



Research Article

Zhihua Zhang*

Deterministic and random approximation by the combination of algebraic polynomials and trigonometric polynomials

<https://doi.org/10.1515/math-2021-0089>

received February 2, 2021; accepted July 21, 2021

Abstract: Fourier approximation plays a key role in qualitative theory of deterministic and random differential equations. In this paper, we will develop a new approximation tool. For an m -order differentiable function f on $[0, 1]$, we will construct an m -degree algebraic polynomial P_m depending on values of f and its derivatives at ends of $[0, 1]$ such that the Fourier coefficients of $R_m = f - P_m$ decay fast. Since the partial sum of Fourier series R_m is a trigonometric polynomial, we can reconstruct the function f well by the combination of a polynomial and a trigonometric polynomial. Moreover, we will extend these results to the case of random processes.

Keywords: Fourier approximation, trigonometric polynomial, differential equations

MSC 2020: 41-xx, 42-xx, 65-xx

1 Introduction

Fourier approximation plays a key role in qualitative theory of deterministic and random differential equations [1]. Given an m -order differentiable function f on $[0, 1]$, when we extend f into a 1-periodic function f^* on the real axis \mathbb{R} , due to discontinuity of f^* at integral points, its Fourier coefficients decay slowly, so we need a lot of Fourier coefficients to reconstruct f [2–5]. Chebyshev polynomial and other orthogonal polynomials can not also overcome boundary discontinuity (i.e., $f(0) \neq f(1)$). In this paper, we construct an m -degree algebraic polynomial P_m uniquely determined by $f^{(v)}(0)$ and $f^{(v)}(1)$ ($v = 0, 1, \dots, m - 1$) such that Fourier coefficients of $f - P_m$ decay fast. We expand $f - P_m$ into a Fourier series and obtain a Fourier expansion of f with a polynomial term:

$$f(x) = P_m(x) + \sum_n c_n e^{2\pi i n x},$$

where Fourier coefficients $c_n = o\left(\frac{1}{n^m}\right)$. Its partial sum:

$$s_N^{(p)}(x) = P_m(x) + \sum_{|n| \leq N} c_n e^{2\pi i n x}$$

is a sum of an m -degree algebraic polynomial $P_m(x)$ and an N -degree trigonometric polynomial. The partial sum $s_N^{(p)}(x)$ can well reconstruct f and can attain the best square approximation order.

Since data often originates from random background in application [6–8], using theory of stochastic calculus [9,10], we can extend the above results to deal with random processes. Suppose that \mathbf{f} is a random

* Corresponding author: Zhihua Zhang, School of Mathematics, Shandong University, Jinan, 250100, China, e-mail: zhangzhihua@sdu.edu.cn

process on $[0, 1]$. Then the corresponding $\mathbf{P}_m(x)$ is a random polynomial, and the corresponding Fourier expansion with polynomial term is

$$\mathbf{f}(x) = \mathbf{P}_m(x) + \sum_n c_n(\mathbf{R})e^{2\pi inx}, \tag{1.1}$$

where Fourier coefficients c_n are random variables whose expectations and variances satisfy, respectively,

$$|E[c_n(\mathbf{R})]| \leq \frac{1}{(2\pi n)^m} \max_{0 \leq x \leq 1} (|E[\mathbf{f}^{(m)}(x)]|) \quad (\mathbf{R} = \mathbf{f} - \mathbf{P}_m),$$

$$\text{Var}(c_n(\mathbf{R})) \leq \frac{1}{(2\pi n)^{2m}} \max_{0 \leq x, y \leq 1} |\text{Cov}(\mathbf{f}^{(m)}(x), \mathbf{f}^{(m)}(y))|,$$

where the notation Cov is the covariance.

Denote by $s_N^{(p)}$ the partial sum of (1.1). We get the mean square error:

$$E[\|\mathbf{f} - s_N^{(p)}\|_2^2] \leq A_m^* \frac{1}{N^{2m-1}},$$

where A_m^* is a constant which is stated in (4.10).

Similarly, for a random process $f \in C^m([0, 1])$, if we directly expand f into Fourier series, we can obtain that the mean square error is $O(\frac{1}{N})$. Therefore, random approximation by the combination of algebraic polynomials and trigonometric polynomials is better than direct Fourier approximation.

The paper is organized as follows. In Section 2, for a real-valued function $f \in C^m([0, 1])$, we construct the end-point polynomial $\mathbf{P}_m(x)$ and give a decomposition formula of differentiable functions on $[0, 1]$. In Section 3, we give a Fourier expansion with a polynomial term and give an estimate of Fourier coefficients and the error estimate of the partial sum for the corresponding expansion. In Section 4, we extend the above results to random processes.

2 End-point polynomials

Let $f(x)$ be a real-valued function on $[0, 1]$ and $f \in C^m([0, 1])$. We try to find a polynomial $P_m(x)$ such that the Fourier coefficients of $f(x) - P_m(x)$ decay fast.

Let $P_m(x) = \sum_{j=1}^m h_j x^j$. We choose h_1, \dots, h_m such that its ν -order derivative satisfies

$$P_m^{(\nu)}(1) - P_m^{(\nu)}(0) = f^{(\nu)}(1) - f^{(\nu)}(0) \quad (\nu = 0, 1, \dots, m - 1). \tag{2.1}$$

Definition 2.1. For $f \in C^m([0, 1])$, the m -degree polynomial $P_m(x)$ satisfying the condition (2.1) is said to be the end-point polynomial of f .

Now we prove the end-point polynomial exists and is unique, and give its representation.

Noticing that

$$P_m^{(\nu)}(1) = \sum_{j=\nu}^m \frac{j!}{(j-\nu)!} h_j, \quad P_m^{(\nu)}(0) = \nu! h_\nu \quad (\nu = 0, 1, \dots, m - 1).$$

Equality (2.1) can be written into the system of linear equations:

$$\sum_{j=\nu+1}^m \frac{j!}{(j-\nu)!} h_j = f^{(\nu)}(1) - f^{(\nu)}(0) \quad (\nu = 0, 1, \dots, m - 1). \tag{2.2}$$

Denote the coefficient matrix $C = (c_{vj})_{v,j=0,1,\dots,m}$, where

$$c_{vj} = \begin{cases} \frac{j!}{(j-v)!}, & v \leq j, \\ 0, & v > j. \end{cases}$$

In detail,

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ 0 & 2! & \frac{3!}{2!} & \frac{4!}{3!} & \dots & \frac{j!}{(j-1)!} & \dots & \frac{m!}{(m-1)!} \\ 0 & 0 & 3! & \frac{4!}{2!} & \dots & \frac{j!}{(j-2)!} & \dots & \frac{m!}{(m-2)!} \\ 0 & 0 & 0 & 4! & \dots & \frac{j!}{(j-3)!} & \dots & \frac{m!}{(m-3)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & j! & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & m! \end{pmatrix}$$

and the vectors

$$\mathbf{h} = (h_1, h_2, \dots, h_m)^T, \quad \mathbf{F} = (f(1) - f(0), f'(1) - f'(0), \dots, f^{(m-1)}(1) - f^{(m-1)}(0))^T,$$

where T means transpose of a vector. The matrix form of (2.2) is

$$C\mathbf{h} = \mathbf{F}.$$

The determinant of the matrix C is equal to $\prod_{k=1}^m k!$. Therefore, based on Cramer's rule, the system of linear equations (2.2) has a unique solution:

$$h_j = \frac{\Delta_j}{\prod_{k=1}^m k!} \quad (j = 1, \dots, m), \tag{2.3}$$

where Δ_j is the determinant obtained by replacing j th column of the determinant $\det C$ by the vector \mathbf{F} . Since the coefficient matrix C is a triangular matrix, it is easy to be solved out recurrently:

$$\begin{aligned} h_m &= \frac{1}{m!}(f^{(m-1)}(1) - f^{(m-1)}(0)), \\ h_{m-1} &= \frac{1}{(m-1)!}(f^{(m-2)}(1) - f^{(m-2)}(0)) - \frac{1}{2}(f^{(m-1)}(1) - f^{(m-1)}(0)), \\ h_{m-2} &= \frac{1}{(m-2)!}((f^{(m-3)}(1) - f^{(m-3)}(0)) - \frac{1}{2}(f^{(m-2)}(1) - f^{(m-2)}(0)) + \frac{1}{6}(f^{(m-1)}(1) - f^{(m-1)}(0))), \\ &\vdots \end{aligned}$$

i.e., each h_j is a linear combination of

$$f^{(j-1)}(1) - f^{(j-1)}(0), \quad f^{(j)}(1) - f^{(j)}(0), \quad \dots, \quad f^{(m-1)}(1) - f^{(m-1)}(0).$$

For example, in the case $m = 3$, $f \in C^3([0, 1])$ and $P_3(x) = h_1x + h_2x^2 + h_3x^3$, where

$$\begin{aligned} h_1 &= (f(1) - f(0)) - \frac{1}{2}(f'(1) - f'(0)) + \frac{1}{6}(f''(1) - f''(0)), \\ h_2 &= \frac{1}{2}(f'(1) - f'(0)) - \frac{1}{2}(f''(1) - f''(0)), \\ h_3 &= \frac{1}{6}(f''(1) - f''(0)) \end{aligned}$$

satisfy

$$\begin{aligned} P_3(1) - P_3(0) &= f(1) - f(0), \\ P'_3(1) - P'_3(0) &= f'(1) - f'(0), \\ P''_3(1) - P''_3(0) &= f''(1) - f''(0). \end{aligned}$$

Denote $R(x) = f(x) - P_m(x)$. By (2.1), we have that

Theorem 2.1. *Suppose that $f \in C^m([0, 1])$ and $P_m(x)$ is its end-point polynomial. Then*

$$P_m(x) = \sum_{j=1}^m h_j x^j, \tag{2.4}$$

where coefficients h_j are stated in (2.3), and the decomposition formula:

$$f(x) = P_m(x) + R(x) \tag{2.5}$$

holds, where the remainder term $R(x) \in C^m([0, 1])$ satisfy

$$R^{(v)}(1) = R^{(v)}(0) \quad (v = 0, 1, \dots, m - 1). \tag{2.6}$$

3 Fourier expansion with end-point polynomial

We expand the remainder term of decomposition formula (2.5) into the Fourier series:

$$R(x) = \sum_n c_n(R) e^{2\pi i n x},$$

where the Fourier coefficients $c_n(R) = \int_0^1 R(x) e^{-2\pi i n x} dx$. By (2.6),

$$c_n(R) = \frac{1}{(2\pi i n)^m} \int_0^1 R^{(m)}(x) e^{-2\pi i n x} dx = o\left(\frac{1}{n^m}\right). \tag{3.1}$$

By (2.5), the Fourier expansion with end-point polynomial is

$$f(x) = P_m(x) + \sum_n c_n(R) e^{2\pi i n x}$$

whose Fourier coefficients decay fast. Since $P_m(x)$ is a polynomial of degree m , by the decomposition formula: $f(x) = P_m(x) + R(x)$ and (3.1), we deduce that

$$c_n(R) = \frac{1}{(2\pi i n)^m} \int_0^1 (f^{(m)}(x) - P_m^{(m)}(x)) e^{-2\pi i n x} dx \quad (n \neq 0).$$

Since P_m is an m -order polynomial, its m -order derivative is a constant. So

$$c_n(R) = \frac{1}{(2\pi i n)^m} \int_0^1 f^{(m)}(x) e^{-2\pi i n x} dx = \frac{1}{(2\pi i n)^m} c_n(f^{(m)}) \quad (n \neq 0).$$

This gives a relationship between the Fourier coefficients of the remainder and the Fourier coefficients of the original function f . So we get a Fourier expansion with the end-point polynomial of $f \in C^m([0, 1])$ as follows.

Theorem 3.1. *Let $f \in C^m([0, 1])$ and P_m be the end-point polynomial. Then the expansion:*

$$f(x) = P_m(x) + c_0 + \sum_{n \neq 0} \frac{1}{(2\pi i n)^m} c_n(f) e^{2\pi i n x} \tag{3.2}$$

holds, where

$$c_0 = \int_0^1 (f(x) - P_m(x)) dx, \quad c_n(f^{(m)}) = \int_0^1 f^{(m)}(x) e^{-2\pi i n x} dx.$$

Now we consider the partial sum approximation of Fourier expansion (3.2) with the end-point polynomial. Denote by $s_N^{(p)}(f; x)$ its partial sum, i.e.,

$$s_N^{(p)}(f; x) = P_m(x) + c_0 + \sum_{0 < |n| \leq N} \frac{1}{(2\pi in)^m} c_n(f^{(m)}) e^{2\pi inx}. \quad (3.3)$$

Then

$$f(x) - s_N^{(p)}(f; x) = \sum_{|n| \geq N+1} \frac{1}{(2\pi in)^m} c_n(f^{(m)}) e^{2\pi inx}.$$

By the Parseval identity,

$$\|f - s_N^{(p)}(f)\|_2^2 = \int_0^1 |f(x) - s_N^{(p)}(f; x)|^2 dx = \sum_{|n| \geq N+1} \frac{1}{(2\pi)^{2m} n^{2m}} |c_n(f^{(m)})|^2 = \sum_{|n| \geq N+1} o\left(\frac{1}{n^{2m}}\right) = o\left(\frac{1}{N^{2m-1}}\right).$$

From this and $|c_n(f^{(m)})| \leq \max_{0 \leq x \leq 1} |f^{(m)}(x)|$,

$$\|f - s_N^{(p)}(f)\|_2^2 \leq \frac{1}{(2\pi)^{2m}} \max_{0 \leq x \leq 1} |f^{(m)}(x)|^2 \sum_{|n| \geq N+1} \frac{1}{n^{2m}}.$$

However,

$$\sum_{|n| \geq N+1} \frac{1}{n^{2m}} \leq 2 \int_N^\infty \frac{1}{t^{2m}} dt = \frac{2}{2m-1} N^{2m-1}.$$

So we get the following:

Theorem 3.2. *Let $f \in C^m([0, 1])$ and the partial sum $s_N^{(p)}(f)$ be stated as in (3.3). Then the square error:*

$$\|f - s_N^{(p)}(f)\|_2^2 = o\left(\frac{1}{N^{2m-1}}\right), \quad \|f - s_N^{(p)}(f)\|_2^2 \leq \frac{A_m}{N^{2m-1}},$$

where $A_m(f) = \frac{2}{2m-1} (2\pi)^{-2m} \max_{0 \leq x \leq 1} |f^{(m)}(x)|^2$.

The partial sum $s_N^{(p)}(f; x)$ is a good approximation tool, which is a sum of an algebraic polynomial $P_m(x)$ of degree m and a trigonometric polynomial of degree N , where $P_m(x)$ is determined by values of derivatives of f at end points 0 and 1, and N is determined by predictive error ε . For this purpose, we take N such that $\frac{A_m}{N^{2m-1}} \leq \varepsilon$, i.e.,

$$N \geq \left(\frac{A_m(f)}{\varepsilon}\right)^{\frac{1}{2m-1}}.$$

Especially, the case $m = 2$, $f \in C^2([0, 1])$, and

$$P_2(x) = (f(1) - f(0)) - \frac{1}{2}(f'(1) - f'(0))x + \frac{1}{2}(f'(1) - f'(0))x^2,$$

the Fourier expansion with the end-point polynomial is

$$f(x) = P_2(x) + \int_0^1 (f(x) - P_2(x)) dx + \sum_{n \neq 0} \frac{1}{(2\pi in)^m} c_n(f'') e^{2\pi inx},$$

and the square error of its partial sum is

$$\|f - s_N^{(p)}(f)\|_2^2 \leq \frac{1}{24\pi^4} \max_{0 \leq x \leq 1} |f''(x)|^2 \frac{1}{N^3}.$$

4 Random processes on [0, 1]

Finally, we extend the above results to random processes.

For a random variable ξ , we denote its expectation and variance by $E[\xi]$ and $\text{Var}(\xi)$, respectively. For two random variables ξ and η , we denote their covariance by $\text{Cov}(\xi, \eta)$. We always assume that a random variable ξ satisfies $E[|\xi|^2] < \infty$, i.e., assume that ξ is a second-order random variable. If $\mathbf{f}(x)$ is a random variable for each $x \in [0, 1]$, we say $\mathbf{f}(x)$ is a random process on $[0, 1]$.

Let $\{\xi_n\}_1^\infty$ be a sequence of random variables and ξ be a random variable. If

$$\lim_{n \rightarrow \infty} E[|\xi_n - \xi|^2] = 0,$$

we say ξ is the limit of this sequence $\{\xi_n\}_1^\infty$ [9,10]. Based on this limit concept, the concepts of continuity and derivatives, and integrals for random processes are established (see the details in [9,10]).

Let \mathbf{f} be a real-valued random process and $\mathbf{f} \in C^m([0, 1])$ [9,10], and random variables $\mathbf{h}_1, \dots, \mathbf{h}_m$ satisfy the system of linear equations:

$$\sum_{j=v+1}^m \frac{j!}{(j-v)!} \mathbf{h}_j = \mathbf{f}^{(v)}(1) - \mathbf{f}^{(v)}(0) \quad (v = 0, 1, \dots, m-1).$$

Then $\mathbf{P}_m(x) = \sum_{j=1}^m \mathbf{h}_j x^j$ is called the end-point random polynomial and satisfies

$$\mathbf{P}_m^{(v)}(1) - \mathbf{P}_m^{(v)}(0) = \mathbf{f}^{(v)}(1) - \mathbf{f}^{(v)}(0) \quad (v = 0, 1, \dots, m-1).$$

We get the decomposition formula:

$$\mathbf{f}(x) = \mathbf{P}_m(x) + \mathbf{R}(x),$$

where $\mathbf{R}(x)$ is a real-valued random process on $[0, 1]$ and $\mathbf{R} \in C^m([0, 1])$, and

$$\mathbf{R}^{(v)}(1) = \mathbf{R}^{(v)}(0) \quad (v = 0, 1, \dots, m-1).$$

The corresponding Fourier expansion with the end-point random polynomial is

$$\mathbf{f}(x) = \mathbf{P}_m(x) + \sum_n c_n(\mathbf{R}) e^{2\pi i n x} \quad (\mathbf{R} = \mathbf{f} - \mathbf{P}_m), \tag{4.1}$$

where the Fourier coefficients:

$$c_n(\mathbf{R}) = \int_0^1 \mathbf{R}(x) e^{-2\pi i n x} dx$$

are random variables. Consider their expectations and variances of $c_n(\mathbf{R})$. Since the expectation and the integral can be exchanged,

$$E[c_n(\mathbf{R})] = \int_0^1 E[\mathbf{R}(x)] e^{-2\pi i n x} dx. \tag{4.2}$$

Since the expectation and the derivative can be exchanged, from $\mathbf{R} \in C^m([0, 1])$, we deduce that the deterministic function $E[\mathbf{R}(x)]$ satisfies that

$$E[\mathbf{R}(x)] \in C^m([0, 1]), \quad (E[\mathbf{R}(x)])^{(v)} = E[\mathbf{R}^{(v)}(x)] \quad (v = 0, 1, \dots, m-1).$$

Again, by $\mathbf{R}^{(v)}(1) = \mathbf{R}^{(v)}(0)$,

$$(E[\mathbf{R}(x)])^{(v)}|_{x=1} = E[\mathbf{R}^{(v)}(1)] = E[\mathbf{R}^{(v)}(0)] = (E[\mathbf{R}(x)])^{(v)}|_{x=0} \quad (v = 0, 1, \dots, m-1). \tag{4.3}$$

Using integration by parts, it follows by (4.2) and (4.3) that

$$E[c_n(\mathbf{R})] = \frac{1}{(2\pi in)^m} \int_0^1 (E[\mathbf{R}(x)])^{(m)} e^{-2\pi inx} dx = \frac{1}{(2\pi in)^m} \int_0^1 E[\mathbf{R}^{(m)}(x)] e^{-2\pi inx} dx = o\left(\frac{1}{n^m}\right).$$

Since $\mathbf{R}^{(m)}(x) = \mathbf{f}^{(m)}(x) - \mathbf{P}_m^{(m)}(x) = \mathbf{f}^{(m)}(x) - m! \mathbf{h}_m$, we have

$$E[c_n(\mathbf{R})] = \frac{1}{(2\pi in)^m} \left(\int_0^1 E[\mathbf{f}^{(m)}(x)] e^{-2\pi inx} dx - m! \mathbf{h}_m \int_0^1 e^{2\pi inx} dx \right) = \frac{1}{(2\pi in)^m} \int_0^1 E[\mathbf{f}^{(m)}(x)] e^{-2\pi inx} dx. \quad (4.4)$$

From this, we deduce the estimate:

$$|E[c_n(\mathbf{R})]| \leq \frac{1}{(2\pi m)^m} \max_{0 \leq x \leq 1} (|E[\mathbf{f}^{(m)}(x)]|).$$

Consider the variance of the Fourier coefficients. Note that the variance of random variable $c_n(\mathbf{R})$:

$$\text{Var}(c_n(\mathbf{R})) = E[|c_n(\mathbf{R})|^2] - |E[c_n(\mathbf{R})]|^2.$$

First, we compute $E[|c_n(\mathbf{R})|^2]$. From

$$E[|c_n(\mathbf{R})|^2] = E \left[\left| \int_0^1 \mathbf{R}(x) e^{-2\pi inx} dx \right|^2 \right],$$

$$|c_n(\mathbf{R})|^2 = \int_0^1 \mathbf{R}(x) e^{-2\pi inx} dx \int_0^1 \mathbf{R}(y) e^{-2\pi iny} dy = \int_0^1 \int_0^1 \mathbf{R}(x) \mathbf{R}(y) e^{-2\pi in(x-y)} dx dy,$$

it follows that

$$E[|c_n(\mathbf{R})|^2] = \int_0^1 \int_0^1 E[\mathbf{R}(x) \mathbf{R}(y)] e^{-2\pi in(x-y)} dx dy.$$

By $\mathbf{R} \in C^m([0, 1])$,

$$\mathbf{R}(x) \mathbf{R}(y) \in C^{(m,m)}([0, 1]^2), \quad E[\mathbf{R}(x) \mathbf{R}(y)] \in C^{(m,m)}([0, 1]^2).$$

Using integration by parts, we get

$$\int_0^1 E[\mathbf{R}(x) \mathbf{R}(y)] e^{-2\pi inx} dx = \frac{1}{(2\pi in)^m} \int_0^1 E[\mathbf{R}^{(m)}(x) \mathbf{R}(y)] e^{-2\pi inx} dx.$$

From

$$\mathbf{R}^{(m)}(x) = \mathbf{f}^{(m)}(x) - m! \mathbf{h}_m, \quad E[\mathbf{R}^{(m)}(x) \mathbf{R}(y)] = E[\mathbf{f}^{(m)}(x) \mathbf{R}(y)] - m! \mathbf{h}_m E[\mathbf{R}(y)],$$

it follows that

$$\int_0^1 E[\mathbf{R}(x) \mathbf{R}(y)] e^{-2\pi inx} dx = \frac{1}{(2\pi in)^m} \int_0^1 E[\mathbf{f}^{(m)}(x) \mathbf{R}(y)] e^{-2\pi inx} dx,$$

and so

$$E[|c_n(\mathbf{R})|^2] = \frac{1}{(2\pi in)^m} \int_0^1 \left(\int_0^1 E[\mathbf{f}^{(m)}(x) \mathbf{R}(y)] e^{2\pi iny} dy \right) e^{-2\pi inx} dx.$$

Again, by $\mathbf{R}^{(m)}(y) = \mathbf{f}^{(m)}(y) - m! \mathbf{h}_m$ and

$$\int_0^1 E[\mathbf{f}^{(m)}(x)\mathbf{R}^{(m)}(y)]e^{2\pi i n y} dy = \frac{1}{(2\pi i n)^m} \int_0^1 E[\mathbf{f}^{(m)}(x)\mathbf{R}^{(m)}(y)]e^{2\pi i n y} dy,$$

we get

$$E[|\mathbf{c}_n(\mathbf{R})|^2] = \frac{1}{(2\pi m)^{2m}} \int_0^1 \int_0^1 E[\mathbf{f}^{(m)}(x)\mathbf{f}^{(m)}(y)]e^{2\pi i n(x-y)} dx dy = o\left(\frac{1}{n^{2m}}\right). \tag{4.5}$$

By the Schwarz inequality in the probability theory, we get

$$|E[\mathbf{f}^{(m)}(x)\mathbf{f}^{(m)}(y)]|^2 \leq E[|\mathbf{f}^{(m)}(x)|^2]E[|\mathbf{f}^{(m)}(y)|^2].$$

So

$$E[|\mathbf{c}_n(\mathbf{R})|^2] \leq \frac{1}{(2\pi m)^{2m}} \max_{0 \leq x, y \leq 1} |E[|\mathbf{f}^{(m)}(x)|^2]E[|\mathbf{f}^{(m)}(y)|^2]| \leq \frac{1}{(2\pi m)^{2m}} \max_{0 \leq x \leq 1} (E[|\mathbf{f}^{(m)}(x)|^2])^2. \tag{4.6}$$

Secondly, we compute $E[|\mathbf{c}_n(\mathbf{R})|^2]$. By (4.4),

$$|E[\mathbf{c}_n(\mathbf{R})]|^2 = \frac{1}{(2\pi m)^{2m}} \left| \int_0^1 E[\mathbf{f}^{(m)}(x)]e^{-2\pi i n x} dx \right|^2 = \frac{1}{(2\pi m)^m} \int_0^1 \int_0^1 E[\mathbf{f}^{(m)}(x)]E[\mathbf{f}^{(m)}(y)]e^{2\pi i n(x-y)} dx dy.$$

From this and (4.5), it follows that

$$\begin{aligned} \text{Var}(\mathbf{c}_n(\mathbf{R})) &= E[|\mathbf{c}_n(\mathbf{R})|^2] - |E[\mathbf{c}_n(\mathbf{R})]|^2 \\ &= \frac{1}{(2\pi m)^m} \int_0^1 \int_0^1 (E[\mathbf{f}^{(m)}(x)\mathbf{f}^{(m)}(y)] - E[\mathbf{f}^{(m)}(x)]E[\mathbf{f}^{(m)}(y)])e^{-2\pi i n(x-y)} dx dy. \end{aligned} \tag{4.7}$$

By the definition of the covariance,

$$\text{Cov}(\mathbf{f}^{(m)}(x), \mathbf{f}^{(m)}(y)) = E[\mathbf{f}^{(m)}(x)\mathbf{f}^{(m)}(y)] - E[\mathbf{f}^{(m)}(x)]E[\mathbf{f}^{(m)}(y)].$$

By (4.7),

$$\text{Var}(\mathbf{c}_n(\mathbf{R})) = \frac{1}{(2\pi m)^m} \int_0^1 \int_0^1 \text{Cov}(\mathbf{f}^{(m)}(x), \mathbf{f}^{(m)}(y))e^{-2\pi i n(x-y)} dx dy.$$

So

$$\text{Var}(\mathbf{c}_n(\mathbf{R})) \leq \frac{1}{(2\pi m)^{2m}} \max_{0 \leq x, y \leq 1} \text{Cov}(\mathbf{f}^{(m)}(x), \mathbf{f}^{(m)}(y)), \quad \text{Var}(\mathbf{c}_n(\mathbf{R})) = o\left(\frac{1}{n^{2m}}\right).$$

Theorem 4.1. *Let \mathbf{f} be a random process on $[0, 1]$ and $\mathbf{f} \in C^m([0, 1])$, and $\mathbf{P}_m(x)$ be its end-point polynomial. Then*

$$\mathbf{f}(x) = \mathbf{P}_m(x) + \sum_n \mathbf{c}_n(\mathbf{R})e^{2\pi i n x}, \tag{4.8}$$

where $\mathbf{R}(x) = \mathbf{f}(x) - \mathbf{P}_m(x)$ and the random Fourier coefficients $\mathbf{c}_n(\mathbf{R}) = \int_0^1 \mathbf{R}(x)e^{-2\pi i n x} dx$ satisfy the following:

- (i) $|E[\mathbf{c}_n(\mathbf{R})]| \leq \frac{1}{(2\pi m)^m} \max_{0 \leq x \leq 1} (|E[\mathbf{f}^{(m)}(x)]|)$, $E[\mathbf{c}_n(\mathbf{R})] = o\left(\frac{1}{n^m}\right)$;
- (ii) $E[|\mathbf{c}_n(\mathbf{R})|^2] \leq \frac{1}{(2\pi m)^{2m}} \max_{0 \leq x \leq 1} (E[|\mathbf{f}^{(m)}(x)|^2])$;

$$(iii) \quad \text{Var}(\mathbf{c}_n(\mathbf{R})) \leq \frac{1}{(2m)^{2m}} \max_{0 \leq x, y \leq 1} |\text{Cov}(\mathbf{f}^{(m)}(x), \mathbf{f}^{(m)}(y))|, \quad \text{Var}(\mathbf{c}_n(\mathbf{R})) = o\left(\frac{1}{n^{2m}}\right),$$

where $\text{Cov}(\xi, \eta)$ is the covariance of random variables ξ and η .

Take the partial sum of the expansion (4.1):

$$s_N^{(p)}(\mathbf{f}; x) = \mathbf{P}_m(x) + \sum_{|n| \leq N} \mathbf{c}_n(\mathbf{R}) e^{2\pi i n x}.$$

By the Parseval identity of random Fourier series and Theorem 4.1(ii), we get

$$E[\|\mathbf{f} - s_N^{(p)}\|_2^2] = \sum_{|n| > N} E[|\mathbf{c}_n(\mathbf{R})|^2] \leq \max_{0 \leq x \leq 1} (E[|\mathbf{f}^{(m)}(x)|^2]) \sum_{|n| > N} \frac{1}{(2\pi m)^{2m}} \leq \frac{A_m^*}{N^{2m-1}},$$

where

$$A_m^*(f) = \frac{2}{(2m-1)(2\pi)^{2m}} \max_{0 \leq x \leq 1} (E[|\mathbf{f}^{(m)}(x)|^2]). \quad (4.9)$$

Theorem 4.2. Let $\mathbf{f} \in C^m([0, 1])$ and the partial sum $s_N^{(p)}(\mathbf{f}; x)$ of its Fourier expansion with the end-point polynomial be stated in (4.8). Then the mean square error:

$$E[\|\mathbf{f} - s_N^{(p)}\|_2^2] \leq \frac{A_m^*}{N^{2m-1}},$$

where $A_m^*(f)$ is stated in (4.9).

Remark. Using the similar argument of Theorem 4.2, for a random process $f \in C^m([0, 1])$, if we directly expand f into Fourier series, we can obtain that the mean square error is $O\left(\frac{1}{N}\right)$. Therefore, random approximation by the combination of algebraic polynomials and trigonometric polynomials is better than direct Fourier approximation.

Funding information: This research was supported by European Commission Horizon 2020's Flagship Project "ePcenter" and National Key Science Programme No. 2019QZKK0906 and No. 2015CB953602.

Conflict of interest: Author states no conflict of interest.

References

- [1] B. Boashash, *Time-frequency Signal Analysis and Processing*, 2nd edition, Academic Press, San Diego, 2016.
- [2] A. DeVore and G. G. Lorentz, *Constructive Approximation*, vol. 303 of Grundlehren, Springer, Heidelberg, 1993.
- [3] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, 1971.
- [4] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Reprint Edition, Dover Publications, Mineola, 1994.
- [5] Z. Zhang, *Approximation of bivariate functions via smooth extensions*, *Sci. World J.* **2014** (2014), 102062, DOI: <https://doi.org/10.1155/2014/102062>.
- [6] Z. Zhang and J. Moore, *Mathematical and Physical Fundamentals of Climate Change*, Elsevier, Boston, 2015.
- [7] Z. Zhang, *Environmental Data Analysis*, De Gruyter, Berlin, Boston, 2016, DOI: <https://doi.org/10.1515/9783110424904>.
- [8] Z. Zhang and P. Jorgensen, *Modulated Haar wavelet analysis of climatic background noise*, *Acta Appl. Math.* **140** (2015), 71–93.
- [9] B. Hajek, *An Exploration of Random Processes for Engineers*, 2014, <http://www.ifp.illinois.edu/~hajek/Papers/randomprocDec11.pdf>.
- [10] J.-P. Kahane, *Some Random Series of Functions*, 2nd edition, Cambridge University Press, Cambridge, 1994.