Research Article

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Some new parameterized inequalities for co-ordinated convex functions involving generalized fractional integrals

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Abstract: In this study, we first obtain a new identity for generalized fractional integrals which contains some parameters. Then by this equality, we establish some new parameterized inequalities for co-ordinated convex functions involving generalized fractional integrals. Moreover, we show that the results proved in the main section reduce to several Simpson-, trapezoid- and midpoint-type inequalities for various values of parameters.

Keywords: Simpson's 1/3 formula, integral inequalities, fractional calculus, co-ordinated convex functions

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1 Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considered significant in the literature. These inequalities state that if $\mathbb{F} : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$, then

$$\mathbb{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathbb{F}(\tau_1)\mathrm{d}\tau_1 \leq \frac{\mathbb{F}(\kappa_1) + \mathbb{F}(\kappa_2)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if $\mathbb{F}$ is concave.

Over the last 20 years, numerous studies have focused on obtaining trapezoid- and midpoint-type inequalities which give bounds for the right-hand side and left-hand side of the inequality (1.1), respectively. For example, Dragomir and Agarwal first obtained trapezoid inequalities for convex functions in [1], whereas Kirmacı first, established midpoint inequalities for convex functions in [2]. In [3], Sarikaya et al. generalized the inequalities (1.1) for fractional integrals and the authors also proved some corresponding trapezoid-type inequalities. Iqbal et al. presented some fractional midpoint-type inequalities for convex functions in [4]. On the other hand, Dragomir proved Hermite-Hadamard inequalities for co-ordinated convex mappings in [5]. In [6] and [7], the authors proved midpoint- and trapezoid-type inequalities for

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co-ordinated convex functions, respectively. Moreover, Sarikaya obtained fractional Hermite-Hadamard inequalities and fractional trapezoid for functions with two variables in [8]. Tunç et al. presented some fractional midpoint-type inequalities for co-ordinated convex functions in [9]. In [10], Sarikaya and Ertuğral first introduced new fractional integrals which are called generalized fractional integrals, and then, they proved Hermite-Hadamard inequalities and several trapezoid- and midpoint-type inequalities for generalized fractional integrals. In addition, Turkay et al. defined the generalized fractional integrals for functions with two variables and they presented Hermite-Hadamard- and trapezoid-type inequalities for this kind of fractional integrals in [11]. For the other similar inequalities, please refer to [12–19].

On the other hand, the following inequality is well known in the literature as Simpson's inequality.

Theorem 1. *Suppose that* $\mathbb{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ *is a four times continuously differentiable mapping on* (κ_1, κ_2), *and let* $\|\mathbb{F}^{(4)}\|_\infty = \sup_{\tau_1 \in (\kappa_1, \kappa_2)} |\mathbb{F}^{(4)}(\tau_1)| < \infty$. *Then, one has the inequality*

$$\left| \frac{1}{3}\left[\frac{\mathbb{F}(\kappa_1) + \mathbb{F}(\kappa_2)}{2} + 2\mathbb{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right)\right] - \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathbb{F}(\tau_1)d\tau_1 \right| \le \frac{1}{2880} \|\mathbb{F}^{(4)}\|_\infty (\kappa_2 - \kappa_1)^4.$$

In recent years, many authors have focused on Simpson-type inequalities for various classes of functions. For example, Dragomir et al. presented new Simpson-type results and their applications to quadrature formulas in numerical integration in [20]. In addition, some inequalities of Simpson-type for η-convex functions are deduced by Alomari et al. in [21]. Afterward, Sarikaya et al. observed the variants of Simpson-type inequalities based on convexity in [22]. Moreover, some papers are devoted to Simpson inequalities for co-ordinated convex functions [23–26]. On the other hand, some authors proved several Simpson-type inequalities for fractional integrals in [27–30]. In addition, in [31], Ertugral and Sarikaya obtained some inequalities of Simpson-type for generalized fractional integrals. For more recent developments, one can refer to [32–38].

The aim of this paper is to obtain some parameterized inequalities for co-ordinated convex functions via generalized fractional integrals. These inequalities reduce to Simpson, trapezoid and midpoint inequalities in the case of special choice of parameters. The overall structure of the study takes the form of six sections including an introduction. The remaining part of the paper proceeds as follows: In Section 2, the generalized fractional integral operators is summarized, along with some related theorems. In Section 3, an identity involving generalized fractional integrals is presented for partial differentiable functions. Then we prove several parameterized inequalities for functions whose partial derivatives in absolute value are co-ordinated convex in Section 4. Moreover, some special cases of the results in Section 4 are presented in Section 5. Finally, some conclusions and further directions of research are discussed in Section 6.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. A function $\mathbb{F} : \Delta := [\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4] \to \mathbb{R}$ is called co-ordinated convex on Δ, for all (x, u), $(y, v) \in \Delta$ and $t, s \in [0, 1]$, if it satisfies the following inequality:

$$\mathbb{F}(tx + (1-t)y, su + (1-s)v) \le ts\mathbb{F}(x, u) + t(1-s)\mathbb{F}(x, v) + s(1-t)\mathbb{F}(y, u) + (1-t)(1-s)\mathbb{F}(y, v). \tag{1.2}$$

The mapping $\mathbb{F}$ is a co-ordinated concave on Δ if the inequality (1.2) holds in the reversed direction for all $t, s \in [0, 1]$ and $(x, u), (y, v) \in \Delta$.

2 Generalized fractional integrals

Fractional calculus and applications have application areas in many different fields such as physics, chemistry and engineering as well as mathematics. The application of arithmetic carried out in classical analysis in fractional analysis is very important in terms of obtaining more realistic results in the solution of many problems. Many real dynamical systems are better characterized by using non-integer order dynamic models based on fractional computation. While integer orders are a model that is not suitable for nature in classical analysis, fractional computation in which arbitrary orders are examined enables us to obtain more realistic approaches.

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [10].

Definition 2. Let $\mathbb{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an integrable function. The left-sided and right-sided generalized fractional integral operators are given by

$$_{\kappa_1+}I_{\varphi}\mathbb{F}(\tau_1) = \int_{\kappa_1}^{\tau_1} \frac{\varphi(\tau_1 - \xi)}{\tau_1 - \xi}\mathbb{F}(\xi)\mathrm{d}\xi, \quad \tau_1 > \kappa_1, \tag{2.1}$$

and

$$_{\kappa_2-}I_{\varphi}\mathbb{F}(\tau_1) = \int_{\tau_1}^{\kappa_2} \frac{\varphi(\xi - \tau_1)}{\xi - \tau_1}\mathbb{F}(\xi)\mathrm{d}\xi, \quad \tau_1 < \kappa_2, \tag{2.2}$$

respectively. Here, the function $\varphi : [0, \infty) \to [0, \infty)$ satisfies the condition

$$\int_0^1 \frac{\varphi(\xi)}{\xi}\mathrm{d}\xi < \infty.$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (2.1) and (2.2) are mentioned below.

(i) If we take $\varphi(\xi) = \xi$, the operators (2.1) and (2.2) reduce to the Riemann integral as follows:

$$I_{\kappa_1+}\mathbb{F}(\tau_1) = \int_{\kappa_1}^{\tau_1} \mathbb{F}(\xi)\mathrm{d}\xi, \quad \tau_1 > \kappa_1,$$

$$I_{\kappa_2-}\mathbb{F}(\tau_1) = \int_{\tau_1}^{\kappa_2} \mathbb{F}(\xi)\mathrm{d}\xi, \quad \tau_1 < \kappa_2.$$

(ii) If we take $\varphi(\xi) = \frac{\xi^{\alpha}}{\Gamma(\alpha)}$, the operators (2.1) and (2.2) reduce to the Riemann-Liouville fractional integral as follows:

$$J_{\kappa_1+}^{\alpha}\mathbb{F}(\tau_1) = \frac{1}{\Gamma(\alpha)}\int_{\kappa_1}^{\tau_1} (\tau_1 - \xi)^{\alpha-1}\mathbb{F}(\xi)\mathrm{d}\xi, \quad \tau_1 > \kappa_1,$$

$$J_{\kappa_2-}^{\alpha}\mathbb{F}(\tau_1) = \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\kappa_2} (\xi - \tau_1)^{\alpha-1}\mathbb{F}(\xi)\mathrm{d}\xi, \quad \tau_1 < \kappa_2.$$

(iii) If we take $\varphi(\xi) = \frac{1}{k\Gamma_k(\alpha)}\xi^{\frac{\alpha}{k}}$, the operators (2.1) and (2.2) reduce to the k-Riemann-Liouville fractional integral as follows:

$$J_{\kappa_1+,k}^{\alpha}\mathbb{F}(\tau_1) = \frac{1}{k\Gamma_k(\alpha)}\int_{\kappa_1}^{\tau_1} (\tau_1 - \xi)^{\frac{\alpha}{k}-1}\mathbb{F}(\xi)\mathrm{d}\xi, \quad \tau_1 > \kappa_1,$$

$$J_{\kappa_2-,k}^{\alpha}\mathbb{F}(\tau_1) = \frac{1}{k\Gamma_k(\alpha)}\int_{\tau_1}^{\kappa_2} (\xi - \tau_1)^{\frac{\alpha}{k}-1}\mathbb{F}(\xi)\mathrm{d}\xi, \quad \tau_1 < \kappa_2,$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty} \xi^{\alpha-1} e^{-\frac{\xi^k}{k}} \mathrm{d}\xi, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0;\ k > 0$$

are given by Mubeen and Habibullah in [39].

In the literature, there are several papers on inequalities for generalized fractional integrals. Some of them please refer to [31,40–47].

Inspired by this definition, Turkay et al. [11] give the following definitions:

Definition 3. Let $\mathbb{F} \in L_1([\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4])$. The generalized Riemann-Liouville fractional integrals ${}_{\kappa_1+,\kappa_3+}I_{\varphi,\psi}$, ${}_{\kappa_1+,\kappa_4-}I_{\varphi,\psi}$, ${}_{\kappa_2-,\kappa_3+}I_{\varphi,\psi}$, ${}_{\kappa_2-,\kappa_4-}I_{\varphi,\psi}$ are defined by

$${}_{\kappa_1+,\kappa_3+}I_{\varphi,\psi}\mathbb{F}(\tau_1, \tau_2) = \int_{\kappa_1}^{\tau_1}\int_{\kappa_3}^{\tau_2} \frac{\varphi(\tau_1 - \xi)}{\tau_1 - \xi}\frac{\psi(\tau_2 - \eta)}{\tau_2 - \eta}\mathbb{F}(\xi, \eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1 > \kappa_1,\ \tau_2 > \kappa_3, \tag{2.3}$$

$${}_{\kappa_1+,\kappa_4-}I_{\varphi,\psi}\mathbb{F}(\tau_1, \tau_2) = \int_{\kappa_1}^{\tau_1}\int_{\tau_2}^{\kappa_4} \frac{\varphi(\tau_1 - \xi)}{\tau_1 - \xi}\frac{\psi(\eta - \tau_2)}{\eta - \tau_2}\mathbb{F}(\xi, \eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1 > \kappa_1,\ \tau_2 < \kappa_4, \tag{2.4}$$

$${}_{\kappa_2-,\kappa_3+}I_{\varphi,\psi}\mathbb{F}(\tau_1, \tau_2) = \int_{\tau_1}^{\kappa_2}\int_{\kappa_3}^{\tau_2} \frac{\varphi(\xi - \tau_1)}{\xi - \tau_1}\frac{\psi(\tau_2 - \eta)}{\tau_2 - \eta}\mathbb{F}(\xi, \eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1 < \kappa_2,\ \tau_2 > \kappa_3 \tag{2.5}$$

and

$${}_{\kappa_2-,\kappa_4-}I_{\varphi,\psi}\mathbb{F}(\tau_1, \tau_2) = \int_{\tau_1}^{\kappa_2}\int_{\tau_2}^{\kappa_4} \frac{\varphi(\xi - \tau_1)}{\xi - \tau_1}\frac{\psi(\eta - \tau_2)}{\eta - \tau_2}\mathbb{F}(\xi, \eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1 < \kappa_2,\ \tau_2 < \kappa_4, \tag{2.6}$$

where $\varphi, \psi : [0, \infty) \to [0, \infty)$ are functions which satisfy $\int_0^1 \frac{\varphi(\xi)}{\xi}\mathrm{d}\xi < \infty$ and $\int_0^1 \frac{\psi(\eta)}{\eta}\mathrm{d}\eta < \infty$, respectively.

In this definition, known fractional integrals can be obtained by some special choices. For example,

(i) If we take $\varphi(\xi) = \xi$ and $\psi(\eta) = \eta$, then the operators (2.3), (2.4), (2.5) and (2.6) transform into the Riemann integrals on coordinates, respectively, as follows:

$$I_{\kappa_1+,\kappa_3+}\mathbb{F}(\tau_1, \tau_2) = \int_{\kappa_1}^{\tau_1}\int_{\kappa_3}^{\tau_2}\mathbb{F}(\xi, \eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1 > \kappa_1,\ \tau_2 > \kappa_3,$$

$$I_{\kappa_1+,\kappa_4}\mathbb{F}(\tau_1, \tau_2) = \int_{\kappa_1}^{\tau_1}\int_{\tau_2}^{\kappa_4}\mathbb{F}(\xi, \eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1 > \kappa_1,\ \tau_2 < \kappa_4,$$

$$I_{\kappa_2-,\kappa_3+}\mathbb{F}(\tau_1, \tau_2) = \int_{\tau_1}^{\kappa_2}\int_{\kappa_3}^{\tau_2}\mathbb{F}(\xi, \eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1 < \kappa_2,\ \tau_2 > \kappa_3$$

and

$$I_{\kappa_2-,\kappa_4-}\mathbb{F}(\tau_1, \tau_2) = \int_{\tau_1}^{\kappa_2}\int_{\tau_2}^{\kappa_4}\mathbb{F}(\xi, \eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1 < \kappa_2,\ \tau_2 < \kappa_4.$$

(ii) If we take $\varphi(\xi) = \frac{\xi^\alpha}{\Gamma(\alpha)}$, $\psi(\eta) = \frac{\eta^\beta}{\Gamma(\beta)}$, then for $\alpha, \beta > 0$ the operators (2.3), (2.4), (2.5) and (2.6) transform into the Riemann-Liouville fractional integrals on coordinates [8], respectively, as follows:

$$J^{\alpha,\beta}_{\kappa_1+,\kappa_3+}\mathbb{F}(\tau_1,\tau_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\tau_1}\int_{\kappa_3}^{\tau_2}(\tau_1-\xi)^{\alpha-1}(\tau_2-\eta)^{\beta-1}\mathbb{F}(\xi,\eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1>\kappa_1,\ \tau_2>\kappa_3,$$

$$J^{\alpha,\beta}_{\kappa_1+,\kappa_4}\mathbb{F}(\tau_1,\tau_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\tau_1}\int_{\tau_2}^{\kappa_4}(\tau_1-\xi)^{\alpha-1}(\eta-\tau_2)^{\beta-1}\mathbb{F}(\xi,\eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1>\kappa_1,\ \tau_2<\kappa_4,$$

$$J^{\alpha,\beta}_{\kappa_2-,\kappa_3+}\mathbb{F}(\tau_1,\tau_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\tau_1}^{\kappa_2}\int_{\kappa_3}^{\tau_2}(\xi-\tau_1)^{\alpha-1}(\tau_2-\eta)^{\beta-1}\mathbb{F}(\xi,\eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1<\kappa_2,\ \tau_2>\kappa_3$$

and

$$J^{\alpha,\beta}_{\kappa_2-,\kappa_4-}\mathbb{F}(\tau_1,\tau_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\tau_1}^{\kappa_2}\int_{\tau_2}^{\kappa_4}(\xi-\tau_1)^{\alpha-1}(\eta-\tau_2)^{\beta-1}\mathbb{F}(\xi,\eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1<\kappa_2,\ \tau_2<\kappa_4,$$

where Γ is the gamma function.

(iii) If we take $\varphi(\xi) = \frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(\eta) = \frac{\eta^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, for $\alpha, \beta, k > 0$, then the operators (2.3), (2.4), (2.5) and (2.6) transform into the Riemann-Liouville k-fractional integrals on coordinates [48], respectively, as follows:

$$J^{\alpha,\beta,k}_{\kappa_1+,\kappa_3+}\mathbb{F}(\tau_1,\tau_2) = \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\kappa_1}^{\tau_1}\int_{\kappa_3}^{\tau_2}(\tau_1-\xi)^{\frac{\alpha}{k}-1}(\tau_2-\eta)^{\frac{\beta}{k}-1}\mathbb{F}(\xi,\eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1>\kappa_1,\ \tau_2>\kappa_3,$$

$$J^{\alpha,\beta,k}_{\kappa_1+,\kappa_4-}\mathbb{F}(\tau_1,\tau_2) = \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\kappa_1}^{\tau_1}\int_{\tau_2}^{\kappa_4}(\tau_1-\xi)^{\frac{\alpha}{k}-1}(\eta-\tau_2)^{\frac{\beta}{k}-1}\mathbb{F}(\xi,\eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1>\kappa_1,\ \tau_2<\kappa_4,$$

$$J^{\alpha,\beta,k}_{\kappa_2-,\kappa_3+}\mathbb{F}(\tau_1,\tau_2) = \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\tau_1}^{\kappa_2}\int_{\kappa_3}^{\tau_2}(\xi-\tau_1)^{\frac{\alpha}{k}-1}(\tau_2-\eta)^{\frac{\beta}{k}-1}\mathbb{F}(\xi,\eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1<\kappa_2,\ \tau_2>\kappa_3$$

and

$$J^{\alpha,\beta,k}_{\kappa_2-,\kappa_4-}\mathbb{F}(\tau_1,\tau_2) = \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{\tau_1}^{\kappa_2}\int_{\tau_2}^{\kappa_4}(\xi-\tau_1)^{\frac{\alpha}{k}-1}(\eta-\tau_2)^{\frac{\beta}{k}-1}\mathbb{F}(\xi,\eta)\mathrm{d}\eta\mathrm{d}\xi, \quad \tau_1<\kappa_2,\ \tau_2<\kappa_4,$$

where Γ_k is the k-gamma function.

3 An Identity

Throughout this study, for brevity, we define

$$\Delta(\xi) = \int_0^\xi \frac{\varphi((\kappa_2-\kappa_1)u)}{u}\mathrm{d}u, \quad \Lambda(\eta) = \int_0^\eta \frac{\psi((\kappa_4-\kappa_3)u)}{u}\mathrm{d}u. \tag{3.1}$$

Now we give an identity for generalized fractional integrals.

Lemma 1. *Let* $\mathbb{F}:\Delta\to\mathbb{R}$ *be a twice partially differentiable mapping on* Δ°. If $\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}\in L(\Delta)$, *then we have the following equality*:

$$\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)=(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^1\int_0^1 w(\xi,\eta)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\eta\mathrm{d}\xi$$

for all $\lambda_i,\mu_i\geq 0$, $i=1,2$, *where the mapping* $w:[0,1]\times[0,1]\to\mathbb{R}$ *is defined by*

$$w(\xi,\eta)=\begin{cases}(\Delta(\xi)-\Delta(1)\lambda_1)(\Lambda(\eta)-\Lambda(1)\lambda_2), & 0\leq\xi\leq\frac{1}{2},\ 0\leq\eta\leq\frac{1}{2},\\ (\Delta(\xi)-\Delta(1)\lambda_1)(\Lambda(\eta)-\Lambda(1)\mu_2), & 0\leq\xi\leq\frac{1}{2},\ \frac{1}{2}\leq\eta\leq 1,\\ (\Delta(\xi)-\Delta(1)\mu_1)(\Lambda(\eta)-\Lambda(1)\lambda_2), & \frac{1}{2}\leq\xi\leq 1,\ 0\leq\eta\leq\frac{1}{2},\\ (\Delta(\xi)-\Delta(1)\mu_1)(\Lambda(\eta)-\Lambda(1)\mu_2), & \frac{1}{2}\leq\xi\leq 1,\ \frac{1}{2}\leq\eta\leq 1\end{cases}$$

and

$$\begin{aligned}&\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)\\ &=\Delta(1)(\mu_1-\lambda_1)\Lambda(1)(\mu_2-\lambda_2)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+\Delta(1)\lambda_1\Lambda(1)(\mu_2-\lambda_2)\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\\ &\quad+\Delta(1)(1-\mu_1)\Lambda(1)(\mu_2-\lambda_2)\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+\Delta(1)(\mu_1-\lambda_1)\Lambda(1)\lambda_2\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)\\ &\quad+\Delta(1)(\mu_1-\lambda_1)\Lambda(1)(1-\mu_2)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)+\Delta(1)\lambda_1\Lambda(1)\lambda_2\mathbb{F}(\kappa_1,\kappa_3)\\ &\quad+\Delta(1)(1-\mu_1)\Lambda(1)\lambda_2\mathbb{F}(\kappa_2,\kappa_3)+\Delta(1)\lambda_1\Lambda(1)(1-\mu_2)\mathbb{F}(\kappa_1,\kappa_4)+\Delta(1)(1-\mu_1)\Lambda(1)(1-\mu_2)\mathbb{F}(\kappa_2,\kappa_4)\\ &\quad-\Lambda(1)(\mu_2-\lambda_2)\left[{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right]\\ &\quad-\Delta(1)(\mu_1-\lambda_1)\left[{}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right]\\ &\quad-\Delta(1)\lambda_1\left[{}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}(\kappa_1,\kappa_4)\right]-\Lambda(1)\lambda_2\left[{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}(\kappa_2,\kappa_3)\right]\\ &\quad-\Lambda(1)(1-\mu_2)\left[{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}(\kappa_1,\kappa_4)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}(\kappa_2,\kappa_4)\right]-\Delta(1)(1-\mu_1)\left[{}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}(\kappa_2,\kappa_4)\right]\\ &\quad+{}_{\frac{\kappa_1+\kappa_2}{2}-\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}-\frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_4)+{}_{\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_2,\kappa_4).\end{aligned}$$

Proof. From the definition of the mapping $w(\xi,\eta)$, we have

$$\begin{aligned}&(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^1\int_0^1 w(\xi,\eta)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\eta\mathrm{d}\xi\\ &=(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\left[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}(\Delta(\xi)-\Delta(1)\lambda_1)(\Lambda(\eta)-\Lambda(1)\lambda_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\right.\\ &\quad+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1(\Delta(\xi)-\Delta(1)\lambda_1)(\Lambda(\eta)-\Lambda(1)\mu_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\end{aligned}$$

$$+\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}(\Delta(\xi)-\Delta(1)\mu_1)(\Lambda(\eta)-\Lambda(1)\lambda_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta$$

$$\left.+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}(\Delta(\xi)-\Delta(1)\mu_1)(\Lambda(\eta)-\Lambda(1)\mu_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\right]$$

$$=(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)[J_1+J_2+J_3+J_4].$$

Integrating by parts, one can easily obtain

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}(\Delta(\xi)-\Delta(1)\lambda_1)(\Lambda(\eta)-\Lambda(1)\lambda_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&= \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\Bigg[\left(\Delta\left(\frac{1}{2}\right)-\Delta(1)\lambda_1\right)\left(\Lambda\left(\frac{1}{2}\right)-\Lambda(1)\lambda_2\right)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\\
&\quad+\Delta(1)\lambda_1\left(\Lambda\left(\frac{1}{2}\right)-\Lambda(1)\lambda_2\right)\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)-\left(\Lambda\left(\frac{1}{2}\right)-\Lambda(1)\lambda_2\right)\\
&\quad\times\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\mathbb{F}\left(\tau_1,\frac{\kappa_3+\kappa_4}{2}\right)\frac{\varphi(\tau_1-\kappa_1)}{\tau_1-\kappa_1}\mathrm{d}\tau_1+\left(\Delta\left(\frac{1}{2}\right)-\Delta(1)\lambda_1\right)\Lambda(1)\lambda_2\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)\\
&\quad+\Delta(1)\lambda_1\Lambda(1)\lambda_2\mathbb{F}(\kappa_1,\kappa_3)-\Lambda(1)\lambda_2\int_0^{\frac{\kappa_1+\kappa_2}{2}}\frac{\varphi(\tau_1-\kappa_1)}{\tau_1-\kappa_1}\mathbb{F}(\tau_1,\kappa_3)\mathrm{d}\tau_1-\left(\Delta\left(\frac{1}{2}\right)-\Delta(1)\lambda_1\right)\\
&\quad\times\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\tau_2\right)\frac{\psi(\tau_2-\kappa_3)}{\tau_2-\kappa_3}\mathrm{d}\tau_2-\Delta(1)\lambda_1\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}\mathbb{F}(\kappa_1,\tau_2)\frac{\psi(\tau_2-\kappa_3)}{\tau_2-\kappa_3}\mathrm{d}\tau_2\\
&\quad+\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\kappa_3}^{\frac{\kappa_3+\kappa_4}{2}}\mathbb{F}(\tau_1,\tau_2)\frac{\varphi(\tau_1-\kappa_1)}{\tau_1-\kappa_1}\frac{\psi(\tau_2-\kappa_3)}{\tau_2-\kappa_3}\mathrm{d}\tau_2\mathrm{d}\tau_1\Bigg].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
J_1 &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}(\Delta(\xi)-\Delta(1)\lambda_1)(\Lambda(\eta)-\Lambda(1)\lambda_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&= \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\Bigg[\left(\Delta\left(\frac{1}{2}\right)-\Delta(1)\lambda_1\right)\left(\Lambda\left(\frac{1}{2}\right)-\Lambda(1)\lambda_2\right)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)\\
&\quad+\Delta(1)\lambda_1\left(\Lambda\left(\frac{1}{2}\right)-\Lambda(1)\lambda_2\right)\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)-\left(\Lambda\left(\frac{1}{2}\right)-\Lambda(1)\lambda_2\right)_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\\
&\quad+\left(\Delta\left(\frac{1}{2}\right)-\Delta(1)\lambda_1\right)\Lambda(1)\lambda_2\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+\Delta(1)\lambda_1\Lambda(1)\lambda_2\mathbb{F}(\kappa_1,\kappa_3)-\Lambda(1)\lambda_2\,{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}(\kappa_1,\kappa_3)\\
&\quad-\left(\Delta\left(\frac{1}{2}\right)-\Delta(1)\lambda_1\right)_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)-\Delta(1)\lambda_1\,{}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}-\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_3)\Bigg].
\end{aligned}\tag{3.2}$$

Similarly, we get

$$\begin{aligned} J_2 &= \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} (\Delta(\xi) - \Delta(1)\lambda_1)(\Lambda(\eta) - \Lambda(1)\mu_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1-\xi)\kappa_1, \eta\kappa_4 + (1-\eta)\kappa_3)\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\Bigg[\left(\Delta\left(\frac{1}{2}\right) - \Delta(1)\lambda_1\right)(\Lambda(1) - \Lambda(1)\mu_2)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_4\right) \\ &\quad + \Delta(1)\lambda_1(\Lambda(1) - \Lambda(1)\mu_2)\mathbb{F}(\kappa_1, \kappa_4) - (\Lambda(1) - \Lambda(1))\mu_2\, {}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}(\kappa_1, \kappa_4) \\ &\quad - \left(\Delta\left(\frac{1}{2}\right) - \Delta(1)\lambda_1\right)\left(\Lambda\left(\frac{1}{2}\right) - \Lambda(1)\mu_2\right)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) \\ &\quad - \Delta(1)\lambda_1\left(\Lambda\left(\frac{1}{2}\right) - \Lambda(1)\mu_2\right)\mathbb{F}\left(\kappa_1, \frac{\kappa_3+\kappa_4}{2}\right) + \left(\Lambda\left(\frac{1}{2}\right) - \Lambda(1)\mu_2\right){}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}\left(\kappa_1, \frac{\kappa_3+\kappa_4}{2}\right) \\ &\quad - \left(\Delta\left(\frac{1}{2}\right) - \Delta(1)\lambda_1\right){}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_4\right) - \Delta(1)\lambda_1\, {}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}(\kappa_1, \kappa_4) + {}_{\frac{\kappa_1+\kappa_2}{2}-, \frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_1, \kappa_4)\Bigg], \end{aligned} \tag{3.3}$$

$$\begin{aligned} J_3 &= \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (\Delta(\xi) - \Delta(1)\mu_1)(\Lambda(\eta) - \Lambda(1)\lambda_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1-\xi)\kappa_1, \eta\kappa_4 + (1-\eta)\kappa_3)\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\Bigg[(\Delta(1) - \Delta(1)\mu_1)\left(\Lambda\left(\frac{1}{2}\right) - \Lambda(1)\lambda_2\right)\mathbb{F}\left(\kappa_2, \frac{\kappa_3+\kappa_4}{2}\right) \\ &\quad - \left(\Delta\left(\frac{1}{2}\right) - \Delta(1)\mu_1\right)\left(\Lambda\left(\frac{1}{2}\right) - \Lambda(1)\lambda_2\right)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) \\ &\quad - \left(\Lambda\left(\frac{1}{2}\right) - \Lambda(1)\lambda_2\right){}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}\left(\kappa_2, \frac{\kappa_3+\kappa_4}{2}\right) + (\Delta(1) - \Delta(1)\mu_1)\Lambda(1)\lambda_2\mathbb{F}(\kappa_2, \kappa_3) \\ &\quad - \left(\Delta\left(\frac{1}{2}\right) - \Delta(1)\mu_1\right)\Lambda(1)\lambda_2\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_3\right) - \Lambda(1)\lambda_2\, {}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}(\kappa_2, \kappa_3) \\ &\quad - (\Delta(1) - \Delta(1)\mu_1){}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}(\kappa_2, \kappa_3) + \left(\Delta\left(\frac{1}{2}\right) - \Delta(1)\mu_1\right){}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_3\right) \\ &\quad + {}_{\frac{\kappa_1+\kappa_2}{2}+, \frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_2, \kappa_3)\Bigg] \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} J_4 &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (\Delta(\xi) - \Delta(1)\mu_1)(\Lambda(\eta) - \Lambda(1)\mu_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1-\xi)\kappa_1, \eta\kappa_4 + (1-\eta)\kappa_3)\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}[(\Delta(1) - \Delta(1)\mu_1)(\Lambda(1) - \Lambda(1)\mu_2)\mathbb{F}(\kappa_2, \kappa_4) \\ &\quad - \left(\Delta\left(\frac{1}{2}\right) - \Delta(1)\mu_1\right)(\Lambda(1) - \Lambda(1)\mu_2)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_4\right) \\ &\quad - (\Lambda(1) - \Lambda(1)\mu_2){}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}(\kappa_2, \kappa_4) - (\Delta(1) - \Delta(1)\mu_1)\left(\Lambda\left(\frac{1}{2}\right) - \Lambda(1)\mu_2\right)\mathbb{F}\left(\kappa_2, \frac{\kappa_3+\kappa_4}{2}\right) \\ &\quad + \left(\Delta\left(\frac{1}{2}\right) - \Delta(1)\mu_1\right)\left(\Lambda\left(\frac{1}{2}\right) - \Lambda(1)\mu_2\right)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) \\ &\quad + \left(\Lambda\left(\frac{1}{2}\right) - \Lambda(1)\mu_2\right){}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}\left(\kappa_2, \frac{\kappa_3+\kappa_4}{2}\right) - (\Delta(1) - \Delta(1)\mu_1){}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}(\kappa_2, \kappa_4) \\ &\quad + \left(\Delta\left(\frac{1}{2}\right) - \Delta(1)\mu_1\right){}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_4\right) + {}_{\frac{\kappa_1+\kappa_2}{2}+, \frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_2, \kappa_4)\Bigg]. \end{aligned} \tag{3.5}$$

By the equalities (3.2)–(3.5), we have

$$
\begin{aligned}
&(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)[J_1+J_2+J_3+J_4]\\
&\quad=(\Delta(1)\mu_1-\Delta(1)\lambda_1)(\Lambda(1)\mu_2-\Lambda(1)\lambda_2)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+\Delta(1)\lambda_1(\Lambda(1)\mu_2-\Lambda(1)\lambda_2)\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\\
&\qquad+(\Delta(1)-\Delta(1)\mu_1)(\Lambda(1)\mu_2-\Lambda(1)\lambda_2)\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+(\Delta(1)\mu_1-\Delta(1)\lambda_1)\Lambda(1)\lambda_2\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)\\
&\qquad+(\Delta(1)\mu_1-\Delta(1)\lambda_1)(\Lambda(1)-\Lambda(1)\mu_2)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)+\Delta(1)\lambda_1\Lambda(1)\lambda_2\mathbb{F}(\kappa_1,\kappa_3)+(\Delta(1)-\Delta(1)\mu_1)\Lambda(1)\lambda_2\mathbb{F}(\kappa_2,\kappa_3)\\
&\qquad+\Delta(1)\lambda_1(\Lambda(1)-\Lambda(1)\mu_2)\mathbb{F}(\kappa_1,\kappa_4)+(\Delta(1)-\Delta(1)\mu_1)(\Lambda(1)-\Lambda(1)\mu_2)\mathbb{F}(\kappa_2,\kappa_4)\\
&\qquad-(\Lambda(1)\mu_2-\Lambda(1)\lambda_2)\left[{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right]\\
&\qquad-(\Delta(1)\mu_1-\Delta(1)\lambda_1)\left[{}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right]\\
&\qquad-\Delta(1)\lambda_1\left[{}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}(\kappa_1,\kappa_4)\right]-\Lambda(1)\lambda_2\left[{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}(\kappa_2,\kappa_3)\right]\\
&\qquad-(\Lambda(1)-\Lambda(1)\mu_2)\left[{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}(\kappa_1,\kappa_4)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}(\kappa_2,\kappa_4)\right]-(\Delta(1)\\
&\qquad-\Delta(1)\mu_1)\left[{}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}(\kappa_2,\kappa_4)\right]\\
&\qquad+{}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_4)\\
&\qquad+{}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_2,\kappa_4).
\end{aligned}
$$

which completes the proof. □

Remark 1. If we assume $\varphi(\xi)=\xi$ and $\psi(\eta)=\eta$ in Lemma 1, we obtain,

$$
\begin{aligned}
\Phi(\kappa_1,\kappa_2;\kappa_3,\kappa_4)&=(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^{\frac12}\int_0^{\frac12}(\xi-\lambda_1)(\eta-\lambda_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&\quad+(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^{\frac12}\int_{\frac12}^{1}(\xi-\lambda_1)(\eta-\mu_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&\quad+(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_{\frac12}^{1}\int_0^{\frac12}(\xi-\mu_1)(\eta-\lambda_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&\quad+(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_{\frac12}^{1}\int_{\frac12}^{1}(\xi-\mu_1)(\eta-\mu_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi(\kappa_1,\kappa_2;\kappa_3,\kappa_4)&=(\mu_1-\lambda_1)(\mu_2-\lambda_2)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+\lambda_1(\mu_2-\lambda_2)\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\\
&\quad+(1-\mu_1)(\mu_2-\lambda_2)\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+(\mu_1-\lambda_1)\lambda_2\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+(\mu_1-\lambda_1)(1-\mu_2)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\\
&\quad+\lambda_1\lambda_2\mathbb{F}(\kappa_1,\kappa_3)+(1-\mu_1)\lambda_2\mathbb{F}(\kappa_2,\kappa_3)+\lambda_1(1-\mu_2)\mathbb{F}(\kappa_1,\kappa_4)+(1-\mu_1)(1-\mu_2)\mathbb{F}(\kappa_2,\kappa_4)
\end{aligned}
$$

$$- \frac{\lambda_2}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2}\mathbb{F}(\tau_1, \kappa_3)\mathrm{d}\tau_1 - \frac{\mu_2 - \lambda_2}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2}\mathbb{F}\left(\tau_1, \frac{\kappa_3 + \kappa_4}{2}\right)\mathrm{d}\tau_1 - \frac{1 - \mu_2}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2}\mathbb{F}(\tau_1, \kappa_4)\mathrm{d}\tau_1 - \frac{\lambda_1}{\kappa_4 - \kappa_3}\int_{\kappa_3}^{\kappa_4}\mathbb{F}(\kappa_1, \tau_2)\mathrm{d}\tau_2$$
$$- \frac{\mu_1 - \lambda_1}{\kappa_4 - \kappa_3}\int_{\kappa_3}^{\kappa_4}\mathbb{F}\left(\frac{\kappa_1 + \kappa_2}{2}, \tau_2\right)\mathrm{d}\tau_2 - \frac{1 - \mu_1}{\kappa_4 - \kappa_3}\int_{\kappa_3}^{\kappa_4}\mathbb{F}(\kappa_2, \tau_2)\mathrm{d}\tau_2 + \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\mathbb{F}(\tau_1, \tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1,$$

which is given by Budak and Ali in [49].

Corollary 1. *In* Lemma 1, *if we set* $\varphi(\xi) = \frac{\xi^\alpha}{\Gamma(\alpha)}$ *and* $\psi(\eta) = \frac{\xi^\beta}{\Gamma(\beta)}$, *then we obtain the following new Riemann-Liouville fractional integral identity*:

$$\begin{aligned}\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4) &= (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}(\xi^\alpha - \lambda_1)(\eta^\beta - \lambda_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1 - \xi)\kappa_1, \eta\kappa_4 + (1 - \eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&+ (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}(\xi^\alpha - \lambda_1)(\eta^\beta - \mu_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1 - \xi)\kappa_1, \eta\kappa_4 + (1 - \eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&+ (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}(\xi^\alpha - \mu_1)(\eta^\beta - \lambda_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1 - \xi)\kappa_1, \eta\kappa_4 + (1 - \eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&+ (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}(\xi^\alpha - \mu_1)(\eta^\beta - \mu_2)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1 - \xi)\kappa_1, \eta\kappa_4 + (1 - \eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta,\end{aligned}$$

where

$$\begin{aligned}\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4) &= (\mu_1 - \lambda_1)(\mu_2 - \lambda_2)\mathbb{F}\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) + \lambda_1(\mu_2 - \lambda_2)\mathbb{F}\left(\kappa_1, \frac{\kappa_3 + \kappa_4}{2}\right)\\
&+ (1 - \mu_1)(\mu_2 - \lambda_2)\mathbb{F}\left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2}\right) + (\mu_1 - \lambda_1)\lambda_2\mathbb{F}\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3\right) + (\mu_1 - \lambda_1)(1 - \mu_2)\mathbb{F}\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_4\right)\\
&+ \lambda_1\lambda_2\mathbb{F}(\kappa_1, \kappa_3) + (1 - \mu_1)\lambda_2\mathbb{F}(\kappa_2, \kappa_3) + \lambda_1(1 - \mu_2)\mathbb{F}(\kappa_1, \kappa_4) + (1 - \mu_1)(1 - \mu_2)\mathbb{F}(\kappa_2, \kappa_4)\\
&- (\mu_2 - \lambda_2)\frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha}\left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}\left(\kappa_1, \frac{\kappa_3 + \kappa_4}{2}\right) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}\left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2}\right)\right]\\
&- (\mu_1 - \lambda_1)\frac{\Gamma(\beta + 1)}{(\kappa_4 - \kappa_3)^\beta}\left[J^\beta_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3\right) + J^\beta_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_4\right)\right]\\
&- \lambda_1\frac{\Gamma(\beta + 1)}{(\kappa_4 - \kappa_3)^\beta}\left[J^\beta_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1, \kappa_3) + J^\beta_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1, \kappa_4)\right]\\
&- \lambda_2\frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha}\left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}(\kappa_1, \kappa_3) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}(\kappa_2, \kappa_3)\right]\\
&- (1 - \mu_2)\frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha}\left[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}(\kappa_1, \kappa_4) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}(\kappa_2, \kappa_4)\right]\\
&- (1 - \mu_1)\frac{\Gamma(\beta + 1)}{(\kappa_4 - \kappa_3)^\beta}\left[J^\beta_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2, \kappa_3) + J^\beta_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2, \kappa_4)\right]\\
&+ \frac{\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha}\frac{\Gamma(\beta + 1)}{(\kappa_4 - \kappa_3)^\beta}\left[J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}-\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1, \kappa_3) + J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}-\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1, \kappa_4)\right.\\
&\left.+ J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2, \kappa_3) + J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2, \kappa_4)\right],\end{aligned}$$

where $\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ *is defined as in* Corollary 1.

Corollary 2. *In* Lemma 1, *if we set* $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $\psi(\eta)=\frac{\xi^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, *then we obtain the following new k-Riemann-Liouville fractional integral identity*:

$$\begin{aligned}\Im(\kappa_1,\kappa_2;\kappa_3,\kappa_4)&=(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^{\frac12}\int_0^{\frac12}\left(\xi^{\frac{\alpha}{k}}-\lambda_1\right)\left(\eta^{\frac{\beta}{k}}-\lambda_2\right)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&+(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^{\frac12}\int_{\frac12}^{1}\left(\xi^{\frac{\alpha}{k}}-\lambda_1\right)\left(\eta^{\frac{\beta}{k}}-\mu_2\right)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&+(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_{\frac12}^{1}\int_0^{\frac12}\left(\xi^{\frac{\alpha}{k}}-\mu_1\right)\left(\eta^{\frac{\beta}{k}}-\lambda_2\right)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta\\
&+(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_{\frac12}^{1}\int_{\frac12}^{1}\left(\xi^{\frac{\alpha}{k}}-\mu_1\right)\left(\eta^{\frac{\beta}{k}}-\mu_2\right)\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\xi\mathrm{d}\eta,\end{aligned}$$

where

$$\begin{aligned}\Im(\kappa_1,\kappa_2;\kappa_3,\kappa_4)&=(\mu_1-\lambda_1)(\mu_2-\lambda_2)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+\lambda_1(\mu_2-\lambda_2)\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)\\
&+(1-\mu_1)(\mu_2-\lambda_2)\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+(\mu_1-\lambda_1)\lambda_2\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+(\mu_1-\lambda_1)(1-\mu_2)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\\
&+\lambda_1\lambda_2\mathbb{F}(\kappa_1,\kappa_3)+(1-\mu_1)\lambda_2\mathbb{F}(\kappa_2,\kappa_3)+\lambda_1(1-\mu_2)\mathbb{F}(\kappa_1,\kappa_4)+(1-\mu_1)(1-\mu_2)\mathbb{F}(\kappa_2,\kappa_4)\\
&-(\mu_2-\lambda_2)\frac{\Gamma_k(\alpha+1)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right]\\
&-(\mu_1-\lambda_1)\frac{\Gamma_k(\beta+1)}{(\kappa_4-\kappa_3)^{\frac{\beta}{k}}}\left[J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}-,k}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}+,k}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right]\\
&-\lambda_1\frac{\Gamma_k(\beta+1)}{(\kappa_4-\kappa_3)^{\beta}}\left[J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}-,k}\mathbb{F}(\kappa_1,\kappa_3)+J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}+,k}\mathbb{F}(\kappa_1,\kappa_4)\right]\\
&-\lambda_2\frac{\Gamma_k(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathbb{F}(\kappa_1,\kappa_3)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathbb{F}(\kappa_2,\kappa_3)\right]\\
&-(1-\mu_2)\frac{\Gamma_k(\alpha+1)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathbb{F}(\kappa_1,\kappa_4)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathbb{F}(\kappa_2,\kappa_4)\right]\\
&-(1-\mu_1)\frac{\Gamma_k(\beta+1)}{(\kappa_4-\kappa_3)^{\frac{\beta}{k}}}\left[J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}-,k}\mathbb{F}(\kappa_2,\kappa_3)+J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}+,k}\mathbb{F}(\kappa_2,\kappa_4)\right]\\
&+\frac{\Gamma_k(\alpha+1)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\frac{\Gamma_k(\beta+1)}{(\kappa_4-\kappa_3)^{\frac{\beta}{k}}}\left[J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}-\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}-\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1,\kappa_4)\right.\\
&\left.+J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2,\kappa_3)+J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2,\kappa_4)\right],\end{aligned}$$

where $\Im(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ *is defined as in* Corollary 2.

4 Some new inequalities for generalized fractional integrals

In this section, we establish some new Simpson-type inequalities for differentiable co-ordinated convex functions via generalized fractional integrals.

Theorem 2. *We assume that the conditions of* Lemma 1 *hold. If the mapping* $\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}\right|$ *is co-ordinated convex on* $[\kappa_1, \kappa_2]$, *then the following inequality holds for generalized fractional integrals*:

$$\begin{aligned}|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| &\leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\\ &\times \Bigg[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|(\Pi_1^{\varphi}(\lambda_1)\nabla_1^{\psi}(\lambda_2) + \Pi_1^{\varphi}(\lambda_1)\nabla_3^{\psi}(\mu_2) + \Pi_3^{\varphi}(\mu_1)\nabla_1^{\psi}(\lambda_2) + \Pi_3^{\varphi}(\mu_1)\nabla_3^{\psi}(\mu_2))\\ &+ \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|(\Pi_1^{\varphi}(\lambda_1)\nabla_2^{\psi}(\lambda_2) + \Pi_1^{\varphi}(\lambda_1)\nabla_4^{\psi}(\mu_2) + \Pi_3^{\varphi}(\mu_1)\nabla_2^{\psi}(\lambda_2) + \Pi_3^{\varphi}(\mu_1)\nabla_4^{\psi}(\mu_2))\\ &+ \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|(\Pi_2^{\varphi}(\lambda_1)\nabla_1^{\psi}(\lambda_2) + \Pi_2^{\varphi}(\lambda_1)\nabla_3^{\psi}(\mu_2) + \Pi_4^{\varphi}(\mu_1)\nabla_1^{\psi}(\lambda_2) + \Pi_4^{\varphi}(\mu_1)\nabla_3^{\psi}(\mu_2))\\ &+ \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|(\Pi_2^{\varphi}(\lambda_1)\nabla_2^{\psi}(\lambda_2) + \Pi_2^{\varphi}(\lambda_1)\nabla_4^{\psi}(\mu_2) + \Pi_4^{\varphi}(\mu_1)\nabla_2^{\psi}(\lambda_2) + \Pi_4^{\varphi}(\mu_1)\nabla_4^{\psi}(\mu_2))\Bigg],\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}\Pi_1^{\varphi}(\lambda_1) &= \int_0^{\frac{1}{2}} \xi|\Delta(\xi) - \Delta(1)\lambda_1|\mathrm{d}\xi, \quad \Pi_2^{\varphi}(\lambda_1) = \int_0^{\frac{1}{2}} (1 - \xi)|\Delta(\xi) - \Delta(1)\lambda_1|\mathrm{d}\xi,\\ \Pi_3^{\varphi}(\mu_1) &= \int_{\frac{1}{2}}^{1} \xi|\Delta(\xi) - \Delta(1)\mu_1|\mathrm{d}\xi, \quad \Pi_4^{\varphi}(\mu_1) = \int_{\frac{1}{2}}^{1} (1 - \xi)|\Delta(\xi) - \Delta(1)\mu_1|\mathrm{d}\xi,\\ \nabla_1^{\psi}(\lambda_2) &= \int_0^{\frac{1}{2}} \eta|\Lambda(\eta) - \Lambda(1)\lambda_2|\mathrm{d}\eta, \quad \nabla_2^{\psi}(\lambda_2) = \int_0^{\frac{1}{2}} (1 - \eta)|\Lambda(\eta) - \Lambda(1)\lambda_2|\mathrm{d}\eta,\\ \nabla_3^{\psi}(\mu_2) &= \int_{\frac{1}{2}}^{1} \eta|\Lambda(\eta) - \Lambda(1)\mu_2|\mathrm{d}\eta, \quad \nabla_4^{\psi}(\mu_2) = \int_{\frac{1}{2}}^{1} (1 - \eta)|\Lambda(\eta) - \Lambda(1)\mu_2|\mathrm{d}\eta.\end{aligned} \tag{4.2}$$

Proof. By taking the modulus in Lemma 1 and using the properties of the modulus, we obtain that

$$\begin{aligned}|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| &\leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\\ &\times \Bigg[\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} |\Delta(\xi) - \Delta(1)\lambda_1||\Lambda(\eta) - \Lambda(1)\lambda_2|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1 - \xi)\kappa_1, \eta\kappa_4 + (1 - \eta)\kappa_3)\right|\mathrm{d}\xi\mathrm{d}\eta\\ &+ \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} |\Delta(\xi) - \Delta(1)\lambda_1||\Lambda(\eta) - \Lambda(1)\mu_2|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1 - \xi)\kappa_1, \eta\kappa_4 + (1 - \eta)\kappa_3)\right|\mathrm{d}\xi\mathrm{d}\eta\\ &+ \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} |\Delta(\xi) - \Delta(1)\mu_1||\Lambda(\eta) - \Lambda(1)\lambda_2|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1 - \xi)\kappa_1, \eta\kappa_4 + (1 - \eta)\kappa_3)\right|\mathrm{d}\xi\mathrm{d}\eta\\ &+ \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} |\Delta(\xi) - \Delta(1)\mu_1||\Lambda(\eta) - \Lambda(1)\mu_2|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1 - \xi)\kappa_1, \eta\kappa_4 + (1 - \eta)\kappa_3)\right|\mathrm{d}\xi\mathrm{d}\eta\Bigg].\end{aligned} \tag{4.3}$$

Since the mapping $\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}\right|$ is co-ordinated convex on Δ, therefore, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} |\Delta(\xi) - \Delta(1)\lambda_1||\Lambda(\eta) - \Lambda(1)\lambda_2|\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1-\xi)\kappa_1, \eta\kappa_4 + (1-\eta)\kappa_3)\right| d\xi\, d\eta \\
&\le \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} |\Delta(\xi) - \Delta(1)\lambda_1||\Lambda(\eta) - \Lambda(1)\lambda_2|\left(\xi\eta\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right| + \xi(1-\eta)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|\right. \\
&\quad \left. + (1-\xi)\eta\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right| + (1-\xi)(1-\eta)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|\right) d\xi\, d\eta \\
&= \Pi_1^{\varphi}(\lambda_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right| + \Pi_1^{\varphi}(\lambda_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right| + \Pi_2^{\varphi}(\lambda_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right| \\
&\quad + \Pi_2^{\varphi}(\lambda_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|.
\end{aligned} \tag{4.4}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} |\Delta(\xi) - \Delta(1)\lambda_1||\Lambda(\eta) - \Lambda(1)\mu_2|\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1-\xi)\kappa_1, \eta\kappa_4 + (1-\eta)\kappa_3)\right| d\xi\, d\eta \\
&= \Pi_1^{\varphi}(\lambda_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right| + \Pi_1^{\varphi}(\lambda_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right| \\
&\quad + \Pi_2^{\varphi}(\lambda_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right| + \Pi_2^{\varphi}(\lambda_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|,
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} |\Delta(\xi) - \Delta(1)\mu_1||\Lambda(\eta) - \Lambda(1)\lambda_2|\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1-\xi)\kappa_1, \eta\kappa_4 + (1-\eta)\kappa_3)\right| d\xi\, d\eta \\
&= \Pi_3^{\varphi}(\mu_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right| + \Pi_3^{\varphi}(\mu_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right| \\
&\quad + \Pi_4^{\varphi}(\mu_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right| + \Pi_4^{\varphi}(\mu_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|,
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} |\Delta(\xi) - \Delta(1)\mu_1||\Lambda(\eta) - \Lambda(1)\mu_2|\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2 + (1-\xi)\kappa_1, \eta\kappa_4 + (1-\eta)\kappa_3)\right| d\xi\, d\eta \\
&= \Pi_3^{\varphi}(\mu_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right| + \Pi_3^{\varphi}(\mu_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right| \\
&\quad + \Pi_4^{\varphi}(\mu_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right| + \Pi_4^{\varphi}(\mu_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|.
\end{aligned} \tag{4.7}$$

By the inequalities (4.4)–(4.7), the proof is completed. □

Remark 2. In Theorem 2, if we take $\varphi(\xi) = \xi$ and $\psi(\eta) = \eta$, we obtain the inequality

$$\begin{aligned}
|\Phi(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| &\le (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \\
&\quad \times \left[\Psi_1\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right| + \Psi_2\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right| + \Psi_3\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right| + \Psi_3\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|\right],
\end{aligned}$$

where $\Phi(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ is defined as in Remark 1 and

$$\Psi_1 = \left[\frac{\lambda_1^3 + \mu_1^3}{3} - \frac{\lambda_1}{8} - \frac{5\mu_1}{8} + \frac{5}{12}\right]\left[\frac{\lambda_2^3 + \mu_2^3}{3} - \frac{\lambda_2}{8} - \frac{5\mu_2}{8} + \frac{5}{12}\right],$$
$$\Psi_2 = \left[\frac{\lambda_1^3 + \mu_1^3}{3} - \frac{\lambda_1}{8} - \frac{5\mu_1}{8} + \frac{5}{12}\right]\left[-\frac{\lambda_2^3 + \mu_2^3}{3} + \lambda_2^2 + \mu_2^2 - \frac{7\mu_2 + 3\lambda_2}{8} + \frac{1}{3}\right],$$
$$\Psi_3 = \left[-\frac{\lambda_1^3 + \mu_1^3}{3} + \lambda_1^2 + \mu_1^2 - \frac{7\mu_1 + 3\lambda_1}{8} + \frac{1}{3}\right]\left[\frac{\lambda_2^3 + \mu_2^3}{3} - \frac{\lambda_2}{8} - \frac{5\mu_2}{8} + \frac{5}{12}\right]$$

and

$$\Psi_4 = \left[-\frac{\lambda_1^3 + \mu_1^3}{3} + \lambda_1^2 + \mu_1^2 - \frac{7\mu_1 + 3\lambda_1}{8} + \frac{1}{3}\right]\left[-\frac{\lambda_2^3 + \mu_2^3}{3} + \lambda_2^2 + \mu_2^2 - \frac{7\mu_2 + 3\lambda_2}{8} + \frac{1}{3}\right],$$

which is given by Budak and Ali in [49].

Corollary 3. *In* Theorem 2, *if we use* $\varphi(\xi) = \frac{\xi^\alpha}{\Gamma(\kappa_1)}$ *and* $\psi(\eta) = \frac{\xi^\beta}{\Gamma(\beta)}$, *then we obtain the following parameterized Simpson-type inequality for Riemann-Liouville fractional integrals*:

$$\begin{aligned}|\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\\ \times \Bigg[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|(\Pi_1^\alpha(\lambda_1)\nabla_1^\beta(\lambda_2) + \Pi_1^\alpha(\lambda_1)\nabla_3^\beta(\mu_2) + \Pi_3^\alpha(\mu_1)\nabla_1^\beta(\lambda_2) + \Pi_3^\alpha(\mu_1)\nabla_3^\beta(\mu_2))\\ + \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|(\Pi_1^\alpha(\lambda_1)\nabla_2^\beta(\lambda_2) + \Pi_1^\alpha(\lambda_1)\nabla_4^\beta(\mu_2) + \Pi_3^\alpha(\mu_1)\nabla_2^\beta(\lambda_2) + \Pi_3^\alpha(\mu_1)\nabla_4^\beta(\mu_2))\\ + \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|(\Pi_2^\alpha(\lambda_1)\nabla_1^\beta(\lambda_2) + \Pi_2^\alpha(\lambda_1)\nabla_3^\beta(\mu_2) + \Pi_4^\alpha(\mu_1)\nabla_1^\beta(\lambda_2) + \Pi_4^\alpha(\mu_1)\nabla_3^\beta(\mu_2))\\ + \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|(\Pi_2^\alpha(\lambda_1)\nabla_2^\beta(\lambda_2) + \Pi_2^\alpha(\lambda_1)\nabla_4^\beta(\mu_2) + \Pi_4^\alpha(\mu_1)\nabla_2^\beta(\lambda_2) + \Pi_4^\alpha(\mu_1)\nabla_4^\beta(\mu_2))\Bigg],\end{aligned}$$

where $\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ *is defined as in* Corollary 1 *and*

$$\Pi_1^\alpha(\lambda_1) = \frac{\alpha}{\alpha + 2}\lambda_1^{\frac{\alpha+2}{\alpha}} - \frac{\lambda_1}{8} + \frac{1}{2^{\alpha+2}(\alpha + 2)},$$
$$\Pi_2^\alpha(\lambda_1) = \frac{2\alpha}{\alpha + 1}\lambda_1^{\frac{\alpha+1}{\alpha}} - \frac{\lambda_1}{2} + \frac{1}{2^{\alpha+1}(\alpha + 1)} - \Pi_1^\alpha(\lambda_1),$$
$$\Pi_3^\alpha(\mu_1) = \frac{\alpha}{\alpha + 2}\mu_1^{\frac{\alpha+2}{\alpha}} - \frac{5}{8}\mu_1 + \frac{2^{\alpha+2} + 1}{2^{\alpha+2}(\alpha + 2)},$$
$$\Pi_4^\alpha(\mu_1) = \frac{2\alpha}{\alpha + 1}\mu_1^{\frac{\alpha+1}{\alpha}} - \frac{3}{2}\mu_1 + \frac{2^{\alpha+1} + 1}{2^{\alpha+1}(\alpha + 1)} - \Pi_3^\alpha(\mu_1)$$

and

$$\nabla_1^\beta(\lambda_2) = \frac{\beta}{\beta + 2}\lambda_2^{\frac{\beta+2}{\beta}} - \frac{\lambda_2}{8} + \frac{1}{2^{\beta+2}(\beta + 2)},$$
$$\nabla_2^\beta(\lambda_2) = \frac{2\beta}{\beta + 1}\lambda_2^{\frac{\beta+1}{\beta}} - \frac{\lambda_2}{2} + \frac{1}{2^{\beta+1}(\beta + 1)} - \nabla_1^\beta(\lambda_2),$$
$$\nabla_3^\beta(\mu_2) = \frac{\beta}{\beta + 2}\mu_2^{\frac{\beta+2}{\beta}} - \frac{5}{8}\mu_2 + \frac{2^{\beta+2} + 1}{2^{\beta+2}(\beta + 2)},$$
$$\nabla_4^\beta(\mu_2) = \frac{2\beta}{\beta + 1}\mu_2^{\frac{\beta+1}{\beta}} - \frac{3}{2}\mu_2 + \frac{2^{\beta+1} + 1}{2^{\beta+1}(\beta + 1)} - \nabla_3^\beta(\mu_2).$$

Corollary 4. *In* Theorem 2, *if we use* $\varphi(\xi) = \frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $\psi(\eta) = \frac{\eta^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, *then we obtain the following parameterized Simpson-type inequality for k-Riemann-Liouville fractional integrals*:

$$
\begin{aligned}
|\Im(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| &\leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \\
&\times \Bigg[\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right| \left(\Pi_1^{\frac{\alpha}{k}}(\lambda_1)\nabla_1^{\frac{\beta}{k}}(\lambda_2) + \Pi_1^{\frac{\alpha}{k}}(\lambda_1)\nabla_3^{\frac{\beta}{k}}(\mu_2) + \Pi_3^{\frac{\alpha}{k}}(\mu_1)\nabla_1^{\frac{\beta}{k}}(\lambda_2) + \Pi_3^{\frac{\alpha}{k}}(\mu_1)\nabla_3^{\frac{\beta}{k}}(\mu_2)\right) \\
&+ \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right| \left(\Pi_1^{\frac{\alpha}{k}}(\lambda_1)\nabla_2^{\frac{\beta}{k}}(\lambda_2) + \Pi_1^{\frac{\alpha}{k}}(\lambda_1)\nabla_4^{\frac{\beta}{k}}(\mu_2) + \Pi_3^{\frac{\alpha}{k}}(\mu_1)\nabla_2^{\frac{\beta}{k}}(\lambda_2) + \Pi_3^{\frac{\alpha}{k}}(\mu_1)\nabla_4^{\frac{\beta}{k}}(\mu_2)\right) \\
&+ \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right| \left(\Pi_2^{\frac{\alpha}{k}}(\lambda_1)\nabla_1^{\frac{\beta}{k}}(\lambda_2) + \Pi_2^{\frac{\alpha}{k}}(\lambda_1)\nabla_3^{\frac{\beta}{k}}(\mu_2) + \Pi_4^{\frac{\alpha}{k}}(\mu_1)\nabla_1^{\frac{\beta}{k}}(\lambda_2) + \Pi_4^{\frac{\alpha}{k}}(\mu_1)\nabla_3^{\frac{\beta}{k}}(\mu_2)\right) \\
&+ \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right| \left(\Pi_2^{\frac{\alpha}{k}}(\lambda_1)\nabla_2^{\frac{\beta}{k}}(\lambda_2) + \Pi_2^{\frac{\alpha}{k}}(\lambda_1)\nabla_4^{\frac{\beta}{k}}(\mu_2) + \Pi_4^{\frac{\alpha}{k}}(\mu_1)\nabla_2^{\frac{\beta}{k}}(\lambda_2) + \Pi_4^{\frac{\alpha}{k}}(\mu_1)\nabla_4^{\frac{\beta}{k}}(\mu_2)\right)\Bigg],
\end{aligned}
$$

where $\Im(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ *is defined as in* Corollary 2 *and*

$$
\begin{aligned}
\Pi_1^{\frac{\alpha}{k}}(\lambda_1) &= \frac{\alpha}{\alpha + 2k}\lambda_1^{\frac{\alpha+2k}{\alpha}} - \frac{\lambda_1}{8} + \frac{1}{2^{\frac{\alpha+2k}{k}}\left(\frac{\alpha+2k}{k}\right)}, \\
\Pi_2^{\frac{\alpha}{k}}(\lambda_1) &= \frac{2\alpha}{\alpha + k}\lambda_1^{\frac{\alpha+k}{\alpha}} - \frac{\lambda_1}{2} + \frac{k}{2^{\frac{\alpha+k}{k}}(\alpha + k)} - \Pi_1^{\frac{\alpha}{k}}, \\
\Pi_3^{\frac{\alpha}{k}}(\mu_1) &= \frac{\alpha}{\alpha + 2k}\mu_1^{\frac{\alpha+2k}{\alpha}} - \frac{5}{8}\mu_1 + \frac{2^{\frac{\alpha+2k}{k}} + 1}{2^{\frac{\alpha+2k}{k}}\left(\frac{\alpha+2k}{k}\right)}, \\
\Pi_4^{\frac{\alpha}{k}}(\mu_1) &= \frac{2\alpha}{\alpha + k}\mu_1^{\frac{\alpha+k}{\alpha}} - \frac{3}{2}\mu_1 + \frac{2^{\frac{\alpha+k}{k}}k + k}{2^{\frac{\alpha+k}{k}}(\alpha + k)} - \Pi_3^{\frac{\alpha}{k}}
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_1^{\frac{\beta}{k}}(\lambda_2) &= \frac{\beta}{\beta + 2k}\lambda_2^{\frac{\beta+2k}{\beta}} - \frac{\lambda_2}{8} + \frac{1}{2^{\frac{\beta+2k}{k}}\left(\frac{\beta+2k}{k}\right)}, \\
\nabla_2^{\frac{\beta}{k}}(\lambda_2) &= \frac{2\beta}{\beta + k}\lambda_2^{\frac{\beta+k}{\beta}} - \frac{\lambda_2}{2} + \frac{k}{2^{\frac{\beta+k}{k}}(\beta + k)} - \nabla_1^{\frac{\beta}{k}}, \\
\nabla_3^{\frac{\beta}{k}}(\mu_2) &= \frac{\beta}{\beta + 2k}\mu_2^{\frac{\beta+2k}{\beta}} - \frac{5}{8}\mu_2 + \frac{2^{\frac{\beta+2k}{k}} + 1}{2^{\frac{\beta+2k}{k}}\left(\frac{\beta+2k}{k}\right)}, \\
\nabla_4^{\frac{\beta}{k}}(\mu_2) &= \frac{2\beta}{\beta + k}\mu_2^{\frac{\beta+k}{\beta}} - \frac{3}{2}\mu_2 + \frac{2^{\frac{\beta+k}{k}}k + k}{2^{\frac{\beta+k}{k}}(\beta + k)} - \nabla_3^{\frac{\beta}{k}}.
\end{aligned}
$$

Theorem 3. *We assume that the conditions of* Lemma 1 *hold. If the mapping* $\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}\right|^q$ *is co-ordinated convex on* Δ, $q \geq 1$, *then we have the following inequality*:

$$
\begin{aligned}
|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) &\Bigg[\left(\int_0^{\frac{1}{2}} |\Delta(\xi) - \Delta(1)\lambda_1| d\xi\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} |\Lambda(\eta) - \Lambda(1)\lambda_2| d\eta\right)^{1-\frac{1}{q}} \\
&\times \left(\Pi_1^{\varphi}(\lambda_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_1^{\varphi}(\lambda_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right.
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
&+ \Pi_2^{\varphi}(\lambda_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q + \Pi_2^{\varphi}(\lambda_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\mu_2|\mathrm{d}\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}|\Delta(\xi)-\Delta(1)\lambda_1|\mathrm{d}\xi\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\Pi_1^{\varphi}(\lambda_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q + \Pi_1^{\varphi}(\lambda_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q\\
&+ \Pi_2^{\varphi}(\lambda_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q + \Pi_2^{\varphi}(\lambda_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}|\Lambda(\eta)-\Lambda(1)\lambda_2|\mathrm{d}\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\mu_1|\mathrm{d}\xi\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\Pi_3^{\varphi}(\mu_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q + \Pi_3^{\varphi}(\mu_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q\\
&+ \Pi_4^{\varphi}(\mu_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q + \Pi_4^{\varphi}(\mu_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}|\Lambda(\eta)-\Lambda(1)\mu_2|\mathrm{d}\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}|\Delta(\xi)-\Delta(1)\mu_1|\mathrm{d}\xi\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\Pi_3^{\varphi}(\mu_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q + \Pi_3^{\varphi}(\mu_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q\\
&+ \Pi_4^{\varphi}(\mu_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q + \Pi_4^{\varphi}(\mu_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\Bigg)^{\frac{1}{q}}\Bigg],
\end{aligned}
\tag{4.8}
$$

where Π_i^{φ} *and* ∇_i^{ψ}, $i = 1, 2, 3, 4$ *are defined as in* Theorem 2.

Proof. By using power mean inequality in (4.3) and co-ordinated convexity of $\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}\right|^q$, we have

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\lambda_1||\Lambda(\eta)-\Lambda(1)\lambda_2|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\right|\mathrm{d}\xi\,\mathrm{d}\eta\\
&\le\left(\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\lambda_1|\mathrm{d}\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\lambda_2|\mathrm{d}\eta\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}|(\Delta(\xi)-\Delta(1)\lambda_1)||(\Lambda(\eta)-\Lambda(1)\lambda_2)|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\right|^q\mathrm{d}\xi d\eta\right)^{\frac{1}{q}}\\
&\le\left(\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\lambda_1|\mathrm{d}\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\lambda_2|\mathrm{d}\eta\right)^{1-\frac{1}{q}}
\end{aligned}
\tag{4.9}
$$

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\lambda_1||\Lambda(\eta)-\Lambda(1)\lambda_2|\Bigg(\xi\eta\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+\xi(1-\eta)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q$$

$$+(1-\xi)\eta\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+(1-\xi)(1-\eta)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\Bigg)^{\frac{1}{q}}\mathrm{d}\xi\,\mathrm{d}\eta$$

$$=\left(\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\lambda_1|\mathrm{d}\xi\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\lambda_2|\mathrm{d}\eta\right)^{1-\frac{1}{q}}\Bigg(\Pi_1^{\varphi}(\lambda_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q \tag{4.9}$$

$$+\Pi_1^{\varphi}(\lambda_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+\Pi_2^{\varphi}(\lambda_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\Pi_2^{\varphi}(\lambda_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\Bigg)^{\frac{1}{q}}.$$

Similarly, we obtain

$$\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1|(\Delta(\xi)-\Delta(1)\lambda_1)||(\Lambda(\eta)-\Lambda(1)\mu_2)|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\right|\mathrm{d}\xi\,\mathrm{d}\eta$$

$$\leq\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\mu_2|\mathrm{d}\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1|\Delta(\xi)-\Delta(1)\lambda_1|\mathrm{d}\xi\right)^{1-\frac{1}{q}}\Bigg(\Pi_1^{\varphi}(\lambda_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q \tag{4.10}$$

$$+\Pi_1^{\varphi}(\lambda_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+\Pi_2^{\varphi}(\lambda_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\Pi_2^{\varphi}(\lambda_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\Bigg)^{\frac{1}{q}},$$

$$\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}|(\Delta(\xi)-\Delta(1)\mu_1)||(\Lambda(\eta)-\Lambda(1)\lambda_2)|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\right|\mathrm{d}\xi\,\mathrm{d}\eta$$

$$\leq\left(\int_{\frac{1}{2}}^1|\Lambda(\eta)-\Lambda(1)\lambda_2|\mathrm{d}\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\mu_1|\mathrm{d}\xi\right)^{1-\frac{1}{q}}\Bigg(\Pi_3^{\varphi}(\mu_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q \tag{4.11}$$

$$+\Pi_3^{\varphi}(\mu_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+\Pi_4^{\varphi}(\mu_1)\nabla_1^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\Pi_4^{\varphi}(\mu_1)\nabla_2^{\psi}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\Bigg)^{\frac{1}{q}}$$

and

$$\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1|(\Delta(\xi)-\Delta(1)\mu_1)||(\Lambda(\eta)-\Lambda(1)\mu_2)|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\right|\mathrm{d}\xi\,\mathrm{d}\eta$$

$$\leq\left(\int_{\frac{1}{2}}^1|\Lambda(\eta)-\Lambda(1)\mu_2|\mathrm{d}\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1|\Delta(\xi)-\Delta(1)\mu_1|\mathrm{d}\xi\right)^{1-\frac{1}{q}}\Bigg(\Pi_3^{\varphi}(\mu_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q \tag{4.12}$$

$$+\Pi_3^{\varphi}(\mu_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+\Pi_4^{\varphi}(\mu_1)\nabla_3^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\Pi_4^{\varphi}(\mu_1)\nabla_4^{\psi}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\Bigg)^{\frac{1}{q}}.$$

By the inequalities (4.9)–(4.12), the proof is completed. □

Corollary 5. *In* Theorem 3, *if we take* $\varphi(\xi) = \xi$ *and* $\psi(\eta) = \eta$, *we have*

$$
\begin{aligned}
|\Phi(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\Bigg[& (\Pi_1(\lambda_1) + \Pi_2(\lambda_1))^{1-\frac{1}{q}}(\nabla_1(\lambda_2) + \nabla_2(\lambda_2))^{1-\frac{1}{q}} \\
& \times \left(\Pi_1(\lambda_1)\nabla_1(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_1(\lambda_1)\nabla_2(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right. \\
& \left. + \Pi_2(\lambda_1)\nabla_1(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \Pi_2(\lambda_1)\nabla_2(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q\right)^{\frac{1}{q}} \\
& + (\nabla_3(\mu_2) + \nabla_4(\mu_2))^{1-\frac{1}{q}}(\Pi_1(\lambda_1) + \Pi_2(\lambda_1))^{1-\frac{1}{q}} \\
& \times \left(\Pi_1(\lambda_1)\nabla_3(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_1(\lambda_1)\nabla_4(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right. \\
& \left. + \Pi_2(\lambda_1)\nabla_3(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \Pi_2(\lambda_1)\nabla_4(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q\right)^{\frac{1}{q}} \\
& + (\nabla_1(\lambda_2) + \nabla_2(\lambda_2))^{1-\frac{1}{q}}(\Pi_3(\mu_1) + \Pi_4(\mu_1))^{1-\frac{1}{q}} \\
& \times \left(\Pi_3(\mu_1)\nabla_1(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_3(\mu_1)\nabla_2(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right. \\
& \left. + \Pi_4(\mu_1)\nabla_1(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \Pi_4(\mu_1)\nabla_2(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q\right)^{\frac{1}{q}} \\
& + (\nabla_3(\mu_2) + \nabla_4(\mu_2))^{1-\frac{1}{q}}(\Pi_3(\mu_1) + \Pi_4(\mu_1))^{1-\frac{1}{q}} \\
& \times \left(\Pi_3(\mu_1)\nabla_3(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_3(\mu_1)\nabla_4(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right. \\
& \left. + \Pi_4(\mu_1)\nabla_3(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \Pi_4(\mu_1)\nabla_4(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q\right)^{\frac{1}{q}}\Bigg],
\end{aligned}
$$

where $\Phi(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ *is defined as in* Remark 1,

$$
\begin{aligned}
\Pi_1(\lambda_1) &= \frac{1}{3}\lambda_1^3 - \frac{\lambda_1}{8} + \frac{1}{24}, \\
\Pi_2(\lambda_1) &= -\frac{\lambda_1^3}{3} + \lambda_1^2 - \frac{3\lambda_1}{8} + \frac{1}{12}, \\
\Pi_3(\mu_1) &= \frac{1}{3}\mu_1^3 - \frac{5}{8}\mu_1 + \frac{9}{24}, \\
\Pi_4(\mu_1) &= \frac{1}{3}\mu_1^3 + \mu_1^2 - \frac{7}{8}\mu_1 + \frac{1}{4}
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_1(\lambda_2) &= \frac{1}{3}\lambda_2^3 - \frac{\lambda_2}{8} + \frac{1}{24}, \\
\nabla_2(\lambda_2) &= -\frac{\lambda_2^3}{3} + \lambda_2^2 - \frac{3\lambda_1}{8} + \frac{1}{12}, \\
\nabla_3(\mu_2) &= \frac{1}{3}\mu_2^3 - \frac{5}{8}\mu_2 + \frac{9}{24}, \\
\nabla_4(\mu_2) &= \frac{1}{3}\mu_2^3 + \mu_2^2 - \frac{7}{8}\mu_2 + \frac{1}{4}.
\end{aligned}
$$

Corollary 6. *In* Theorem 3, *if we use* $\varphi(\xi) = \frac{\xi^\alpha}{\Gamma(\alpha)}$ *and* $\psi(\eta) = \frac{\xi^\beta}{\Gamma(\beta)}$, *then we obtain the following parameterized Simpson-type inequality for Riemann-Liouville fractional integrals*:

$$\begin{aligned}
|\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \le (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\Bigg[&(\Pi_1^\alpha(\lambda_1) + \Pi_2^\alpha(\lambda_1))^{1-\frac{1}{q}}(\nabla_1^\beta(\lambda_2) + \nabla_2^\beta(\lambda_2))^{1-\frac{1}{q}} \\
&\times \left(\Pi_1^\alpha(\lambda_1)\nabla_1^\beta(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_1^\alpha(\lambda_1)\nabla_2^\beta(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right. \\
&\left. + \Pi_2^\alpha(\lambda_1)\nabla_1^\beta(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \Pi_2^\alpha(\lambda_1)\nabla_2^\beta(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q\right)^{\frac{1}{q}} \\
&+ (\nabla_3^\beta(\mu_2) + \nabla_4^\beta(\mu_2))^{1-\frac{1}{q}}(\Pi_1^\alpha(\lambda_1) + \Pi_2^\alpha(\lambda_1))^{1-\frac{1}{q}} \\
&\times \left(\Pi_1^\alpha(\lambda_1)\nabla_3^\beta(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_1^\alpha(\lambda_1)\nabla_4^\beta(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right. \\
&\left. + \Pi_2^\alpha(\lambda_1)\nabla_3^\beta(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \Pi_2^\alpha(\lambda_1)\nabla_4^\beta(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q\right)^{\frac{1}{q}} \\
&+ (\nabla_1^\beta(\lambda_2) + \nabla_2^\beta(\lambda_2))^{1-\frac{1}{q}}(\Pi_3^\alpha(\mu_1) + \Pi_4^\alpha(\mu_1))^{1-\frac{1}{q}} \\
&\times \left(\Pi_3^\alpha(\mu_1)\nabla_1^\beta(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_3^\alpha(\mu_1)\nabla_2^\beta(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right. \\
&\left. + \Pi_4^\alpha(\mu_1)\nabla_1^\beta(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \Pi_4^\alpha(\mu_1)\nabla_2^\beta(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q\right)^{\frac{1}{q}} \\
&+ (\nabla_3^\beta(\mu_2) + \nabla_4^\beta(\mu_2))^{1-\frac{1}{q}}(\Pi_3^\alpha(\mu_1) + \Pi_4^\alpha(\mu_1))^{1-\frac{1}{q}} \\
&\times \left(\Pi_3^\alpha(\mu_1)\nabla_3^\beta(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_3^\alpha(\mu_1)\nabla_4^\beta(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right. \\
&\left. + \Pi_4^\alpha(\mu_1)\nabla_3^\beta(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \Pi_4^\alpha(\mu_1)\nabla_4^\beta(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q\right)^{\frac{1}{q}}\Bigg],
\end{aligned}$$

where $\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ *is defined as in* Corollary 1 *and* Π_i^α, ∇_i^β, $i = 1, 2, 3, 4$ *are defined in* Corollary 3.

Corollary 7. *In* Theorem 3, *if we use* $\varphi(\xi) = \frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $\psi(\eta) = \frac{\eta^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, *then we obtain the following parameterized Simpson-type inequality for k-Riemann-Liouville fractional integrals*:

$$\begin{aligned}
|\Im(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \le (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\Bigg[&\left(\Pi_1^{\frac{\alpha}{k}}(\lambda_1) + \Pi_2^{\frac{\alpha}{k}}(\lambda_1)\right)^{1-\frac{1}{q}}\left(\nabla_1^{\frac{\beta}{k}}(\lambda_2) + \nabla_2^{\frac{\beta}{k}}(\lambda_2)\right)^{1-\frac{1}{q}} \\
&\times \left(\Pi_1^{\frac{\alpha}{k}}(\lambda_1)\nabla_1^{\frac{\beta}{k}}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_1^{\frac{\alpha}{k}}(\lambda_1)\nabla_2^{\frac{\beta}{k}}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right. \\
&\left. + \Pi_2^{\frac{\alpha}{k}}(\lambda_1)\nabla_1^{\frac{\beta}{k}}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \Pi_2^{\frac{\alpha}{k}}(\lambda_1)\nabla_2^{\frac{\beta}{k}}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q\right)^{\frac{1}{q}} \\
&+ \left(\nabla_3^{\frac{\beta}{k}}(\mu_2) + \nabla_4^{\frac{\beta}{k}}(\mu_2)\right)^{1-\frac{1}{q}}\left(\Pi_1^{\frac{\alpha}{k}}(\lambda_1) + \Pi_2^{\frac{\alpha}{k}}(\lambda_1)\right)^{1-\frac{1}{q}} \\
&\times \left(\Pi_1^{\frac{\alpha}{k}}(\lambda_1)\nabla_3^{\frac{\beta}{k}}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + \Pi_1^{\frac{\alpha}{k}}(\lambda_1)\nabla_4^{\frac{\beta}{k}}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q\right. \\
&\left. + \Pi_2^{\frac{\alpha}{k}}(\lambda_1)\nabla_3^{\frac{\beta}{k}}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \Pi_2^{\frac{\alpha}{k}}(\lambda_1)\nabla_4^{\frac{\beta}{k}}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\overset{\beta}{\nabla}_1^{k}(\lambda_2)+\overset{\beta}{\nabla}_2^{k}(\lambda_2)\right)^{1-\frac{1}{q}}\left(\overset{\alpha}{\Pi}_3^{k}(\mu_1)+\overset{\alpha}{\Pi}_4^{k}(\mu_1)\right)^{1-\frac{1}{q}}$$
$$\times\left(\overset{\alpha}{\Pi}_3^{k}(\mu_1)\overset{\beta}{\nabla}_1^{k}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+\overset{\alpha}{\Pi}_3^{k}(\mu_1)\overset{\beta}{\nabla}_2^{k}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q\right.$$
$$\left.+\overset{\alpha}{\Pi}_4^{k}(\mu_1)\overset{\beta}{\nabla}_1^{k}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\overset{\alpha}{\Pi}_4^{k}(\mu_1)\overset{\beta}{\nabla}_2^{k}(\lambda_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\overset{\beta}{\nabla}_3^{k}(\mu_2)+\overset{\beta}{\nabla}_4^{k}(\mu_2)\right)^{1-\frac{1}{q}}\left(\overset{\alpha}{\Pi}_3^{k}(\mu_1)+\overset{\alpha}{\Pi}_4^{k}(\mu_1)\right)^{1-\frac{1}{q}}$$
$$\times\left(\overset{\alpha}{\Pi}_3^{k}(\mu_1)\overset{\beta}{\nabla}_3^{k}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+\overset{\alpha}{\Pi}_3^{k}(\mu_1)\overset{\beta}{\nabla}_4^{k}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q\right.$$
$$\left.\left.+\overset{\alpha}{\Pi}_4^{k}(\mu_1)\overset{\beta}{\nabla}_3^{k}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\overset{\alpha}{\Pi}_4^{k}(\mu_1)\overset{\beta}{\nabla}_4^{k}(\mu_2)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\right)^{\frac{1}{q}}\right],$$

where $\Im(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ *is defined as in* Corollary 2 *and* $\overset{\alpha}{\Pi}_i^{k}$, $\overset{\beta}{\nabla}_i^{k}$, $i=1,2,3,4$ *are defined in* Corollary 4.

Theorem 4. *We assume that the conditions of* Lemma 1 *hold. If the mapping* $\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}\right|^q$ *is co-ordinated convex on* Δ, *then we have the following inequality*:

$$|\Theta(\kappa_1,\kappa_2;\kappa_3,\kappa_4)|\le(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\left[\left(\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\lambda_1|\mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\lambda_2|\mathrm{d}\eta\right)^{\frac{1}{p}}\right.$$
$$\times\left(\frac{\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\mu_2|\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|\Delta(\xi)-\Delta(1)\lambda_1|\mathrm{d}\xi\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{1}|\Lambda(\eta)-\Lambda(1)\lambda_2|\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\mu_1|\mathrm{d}\xi\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{1}|\Lambda(\eta)-\Lambda(1)\mu_2|\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|\Delta(\xi)-\Delta(1)\mu_1|\mathrm{d}\xi\right)^{\frac{1}{p}}$$

$$\times\left(\frac{9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q}{64}\right)^{\frac{1}{q}}\Bigg],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using the Hölder inequality in (4.3) and co-ordinated convexity of $\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}\right|^q$, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}|(\Delta(\xi)-\Delta(1)\lambda_1)||(\Lambda(\eta)-\Lambda(1)\lambda_2)|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\right|d\xi d\eta\\
&\le\left(\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\lambda_1|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\lambda_2|^p d\eta\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\right|^q d\xi d\eta\right)^{\frac{1}{q}}\\
&\le\left(\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\lambda_1|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\lambda_2|^p d\eta\right)^{\frac{1}{p}}\\
&\quad\times\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\xi\eta\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+\xi(1-\eta)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q\right.\\
&\quad\left.+(1-\xi)\eta\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+(1-\xi)(1-\eta)\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\right)^{\frac{1}{q}}d\xi d\eta\\
&=\left(\int_0^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\lambda_1|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\lambda_2|^p d\eta\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q}{64}\right)^{\frac{1}{q}}.
\end{aligned}\tag{4.13}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}|\Delta(\xi)-\Delta(1)\lambda_1||\Lambda(\eta)-\Lambda(1)\mu_2|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\right|d\xi d\eta\\
&\le\left(\int_0^{\frac{1}{2}}|\Lambda(\eta)-\Lambda(1)\mu_2|d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|\Delta(\xi)-\Delta(1)\lambda_1|d\xi\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q}{64}\right)^{\frac{1}{q}},
\end{aligned}\tag{4.14}$$

$$\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\mu_1||\Lambda(\eta)-\Lambda(1)\lambda_2|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\right|\mathrm{d}\xi\,\mathrm{d}\eta$$
$$\leq\left(\int_{\frac{1}{2}}^{1}|\Lambda(\eta)-\Lambda(1)\lambda_2|\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}|\Delta(\xi)-\Delta(1)\mu_1|\mathrm{d}\xi\right)^{\frac{1}{p}} \tag{4.15}$$
$$\times\left(\frac{3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q}{64}\right)^{\frac{1}{q}}$$

and

$$\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}|\Delta(\xi)-\Delta(1)\mu_1||\Lambda(\eta)-\Lambda(1)\mu_2|\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1,\eta\kappa_4+(1-\eta)\kappa_3)\right|\mathrm{d}\xi\,\mathrm{d}\eta$$
$$\leq\left(\int_{\frac{1}{2}}^{1}|\Lambda(\eta)-\Lambda(1)\mu_2|\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|\Delta(\xi)-\Delta(1)\mu_1|\mathrm{d}\xi\right)^{\frac{1}{p}} \tag{4.16}$$
$$\times\left(\frac{9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q}{64}\right)^{\frac{1}{q}}.$$

If we substitute the inequalities (4.13)–(4.16) in (4.3), then we obtain the required result. □

Remark 3. In Theorem 4, if we take $\varphi(\xi)=\xi$ and $\psi(\eta)=\eta$, we obtain

$$|\Phi(\kappa_1,\kappa_2;\kappa_3,\kappa_4)|\leq(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\left[\left(\int_{0}^{\frac{1}{2}}|\xi-\lambda_1|^p\mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}|\eta-\lambda_2|^p\mathrm{d}\eta\right)^{\frac{1}{p}}\right.$$
$$\times\left(\frac{\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\int_{0}^{\frac{1}{2}}|\eta-\mu_2|^p\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|\xi-\lambda_1|^p\mathrm{d}\xi\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{1}|\eta-\lambda_2|^p\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}|\xi-\mu_1|^p\mathrm{d}\xi\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q}{64}\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}|\eta-\mu_2|^p\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|\xi-\mu_1|^p\mathrm{d}\xi\right)^{\frac{1}{p}}$$
$$\left.\times\left(\frac{9\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q}{64}\right)^{\frac{1}{q}}\right],$$

where $\Phi(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ is defined as in Remark 1.

Corollary 8. *In* Theorem 4, *if we use* $\varphi(\xi)=\frac{\xi^\alpha}{\Gamma(\kappa_1)}$ *and* $\psi(\eta)=\frac{\xi^\beta}{\Gamma(\beta)}$, *then we obtain the following parameterized Simpson-type inequality for Riemann-Liouville fractional integrals*:

$$|\Omega(\kappa_1,\kappa_2;\kappa_3,\kappa_4)|\le(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\left[\left(\int_0^{\frac{1}{2}}|\xi^\alpha-\lambda_1|^p\mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\eta^\beta-\lambda_2|^p\mathrm{d}\eta\right)^{\frac{1}{p}}\right.$$
$$\times\left(\frac{\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_1,\kappa_4)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_1,\kappa_3)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\int_0^{\frac{1}{2}}|\eta^\beta-\mu_2|^p\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|\xi^\alpha-\lambda_1|^p\mathrm{d}\xi\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_2,\kappa_3)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_1,\kappa_4)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{1}|\eta^\beta-\lambda_2|^p\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\xi^\alpha-\mu_1|^p\mathrm{d}\xi\right)^{\frac{1}{p}}$$
$$\times\left(\frac{3\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_1,\kappa_3)\right|^q+9\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_1,\kappa_4)\right|^q}{64}\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{1}|\eta^\beta-\mu_2|^p\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|\xi^\alpha-\mu_1|^p\mathrm{d}\xi\right)^{\frac{1}{p}}$$
$$\left.\times\left(\frac{9\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_2,\kappa_4)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_2,\kappa_3)\right|^q+3\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2\mathbb{F}}{\partial\xi^\alpha\partial\eta^\beta}(\kappa_1,\kappa_3)\right|^q}{64}\right)^{\frac{1}{q}}\right],$$

where $\Omega(\kappa_1,\kappa_2;\kappa_3,\kappa_4)$ *is defined as in* Corollary 1.

Corollary 9. *In* Theorem 4, *if we use* $\varphi(\xi) = \frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $\psi(\eta) = \frac{\eta^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, *then we obtain the following parameterized Simpson-type inequality for k-Riemann-Liouville fractional integrals*:

$$
\begin{aligned}
|\Im(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \le (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) &\left[\left(\int_0^{\frac{1}{2}} \left|\xi^{\frac{\alpha}{k}} - \lambda_1\right|^p \mathrm{d}\xi\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|\eta^{\frac{\beta}{k}} - \lambda_2\right|^p d\eta\right)^{\frac{1}{p}}\right. \\
&\times \left.\left(\frac{\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + 3\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q + 3\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + 9\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q}{64}\right)^{\frac{1}{q}}\right] \\
&+ \left(\int_0^{\frac{1}{2}} \left|\eta^{\frac{\beta}{k}} - \mu_2\right|^p \mathrm{d}\eta\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \left|\xi^{\frac{\alpha}{k}} - \lambda_1\right|^p \mathrm{d}\xi\right)^{\frac{1}{p}} \\
&\times \left(\frac{3\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + 3\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q + 9\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q}{64}\right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{1}{2}}^{1} \left|\eta^{\frac{\beta}{k}} - \lambda_2\right|^p \mathrm{d}\eta\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|\xi^{\frac{\alpha}{k}} - \mu_1\right|^P \mathrm{d}\xi\right)^{\frac{1}{p}} \\
&\times \left(\frac{3\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + 3\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q + 9\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q}{64}\right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{1}{2}}^{1} \left|\eta^{\frac{\beta}{k}} - \mu_2\right|^P \mathrm{d}\eta\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \left|\xi^{\frac{\alpha}{k}} - \mu_1\right|^p \mathrm{d}\xi\right)^{\frac{1}{p}} \\
&\times \left.\left(\frac{9\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q + 3\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q + 3\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q}{64}\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where $\Im(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ *is defined as in* Corollary 2.

Theorem 5. *We assume that the conditions of* Lemma 1 *hold. If the mapping* $\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}\right|^q$ *is co-ordinated convex on* Δ, *then we have the following inequality*:

$$
\begin{aligned}
|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \le (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) &\left(\int_0^1 \int_0^1 |w(\xi, \eta)|^p \mathrm{d}\eta \mathrm{d}\xi\right)^{\frac{1}{p}} \\
&\times \left[\frac{\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q}{4}\right],
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking the modulus in Lemma 1 and using the Hölder inequality,

$$\begin{aligned}
|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| &= \left|(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^1\int_0^1 w(\xi,\eta)\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1, \eta\kappa_4+(1-\eta)\kappa_3)\mathrm{d}\eta\mathrm{d}\xi\right| \\
&\le (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\int_0^1\int_0^1 |w(\xi,\eta)|\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1, \eta\kappa_4+(1-\eta)\kappa_3)\right|\mathrm{d}\eta\mathrm{d}\xi \\
&\le (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\left(\int_0^1\int_0^1 |w(\xi,\eta)|^p\mathrm{d}\eta\mathrm{d}\xi\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1, \eta\kappa_4+(1-\eta)\kappa_3)\right|^q\right)^{\frac{1}{q}} \\
&\le (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\left(\int_0^1\int_0^1 |w(\xi,\eta)|^p\mathrm{d}\eta\mathrm{d}\xi\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1\int_0^1\left[\xi\eta\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q+\xi(1-\eta)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q\right.\right. \\
&\quad\left.\left.+(1-\xi)\eta\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+(1-\xi)(1-\eta)\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q\right]\mathrm{d}\eta\mathrm{d}\xi\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}\right|^q$, $q > 1$, is a co-ordinated convex function on Δ, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\xi\kappa_2+(1-\xi)\kappa_1, \eta\kappa_4+(1-\eta)\kappa_3)\right|^q\mathrm{d}\eta\mathrm{d}\xi \\
&\le \frac{\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q}{4}.
\end{aligned} \tag{4.17}$$

□

Remark 4. In Theorem 5, if we take $\varphi(\xi) = \xi$ and $\psi(\eta) = \eta$, we have

$$\begin{aligned}
|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| &\le (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)(A_p(\lambda,\mu))^{\frac{1}{p}} \\
&\quad\times\left(\frac{\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|^q+\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|^q+\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|^q+\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|^q}{4}\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
A_p(\lambda,\mu) &= \frac{1}{(p+1)^2}\left[\lambda_1^{p+1}+\left(\frac{1}{2}-\lambda_1\right)^{p+1}+\left(\mu_1-\frac{1}{2}\right)^{p+1}+(1-\mu_1)^{p+1}\right] \\
&\quad\times\left[\lambda_2^{p+1}+\left(\frac{1}{2}-\lambda_2\right)^{p+1}+\left(\mu_2-\frac{1}{2}\right)^{p+1}+(1-\mu_2)^{p+1}\right],
\end{aligned}$$

given by Budak and Ali in [49].

Corollary 10. *In* Theorem 5, *if we use* $\varphi(\xi) = \frac{\xi^{\alpha}}{\Gamma(\alpha)}$ *and* $\psi(\eta) = \frac{\xi^{\beta}}{\Gamma(\beta)}$, *then we obtain*

$$|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\left(\int_0^1\int_0^1 |w_1(\xi, \eta)|^p \mathrm{d}\eta \mathrm{d}\xi\right)^{\frac{1}{p}} \times \left(\frac{\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q}{4}\right)^{\frac{1}{q}},$$

where the mapping $w_1 : [0, 1] \times [0, 1] \to \mathbb{R}$ *is defined by*

$$w_1(\xi^{\alpha}, \eta^{\beta}) = \begin{cases} (\xi^{\alpha} - \lambda_1)(\eta^{\beta} - \lambda_2), & 0 \leq \xi \leq \frac{1}{2},\ 0 \leq \eta \leq \frac{1}{2}, \\ (\xi^{\alpha} - \lambda_1)(\eta^{\beta} - \mu_2), & 0 \leq \xi \leq \frac{1}{2},\ \frac{1}{2} \leq \eta \leq 1, \\ (\xi^{\alpha} - \mu_1)(\eta^{\beta} - \lambda_2), & \frac{1}{2} \leq \xi \leq 1,\ 0 \leq \eta \leq \frac{1}{2}, \\ (\xi^{\alpha} - \mu_1)(\eta^{\beta} - \mu_2), & \frac{1}{2} \leq \xi \leq 1,\ \frac{1}{2} \leq \eta \leq 1. \end{cases}$$

Corollary 11. *In* Theorem 5, *if we use* $\varphi(\xi) = \frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *and* $\psi(\eta) = \frac{\eta^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, *then we obtain*

$$|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\left(\int_0^1\int_0^1 |w_2(\xi, \eta)|^p \mathrm{d}\eta \mathrm{d}\xi\right)^{\frac{1}{p}} \times \left(\frac{\left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_3)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_1, \kappa_4)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_3)\right|^q + \left|\frac{\partial^2 \mathbb{F}}{\partial\xi\partial\eta}(\kappa_2, \kappa_4)\right|^q}{4}\right)^{\frac{1}{q}},$$

where the mapping $w_2 : [0, 1] \times [0, 1] \to \mathbb{R}$ *is defined by*

$$w_2(\xi, \eta) = \begin{cases} \left(\xi^{\frac{\alpha}{k}} - \lambda_1\right)\left(\eta^{\frac{\beta}{k}} - \lambda_2\right), & 0 \leq \xi \leq \frac{1}{2},\ 0 \leq \eta \leq \frac{1}{2}, \\ \left(\xi^{\frac{\alpha}{k}} - \lambda_1\right)\left(\eta^{\frac{\beta}{k}} - \mu_2\right), & 0 \leq \xi \leq \frac{1}{2},\ \frac{1}{2} \leq \eta \leq 1, \\ \left(\xi^{\frac{\alpha}{k}} - \mu_1\right)\left(\eta^{\frac{\beta}{k}} - \lambda_2\right), & \frac{1}{2} \leq \xi \leq 1,\ 0 \leq \eta \leq \frac{1}{2}, \\ \left(\xi^{\frac{\alpha}{k}} - \mu_1\right)\left(\eta^{\frac{\beta}{k}} - \mu_2\right), & \frac{1}{2} \leq \xi \leq 1,\ \frac{1}{2} \leq \eta \leq 1. \end{cases}$$

5 Special cases of main results

In this section, we present some special cases of the results proved in the previous section.

Corollary 12. *Under assumptions of* Theorem 2 *with* $\lambda_1 = \lambda_2 = \frac{1}{6}$ *and* $\mu_1 = \mu_2 = \frac{5}{6}$, *we have the following inequality*:

$$\left|\Delta(1)\Lambda(1)\left[\frac{\mathbb{F}\left(\kappa_1, \frac{\kappa_3+\kappa_4}{2}\right) + \mathbb{F}\left(\kappa_2, \frac{\kappa_3+\kappa_4}{2}\right) + 4\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) + \mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_3\right) + \mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_4\right)}{9}\right.\right.$$

$$
\begin{aligned}
&+\left.\frac{\mathbb{F}(\kappa_1,\kappa_3)+\mathbb{F}(\kappa_2,\kappa_3)+\mathbb{F}(\kappa_1,\kappa_4)+\mathbb{F}(\kappa_2,\kappa_4)}{36}\right]\\
&-\frac{2}{3}\left[\Lambda(1)\left({}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right)\right.\\
&\left.+\Delta(1)\left({}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right)\right]\\
&-\frac{1}{6}\Delta(1)\left[{}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}(\kappa_1,\kappa_4)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}(\kappa_2,\kappa_4)\right]\\
&-\frac{1}{6}\Lambda(1)\left[{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}(\kappa_1,\kappa_4)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}(\kappa_2,\kappa_4)\right]\\
&+{}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_4)\\
&\left.+{}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_2,\kappa_4)\right|\\
&\le(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\left[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|\left(\Pi_1^{\varphi}\left(\frac{1}{6}\right)+\Pi_3^{\varphi}\left(\frac{5}{6}\right)\right)\left(\nabla_1^{\psi}\left(\frac{1}{6}\right)+\nabla_3^{\psi}\left(\frac{5}{6}\right)\right)\right.\\
&+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|\left(\Pi_1^{\varphi}\left(\frac{1}{6}\right)+\Pi_3^{\varphi}\left(\frac{5}{6}\right)\right)\left(\nabla_2^{\psi}\left(\frac{1}{6}\right)+\nabla_4^{\psi}\left(\frac{5}{6}\right)\right)\\
&+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|\left(\Pi_2^{\varphi}\left(\frac{1}{6}\right)+\Pi_4^{\varphi}\left(\frac{5}{6}\right)\right)\left(\nabla_1^{\psi}\left(\frac{1}{6}\right)+\nabla_3^{\psi}\left(\frac{5}{6}\right)\right)\\
&\left.+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|\left(\Pi_2^{\varphi}\left(\frac{1}{6}\right)+\Pi_4^{\varphi}\left(\frac{5}{6}\right)\right)\left(\nabla_2^{\psi}\left(\frac{1}{6}\right)+\nabla_4^{\psi}\left(\frac{5}{6}\right)\right)\right],
\end{aligned}
$$

where Π_i^{φ} *and* ∇_i^{ψ}, $i=1,2,3,4$ *are defined as in* (4.2).

Remark 5. In Corollary 12, if we take $\varphi(\xi)=\xi$ and $\psi(\eta)=\eta$, then Corollary 12 reduces to [24, Theorem 3].

Remark 6. In Corollary 12, if we take $\varphi(\xi)=\frac{\xi^{\alpha}}{\Gamma(\alpha)}$ and $\psi(\eta)=\frac{\eta^{\beta}}{\Gamma(\beta)}$, then we obtain

$$
\begin{aligned}
&\left|\frac{(\kappa_2-\kappa_1)^{\alpha}}{\Gamma(\alpha+1)}\frac{(\kappa_4-\kappa_3)^{\beta}}{\Gamma(\beta+1)}\left[\frac{\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+4\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)}{9}\right.\right.\\
&+\left.\frac{\mathbb{F}(\kappa_1,\kappa_3)+\mathbb{F}(\kappa_2,\kappa_3)+\mathbb{F}(\kappa_1,\kappa_4)+\mathbb{F}(\kappa_2,\kappa_4)}{36}\right]\\
&-\frac{2}{3}\left[\frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left(J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right)\right.\\
&\left.+\frac{\Gamma(\alpha+1)}{(\kappa_4-\kappa_3)^{\beta}}\left(J^{\psi}_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+J^{\psi}_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right)\right]\\
&-\frac{1}{6}\frac{\Gamma(\alpha+1)}{(\kappa_4-\kappa_3)^{\beta}}\left[J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1,\kappa_4)+J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2,\kappa_3)+J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2,\kappa_4)\right]\\
&-\frac{1}{6}\frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}(\kappa_2,\kappa_3)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}(\kappa_1,\kappa_4)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}(\kappa_2,\kappa_4)\right]
\end{aligned}
$$

$$+\frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\frac{\Gamma(\beta+1)}{(\kappa_4-\kappa_3)^{\beta}}\Bigg[J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1,\kappa_4)$$

$$+J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2,\kappa_3)+J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2,\kappa_4)\Bigg]\Bigg|$$

$$\leq(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\Bigg[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|\left(\Pi_1^{\alpha}\left(\frac{1}{6}\right)+\Pi_3^{\alpha}\left(\frac{5}{6}\right)\right)\left(\nabla_1^{\beta}\left(\frac{1}{6}\right)+\nabla_3^{\beta}\left(\frac{5}{6}\right)\right)$$

$$+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|\left(\Pi_1^{\alpha}\left(\frac{1}{6}\right)+\Pi_3^{\alpha}\left(\frac{5}{6}\right)\right)\left(\nabla_2^{\beta}\left(\frac{1}{6}\right)+\nabla_4^{\beta}\left(\frac{5}{6}\right)\right)$$

$$+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|\left(\Pi_2^{\alpha}\left(\frac{1}{6}\right)+\Pi_4^{\alpha}\left(\frac{5}{6}\right)\right)\left(\nabla_1^{\beta}\left(\frac{1}{6}\right)+\nabla_3^{\beta}\left(\frac{5}{6}\right)\right)$$

$$+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|\left(\Pi_2^{\alpha}\left(\frac{1}{6}\right)+\Pi_4^{\alpha}\left(\frac{5}{6}\right)\right)\left(\nabla_2^{\beta}\left(\frac{1}{6}\right)+\nabla_4^{\beta}\left(\frac{5}{6}\right)\right)\Bigg].$$

Remark 7. In Corollary 12, if we take $\varphi(\xi)=\frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(\eta)=\frac{\eta^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, then we obtain

$$\Bigg|\frac{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}{\Gamma(\alpha+k)}\frac{(\kappa_4-\kappa_3)^{\frac{\beta}{k}}}{\Gamma(\beta+k)}\Bigg[\frac{\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)+4\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)+\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)}{9}$$

$$+\frac{\mathbb{F}(\kappa_1,\kappa_3)+\mathbb{F}(\kappa_2,\kappa_3)+\mathbb{F}(\kappa_1,\kappa_4)+\mathbb{F}(\kappa_2,\kappa_4)}{36}\Bigg]$$

$$-\frac{2}{3}\Bigg[\frac{\Gamma(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left(J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right)$$

$$+\frac{\Gamma(\alpha+k)}{(\kappa_4-\kappa_3)^{\frac{\beta}{k}}}\left(J^{\psi}_{\frac{\kappa_3+\kappa_4}{2}-,k}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+J^{\psi}_{\frac{\kappa_3+\kappa_4}{2}+,k}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right)\Bigg]$$

$$-\frac{1}{6}\frac{\Gamma(\alpha+k)}{(\kappa_4-\kappa_3)^{\frac{\beta}{k}}}\Bigg[J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}-,k}\mathbb{F}(\kappa_1,\kappa_3)+J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}+,k}\mathbb{F}(\kappa_1,\kappa_4)+J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}+,k}\mathbb{F}(\kappa_2,\kappa_3)+J^{\beta}_{\frac{\kappa_3+\kappa_4}{2}-,k}\mathbb{F}(\kappa_2,\kappa_4)\Bigg]$$

$$-\frac{1}{6}\frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\Bigg[J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathbb{F}(\kappa_1,\kappa_3)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathbb{F}(\kappa_2,\kappa_3)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathbb{F}(\kappa_1,\kappa_4)+J^{\alpha}_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathbb{F}(\kappa_2,\kappa_4)\Bigg]$$

$$+\frac{\Gamma(\alpha+k)}{(\kappa_2-\kappa_1)^{\alpha}}\frac{\Gamma(\beta+k)}{(\kappa_4-\kappa_3)^{\beta}}\Bigg[J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1,\kappa_4)$$

$$+J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2,\kappa_3)+J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2,\kappa_4)\Bigg]\Bigg|$$

$$\leq(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\Bigg[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|\left(\Pi_1^{\frac{\alpha}{k}}\left(\frac{1}{6}\right)+\Pi_3^{\frac{\alpha}{k}}\left(\frac{5}{6}\right)\right)\left(\nabla_1^{\frac{\beta}{k}}\left(\frac{1}{6}\right)+\nabla_3^{\frac{\beta}{k}}\left(\frac{5}{6}\right)\right)$$

$$+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|\left(\Pi_1^{\frac{\alpha}{k}}\left(\frac{1}{6}\right)+\Pi_3^{\frac{\alpha}{k}}\left(\frac{5}{6}\right)\right)\left(\nabla_2^{\frac{\beta}{k}}\left(\frac{1}{6}\right)+\nabla_4^{\frac{\beta}{k}}\left(\frac{5}{6}\right)\right)$$

$$+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|\left(\Pi_2^{\frac{\alpha}{k}}\left(\frac{1}{6}\right)+\Pi_4^{\frac{\alpha}{k}}\left(\frac{5}{6}\right)\right)\left(\nabla_1^{\frac{\beta}{k}}\left(\frac{1}{6}\right)+\nabla_3^{\frac{\beta}{k}}\left(\frac{5}{6}\right)\right)$$

$$+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|\left(\Pi_2^{\frac{\alpha}{k}}\left(\frac{1}{6}\right)+\Pi_4^{\frac{\alpha}{k}}\left(\frac{5}{6}\right)\right)\left(\nabla_2^{\frac{\beta}{k}}\left(\frac{1}{6}\right)+\nabla_4^{\frac{\beta}{k}}\left(\frac{5}{6}\right)\right)\Bigg].$$

Corollary 13. *Under assumptions of* Theorem 2 *with* $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \frac{1}{2}$, *we have the following inequality*:

$$
\begin{aligned}
&\frac{(\kappa_2-\kappa_1)^\alpha}{\Gamma(\alpha+1)}\frac{(\kappa_4-\kappa_3)^\beta}{\Gamma(\beta+1)}\Bigg|\frac{\mathbb{F}(\kappa_1,\kappa_3)+\mathbb{F}(\kappa_2,\kappa_3)+\mathbb{F}(\kappa_1,\kappa_4)+\mathbb{F}(\kappa_2,\kappa_4)}{4}\\
&\quad-\frac{\Delta(1)}{2}\Big[{}_{\frac{\kappa_3+\kappa_4}{2}-}I_\psi\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_\psi\mathbb{F}(\kappa_1,\kappa_4)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_\psi\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_3+\kappa_4}{2}-}I_\psi\mathbb{F}(\kappa_2,\kappa_4)\Big]\\
&\quad-\frac{\Lambda(1)}{2}\Big[{}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}-}I_\varphi\mathbb{F}(\kappa_1,\kappa_4)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_\varphi\mathbb{F}(\kappa_2,\kappa_4)\Big]\\
&\quad+{}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_4)\\
&\quad+{}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_2,\kappa_4)\Bigg|\\
&\le(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\Bigg[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|\left(\Pi_1^\varphi\left(\frac{1}{2}\right)+\Pi_3^\varphi\left(\frac{1}{2}\right)\right)\left(\nabla_1^\psi\left(\frac{1}{2}\right)+\nabla_3^\psi\left(\frac{1}{2}\right)\right)\\
&\quad+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|\left(\Pi_1^\varphi\left(\frac{1}{2}\right)+\Pi_3^\varphi\left(\frac{1}{2}\right)\right)\left(\nabla_2^\psi\left(\frac{1}{2}\right)+\nabla_4^\psi\left(\frac{1}{2}\right)\right)\\
&\quad+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|\left(\Pi_2^\varphi\left(\frac{1}{2}\right)+\Pi_4^\varphi\left(\frac{1}{2}\right)\right)\left(\nabla_1^\psi\left(\frac{1}{2}\right)+\nabla_3^\psi\left(\frac{1}{2}\right)\right)\\
&\quad+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|\left(\Pi_2^\varphi\left(\frac{1}{2}\right)+\Pi_4^\varphi\left(\frac{1}{2}\right)\right)\left(\nabla_2^\psi\left(\frac{1}{2}\right)+\nabla_4^\psi\left(\frac{1}{2}\right)\right)\Bigg],
\end{aligned}
$$

where Π_i^φ *and* ∇_i^ψ, $i = 1, 2, 3, 4$ *are defined as in* (4.2).

Remark 8. In Corollary 13, if we take $\varphi(\xi) = \xi$ and $\psi(\eta) = \eta$, then Corollary 13 reduces to [7, Theorem 2].

Remark 9. In Corollary 13, if we take $\varphi(\xi) = \frac{\xi^\alpha}{\Gamma(\alpha)}$ and $\psi(\eta) = \frac{\eta^\beta}{\Gamma(\beta)}$, then we obtain

$$
\begin{aligned}
&\Bigg|\left[\frac{\mathbb{F}(\kappa_1,\kappa_3)+\mathbb{F}(\kappa_2,\kappa_3)+\mathbb{F}(\kappa_1,\kappa_4)+\mathbb{F}(\kappa_2,\kappa_4)}{4}\right]\\
&\quad-\frac{\Gamma(\alpha+1)}{2(\kappa_4-\kappa_3)^\beta}\Big[J^\beta_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^\beta_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1,\kappa_4)+J^\beta_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2,\kappa_3)+J^\beta_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2,\kappa_4)\Big]\\
&\quad-\frac{\Gamma(\alpha+1)}{2(\kappa_2-\kappa_1)^\alpha}\Big[J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}(\kappa_2,\kappa_3)+J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}(\kappa_1,\kappa_4)+J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}(\kappa_2,\kappa_4)\Big]\\
&\quad+\frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\frac{\Gamma(\beta+1)}{(\kappa_4-\kappa_3)^\beta}\Big[J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1,\kappa_4)\\
&\quad+J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2,\kappa_3)+J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2,\kappa_4)\Big]\Bigg|\\
&\le(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\Bigg[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|\left(\Pi_1^\alpha\left(\frac{1}{2}\right)+\Pi_3^\alpha\left(\frac{1}{2}\right)\right)\left(\nabla_1^\beta\left(\frac{1}{2}\right)+\nabla_3^\beta\left(\frac{1}{2}\right)\right)\\
&\quad+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|\left(\Pi_1^\alpha\left(\frac{1}{2}\right)+\Pi_3^\alpha\left(\frac{1}{2}\right)\right)\left(\nabla_2^\beta\left(\frac{1}{2}\right)+\nabla_4^\beta\left(\frac{1}{2}\right)\right)\\
&\quad+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|\left(\Pi_2^\alpha\left(\frac{1}{2}\right)+\Pi_4^\alpha\left(\frac{1}{2}\right)\right)\left(\nabla_1^\beta\left(\frac{1}{2}\right)+\nabla_3^\beta\left(\frac{1}{2}\right)\right)\\
&\quad+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|\left(\Pi_2^\alpha\left(\frac{1}{2}\right)+\Pi_4^\alpha\left(\frac{1}{2}\right)\right)\left(\nabla_2^\beta\left(\frac{1}{2}\right)+\nabla_4^\beta\left(\frac{1}{2}\right)\right)\Bigg].
\end{aligned}
$$

Remark 10. In Corollary 13, if we take $\varphi(\xi) = \frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(\eta) = \frac{\eta^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, then we obtain

$$\begin{aligned}
&\Bigg|\left[\frac{\mathbb{F}(\kappa_1,\kappa_3)+\mathbb{F}(\kappa_2,\kappa_3)+\mathbb{F}(\kappa_1,\kappa_4)+\mathbb{F}(\kappa_2,\kappa_4)}{4}\right]\\
&-\frac{\Gamma(\alpha+k)}{2(\kappa_4-\kappa_3)^{\frac{\beta}{k}}}\left[J^{\beta,k}_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^{\beta,k}_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1,\kappa_4)+J^{\beta,k}_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2,\kappa_3)+J^{\beta,k}_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2,\kappa_4)\right]\\
&-\frac{\Gamma(\alpha+k)}{2(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left[J^{\alpha,k}_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^{\alpha,k}_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}(\kappa_2,\kappa_3)+J^{\alpha,k}_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}(\kappa_1,\kappa_4)+J^{\alpha,k}_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}(\kappa_2,\kappa_4)\right]\\
&+\frac{\Gamma(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\frac{\Gamma(\beta+k)}{(\kappa_4-\kappa_3)^{\frac{\beta}{k}}}\Bigg[J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1,\kappa_3)+J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1,\kappa_4)\\
&+J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2,\kappa_3)+J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2,\kappa_4)\Bigg]\Bigg|\\
&\le(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\Bigg[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|\left(\Pi_1^{\frac{\alpha}{k}}\left(\frac{1}{2}\right)+\Pi_3^{\frac{\alpha}{k}}\left(\frac{1}{2}\right)\right)\left(\nabla_1^{\frac{\beta}{k}}\left(\frac{1}{2}\right)+\nabla_3^{\frac{\beta}{k}}\left(\frac{1}{2}\right)\right)\\
&+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|\left(\Pi_1^{\frac{\alpha}{k}}\left(\frac{1}{2}\right)+\Pi_3^{\frac{\alpha}{k}}\left(\frac{1}{2}\right)\right)\left(\nabla_2^{\frac{\beta}{k}}\left(\frac{1}{2}\right)+\nabla_4^{\frac{\beta}{k}}\left(\frac{1}{2}\right)\right)\\
&+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|\left(\Pi_2^{\frac{\alpha}{k}}\left(\frac{1}{2}\right)+\Pi_4^{\frac{\alpha}{k}}\left(\frac{1}{2}\right)\right)\left(\nabla_1^{\frac{\beta}{k}}\left(\frac{1}{2}\right)+\nabla_3^{\frac{\beta}{k}}\left(\frac{1}{2}\right)\right)\\
&+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|\left(\Pi_2^{\frac{\alpha}{k}}\left(\frac{1}{2}\right)+\Pi_4^{\frac{\alpha}{k}}\left(\frac{1}{2}\right)\right)\left(\nabla_2^{\frac{\beta}{k}}\left(\frac{1}{2}\right)+\nabla_4^{\frac{\beta}{k}}\left(\frac{1}{2}\right)\right)\Bigg].
\end{aligned}$$

Corollary 14. *Under assumptions of* Theorem 2 *with* $\lambda_1 = \lambda_2 = 0$ *and* $\mu_1 = \mu_2 = 1$, *we have the following inequality*:

$$\begin{aligned}
&\Bigg|\Delta(1)\Lambda(1)\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\frac{\kappa_3+\kappa_4}{2}\right)-\Lambda(1)\left[{}_{\frac{\kappa_1+\kappa_2}{2}-}I_{\varphi}\mathbb{F}\left(\kappa_1,\frac{\kappa_3+\kappa_4}{2}\right)+{}_{\frac{\kappa_1+\kappa_2}{2}+}I_{\varphi}\mathbb{F}\left(\kappa_2,\frac{\kappa_3+\kappa_4}{2}\right)\right]\\
&-\Delta(1)\left[{}_{\frac{\kappa_3+\kappa_4}{2}-}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_3\right)+{}_{\frac{\kappa_3+\kappa_4}{2}+}I_{\psi}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2},\kappa_4\right)\right]+{}_{\frac{\kappa_1+\kappa_2}{2}-\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_3)\\
&+{}_{\frac{\kappa_1+\kappa_2}{2}-\frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_1,\kappa_4)+{}_{\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}-}I_{\varphi,\psi}\mathbb{F}(\kappa_2,\kappa_3)+{}_{\frac{\kappa_1+\kappa_2}{2}+\frac{\kappa_3+\kappa_4}{2}+}I_{\varphi,\psi}\mathbb{F}(\kappa_2,\kappa_4)\Bigg|\\
&\le(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\Bigg[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|(\Pi_1^{\varphi}(0)+\Pi_3^{\varphi}(1))(\nabla_1^{\psi}(0)+\nabla_3^{\psi}(1))\\
&+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|(\Pi_1^{\varphi}(0)+\Pi_3^{\varphi}(1))(\nabla_2^{\psi}(0)+\nabla_4^{\psi}(1))\\
&+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|(\Pi_2^{\varphi}(0)+\Pi_4^{\varphi}(1))(\nabla_1^{\psi}(0)+\nabla_3^{\psi}(1))\\
&+\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|(\Pi_2^{\varphi}(0)+\Pi_4^{\varphi}(1))(\nabla_2^{\psi}(0)+\nabla_4^{\psi}(1))\Bigg],
\end{aligned}$$

where Π_i^{φ} *and* ∇_i^{ψ}, $i = 1, 2, 3, 4$ *are defined as in* (4.2).

Remark 11. In Corollary 14, if we take $\varphi(\xi) = \xi$ and $\psi(\eta) = \eta$, then Corollary 13 reduces to [6, Theorem 2].

Remark 12. In Corollary 14, if we take $\varphi(\xi) = \frac{\xi^\alpha}{\Gamma(\alpha)}$ and $\psi(\eta) = \frac{\eta^\beta}{\Gamma(\beta)}$, then we obtain

$$\begin{aligned}
&\Bigg| \mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) - \frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\left(J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-}\mathbb{F}\left(\kappa_1, \frac{\kappa_3+\kappa_4}{2}\right) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+}\mathbb{F}\left(\kappa_2, \frac{\kappa_3+\kappa_4}{2}\right)\right) \\
&\quad - \frac{\Gamma(\alpha+1)}{(\kappa_4-\kappa_3)^\beta}\left(J^\psi_{\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_3\right) + J^\psi_{\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_4\right)\right) \\
&\quad + \frac{\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^\alpha}\frac{\Gamma(\beta+1)}{(\kappa_4-\kappa_3)^\beta}\Bigg[J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1,\kappa_3) + J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1,\kappa_4) \\
&\quad + J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2,\kappa_3) + J^{\alpha,\beta}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2,\kappa_4)\Bigg]\Bigg| \\
&\leq (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\Bigg[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|(\Pi^\alpha_1(0)+\Pi^\alpha_3(1))(\nabla^\beta_1(0)+\nabla^\beta_3(1)) \\
&\quad + \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|(\Pi^\alpha_1(0)+\Pi^\alpha_3(1))(\nabla^\beta_2(0)+\nabla^\beta_4(1)) \\
&\quad + \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|(\Pi^\alpha_2(0)+\Pi^\alpha_4(1))(\nabla^\beta_1(0)+\nabla^\beta_3(1)) \\
&\quad + \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|(\Pi^\alpha_2(0)+\Pi^\alpha_4(1))(\nabla^\beta_2(0)+\nabla^\beta_4(1))\Bigg].
\end{aligned}$$

Remark 13. In Corollary 13, if we take $\varphi(\xi) = \frac{\xi^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(\eta) = \frac{\eta^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, then we obtain

$$\begin{aligned}
&\Bigg| \mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) - \frac{\Gamma(\alpha+k)}{(\kappa_2-\kappa_1)^{\frac{\alpha}{k}}}\left(J^\alpha_{\frac{\kappa_1+\kappa_2}{2}-,k}\mathbb{F}\left(\kappa_1, \frac{\kappa_3+\kappa_4}{2}\right) + J^\alpha_{\frac{\kappa_1+\kappa_2}{2}+,k}\mathbb{F}\left(\kappa_2, \frac{\kappa_3+\kappa_4}{2}\right)\right) \\
&\quad - \frac{\Gamma(\alpha+k)}{(\kappa_4-\kappa_3)^\beta}\left(J^\psi_{\frac{\kappa_3+\kappa_4}{2}-,k}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_3\right) + J^\psi_{\frac{\kappa_3+\kappa_4}{2}+,k}\mathbb{F}\left(\frac{\kappa_1+\kappa_2}{2}, \kappa_4\right)\right) \\
&\quad + \frac{\Gamma(\alpha+k)}{(\kappa_2-\kappa_1)^\alpha}\frac{\Gamma(\beta+1)}{(\kappa_4-\kappa_3)^\beta}\Bigg[J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_1,\kappa_3) + J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}-,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_1,\kappa_4) \\
&\quad + J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}-}\mathbb{F}(\kappa_2,\kappa_3) + J^{\alpha,\beta,k}_{\frac{\kappa_1+\kappa_2}{2}+,\frac{\kappa_3+\kappa_4}{2}+}\mathbb{F}(\kappa_2,\kappa_4)\Bigg]\Bigg| \\
&\leq (\kappa_2-\kappa_1)(\kappa_4-\kappa_3)\Bigg[\left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_4)\right|\left(\Pi^{\frac{\alpha}{k}}_1(0)+\Pi^{\frac{\alpha}{k}}_3(1)\right)\left(\nabla^{\frac{\beta}{k}}_1(0)+\nabla^{\frac{\beta}{k}}_3(1)\right) \\
&\quad + \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_2,\kappa_3)\right|\left(\Pi^{\frac{\alpha}{k}}_1(0)+\Pi^{\frac{\alpha}{k}}_3(1)\right)\left(\nabla^{\frac{\beta}{k}}_2(0)+\nabla^{\frac{\beta}{k}}_4(1)\right) \\
&\quad + \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_4)\right|\left(\Pi^{\frac{\alpha}{k}}_2(0)+\Pi^{\frac{\alpha}{k}}_4(1)\right)\left(\nabla^{\frac{\beta}{k}}_1(0)+\nabla^{\frac{\beta}{k}}_3(1)\right) \\
&\quad + \left|\frac{\partial^2\mathbb{F}}{\partial\xi\partial\eta}(\kappa_1,\kappa_3)\right|\left(\Pi^{\frac{\alpha}{k}}_2(0)+\Pi^{\frac{\alpha}{k}}_4(1)\right)\left(\nabla^{\frac{\beta}{k}}_2(0)+\nabla^{\frac{\beta}{k}}_4(1)\right)\Bigg].
\end{aligned}$$

Remark 14. By special choices of λ_1, λ_2, μ_1 and μ_2 in Theorems 3, 4 and 5, one can obtain several new Simpson-, trapezoid- and midpoint-type inequalities. Writing these situations is left to the reader as it will make the article too long.

6 Conclusion

In this paper, we present several generalized inequalities for co-ordinated convex functions via generalized fractional integrals. It is also shown that the results given here are the strong generalization of some already published ones. It is an interesting and new problem that the forthcoming researchers can use the techniques of this study and obtain similar inequalities for different kinds of co-ordinated convexity in their next works.

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