.6643	-23.2307	1.5717	-36.5120
86	-0.0664	-23.6643	-23.2307	1.5717	-36.5121
87	-0.0664	-23.6643	-23.2307	1.5717	-36.5121
88	-0.0664	-23.6644	-23.2307	1.5717	-36.5121
89	-0.0664	-23.6644	-23.2307	1.5717	-36.5121
⋮	⋮	⋮	⋮	⋮	⋮
98	-0.0664	-23.6644	-23.2307	1.5717	-36.5122
99	-0.0664	-23.6644	-23.2307	1.5717	-36.5123
100	-0.0664	-23.6644	-23.2307	1.5717	-36.5123

Table 4 shows numerical values of $k(t)$ for each equations in (26). Also, one can see that the curve of $k(t)$ with respect to t in Figure 1 for $t \in (0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{1}{2}]$, $(\frac{1}{2}, \frac{3}{4}]$, $(\frac{3}{4}, 1)$, respectively (Algorithm 12). By using Theorem 3.3, one can see that the singular q -integro-differential problem (23) has a solution.

Table 4: Some numerical results of $T_k(t) = \int_0^1 G_q(t, s)\tilde{\Omega}(k, s)d_q s$ in Example 4.1 for $t \in \bar{J}$

$(0, \frac{1}{4}]$		$(\frac{1}{4}, \frac{1}{2}]$		$(\frac{1}{2}, \frac{3}{4}]$		$(\frac{3}{4}, 1)$	
t	$k(t)$	t	$k(t)$	t	$k(t)$	t	$k(t)$
0	0	0.2500	0	0.500	0	0.7500	0
0.0156	0	0.2656	0	0.5156	0	0.7656	0
0.0313	0	0.2813	0	0.5313	0	0.7813	0
0.0469	0.0001	0.2969	0	0.5469	0	0.7969	0
0.0625	0.0001	0.3125	0.0001	0.5625	0.0001	0.8125	0
0.0781	0.0002	0.3281	0.0001	0.5781	0.0001	0.8281	0
0.0938	0.0004	0.3438	0.0002	0.5938	0.0002	0.8438	0.0001
0.1094	0.0006	0.3594	0.0004	0.6094	0.0003	0.8594	0.0001
0.1250	0.0008	0.375	0.0005	0.6250	0.0005	0.8750	0.0001
0.1406	0.0011	0.3906	0.0007	0.6406	0.0007	0.8906	0.0002
0.1563	0.0015	0.4063	0.0010	0.6563	0.0011	0.9063	0.0002
0.1719	0.0019	0.4219	0.0013	0.6719	0.0015	0.9219	0.0003
0.1875	0.0024	0.4375	0.0017	0.6875	0.0021	0.9375	0.0003
0.2031	0.0029	0.4531	0.0021	0.7031	0.0027	0.9531	0.0004
0.2188	0.0036	0.4688	0.0027	0.7188	0.0035	0.9688	0.0005
0.2344	0.0043	0.4844	0.0033	0.7344	0.0042	0.9844	0.0006

In the next example we consider the discontinuous map $\Omega(t, \cdot, \cdot, \cdot, \cdot)$ at points of a subset of J of measure zero. Then, we obtain solutions of the problem (1) under some different conditions in Theorem 3.3 when the map $\Omega(t, \cdot, \cdot, \cdot, \cdot)$ is discontinuous at $t = 0$.

Example 4.2. Consider the singular fractional q -integro-differential problem

$$\begin{aligned} & \mathcal{D}_q^{\frac{10}{3}}[k](t) + \frac{1}{\sqrt[3]{t}} \left[\frac{1}{2} |k|^{\frac{1}{3}} + \frac{8}{5} |k'|^{\frac{2}{5}} + \frac{1}{10} |\mathcal{D}_q^{\frac{4}{15}}[k](t)|^{\frac{3}{4}} + \frac{15}{6} |z_k(t)|^{\frac{7}{9}} \right] \\ & + \frac{3}{2} \left(t^2 + \Gamma_q \left(\frac{4}{3} \right) \right) \left[\frac{1}{1+k^2(t)} + \frac{1}{2+[k'(t)]^2} + \frac{1}{1+(\mathcal{D}_q^{\frac{4}{15}}[k](t))^2} + \frac{1}{2+[z_k(t)]^2} \right] = 0, \end{aligned} \tag{27}$$

for $t \in J$ and $q \in (0, 1)$, with boundary conditions $k(0) = 0$ and

$$k(1) = \mathcal{D}_q^{\frac{6}{11}}[k] \left(\frac{5}{8} \right).$$

It is clear that $\sigma = \frac{10}{3} \geq 2$, $\zeta = \frac{4}{15} \in (0, 1)$, $\eta = \frac{6}{11} \in (0, 1)$, $\tau = \frac{5}{8} \in (0, 1)$ and

$$z_k(t) = \int_0^t f(r)k(r)dr.$$

Put $g(t) = \frac{1}{\sqrt[3]{t}}$ and

$$\begin{aligned} \Theta_1(k_1, k_2, k_3, k_4) &= \sum_{i=1}^4 \beta_i |k_i|^{p_i}, \\ \Theta_2(k_1, k_2, k_3, k_4) &= \sum_{i=1}^4 \frac{3(t^2 + 5)}{2 + k_i^2}, \end{aligned}$$

for $t \in J$. Hence, we get $m = \|g(t)\|_1 = \frac{5}{4}$,

$$\lim_{k \rightarrow \infty} \frac{\Theta_1(k, k, k, k)}{k} = 0,$$

$$\ell = \lim_{k \rightarrow \infty} \Theta_2(k, k, k, k) = 6 \left(1 + \Gamma_q \left(\frac{4}{3} \right) \right) < \infty$$

and

$$\Omega(t, k_1, k_2, k_3, k_4) \leq \frac{1}{\sqrt[3]{t}} \Theta_1(k_1, k_2, k_3, k_4) + 6 \left(1 + \Gamma_q \left(\frac{4}{3} \right) \right).$$

One can see that in Problem (27) $\gamma = \frac{1}{5} \in (0, 1)$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{8}{5}$, $\beta_3 = \frac{1}{10}$, $\beta_4 = \frac{15}{6} \in [0, \infty)$, $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{5}$, $p_3 = \frac{3}{4}$, $p_4 = \frac{7}{9} \in [0, 1)$. At first by using Eqs (16) and (19), we obtain

$$\lambda = 1 - \frac{\tau^{1-\eta}}{\Gamma_q(2-\eta)} = 1 - \frac{\left(\frac{5}{8}\right)^{\left(\frac{5}{11}\right)}}{\Gamma_q\left(\frac{5}{11}\right)} \cong \begin{cases} 0.1644, & q = \frac{1}{8}, \\ -0.1222, & q = \frac{1}{2}, \\ -0.1074, & q = \frac{9}{13}, \end{cases}$$

$M_1 \cong 11.5855, 11.1921, 11.1201$; $M_2 \cong 11.6995, 11.5296, 11.5264$; $\varepsilon_0 \cong 0.0855, 0.0867, 0.0868$; $\frac{1}{\Gamma_q\left(\frac{5}{11}\right)} = 1.0224, 1.0605, 1.0743$ and

$$\Lambda_0 = \max\{M_1, M_2\} = 11.6995, 11.5206, 11.5264,$$

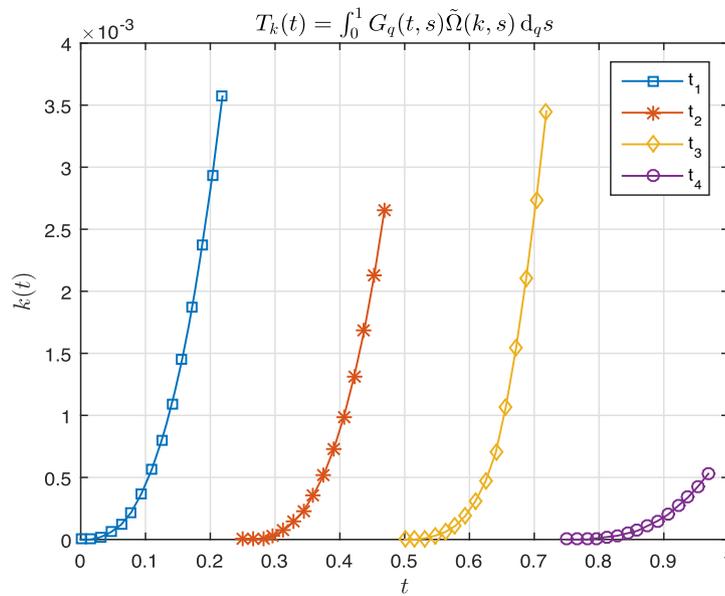


Figure 1: $k(t)$ with respect to t for Equations in (26) in Example 4.1 for $t \in (0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{1}{2}]$, $(\frac{1}{2}, \frac{3}{4}]$, $(\frac{3}{4}, 1)$, respectively, according to Table 4.

Table 5: Some numerical results of λ , M_1 , M_2 , ϵ_0 , $\frac{1}{\Gamma_q(2-\zeta)}$ in Example 4.2 for $t \in \bar{J}$ and $q = \frac{1}{8}$

$(q = \frac{1}{8})n$	λ	M_1	M_2	ϵ_0	$\frac{1}{\Gamma_q(2-\zeta)}$
1	0.1655	11.4862	11.6001	0.0862	1.0206
2	0.1645	11.573	11.6869	0.0856	1.0221
3	0.1644	11.5839	11.6979	0.0855	1.0223
4	0.1644	11.5853	11.6993	0.0855	1.0224
5	0.1644	11.5855	11.6995	0.0855	1.0224
6	0.1644	11.5855	11.6995	0.0855	1.0224

Table 6: Some numerical results of λ , M_1 , M_2 , ϵ_0 , $\frac{1}{\Gamma_q(2-\zeta)}$ in Example 4.2 for $t \in \bar{J}$ and $q = \frac{9}{13}$

$(q = \frac{9}{13})n$	λ	M_1	M_2	ϵ_0	$\frac{1}{\Gamma_q(2-\zeta)}$
1	0.2735	2.3829	2.6859	0.3723	0.7858
2	0.2180	3.6932	4.0335	0.2479	0.8780
3	0.1821	5.0443	5.4078	0.1849	0.9399
⋮	⋮	⋮	⋮	⋮	⋮
21	0.1075	11.1067	11.5129	0.0869	1.0741
22	0.1074	11.1108	11.5170	0.0868	1.0741
23	0.1074	11.1136	11.5199	0.0868	1.0742
24	0.1074	11.1156	11.5218	0.0868	1.0742
⋮	⋮	⋮	⋮	⋮	⋮
31	0.1074	11.1197	11.526	0.0868	1.0742
32	0.1074	11.1198	11.5261	0.0868	1.0743
33	0.1074	11.1200	11.5262	0.0868	1.0743
34	0.1074	11.1200	11.5262	0.0868	1.0743
35	0.1074	11.1200	11.5262	0.0868	1.0743
36	0.1074	11.1200	11.5263	0.0868	1.0743
37	0.1074	11.1200	11.5263	0.0868	1.0743
38	0.1074	11.1200	11.5263	0.0868	1.0743
39	0.1074	11.1201	11.5264	0.0868	1.0743
40	0.1074	11.1201	11.5264	0.0868	1.0743

Table 7: Some numerical results of $\lambda, M_1, M_2, \varepsilon_0, \frac{1}{\Gamma_q(2-\zeta)}$ in Example 4.2 for $t \in \bar{J}$ and $q = \frac{1}{2}$

$(q = \frac{1}{2})n$	λ	M_1	M_2	ε_0	$\frac{1}{\Gamma_q(2-\zeta)}$
1	0.1839	6.2473	6.5585	0.1525	0.9529
2	0.1524	8.2455	8.5706	0.1167	1.0072
3	0.1371	9.5643	9.8958	0.1011	1.034
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
9	0.1224	11.1638	11.5011	0.0869	1.0601
10	0.1223	11.1779	11.5154	0.0868	1.0603
11	0.1222	11.185	11.5225	0.0868	1.0604
12	0.1222	11.1886	11.5260	0.0868	1.0604
13	0.1222	11.1903	11.5278	0.0867	1.0604
14	0.1222	11.1912	11.5286	0.0867	1.0605
15	0.1222	11.1916	11.5291	0.0867	1.0605
16	0.1222	11.1919	11.5294	0.0867	1.0605
17	0.1222	11.1920	11.5295	0.0867	1.0605
18	0.1222	11.1920	11.5295	0.0867	1.0605
19	0.1222	11.1920	11.5295	0.0867	1.0605
20	0.1222	11.1921	11.5296	0.0867	1.0605
21	0.1222	11.1921	11.5296	0.0867	1.0605

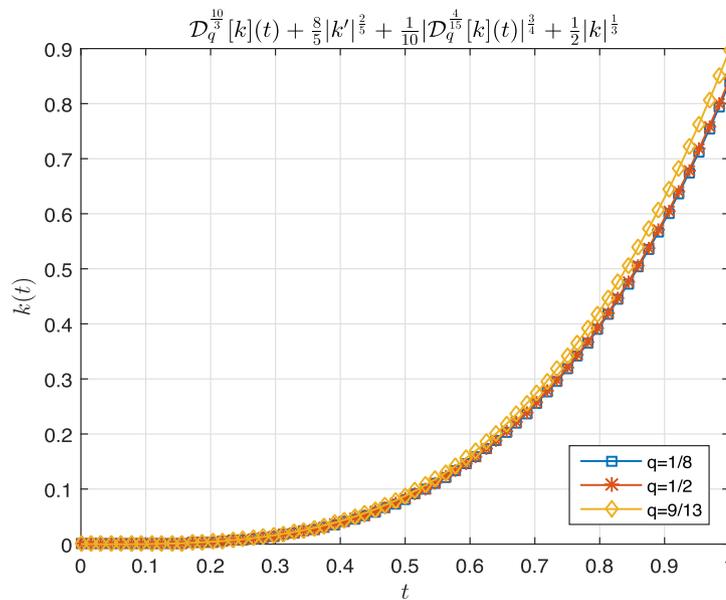


Figure 2: $k(t)$ with respect to t for Equations in (27) in Example 4.2 for $q \in \{\frac{1}{8}, \frac{1}{2}, \frac{9}{13}\}$, respectively, according to Table 8.

for $q = \frac{1}{8}, \frac{1}{2}, \frac{9}{13}$, respectively, which are shown in Tables 5–7. Note that the value of r must be more than

$$r_0 \max \left\{ 1, \frac{1}{\Gamma_q(2-\zeta)}, m \right\} = 1.2500,$$

for $q \in (0, 1)$ according to Tables 5–7. These results are obtained by Algorithm 13. Now, for showing the numerical results, we consider the problem (27) as follows (Figure 2):

$$\begin{aligned} & \mathcal{D}_q^{\frac{10}{3}}[k](t) + \frac{1}{\sqrt[5]{t}} \left[\frac{1}{2} |k|^{\frac{1}{3}} + \frac{8}{5} |k'|^{\frac{2}{5}} + \frac{1}{10} |\mathcal{D}_q^{\frac{4}{15}}[k](t)|^{\frac{3}{4}} + \frac{15}{6} |z_k(t)|^{\frac{7}{9}} \right] \\ & + \frac{3}{2} \left(t^2 + \Gamma_q \left(\frac{4}{3} \right) \right) \left[\frac{1}{1+k^2(t)} + \frac{1}{2+[k'(t)]^2} + \frac{1}{1+(\mathcal{D}_q^{\frac{4}{15}}[k](t))^2} + \frac{1}{2+[z_k(t)]^2} \right] \\ & \leq \mathcal{D}_q^{\frac{10}{3}}[k](t) + \frac{1}{2} |k|^{\frac{1}{3}} + \frac{8}{5} |k'|^{\frac{2}{5}} + \frac{1}{10} |\mathcal{D}_q^{\frac{4}{15}}[k](t)|^{\frac{3}{4}} + \frac{15}{6} |z_k(t)|^{\frac{7}{9}} + 6 \left(t^2 + \Gamma_q \left(\frac{4}{3} \right) \right) = 0. \end{aligned}$$

Thus,

$$\mathcal{D}_q^{\frac{10}{3}}[k](t) + \frac{8}{5} |k'|^{\frac{2}{5}} + \frac{1}{10} |\mathcal{D}_q^{\frac{4}{15}}[k](t)|^{\frac{3}{4}} + \frac{1}{2} |k|^{\frac{1}{3}} = -\frac{15}{6} |z_k(t)|^{\frac{7}{9}} - 6 \left(t^2 + \Gamma_q \left(\frac{4}{3} \right) \right). \tag{28}$$

Table 8: Some numerical results of $k(t)$ in Example 4.2 for $t \in \bar{J}$, $q \in \left\{ \frac{1}{8}, \frac{1}{2}, \frac{9}{13} \right\}$ and $n = 1, 2, \dots, 10$

	$q = \frac{1}{8}$		$q = \frac{1}{2}$		$q = \frac{9}{13}$	
	t	$k(t)$	t	$k(t)$	t	$k(t)$
$(n = 1)$						
1	0	0	0	0	0	0
1	0.0156	0	0.0156	0	0.0156	0
1	0.0313	0	0.0313	0	0.0313	0
1	0.0469	0	0.0469	0	0.0469	0
1	0.0625	0.0001	0.0625	0.0001	0.0625	0.0001
1	0.0781	0.0002	0.0781	0.0002	0.0781	0.0002
⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	0.9531	0.7127	0.9531	0.7175	0.9531	0.7632
1	0.9688	0.7528	0.9688	0.7579	0.9688	0.806
1	0.9844	0.7945	0.9844	0.7999	0.9844	0.8506
$(n = 2)$						
2	0	0	0	0	0	0
2	0.0156	0	0.0156	0	0.0156	0
2	0.0313	0	0.0313	0	0.0313	0
2	0.0469	0	0.0469	0	0.0469	0
2	0.0625	0.0001	0.0625	0.0001	0.0625	0.0001
2	0.0781	0.0002	0.0781	0.0002	0.0781	0.0002
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2	0.9688	0.7523	0.9688	0.7422	0.9688	0.7738
2	0.9844	0.794	0.9844	0.7834	0.9844	0.8166
2	1	0.8373	1	0.8261	1	0.8611
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$(n = 10)$						
10	0	0	0	0	0	0
10	0.0156	0	0.0156	0	0.0156	0
10	0.0313	0	0.0313	0	0.0313	0
10	0.0469	0	0.0469	0	0.0469	0
10	0.0625	0.0001	0.0625	0.0001	0.0625	0.0001
10	0.0781	0.0002	0.0781	0.0002	0.0781	0.0002
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	0.9531	0.7121	0.9531	0.6894	0.9531	0.6842
10	0.9688	0.7522	0.9688	0.7282	0.9688	0.7228
10	0.9844	0.7939	0.9844	0.7686	0.9844	0.7629
10	1	0.8372	1	0.8106	1	0.8045

Table 8 shows numerical values of $k(t)$ in equation (27). Furthermore, one can see that the curve of $k(t)$ with respect to t in Table 8 (Algorithm 14)). We can see that Θ_1 , Θ_2 and g satisfy the conditions of Theorem 3.3. Thus, the problem (27) has a solution.

5 Conclusion

The q -integro-differential boundary equations and their applications represent a matter of high interest in the area of fractional q -calculus due to their various applications in areas of science and technology. Indeed, the q -integro-differential boundary value problems often occur in mathematical modeling of a variety of physical operations. In this context, we prove the existence of a solution for a new class of singular q -integro-differential equations (18) and (27) on a time scale. The results are verified by constructing two examples along with their numerical simulations that demonstrated perfect consistency with the theoretical findings. To this end, the authors investigated a complicated case by utilizing an appropriate basic theory which facilitates a particular interest in this paper.

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