Research Article

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Long time decay for 3D Navier-Stokes equations in Fourier-Lei-Lin spaces

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Abstract: In this paper, we study the long time decay of global solution to the 3D incompressible Navier-Stokes equations. We prove that if $u \in C(\mathbb{R}^+, \mathcal{X}^{-1,\sigma}(\mathbb{R}^3))$ is a global solution to the considered equation, where $\mathcal{X}^{i,\sigma}(\mathbb{R}^3)$ is the Fourier-Lei-Lin space with parameters $i = -1$ and $\sigma \geq -1$, then $\|u(t)\|_{\mathcal{X}^{-1,\sigma}}$ decays to zero as time goes to infinity. The used techniques are based on Fourier analysis.

Keywords: Navier-Stokes equations, critical spaces, long time decay

MSC 2020: 35Q30, 35D35

1 Introduction

The 3D incompressible Navier-Stokes equations (NSEs) are

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u = -\nabla p & \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ u(0,x) = u^0(x) & \text{in } \mathbb{R}^3, \end{cases} \tag{NSE}$$

where $\nu > 0$ is the viscosity of the fluid, $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$, and $p = p(t,x)$ denote, respectively, the unknown velocity and the unknown pressure of the fluid at the point $(t,x) \in \mathbb{R}^+\times\mathbb{R}^3$, $(u\cdot\nabla u) = u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$, and $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is an initial given velocity. If u^0 is quite regular, then the divergence-free condition determines the pressure p.

Several authors have studied the local existence of solutions to the (NSE), for example, Leray [1,2] and Kato [3]. The global existence of weak solutions goes back to Leray [2] and Hopf [4]. The global well-posedness of strong solutions for small initial data in the critical Sobolev space $\dot{H}^{\frac{1}{2}}$ is due to Fujita and Kato [5]. In [6], Chemin considered initial data that belong to the space $\dot{H}^s$ for $s > \frac{1}{2}$. In [7], Kato has proved the case of the Lebesgue space L^3. In [8], Koch and Tataru considered the space BMO^{-1} (see also [9–11]). In all these works, the norms in the corresponding spaces of the initial data are assumed to be very small. More precisely, the norm was supposed to be bounded by the viscosity ν multiplied by some positive constants. More results and details in this direction can be found in the book by Cannone [12].

In [13], the authors consider a new critical space that is contained in $\mathbf{BMO}^{-1}$, where they show it is sufficient to assume that the norms of initial data are less than exactly the viscosity coefficient ν. Then, the space used in [13] is the following

$$\mathcal{X}^{-1}(\mathbb{R}^3) = \left\{ f \in \mathcal{D}'(\mathbb{R}^3);\ \int_{\mathbb{R}^3} \frac{|\hat{u}(\xi)|}{|\xi|}\mathrm{d}\xi < \infty \right\},$$

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which is equipped with the norm

$$\|f\|_{\mathcal{X}^{-1}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \frac{|\hat{u}(\xi)|}{|\xi|} \mathrm{d}\xi.$$

We will also use the notation, for $i = 0, 1$,

$$\mathcal{X}^i(\mathbb{R}^3) = \left\{ f \in \mathcal{D}'(\mathbb{R}^3);\ \hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^3),\ \int_{\mathbb{R}^3} |\xi|^i |\hat{u}(\xi)| \mathrm{d}\xi < \infty \right\}.$$

For the small initial data, the global existence is proved in [13]:

Theorem 1.1. (See [13]). *Let $u^0 \in \mathcal{X}^{-1}(\mathbb{R}^3)$, such that $\|u^0\|_{\mathcal{X}^{-1}(\mathbb{R}^3)} < \nu$. Then, there is a unique $u \in C(\mathbb{R}^+, \mathcal{X}^{-1}(\mathbb{R}^3))$ such that $\Delta u \in L^1(\mathbb{R}^+, \mathcal{X}^{-1}(\mathbb{R}^3))$. Moreover, $\forall t \geq 0$*

$$\sup_{0 \leq t < \infty} \left(\|u(t)\|_{\mathcal{X}^{-1}} + \left(\nu - \|u^0\|_{\mathcal{X}^{-1}}\right) \int_0^t \|\nabla u\|_{L^\infty} \mathrm{d}\tau \right) \leq \|u^0\|_{\mathcal{X}^{-1}}.$$

On the other hand, in [14] the authors proved the local existence for the large initial data and blow-up criteria if the maximal time is finite, precisely:

Theorem 1.2. (See [14]) *Let $u^0 \in \mathcal{X}^{-1}(\mathbb{R}^3)$. There exists time T such that the system* (NSE) *has a unique solution $u \in L^2([0, T], \mathcal{X}^0(\mathbb{R}^3))$ which also belongs to*

$$C([0, T], \mathcal{X}^{-1}(\mathbb{R}^3)) \cap L^1([0, T], \mathcal{X}^1(\mathbb{R}^3)) \cap L^\infty([0, T], \mathcal{X}^{-1}(\mathbb{R}^3)).$$

Let T^* denote the maximal time of existence of such solution. Hence,
if $\|u\|_{\mathcal{X}^{-1}} < \nu$, then

$$T^* = \infty;$$

if T^* is finite, then

$$\int_0^{T^*} \|u(t)\|^2_{\mathcal{X}^0} = \infty.$$

In [15], to improve the result [13,14], we introduced the Fourier Lei-Lin space which is defined as follows:

$$\mathcal{X}^\sigma(\mathbb{R}^3) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3);\ \hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3} |\xi|^\sigma |\hat{f}(\xi)| \mathrm{d}\xi < \infty \right\}, \quad \sigma \in \mathbb{R},$$

which is equipped with the norm

$$\|f\|_{\mathcal{X}^\sigma(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\xi|^\sigma |\hat{f}(\xi)| \mathrm{d}\xi, \quad f \in \mathcal{X}^\sigma(\mathbb{R}^3).$$

In the same study, as $\mathcal{X}^\sigma(\mathbb{R}^3)$ is not a Banach space for $\sigma > 0$, we introduced the following non-homogeneous spaces:

$$\mathcal{X}^{i,\sigma}(\mathbb{R}^3) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3);\ \hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3} (|\xi|^i + |\xi|^\sigma) |\hat{f}(\xi)| \mathrm{d}\xi < \infty \right\}, \quad \sigma \in \mathbb{R},\ i = -1, 0,$$

equipped with the norm

$$\|f\|_{\mathcal{X}^{i,\sigma}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (|\xi|^i + |\xi|^\sigma)|\hat{f}(\xi)|d\xi = \|f\|_{\mathcal{X}^i} + \|f\|_{\mathcal{X}^\sigma}.$$

Precisely, we proved the following theorem:

Theorem 1.3. (See [15]) *Let $\sigma \geq -1$ and $u^0 \in \mathcal{X}^{-1,\sigma}(\mathbb{R}^3)$ be such that $\|u^0\|_{\mathcal{X}^{-1}} < \frac{\nu}{2^{\sigma+3}}$. Then there exists a unique global solution $u \in C(\mathbb{R}^+, \mathcal{X}^{-1,\sigma}(\mathbb{R}^3))$ such that*

$$\|u(t)\|_{\mathcal{X}^{-1,\sigma}} + \frac{\nu}{2}\int_0^t \|\Delta u\|_{\mathcal{X}^{-1,\sigma}} d\tau \leq \|u^0\|_{\mathcal{X}^{-1,\sigma}}, \quad t \geq 0.$$

Our problem is to show that the norm of the global solution to (NSE) in $\mathcal{X}^{-1,\sigma}$ tends to zero when the time grows to infinity. The behavior of the norm of the solution to infinity, in the different Banach spaces, was studied by several researchers. Wiegner proved in [16] that the L^2 norm of the solutions vanishes for any square integrable initial data, as time goes to infinity and gives a decay rate that seems to be optimal for a class of initial data.

In [17] for the critical Sobolev spaces $\dot{H}^{\frac{1}{2}}$, I. Gallagher, D. Iftimie, and F. Planchon proved that $\|u(t)\|_{\dot{H}^{\frac{1}{2}}}$ goes to zero at infinity. Recently, Benameur [18] has proved the following result.

Theorem 1.4. (See [18]). *Let $u \in C(\mathbb{R}^+, \mathcal{X}^{-1}(\mathbb{R}^3))$ be a global solution to* (NSE). *Then*

$$\limsup_{t\to\infty} \|u(t)\|_{\mathcal{X}^{-1}} = 0.$$

Now we are ready to state the main result.

Theorem 1.5. *For $\sigma \geq -1$, let $u \in C(\mathbb{R}^+, \mathcal{X}^{-1,\sigma}(\mathbb{R}^3))$ be the global solution to* (NSE). *Then*

$$\limsup_{t\to\infty} \|u(t)\|_{\mathcal{X}^{-1,\sigma}} = 0.$$

In the following, we give a natural application of Theorem 1.5, which is the stability of global solutions of (NSE) system.

Theorem 1.6. *For $\sigma \geq -1$, let $u \in C(\mathbb{R}^+, \mathcal{X}^{-1,\sigma}(\mathbb{R}^3))$ be the global solution to* (NSE). *Then, for all $v^0 \in \mathcal{X}^{-1,\sigma}(\mathbb{R}^3)$ such that*

$$\|v^0 - u(0)\|_{\mathcal{X}^{-1,\sigma}} < \frac{\nu}{2^{\sigma+7}} e^{-\frac{2^{\sigma+8}}{\nu}\int_0^\infty \|u(s)\|^2_{\mathcal{X}^{0,\sigma+1}} ds}.$$

Then, Navier-Stokes system starting by v^0 has a global solution. Moreover, if v is the corresponding global solution, then

$$\|v(t) - u(t)\|_{\mathcal{X}^{-1,\sigma}} + \frac{\nu}{2}\int_0^t \|v(\tau) - u(\tau)\|_{\mathcal{X}^{1,\sigma+1}} d\tau \leq \|v^0 - u(0)\|_{\mathcal{X}^{-1,\sigma}} e^{\frac{2^{\sigma+8}}{\nu}\int_0^\infty \|u(s)\|^2_{\mathcal{X}^{0,\sigma+1}} ds}.$$

The paper is organized in the following way: in Section 2, we give some notations and important preliminary results. Section 3 is devoted to prove the principal result. In Section 4, we prove the stability result for the global solutions.

2 Notations and preliminary results

2.1 Notations

In this section, we collect some notations and definitions that will be used later.

First, the Fourier transformation is normalized as

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix\cdot\xi) f(x)\,\mathrm{d}x, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3,$$

the inverse Fourier formula is

$$\mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \exp(i\xi\cdot x) g(\xi)\,\mathrm{d}\xi, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

and the convolution product of a suitable pair of functions f and g on $\mathbb{R}^3$ is given by

$$(f * g)(x) := \int_{\mathbb{R}^3} f(y) g(x-y)\,\mathrm{d}y.$$

If $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$ are two vector fields, then we set

$$f \otimes g := (g_1 f, g_2 f, g_3 f),$$

and

$$\mathrm{div}(f \otimes g) := (\mathrm{div}(g_1 f), \mathrm{div}(g_2 f), \mathrm{div}(g_3 f)).$$

We denote by $\mathbb{P}$ the Leray projection operator defined by the formula:

$$\mathcal{F}(\mathbb{P} f)(\xi) = \widehat{f}(\xi) - \frac{(f(\xi)\cdot\xi)}{|\xi|^2}\xi.$$

Finally, let $(B, \|\cdot\|)$ be a Banach space, $1 \le p \le \infty$ and $T > 0$. We define $L^p_T(B)$ to be the space of all measurable functions

$$[0, T] \ni t \mapsto f(t) \in B \text{ such that } t \mapsto \|f(t)\| \in L^p([0, T]).$$

2.2 Preliminary results

In this section, we recall some classical results and we give a few technical lemmas.

Lemma 2.1. *Let $\sigma \in \mathbb{R}$, we have*

$$\|f\|_{\mathcal{X}^{\sigma+1}(\mathbb{R}^3)} \le \|f\|^{1/2}_{\mathcal{X}^{\sigma}(\mathbb{R}^3)} \|f\|^{1/2}_{\mathcal{X}^{\sigma+2}(\mathbb{R}^3)}. \tag{2.1}$$

For $\sigma \ge -1$, we have

$$\|f\|_{\mathcal{X}^{0}(\mathbb{R}^3)} \le \|f\|^{\frac{\sigma+2}{\sigma+3}}_{\mathcal{X}^{-1}(\mathbb{R}^3)} \|f\|^{\frac{1}{\sigma+3}}_{\mathcal{X}^{\sigma+2}(\mathbb{R}^3)} \tag{2.2}$$

and

$$\|f\|_{\mathcal{X}^{\sigma+1}(\mathbb{R}^3)} \le \|f\|^{\frac{1}{\sigma+3}}_{\mathcal{X}^{-1}(\mathbb{R}^3)} \|f\|^{\frac{\sigma+2}{\sigma+3}}_{\mathcal{X}^{\sigma+2}(\mathbb{R}^3)}. \tag{2.3}$$

Proof. First, let $\sigma \in \mathbb{R}$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|f\|_{\mathcal{X}^{\sigma+1}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{\sigma+1}|\hat{f}(\xi)|\mathrm{d}\xi \\ &= \int_{\mathbb{R}^3} (|\xi|^{\sigma}|\hat{f}(\xi)|)^{1/2}(|\xi|^{\sigma+2}|\hat{f}(\xi)|)^{1/2}\mathrm{d}\xi \\ &\le \left(\int_{\mathbb{R}^3} |\xi|^{\sigma}|\hat{f}(\xi)|\mathrm{d}\xi\right)^{1/2}\left(\int_{\mathbb{R}^3} |\xi|^{\sigma+2}|\hat{f}(\xi)|\mathrm{d}\xi\right)^{1/2} \\ &\le \|f\|^{1/2}_{\mathcal{X}^{\sigma}}\|f\|^{1/2}_{\mathcal{X}^{\sigma+2}},\end{aligned}$$

which proves (2.1).

Second, for $\sigma \ge -1$ choosing $(p, q) = \left(\frac{\sigma+3}{\sigma+2}, \sigma + 3\right)$ and applying Hölder inequality, we obtain

$$\begin{aligned}\|f\|_{\mathcal{X}^{0}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\hat{f}(\xi)|\mathrm{d}\xi \\ &= \int_{\mathbb{R}^3} (|\xi|^{-1}|\hat{f}(\xi)|)^{\frac{\sigma+2}{\sigma+3}}(|\xi|^{\sigma+2}|\hat{f}(\xi)|)^{\frac{1}{\sigma+3}}\mathrm{d}\xi \\ &\le \left(\int_{\mathbb{R}^3} |\xi|^{-1}|\hat{f}(\xi)|\mathrm{d}\xi\right)^{\frac{\sigma+2}{\sigma+3}}\left(\int_{\mathbb{R}^3} |\xi|^{\sigma+2}|\hat{f}(\xi)|\mathrm{d}\xi\right)^{\frac{1}{\sigma+3}} \\ &\le \|f\|^{\frac{\sigma+2}{\sigma+3}}_{\mathcal{X}^{-1}}\|f\|^{\frac{1}{\sigma+3}}_{\mathcal{X}^{\sigma+2}}.\end{aligned}$$

Finally, for $\sigma \ge -1$ choosing $(p, q) = \left(\sigma + 3, \frac{\sigma+3}{\sigma+2}\right)$ and applying Hölder inequality, we obtain

$$\begin{aligned}\|f\|_{\mathcal{X}^{\sigma+1}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{\sigma+1}|\hat{f}(\xi)|\mathrm{d}\xi \\ &= \int_{\mathbb{R}^3} (|\xi|^{-1}|\hat{f}(\xi)|)^{\frac{1}{\sigma+3}}(|\xi|^{\sigma+2}|\hat{f}(\xi)|)^{\frac{\sigma+2}{\sigma+3}}\mathrm{d}\xi \\ &\le \left(\int_{\mathbb{R}^3} |\xi|^{-1}|\hat{f}(\xi)|\mathrm{d}\xi\right)^{\frac{1}{\sigma+3}}\left(\int_{\mathbb{R}^3} |\xi|^{\sigma+2}|\hat{f}(\xi)|\mathrm{d}\xi\right)^{\frac{\sigma+2}{\sigma+3}} \\ &\le \|f\|^{\frac{1}{\sigma+3}}_{\mathcal{X}^{-1}}\|f\|^{\frac{\sigma+2}{\sigma+3}}_{\mathcal{X}^{\sigma+2}}.\end{aligned}$$

□

Lemma 2.2. *Let* $\sigma \ge -1$, *we have*

$$\|f\|_{\mathcal{X}^{0,\sigma+1}(\mathbb{R}^3)} \le 2\|f\|^{1/2}_{\mathcal{X}^{-1,\sigma}(\mathbb{R}^3)}\|f\|^{1/2}_{\mathcal{X}^{1,\sigma+2}(\mathbb{R}^3)}.$$

Proof. Let $\sigma \ge -1$, we have

$$\|f\|_{\mathcal{X}^{0,\sigma+1}} = \|f\|_{\mathcal{X}^{0}} + \|f\|_{\mathcal{X}^{\sigma+1}}.$$

Using the inequality (2.1), we obtain

$$\|f\|_{\mathcal{X}^{0,\sigma+1}} \le \|f\|^{1/2}_{\mathcal{X}^{-1}}\|f\|^{1/2}_{\mathcal{X}^{1}} + \|f\|^{1/2}_{\mathcal{X}^{\sigma}}\|f\|^{1/2}_{\mathcal{X}^{\sigma+2}} \le 2\|f\|^{1/2}_{\mathcal{X}^{-1,\sigma}}\|f\|^{1/2}_{\mathcal{X}^{1,\sigma+2}}.$$

□

Lemma 2.3. *Let* $\sigma \ge -1$, *we have*

$$\|fg\|_{\mathcal{X}^{0,\sigma+1}(\mathbb{R}^3)} \le 2^{\sigma+2}\|f\|_{\mathcal{X}^{0,\sigma+1}(\mathbb{R}^3)}\|g\|_{\mathcal{X}^{0,\sigma+1}(\mathbb{R}^3)} \tag{2.4}$$

and

$$\|fg\|_{\mathcal{X}^{0,\sigma+1}(\mathbb{R}^3)} \leq 2^{\sigma+4}\|f\|^{1/2}_{\mathcal{X}^{-1,\sigma}(\mathbb{R}^3)}\|f\|^{1/2}_{\mathcal{X}^{1,\sigma+2}(\mathbb{R}^3)}\|g\|^{1/2}_{\mathcal{X}^{-1,\sigma}(\mathbb{R}^3)}\|g\|^{1/2}_{\mathcal{X}^{1,\sigma+2}(\mathbb{R}^3)}. \tag{2.5}$$

Proof. Let $\sigma \geq -1$. First, we have

$$\begin{aligned}
\|fg\|_{\mathcal{X}^{0,\sigma+1}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}(1+|\xi|^{\sigma+1})|\widehat{fg}(\xi)|\mathrm{d}\xi \\
&\leq \int_{\mathbb{R}^3}(1+|\xi|^{\sigma+1})\int_{\mathbb{R}^3}|\hat{f}(\xi-\eta)||\hat{g}(\eta)|\mathrm{d}\eta\mathrm{d}\xi \\
&\leq \int_{\mathbb{R}^3}(1+|\xi|^{\sigma+1})\left(\int_{|\eta|<|\xi-\eta|}|\hat{f}(\xi-\eta)||\hat{g}(\eta)|\mathrm{d}\eta + \int_{|\eta|>|\xi-\eta|}|\hat{f}(\xi-\eta)||\hat{g}(\eta)|\mathrm{d}\eta\right)\mathrm{d}\xi.
\end{aligned}$$

Using the inequality $(1+|\xi|^{\sigma+1}) \leq 2^{\sigma+1}(1+(\max(|\xi-\eta|,|\eta|))^{\sigma+1})$ and taking $F_1(\xi) = (1+|\xi|^{1+\sigma})|\hat{f}(\xi)|$, $F_2(\xi) = |\hat{f}(\xi)|$, $G_1(\xi) = (1+|\xi|^{1+\sigma})|\hat{g}(\xi)|$, $G_2(\xi) = |\hat{g}(\xi)|$, we get

$$\begin{aligned}
\|fg\|_{\mathcal{X}^{0,\sigma+1}(\mathbb{R}^3)} &\leq 2^{\sigma+1}\|F_1 * G_2\|_{L^1} + 2^{\sigma+1}\|G_1 * F_2\|_{L^1} \\
&\leq 2^{\sigma+1}\|F_1\|_{L^1}\|G_2\|_{L^1} + 2^{\sigma+1}\|F_2\|_{L^1}\|G_2\|_{L^1} \\
&\leq 2^{\sigma+1}\|f\|_{\mathcal{X}^{0,\sigma+1}}\|g\|_{\mathcal{X}^0} + 2^{\sigma+1}\|f\|_{\mathcal{X}^0}\|g\|_{\mathcal{X}^{0,\sigma+1}} \\
&\leq 2^{\sigma+2}\|f\|_{\mathcal{X}^{0,\sigma+1}}\|g\|_{\mathcal{X}^{0,\sigma+1}},
\end{aligned}$$

which proves (2.4).

Second, combining inequality (2.4) and Lemma (2.2), we obtain (2.5). □

Lemma 2.4. *Let* $\sigma \geq -1$, *we have*

$$\|f^2\|_{\mathcal{X}^{\sigma+1}(\mathbb{R}^3)} \leq 2^{\sigma+2}\|f\|_{\mathcal{X}^{-1}(\mathbb{R}^3)}\|f\|_{\mathcal{X}^{\sigma+2}(\mathbb{R}^3)}.$$

Proof. Let $\sigma \geq -1$, we have

$$\begin{aligned}
\|f^2\|_{\mathcal{X}^{\sigma+1}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}|\xi|^{\sigma+1}|\widehat{ff}(\xi)|\mathrm{d}\xi \\
&\leq \int_{\mathbb{R}^3}|\xi|^{\sigma+1}\left(\int_{\mathbb{R}^3}|\hat{f}(\xi-\eta)||\hat{f}(\eta)|\mathrm{d}\eta\right)\mathrm{d}\xi \\
&\leq \int_{\mathbb{R}^3}|\xi|^{\sigma+1}\left(\int_{|\eta|<|\xi-\eta|}|\hat{f}(\xi-\eta)||\hat{f}(\eta)|\mathrm{d}\eta + \int_{|\eta|>|\xi-\eta|}|\hat{f}(\xi-\eta)||\hat{f}(\eta)|\mathrm{d}\eta\right)\mathrm{d}\xi.
\end{aligned}$$

From the inequality $|\xi|^{\sigma+1} \leq 2^{\sigma+1}(\max(|\xi-\eta|,|\eta|))^{\sigma+1}$, we get

$$\begin{aligned}
\|f^2\|_{\mathcal{X}^{\sigma+1}(\mathbb{R}^3)} &\leq 2^{\sigma+1}\int\left(\int_{|\eta|<|\xi-\eta|}|\xi-\eta|^{\sigma+1}|\hat{f}(\xi-\eta)||\hat{f}(\eta)|\mathrm{d}\eta + \int_{|\eta|>|\xi-\eta|}|\hat{f}(\xi-\eta)||\eta|^{\sigma+1}|\hat{f}(\eta)|\mathrm{d}\eta\right)\mathrm{d}\xi \\
&\leq 2^{\sigma+1}\||\xi|^{\sigma+1}|\hat{f}| * |\hat{f}|\|_{L^1} + 2^{\sigma+1}\||\hat{f}| * |\xi|^{\sigma+1}|\hat{f}|\|_{L^1} \\
&\leq 2^{\sigma+2}\||\xi|^{\sigma+1}|\hat{f}|\|_{L^1}\|\hat{f}\|_{L^1} \\
&\leq 2^{\sigma+2}\|f\|_{\mathcal{X}^{\sigma+1}}\|f\|_{\mathcal{X}^0}.
\end{aligned}$$

Using inequalities (2.2) and (2.3), we obtain the desired result. □

3 Long time decay for the global solutions

In this section, we prove the main result Theorem 1.5.

For $\sigma \geq -1$ $\left(\text{we can take } \varepsilon < \frac{\nu}{2^{\sigma+3}}\right)$, let $u \in \mathcal{C}(\mathbb{R}^+, \mathcal{X}^{-1,\sigma}(\mathbb{R}^3))$ be the global solution of (NSE) given by Theorem 1.3.

First, from Theorem 1.4, we have

$$\limsup_{t\to\infty} \|u(t)\|_{\mathcal{X}^{-1}} = 0.$$

Then, for $\varepsilon > 0$ there is $t_0 > 0$ such that

$$\|u(t)\|_{\mathcal{X}^{-1}} < \varepsilon, \quad \forall t \geq t_0. \tag{3.1}$$

As

$$\|u(t)\|_{\mathcal{X}^{-1,\sigma}} = \|u(t)\|_{\mathcal{X}^{-1}} + \|u(t)\|_{\mathcal{X}^{\sigma}},$$

it suffices to prove that

$$\limsup_{t\to\infty} \|u(t)\|_{\mathcal{X}^{\sigma}} = 0.$$

On the other hand, we have

$$\partial_t u - \nu\Delta u + u\cdot\nabla u = -\nabla p.$$

Taking the Fourier transform with respect to the space variable, we obtain

$$\partial_t \hat{u}(t,\xi) + \nu|\xi|^2\hat{u}(t,\xi) + (\widehat{u\cdot\nabla u})(t,\xi) = 0.$$

Multiplying by $\bar{\hat{u}}(t,\xi)$, we get

$$\partial_t \hat{u}(t,\xi)\cdot\bar{\hat{u}}(t,\xi) + \nu|\xi|^2\hat{u}(t,\xi)\cdot\bar{\hat{u}}(t,\xi) + (\widehat{u\cdot\nabla u})(t,\xi)\cdot\bar{\hat{u}}(t,\xi) = 0.$$

This implies

$$\partial_t|\hat{u}(t,\xi)|^2 + 2\nu|\xi|^2|\hat{u}(t,\xi)|^2 + 2\mathrm{Re}((\widehat{u\cdot\nabla u})(t,\xi)\cdot\bar{\hat{u}}(t,\xi)) = 0$$

and

$$\partial_t|\hat{u}(t,\xi)|^2 + 2\nu|\xi|^2|\hat{u}(t,\xi)|^2 \leq 2\mathrm{Re}|(\widehat{u\cdot\nabla u})(t,\xi)||\bar{\hat{u}}(t,\xi)|.$$

Let $\varepsilon > 0$, we have

$$\partial_t|\hat{u}(t,\xi)|^2 = \partial_t(|\hat{u}(t,\xi)|^2 + \varepsilon) = 2\sqrt{|\hat{u}(t,\xi)|^2+\varepsilon}\cdot\partial_t\sqrt{|\hat{u}(t,\xi)|^2+\varepsilon}.$$

Then

$$\partial_t\sqrt{|\hat{u}(t,\xi)|^2+\varepsilon} + \nu|\xi|^2\frac{|\hat{u}(t,\xi)|^2}{\sqrt{|\hat{u}(t,\xi)|^2+\varepsilon}} \leq |(\widehat{u\cdot\nabla u})(t,\xi)|\frac{|\bar{\hat{u}}(t,\xi)|}{\sqrt{|\hat{u}(t,\xi)|^2+\varepsilon}} \leq |(\widehat{u\cdot\nabla u})(t,\xi)|.$$

Integrating over (t_0, t), we obtain

$$\sqrt{|\hat{u}(t,\xi)|^2+\varepsilon} + \nu|\xi|^2\int_{t_0}^{t}\frac{|\hat{u}(\tau,\xi)|^2}{\sqrt{|\hat{u}(\tau,\xi)|^2+\varepsilon}}d\tau \leq \sqrt{|\hat{u}(t_0,\xi)|^2+\varepsilon} + \int_{t_0}^{t}|(\widehat{u\cdot\nabla u})(\tau,\xi)|d\tau.$$

Letting $\varepsilon \to 0$, we get

$$|\hat{u}(t,\xi)| + \nu|\xi|^2\int_{t_0}^{t}|\hat{u}(\tau,\xi)|d\tau \leq |\hat{u}(t_0,\xi)| + \int_{t_0}^{t}|(\widehat{u\cdot\nabla u})(\tau,\xi)|d\tau.$$

Multiplying by $|\xi|^{\sigma}$ and integrating with respect to ξ, we obtain

$$\|u(t)\|_{\mathcal{X}^{\sigma}} + \nu \int_{t_0}^{t} \|\Delta u\|_{\mathcal{X}^{\sigma}} d\tau \leq \|u(t_0)\|_{\mathcal{X}^{\sigma}} + \int_{t_0}^{t} \|(u \cdot \nabla u)(\tau, \xi)\|_{\mathcal{X}^{\sigma}} d\tau.$$

Using the fundamental property $u \cdot \nabla v = \operatorname{div}(u \otimes v)$ if $\operatorname{div} u = 0$, we get

$$\|u(t)\|_{\mathcal{X}^{\sigma}} + \nu \int_{t_0}^{t} \|\Delta u\|_{\mathcal{X}^{\sigma}} d\tau \leq \|u(t_0)\|_{\mathcal{X}^{\sigma}} + \int_{t_0}^{t} \|\operatorname{div}(u \otimes u)\|_{\mathcal{X}^{\sigma}} d\tau \leq \|u(t_0)\|_{\mathcal{X}^{\sigma}} + \int_{t_0}^{t} \|u \otimes u\|_{\mathcal{X}^{\sigma+1}} d\tau.$$

Lemma (2.4) yields

$$\|u(t)\|_{\mathcal{X}^{\sigma}} + \nu \int_{t_0}^{t} \|\Delta u\|_{\mathcal{X}^{\sigma}} d\tau \leq \|u(t_0)\|_{\mathcal{X}^{\sigma}} + 2^{\sigma+2} \int_{t_0}^{t} \|u\|_{\mathcal{X}^{-1}} \|\Delta u\|_{\mathcal{X}^{\sigma}}.$$

From inequality (3.1), we obtain

$$\|u(t)\|_{\mathcal{X}^{\sigma}} + \frac{\nu}{2} \int_{t_0}^{t} \|\Delta u\|_{\mathcal{X}^{\sigma}} d\tau \leq \|u(t_0)\|_{\mathcal{X}^{\sigma}}, \quad \forall t \geq t_0. \tag{3.2}$$

Now, for $t \geq t_0$, we have

$$\|u(t)\|_{\mathcal{X}^{\sigma}} = \int_{\mathbb{R}^3} |\xi|^{\sigma} |\hat{u}(t, \xi)| d\xi = I_1(t) + I_2(t),$$

with

$$I_1(t) = \int_{|\xi|<1} |\xi|^{\sigma} |\hat{u}(t, \xi)| d\xi$$

$$I_2(t) = \int_{|\xi|>1} |\xi|^{\sigma} |\hat{u}(t, \xi)| d\xi.$$

First, using inequality (3.1), we obtain

$$I_1(t) = \int_{|\xi|<1} |\xi|^{\sigma} |\hat{u}(t, \xi)| d\xi < \int_{|\xi|<1} |\xi|^{-1} |\hat{u}(t, \xi)| d\xi \leq \|u(t)\|_{\mathcal{X}^{-1}} < \varepsilon/2. \tag{3.3}$$

Second, we have

$$I_2(t) = \int_{|\xi|>1} |\xi|^{\sigma} |\hat{u}(t, \xi)| d\xi \leq \int_{|\xi|>1} |\xi|^{\sigma+2} |\hat{u}(t, \xi)| d\xi \leq \|\Delta u\|_{\mathcal{X}^{\sigma}},$$

then

$$\int_{t_0}^{\infty} I_2(t) dt \leq \int_{t_0}^{\infty} \|\Delta u\|_{\mathcal{X}^{\sigma}} d\tau \leq \frac{2}{\nu} \|u(t_0)\|_{\mathcal{X}^{\sigma}}.$$

As, $I_2(t) \geq 0$ and $I_2 \in L^1([t_0, \infty)) \cap C([t_0, \infty))$, then there exists $T_0 > t_0$ such that

$$I_2(T_0) < \varepsilon/2.$$

Moreover, from inequality (3.3), there exists $T_0 > t_0$ such that

$$I_1(T_0) < \varepsilon/2.$$

Thus,

$$\|u(T_0)\|_{\mathcal{X}^\sigma} = I_1(T_0) + I_2(T_0) < \varepsilon,$$

and

$$\|u(t)\|_{\mathcal{X}^\sigma} + \frac{\nu}{2}\int_0^t \|\Delta u\|_{\mathcal{X}^\sigma} d\tau \leq \|u(T_0)\|_{\mathcal{X}^\sigma} < \varepsilon, \quad \forall t \geq T_0,$$

which means

$$\|u(t)\|_{\mathcal{X}^\sigma} < \varepsilon, \quad \forall t \geq T_0,$$

and the proof is finished.

4 Stability of global solutions

In this section, we prove Theorem 1.6. This proof is done in two steps.

Step 1: We begin by recalling the following property: (see [15]).

For $\sigma \geq -1$ and $u^0 \in \mathcal{X}^{-1,\sigma}(\mathbb{R}^3)$ such that $\|u^0\|_{\mathcal{X}^{-1}} < \frac{\nu}{2^{\sigma+3}}$. Let $u \in C([0, T^*), \mathcal{X}^{-1,\sigma}(\mathbb{R}^3)) \cap L^1_{loc}([0, T^*), \mathcal{X}^{1,\sigma+2}(\mathbb{R}^3))$ be the maximal solution of (NSE). From Theorem 1.3, there exists a unique global solution $u \in C(\mathbb{R}^+, \mathcal{X}^{-1,\sigma}(\mathbb{R}^3))$ such that $u \in L^1(\mathbb{R}^+, \mathcal{X}^{1,\sigma+2}(\mathbb{R}^3))$. Moreover, $\forall t \geq 0$

$$\|u(t)\|_{\mathcal{X}^{-1,\sigma}} + \frac{\nu}{2}\int_0^t \|\Delta u\|_{\mathcal{X}^{-1,\sigma}} d\tau \leq \|u^0\|_{\mathcal{X}^{-1,\sigma}}.$$

Step 2: For $\sigma \geq -1$, let $v \in C([0, T^*), \mathcal{X}^{-1,\sigma}(\mathbb{R}^3))$ be the maximal solution of (NSE) with the initial condition v^0. We wish to prove $T^* = \infty$.

In fact from Theorem 1.3, we get $v \in L^1_{loc}([0, T^*), \mathcal{X}^{1,\sigma+2}(\mathbb{R}^3))$. Put $w = v - u$ and $w^0 = v^0 - u(0)$. We have

$$\partial_t w - \nu\Delta w + w \cdot \nabla w + u \cdot \nabla w + w \cdot \nabla u = -\nabla P.$$

Using the fundamental property $u \cdot \nabla v = \operatorname{div}(u \otimes v)$ if $\operatorname{div} u = 0$, we get

$$\partial_t w - \nu\Delta w + \operatorname{div}(w \otimes w) + \operatorname{div}(u \otimes w) + \operatorname{div}(w \otimes u) = -\nabla P.$$

Then, for $t \in [0, T^*)$

$$\begin{aligned}
&\|w(t)\|_{\mathcal{X}^{-1,\sigma}} + \nu \int_0^t \|w(\tau)\|_{\mathcal{X}^{1,\sigma+1}} d\tau \\
&\leq \|w^0\|_{\mathcal{X}^{-1,\sigma}} + \int_0^t \|\operatorname{div}(w \otimes w)\|_{\mathcal{X}^{-1,\sigma}} d\tau + \int_0^t \Big(\|\operatorname{div}(u \otimes w)\|_{\mathcal{X}^{-1,\sigma}} + \|\operatorname{div}(w \otimes u)\|_{\mathcal{X}^{-1,\sigma}}\Big) d\tau.
\end{aligned}$$

On the other hand, from inequality (2.5) we get

$$\int_0^t \|\operatorname{div}(w \otimes w)\|_{\mathcal{X}^{-1,\sigma}} d\tau \leq \int_0^t \|w \otimes w\|_{\mathcal{X}^{0,\sigma+1}} d\tau \leq 2^{\sigma+4} \int_0^t \|w\|_{\mathcal{X}^{-1,\sigma}} \|w\|_{\mathcal{X}^{1,\sigma+2}} d\tau.$$

Also, inequality (2.4), Lemma (2.2), and Young inequality yield

$$\begin{aligned}\int_0^t \left(\|\mathrm{div}(u\otimes w)\|_{\mathcal{X}^{-1,\sigma}} + \|\mathrm{div}(w\otimes u)\|_{\mathcal{X}^{-1,\sigma}}\right)\mathrm{d}\tau &\leq \int_0^t \left(\|u\otimes w\|_{\mathcal{X}^{0,\sigma+1}} + \|w\otimes u\|_{\mathcal{X}^{0,\sigma+1}}\right)\mathrm{d}\tau\\ &\leq 2^{\sigma+3}\int_0^t \|u\|_{\mathcal{X}^{0,\sigma+1}}\|w\|_{\mathcal{X}^{0,\sigma+1}}\mathrm{d}\tau\\ &\leq 2^{\sigma+4}\int_0^t \|u\|_{\mathcal{X}^{0,\sigma+1}}\|w\|^{1/2}_{\mathcal{X}^{-1,\sigma}}\|w\|^{1/2}_{\mathcal{X}^{1,\sigma+1}}\mathrm{d}\tau\\ &\leq \frac{2^{\sigma+8}}{\nu}\int_0^t \|u\|^2_{\mathcal{X}^{0,\sigma+1}}\|w\|_{\mathcal{X}^{-1,\sigma}}\mathrm{d}\tau + \frac{\nu}{4}\int_0^t \|w\|_{\mathcal{X}^{1,\sigma+1}}\mathrm{d}\tau.\end{aligned}$$

Then

$$\|w(t)\|_{\mathcal{X}^{-1,\sigma}} + \frac{3\nu}{4}\int_0^t \|w(\tau)\|_{\mathcal{X}^{1,\sigma+1}}\mathrm{d}\tau \leq \|w^0\|_{\mathcal{X}^{-1,\sigma}} + 2^{\sigma+4}\int_0^t \|w\|_{\mathcal{X}^{-1,\sigma}}\|w\|_{\mathcal{X}^{1,\sigma+2}}\mathrm{d}\tau + \frac{2^{\sigma+8}}{\nu}\int_0^t \|u\|^2_{\mathcal{X}^{0,\sigma+1}}\|w\|_{\mathcal{X}^{-1,\sigma}}\mathrm{d}\tau.$$

Put

$$T = \sup\left\{t\in[0,T^*),\ \sup_{z\in[0,t]}\|w(t)\|_{\mathcal{X}^{-1,\sigma}} < \frac{\nu}{2^{\sigma+6}}\right\}.$$

For $t\in[0,T)$, we have

$$\|w(t)\|_{\mathcal{X}^{-1,\sigma}} + \frac{\nu}{2}\int_0^t \|w(\tau)\|_{\mathcal{X}^{1,\sigma+1}}\mathrm{d}\tau \leq \|w^0\|_{\mathcal{X}^{-1,\sigma}} + \frac{2^{\sigma+8}}{\nu}\int_0^t \|u\|^2_{\mathcal{X}^{0,\sigma+1}}\|w\|_{\mathcal{X}^{-1,\sigma}}\mathrm{d}\tau.$$

Using Gronwall Lemma, we can deduce

$$\|w(t)\|_{\mathcal{X}^{-1,\sigma}} + \frac{\nu}{2}\int_0^t \|w(\tau)\|_{\mathcal{X}^{1,\sigma+1}}\mathrm{d}\tau \leq \|w^0\|_{\mathcal{X}^{-1,\sigma}}e^{\frac{2^{\sigma+8}}{\nu}\int_0^\infty \|u\|^2_{\mathcal{X}^{0,\sigma+1}}} < \frac{\nu}{2^{\sigma+7}}.$$

Then $T = T^*$ and $\int_0^{T^*}\|w(\tau)\|_{\mathcal{X}^{1,\sigma+1}}\mathrm{d}\tau < \infty$, therefore, $T^* = \infty$ and the proof is finished.

Some numerical approaches regarding similar problems are given in [19,20].

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