



## Research Article

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# On split quaternion equivalents for Quaternaccis, shortly Split Quaternaccis

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**Abstract:** In this paper, we introduce generalizations of Quaternacci sequences (Quaternaccis), called Split Quaternacci sequences, which arose on a base of split quaternion algebras. Explicit and recurrent formulae for Split Quaternacci sequences are given, as well as generating functions. Also, matrices related to Split Quaternaccis sequences are investigated. Moreover, new identities connecting Horadam sequences with other known sequences are generated. Analogous identities for Horadam quaternions and split Horadam quaternions are proved.

**Keywords:** split quaternions, quaternions, quasi-Fibonacci numbers, Quaternaccis, Split Quaternaccis

**MSC 2020:** 11B83, 11E88, 11R52, 11B37, 11B39

## 1 Introduction

In [1], we introduced new sequences called Quaternaccis. The idea of Quaternaccis arose during research on two-parametric quasi-Fibonacci numbers of 7th and 9th order [2] (see also [3], where some basic ideas were presented). For instance, two-parametric quasi-Fibonacci numbers of 7th order are members of sequences  $(A_{n,7}(\delta, \gamma))$ ,  $(B_{n,7}(\delta, \gamma))$  and  $(C_{n,7}(\delta, \gamma))$  defined by the following relations:

$$(1 + \delta(\zeta^k + \zeta^{6k}) + \gamma(\zeta^{2k} + \zeta^{5k}))^n = A_{n,7}(\delta, \gamma) + B_{n,7}(\delta, \gamma)(\zeta^k + \zeta^{6k}) + C_{n,7}(\delta, \gamma)(\zeta^{2k} + \zeta^{5k}),$$

where  $k \in \mathbb{N} \setminus 7\mathbb{N}$ ,  $n \in \mathbb{N}_0$ ,  $\delta, \gamma \in \mathbb{C}$  and  $\zeta \in \mathbb{C}$  is a primitive 7th root of unity. From one side  $n$ th roots of unity form a cyclic group under multiplication, and from the other some sums of these roots are linearly independent (like  $1$ ,  $\zeta^k + \zeta^{6k}$  and  $\zeta^{2k} + \zeta^{5k}$  used in the above definition). That made us extend the idea of quasi-Fibonacci numbers to real quaternions (more about quaternion algebra can be found e.g. in [4]).

**Definition 1.** [1, Definition 1] Quaternacci sequences (shortly Quaternaccis)  $A_n(b, c, d)$ ,  $B_n(b, c, d)$ ,  $C_n(b, c, d)$ ,  $D_n(b, c, d)$  are defined by the following relations:

$$(1 + bi + cj + dk)^n = A_n(b, c, d) + B_n(b, c, d)i + C_n(b, c, d)j + D_n(b, c, d)k,$$

where  $n \in \mathbb{N}_0$ ,  $b, c, d \in \mathbb{R}$  and the set  $\{1, i, j, k\}$  forms a basis of quaternion algebra.

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Thus, from Definition 1 we obtain an explicit formula for powers of quaternions. During our investigations we obtained a lot of interesting results, also on quaternion structure itself [1]. For example, each of the four families of Quaternacci sequences satisfies the same recurrent relation:

$$\mathbb{X}_n = 2\mathbb{X}_{n-1} - (1 + b^2 + c^2 + d^2)\mathbb{X}_{n-2}, \quad n \geq 2, \tag{1}$$

but with different initial conditions

$$\begin{aligned} A_0(b, c, d) = 1, \quad A_1(b, c, d) = 1, \quad B_0(b, c, d) = 0, \quad B_1(b, c, d) = b, \\ C_0(b, c, d) = 0, \quad C_1(b, c, d) = c, \quad D_0(b, c, d) = 0, \quad D_1(b, c, d) = d, \end{aligned}$$

which led us to explicit formulae for Quaternaccis for  $bcd \neq 0$ :

$$\begin{aligned} A_n &= \frac{(1 - i\mu)^n + (1 + i\mu)^n}{(1 - i\mu) + (1 + i\mu)} = \frac{1}{2}((1 - i\mu)^n + (1 + i\mu)^n), \\ \frac{1}{b}B_n &= \frac{1}{c}C_n = \frac{1}{d}D_n = \frac{(1 - i\mu)^n - (1 + i\mu)^n}{(1 - i\mu) - (1 + i\mu)} = \frac{i}{2\mu}((1 - i\mu)^n - (1 + i\mu)^n), \end{aligned}$$

where  $\mu = \sqrt{b^2 + c^2 + d^2}$ .

We decided to introduce the aforementioned notions in the context of (real) split quaternions.

The aim of the first part of this paper is to present the results of our investigations. In the second part, we generate some new identities, called “bridges,” connecting sequences in question with the other known sequences (we give analogous identities for Quaternaccis as well).

## 2 Preliminaries

The split quaternion algebra (see [5]) was introduced by James Cockle in 1849 and it can be defined as follows.

**Definition 2.** Let  $\widehat{H}$  be a four-dimensional vector space over  $\mathbb{R}$  with the basis  $\{1, i, j, k\}$ . Then a (real) split quaternion  $q$  is an element of  $\widehat{H}$  written with respect to this basis, that is,

$$q := a_0 + a_1i + b_1j + c_1k,$$

where  $a_0, a_1, a_2, a_3 \in \mathbb{R}$ .

Multiplication of basis vectors is given by the following rules:

$$\begin{aligned} i^2 = -1, \quad j^2 = 1, \quad k^2 = 1 \quad j^2 = k^2 = ijk = 1, \\ ij = -ji = k, \quad jk = -kj = -i, \quad ki = -ik = j, \end{aligned}$$

and is extended to  $\widehat{H}$  by assuming that it is associative, distributive over addition and commutative with respect to scalar multiplication. Moreover, an operation  $\bar{\omega} : \widehat{H} \rightarrow \widehat{H}$ , called conjugation, is defined as follows:

$$\bar{q} = \overline{a_0 + a_1i + a_2j + a_3k} = a_0 - a_1i - a_2j - a_3k.$$

We also define norm of any split quaternion  $q$  in the following way:

$$N = N(q) = \sqrt{|q\bar{q}|} = \sqrt{|a_0^2 + a_1^2 - a_2^2 - a_3^2|}.$$

If  $N(q)$  is equal to 1, then  $q$  is called a unit split quaternion. Split quaternion algebra is an associative, non-commutative, nondivision ring. Unlike quaternion algebra, split quaternion algebra contains zero divisors, nilpotent elements and nontrivial idempotents.

The split quaternion  $q$  is called spacelike, timelike or lightlike, if  $q\bar{q} < 0$ ,  $q\bar{q} > 0$  or  $q\bar{q} = 0$ , respectively. Polar form of the split quaternion  $q = a_0 + a_1i + a_2j + a_3k$  is known (see [5]). Let us denote scalar and vector part of the split quaternion  $q$  by  $S_q = a_0$  and  $V_q = a_1i + a_2j + a_3k$ , respectively, then:

1. Every spacelike quaternion can be written in the form:

$$q = N(q)(\sinh \theta + \vec{w} \cosh \theta),$$

where  $\sinh \theta = \frac{a_0}{N(q)}$ ,  $\cosh \theta = \frac{\sqrt{-a_1^2 + a_2^2 + a_3^2}}{N(q)}$  and  $\vec{w} = \frac{a_1i + a_2j + a_3k}{\sqrt{-a_1^2 + a_2^2 + a_3^2}}$ .

2. Every timelike quaternion with spacelike vector part (that is  $V_q \overline{V_q} < 0$ ) can be written in the form:

$$q = N(q)(\cosh \theta + \vec{w} \sinh \theta),$$

where  $\cosh \theta = \frac{a_0}{N(q)}$ ,  $\sinh \theta = \frac{\sqrt{-a_1^2 + a_2^2 + a_3^2}}{N(q)}$  and  $\vec{w} = \frac{a_1i + a_2j + a_3k}{\sqrt{-a_1^2 + a_2^2 + a_3^2}}$ .

3. Every timelike quaternion with timelike vector part (that is  $V_q \overline{V_q} > 0$ ) can be written in the form:

$$q = N(q)(\cos \theta + \vec{w} \sin \theta),$$

where  $\cos \theta = \frac{a_0}{N(q)}$ ,  $\sin \theta = \frac{\sqrt{-a_1^2 + a_2^2 + a_3^2}}{N(q)}$  and  $\vec{w} = \frac{a_1i + a_2j + a_3k}{\sqrt{a_1^2 - a_2^2 - a_3^2}}$ .

The concept of real quaternions and real split quaternions can be generalized – the idea is to extend the algebra of coefficients beyond real numbers. For example, in [6] complex quaternions and complex split quaternions were investigated and in [7] split quaternions with quaternion coefficients and quaternions with dual coefficients were discussed.

### 3 Split Quaternaccis and their basic properties

In this section, we define Split Quaternacci sequences and we present their basic properties. It is worth pointing out that our ideas can be transferred to other similar algebras, such as e.g. dual quaternions, bicomplex numbers, octonions, Clifford algebras (see Section 5).

**Definition 3.** Split Quaternacci sequences (shortly Split Quaternaccis) are members of sequences  $(A_n(\beta, \gamma, \delta))$ ,  $(B_n(\beta, \gamma, \delta))$ ,  $(C_n(\beta, \gamma, \delta))$  and  $(D_n(\beta, \gamma, \delta))$  defined by the following relation:

$$(1 + \beta i + \gamma j + \delta k)^n = A_n(\beta, \gamma, \delta) + B_n(\beta, \gamma, \delta)i + C_n(\beta, \gamma, \delta)j + D_n(\beta, \gamma, \delta)k,$$

where  $n \in \mathbb{N}_0$ ,  $\beta, \gamma, \delta \in \mathbb{R}$  and the set  $\{1, i, j, k\}$  forms a basis of split quaternion algebra (Table 1).

**Table 1:** Table of Split Quaternaccis for  $n \leq 5$ ,  $(x, X) \in \{(\beta, B), (\gamma, C), (\delta, D)\}$

$n$	$A_n(\beta, \gamma, \delta)$	$X_n(\beta, \gamma, \delta)$
0	1	0
1	1	x
2	$1 + \beta^2 - \gamma^2 + \delta^2$	2x
3	$1 + 3\beta^2 - 3\gamma^2 + 3\delta^2$	$x(3 + \beta^2 - \gamma^2 + \delta^2)$
4	$1 + 6\beta^2 + \beta^4 - 6\gamma^2 - 2\beta^2\gamma^2 + \gamma^4$ $+ 6\delta^2 + 2\beta^2\delta^2 - 2\gamma^2\delta^2 + \delta^4$	$4x(1 + \beta^2 - \gamma^2 + \delta^2)$
5	$1 + 10\beta^2 + 5\beta^4 - 10\gamma^2 - 10\beta^2\gamma^2 + 5\gamma^4$ $+ 10\delta^2 + 10\beta^2\delta^2 - 10\gamma^2\delta^2 + 5\delta^4$	$x(5 + 10\beta^2 + \beta^4 - 10\gamma^2 - 2\beta^2\gamma^2 + \gamma^4$ $+ 10\delta^2 + 2\beta^2\delta^2 - 2\gamma^2\delta^2 + \delta^4)$

**Notation.** From now on, we will use  $(\beta, \gamma, \delta)$  only to denote arguments of Split Quaternaccis and  $i, j, k$  to denote the respective basis vectors of split quaternion algebra. Moreover, while describing properties valid for any argument  $(\beta, \gamma, \delta)$  we shall omit it in formulation of results, in particular we will use the following abbreviations:

$$\begin{aligned} \mathbf{A}_n &= \mathbf{A}_n(\beta, \gamma, \delta), & \mathbf{B}_n &= \mathbf{B}_n(\beta, \gamma, \delta), \\ \mathbf{C}_n &= \mathbf{C}_n(\beta, \gamma, \delta), & \mathbf{D}_n &= \mathbf{D}_n(\beta, \gamma, \delta). \end{aligned}$$

Also, we set

$$\lambda := \sqrt{-\beta^2 + \gamma^2 + \delta^2}. \quad (2)$$

Some results in this section are pretty straightforward generalizations of those from [1], so we shall just give sketches of proofs.

**Theorem 1.** For all  $n \in \mathbb{N}_0$  the following recurrence relation holds:

$$\begin{bmatrix} \mathbf{A}_{n+1} \\ \mathbf{B}_{n+1} \\ \mathbf{C}_{n+1} \\ \mathbf{D}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -\beta & \gamma & \delta \\ \beta & 1 & -\delta & \gamma \\ \gamma & -\delta & 1 & \beta \\ \delta & \gamma & -\beta & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_n \\ \mathbf{B}_n \\ \mathbf{C}_n \\ \mathbf{D}_n \end{bmatrix}, \quad (3)$$

where  $\beta, \gamma, \delta \in \mathbb{R}$  and  $\mathbf{A}_0 = 1, \mathbf{B}_0 = 0, \mathbf{C}_0 = 0, \mathbf{D}_0 = 0$ .

**Proof.** By induction on  $n$ . □

**Proposition 1.** For all  $m, n \in \mathbb{N}_0$ , the following relations hold:

$$\beta \mathbf{D}_n = \delta \mathbf{B}_n, \quad \gamma \mathbf{B}_n = \beta \mathbf{C}_n, \quad \delta \mathbf{C}_n = \gamma \mathbf{D}_n, \quad (4)$$

and

$$\mathbf{B}_m \mathbf{D}_n = \mathbf{D}_m \mathbf{B}_n, \quad \mathbf{C}_m \mathbf{B}_n = \mathbf{B}_m \mathbf{C}_n, \quad \mathbf{D}_m \mathbf{C}_n = \mathbf{C}_m \mathbf{D}_n. \quad (5)$$

**Proof.** Since split quaternion algebra is associative, for every  $w \in \widehat{\mathbb{H}}$  we have  $w^{n+1} = w^n w = w w^n$  so the result follows from Definition 3 and rules of multiplication in  $\widehat{\mathbb{H}}$ . □

From (3) by (4) we obtain two more recurrence relations for Split Quaternaccis:

**Proposition 2.** For all  $n \in \mathbb{N}_0$ , the following relations hold:

$$\begin{bmatrix} \mathbf{A}_{n+1} \\ \mathbf{B}_{n+1} \\ \mathbf{C}_{n+1} \\ \mathbf{D}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -\beta & \gamma & \delta \\ \beta & 1 & \delta & -\gamma \\ \gamma & \delta & 1 & -\beta \\ \delta & -\gamma & \beta & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_n \\ \mathbf{B}_n \\ \mathbf{C}_n \\ \mathbf{D}_n \end{bmatrix} \quad (6)$$

and

$$\begin{bmatrix} \mathbf{A}_{n+1} \\ \mathbf{B}_{n+1} \\ \mathbf{C}_{n+1} \\ \mathbf{D}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -\beta & \gamma & \delta \\ \beta & 1 & 0 & 0 \\ \gamma & 0 & 1 & 0 \\ \delta & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_n \\ \mathbf{B}_n \\ \mathbf{C}_n \\ \mathbf{D}_n \end{bmatrix}. \quad (7)$$

**Remark 1.** In view of Proposition 1, transition matrices in (3), (6) and (7) can be generalized, namely for any parameters  $a_1, a_2, a_3 \in \mathbb{R}$  we have:

$$\begin{bmatrix} \mathbf{A}_{n+1} \\ \mathbf{B}_{n+1} \\ \mathbf{C}_{n+1} \\ \mathbf{D}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -\beta & \gamma & \delta \\ \beta & 1 & -a_1 & a_2 \\ \gamma & -a_1 & 1 & -a_3 \\ \delta & a_2 & a_3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_n \\ \mathbf{B}_n \\ \mathbf{C}_n \\ \mathbf{D}_n \end{bmatrix}. \tag{8}$$

In the next theorem, we collect the basic properties of transition matrices in (3) and (6). Note that they differ only by signs of some entries.

**Theorem 2.** For all  $m, n \in \mathbb{N}_0$ , the following relations hold:

(a)

$$\begin{bmatrix} \mathbf{A}_{n+m} \\ \mathbf{B}_{n+m} \\ \mathbf{C}_{n+m} \\ \mathbf{D}_{n+m} \end{bmatrix} = \begin{bmatrix} 1 & -\beta & \gamma & \delta \\ \beta & 1 & \mp\delta & \pm\gamma \\ \gamma & \mp\delta & 1 & \pm\beta \\ \delta & \pm\gamma & \mp\beta & 1 \end{bmatrix}^n \begin{bmatrix} \mathbf{A}_m \\ \mathbf{B}_m \\ \mathbf{C}_m \\ \mathbf{D}_m \end{bmatrix},$$

(b)

$$\begin{bmatrix} 1 & -\beta & \gamma & \delta \\ \beta & 1 & \mp\delta & \pm\gamma \\ \gamma & \mp\delta & 1 & \pm\beta \\ \delta & \pm\gamma & \mp\beta & 1 \end{bmatrix}^n = \begin{bmatrix} \mathbf{A}_n & -\mathbf{B}_n & \mathbf{C}_n & \mathbf{D}_n \\ \mathbf{B}_n & \mathbf{A}_n & \mp\mathbf{D}_n & \pm\mathbf{C}_n \\ \mathbf{C}_n & \mp\mathbf{D}_n & \mathbf{A}_n & \pm\mathbf{B}_n \\ \mathbf{D}_n & \pm\mathbf{C}_n & \mp\mathbf{B}_n & \mathbf{A}_n \end{bmatrix}, \tag{9}$$

(c) If  $\lambda \neq 0$  (defined in (2)), then we have the following Jordan decomposition:

$$\begin{bmatrix} 1 & -\beta & \gamma & \delta \\ \beta & 1 & \mp\delta & \pm\gamma \\ \gamma & \mp\delta & 1 & \pm\beta \\ \delta & \pm\gamma & \mp\beta & 1 \end{bmatrix} = P \operatorname{diag} \{1 - \lambda, 1 - \lambda, 1 + \lambda, 1 + \lambda\} P^{-1}, \tag{10}$$

where:

$$P = \begin{bmatrix} \frac{\mp\beta\gamma - \delta\lambda}{\gamma^2 + \delta^2} & \frac{\pm\beta\delta - \gamma\lambda}{\gamma^2 + \delta^2} & \frac{\mp\beta\gamma + \delta\lambda}{\gamma^2 + \delta^2} & \frac{\pm\beta\delta + \gamma\lambda}{\gamma^2 + \delta^2} \\ \frac{\beta\delta \mp \gamma\lambda}{\gamma^2 + \delta^2} & \frac{\beta\gamma \pm \delta\lambda}{\gamma^2 + \delta^2} & \frac{\beta\delta \pm \gamma\lambda}{\gamma^2 + \delta^2} & \frac{\beta\gamma \mp \delta\lambda}{\gamma^2 + \delta^2} \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

**Proof.**

(a) Follows from  $n$  successive applications of (3) or (6).

(b) By induction on  $n$ , using Proposition 1 and (a).

(c) From definition. □

**Remark 2.** Note that transition matrices in (3) and (6) have an interesting property which can be seen in point (b) of the aforementioned theorem. Namely, the matrix on the right results from that on the left by substituting  $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ , for  $1, \beta, \gamma, \delta$ , respectively. We noted this phenomenon for Quaternaccis [1]. Moreover, an analogous property is true for two-parameter quasi-Fibonacci numbers of order 7 and 9 (see [2]).

In particular that means that each entry in the first column (and row) of every power of these matrices contains only members of one of the sequences. Thus, we can separate the sequences using minimal poly-

nomials of these matrices (which are of degree 2 by (c)). More precisely, thanks to minimal polynomials and the above remark we obtain a linear relation between members of each sequence separately (the same for each of them), which proves the following theorem.

**Theorem 3.** For  $\lambda \neq 0$  (defined in (2)), all Split Quaternacci sequences  $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$  satisfy (the same) recurrence relation:

$$\mathbb{X}_n = 2\mathbb{X}_{n-1} - (1 - \lambda^2)\mathbb{X}_{n-2}, \quad n > 2, \tag{11}$$

with the following initial conditions:

$$\mathbf{A}_0 = 1, \quad \mathbf{A}_1 = 1, \quad \mathbf{B}_0 = 0, \quad \mathbf{B}_1 = \beta, \quad \mathbf{C}_0 = 0, \quad \mathbf{C}_1 = \gamma, \quad \mathbf{D}_0 = 0, \quad \mathbf{D}_1 = \delta.$$

**Corollary 1.** For  $\beta\gamma\delta \neq 0, \lambda \neq 0$  we have the following Binet formulae for Split Quaternaccis

$$\mathbf{A}_n = \frac{1}{2}((1 - \lambda)^n + (1 + \lambda)^n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \lambda^{2k}, \tag{12}$$

$$\frac{1}{\beta}\mathbf{B}_n = \frac{1}{\gamma}\mathbf{C}_n = \frac{1}{\delta}\mathbf{D}_n = \frac{1}{2\lambda}((1 + \lambda)^n - (1 - \lambda)^n) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \lambda^{2k}. \tag{13}$$

Moreover, for  $(x, X) \in \{(\beta, B), (\gamma, C), (\delta, D)\}$ , if  $\lambda = 0$ , then  $\mathbf{A}_n = 1$  for any positive integer  $n$  and  $\mathbf{X}_n = nx$ , and if  $\lambda \neq 0$  but  $x = 0$ , then  $\mathbf{X}_n = 0$  for any positive integer  $n$ .

We can obtain the aforementioned formulae (and analogous formulae for Quaternaccis) also using generating functions.

**Proposition 3.** The ordinary generating functions  $F_A, F_B, F_C, F_D$  of Split Quaternaccis  $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ , respectively (or Quaternaccis  $A_n(\beta, \gamma, \delta), B_n(\beta, \gamma, \delta), C_n(\beta, \gamma, \delta), D_n(\beta, \gamma, \delta)$ , respectively), are

$$F_A(t) = \frac{1 - t}{(1 - \lambda^2)t^2 - 2t + 1}, \tag{14}$$

$$F_B(t) = \beta \cdot g(t), \quad F_C(t) = \gamma \cdot g(t), \quad F_D(t) = \delta \cdot g(t), \tag{15}$$

with  $g(t) = \frac{t}{(1 - \lambda^2)t^2 - 2t + 1}$  and  $\lambda = \sqrt{-\beta^2 + \gamma^2 + \delta^2}$  (or  $\lambda = \sqrt{-\beta^2 - \gamma^2 - \delta^2}$  in case of Quaternaccis).

**Proof.** From (11) we get

$$\begin{aligned} F_A(t) &= 1 + t + \sum_{n=2}^{\infty} (2\mathbf{A}_{n-1} - (1 - \lambda^2)\mathbf{A}_{n-2})t^n \\ &= 1 + t + 2t \sum_{n=2}^{\infty} \mathbf{A}_{n-1}t^{n-1} - (1 - \lambda^2)t^2 \sum_{n=2}^{\infty} \mathbf{A}_{n-2}t^{n-2} \\ &= 1 + t + 2t(F_A(t) - 1) - (1 - \lambda^2)t^2F_A(t), \end{aligned}$$

from which we get (14). Formulae (15) can be derived in the same manner. □

**Remark 3.** Theorem 3 also gives the connection between Split Quaternacci and Lucas sequences. Recall that (general) Lucas sequences of the first kind  $(U_n(P, Q))$  and the second kind  $(V_n(P, Q))$  where  $P, Q \in \mathbb{C}$  are defined by the same recurrence relation

$$\mathbb{X}_n(P, Q) = P\mathbb{X}_{n-1}(P, Q) - Q\mathbb{X}_{n-2}(P, Q), \tag{16}$$

with initial conditions  $U_0(P, Q) = 0, U_1(P, Q) = 1, V_0(P, Q) = 2, V_1(P, Q) = P$ .

First, under proper assumptions on  $\gamma$  and  $\delta$  we have

$$\mathbf{B}_n(1, \gamma, \delta) = U_n(2, 2 + \gamma + \delta).$$

Second, for  $\lambda = \sqrt{5}$  and  $(x, X) \in \{(\beta, B), (\gamma, C), (\delta, D)\}$  we have

$$x2^{n-1}F_n = \mathbf{X}_n \quad \text{and} \quad 2^{n-1}L_n = \mathbf{A}_n,$$

where  $F_n$  are Fibonacci numbers and  $L_n$  – Lucas numbers.

To prove the analogue of Theorem 2 for matrix in (7) we need the following fact.

**Proposition 4.** For all  $\beta, \gamma, \delta \in \mathbb{R}$  such that  $\beta\gamma\delta \neq 0, \lambda \neq 0$  and  $n \in \mathbb{N}$  the following summation formulae are valid:

$$\frac{1}{\beta} \sum_{k=0}^{n-1} \mathbf{B}_k = \frac{1}{\gamma} \sum_{k=0}^{n-1} \mathbf{C}_k = \frac{1}{\delta} \sum_{k=0}^{n-1} \mathbf{D}_k = \frac{\mathbf{A}_n - 1}{\lambda^2}, \tag{17}$$

$$\sum_{k=0}^{n-1} \mathbf{A}_k = \frac{1}{\beta} \mathbf{B}_n = \frac{1}{\gamma} \mathbf{C}_n = \frac{1}{\delta} \mathbf{D}_n. \tag{18}$$

**Proof.** By induction on  $n$  using (3) and (4). It can also be proved in an elegant way on a base of Theorem 3 and telescoping summing (cf. [1, Corollary 2]). □

**Theorem 4.**

(a) For all  $n \in \mathbb{N}, \beta, \gamma, \delta \in \mathbb{R}, \lambda \neq 0$  the following identity holds:

$$\begin{bmatrix} 1 & -\beta & \gamma & \delta \\ \beta & 1 & 0 & 0 \\ \gamma & 0 & 1 & 0 \\ \delta & 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} \mathbf{A}_n & -\mathbf{B}_n & \mathbf{C}_n & \mathbf{D}_n \\ \mathbf{B}_n & 1 - \beta^2 \mathbf{E}_n & \beta\gamma \mathbf{E}_n & \beta\delta \mathbf{E}_n \\ \mathbf{C}_n & -\beta\gamma \mathbf{E}_n & 1 + \gamma^2 \mathbf{E}_n & \gamma\delta \mathbf{E}_n \\ \mathbf{D}_n & -\beta\delta \mathbf{E}_n & \gamma\delta \mathbf{E}_n & 1 + \delta^2 \mathbf{E}_n \end{bmatrix}, \tag{19}$$

where  $\mathbf{E}_n := \begin{cases} \frac{\mathbf{A}_n - 1}{\lambda^2}, & \beta\gamma\delta \neq 0, \\ 0, & \beta\gamma\delta = 0. \end{cases}$

(b) The transition matrix in (7) is diagonalizable and have three eigenvalues:  $1 - \lambda, 1 + \lambda$  (single) and  $1$  (double).

Although there are explicit forms of Split Quaternaccis, some reduction formulae can also be useful.

**Theorem 5.** For all  $n, m \in \mathbb{N}$  and  $X \in \{B, C, D\}$  the following reduction formulae are valid

$$\mathbf{A}_{m+n} = \mathbf{A}_m \mathbf{A}_n - \mathbf{B}_m \mathbf{B}_n + \mathbf{C}_m \mathbf{C}_n + \mathbf{D}_m \mathbf{D}_n, \tag{20}$$

$$\mathbf{X}_{m+n} = \mathbf{A}_m \mathbf{X}_n + \mathbf{A}_n \mathbf{X}_m, \tag{21}$$

$$\mathbf{X}_{m \cdot n} = (\mathbf{A}_m)^n \mathbf{X}_n \begin{pmatrix} \mathbf{B}_m & \mathbf{C}_m & \mathbf{D}_m \\ \mathbf{A}_m & \mathbf{A}_m & \mathbf{A}_m \end{pmatrix}. \tag{22}$$

**Remark 4.** Formulae given in Theorem 5 can be used to simplify a problem of finding values of different number sequences for large indices. For instance, Professor W. Webb in 2008, during the 13th International Conference on Fibonacci Numbers and Their Applications in Patras, posed a question of finding compact formulae for sums of the form  $\sum_{k=1}^N F_{k^r}$ , where  $r \in \mathbb{N}$ . Because of Remark 3, formulae in Theorem 5 can be used to simplify the above and other similar problems.

**Corollary 2.** For  $X \in \{B, C, D\}$ ,  $n \in \mathbb{N}$  in particular we have

$$\mathbf{A}_{2n} = (1 + \lambda^2)^n \mathbf{A}_n \left( \frac{2\beta}{1 + \lambda^2}, \frac{2\gamma}{1 + \lambda^2}, \frac{2\delta}{1 + \lambda^2} \right) = (\mathbf{A}_n)^2 - (\mathbf{B}_n)^2 + (\mathbf{C}_n)^2 + (\mathbf{D}_n)^2, \quad (23)$$

$$\mathbf{X}_{2n} = (1 + \lambda^2)^n \mathbf{X}_n \left( \frac{2\beta}{1 + \lambda^2}, \frac{2\gamma}{1 + \lambda^2}, \frac{2\delta}{1 + \lambda^2} \right) = 2\mathbf{A}_n \mathbf{X}_n. \quad (24)$$

From Binet formulae for Split Quaternaccis given in Corollary 1 we can derive some reduction formulae also for parameters.

**Theorem 6.** If  $\beta\gamma\delta \neq 0$ ,  $\beta, \gamma > 0$ ,  $\lambda \neq 0$ ,  $n \in \mathbb{N}$  and  $X \in \{B, C, D\}$ , then we have for instance

$$\mathbf{A}_n(\gamma - \beta, \delta, \gamma - \beta) = \mathbf{A}_n(\sqrt{2}\beta, \delta, \sqrt{\gamma}), \quad (25)$$

$$\mathbf{B}_n(\gamma - \beta, \delta, \gamma - \beta) = \frac{\gamma - \beta}{\sqrt{2}\beta} \mathbf{B}_n(\sqrt{2}\beta, \delta, \sqrt{\gamma}), \quad (26)$$

$$\mathbf{C}_n(\gamma - \beta, \delta, \gamma - \beta) = \mathbf{C}_n(\sqrt{2}\beta, \delta, \sqrt{\gamma}), \quad (27)$$

$$\mathbf{D}_n(\gamma - \beta, \delta, \gamma - \beta) = \frac{\gamma - \beta}{\sqrt{2}\gamma} \mathbf{D}_n(\sqrt{2}\beta, \delta, \sqrt{\gamma}). \quad (28)$$

Moreover, by Binet formulae for Split Quaternaccis we obtained the following interesting identities:

$$\mathbf{A}_n \mathbf{A}_m - \mathbf{A}_{n-r} \mathbf{A}_{m+r} = \begin{cases} \frac{1}{2}(1 - \lambda^2)^m (\mathbf{A}_{n-m} - (1 - \lambda^2)^{n-r-m} \mathbf{A}_{n+2r-m}), & n < m + r, \\ \frac{1}{2}(1 - \lambda^2)^m (\mathbf{A}_{n-m} - (1 - \lambda^2)^r \mathbf{A}_{n-2r-m}), & n \geq m + r, \end{cases} \quad (29)$$

$$\mathbf{X}_n \mathbf{X}_m - \mathbf{X}_{n-r} \mathbf{X}_{m+r} = \begin{cases} (1 - \lambda^2)^{n-r} \mathbf{X}_r \mathbf{X}_{m+r-n}, & n < m + r, \\ -(1 - \lambda^2)^m \mathbf{X}_r \mathbf{X}_{n-r-m}, & n \geq m + r, \end{cases} \quad (30)$$

where  $n, m, r \in \mathbb{N}$ ,  $n > m$  and  $X \in \{B, C, D\}$ . As a special case of (29) and (30) we derive analogues of known identities

1. (Catalan identity) for  $n = m$ :

$$\begin{aligned} (\mathbf{A}_n)^2 - \mathbf{A}_{n+r} \mathbf{A}_{n-r} &= \frac{1}{2}(1 - \lambda^2)^{n-r} ((1 - \lambda^2)^r - \mathbf{A}_{2r}), \\ (\mathbf{X}_n)^2 - \mathbf{X}_{n+r} \mathbf{X}_{n-r} &= (1 - \lambda^2)^{n-r} (\mathbf{X}_r)^2; \end{aligned}$$

2. (Cassini identity) for  $n = m$  and  $r = 1$ :

$$\begin{aligned} (\mathbf{A}_n)^2 - \mathbf{A}_{n+1} \mathbf{A}_{n-1} &= -\lambda^2 (1 - \lambda^2)^{n-1}, \\ (\mathbf{X}_n)^2 - \mathbf{X}_{n+1} \mathbf{X}_{n-1} &= -(1 - \lambda^2)^{n-1}; \end{aligned}$$

3. (d'Ocagne identity) for  $n = n + 1$  and  $r = 1$ :

$$\begin{aligned} \mathbf{A}_{n+1} \mathbf{A}_m - \mathbf{A}_{m+1} \mathbf{A}_n &= \frac{1}{2}(1 - \lambda^2)^m (\mathbf{A}_{n-m+1} - \mathbf{A}_{n-m}), \\ \mathbf{X}_{n+1} \mathbf{X}_m - \mathbf{X}_{m+1} \mathbf{X}_n &= -x(1 - \lambda^2)^m \mathbf{X}_{n-m}. \end{aligned}$$

**Remark 5.** It is worth mentioning that in [8] A. F. Horadam introduced Fibonacci quaternions developed later by many authors, see e.g. [9–22]. They are somehow comparable to that of our paper but still not so very close. However, there exists a connection between Quaternaccis and Fibonacci quaternions [1].

The Horadam idea was also developed in the case of split quaternions. The so-called Horadam sequence  $(H_n)$  is defined by the following recurrence relation:

$$H_0 = a, \quad H_1 = b, \quad H_n = pH_{n-1} + qH_{n-2}, \quad n \geq 2, \quad (31)$$

where  $a, b, p, q \in \mathbb{Z}$ . Then we can define split Horadam quaternions  $QH_n$  as follows:

$$QH_n = H_n + H_{n+1}i + H_{n+2}j + H_{n+3}k,$$

where  $H_n$  is the  $n$ th Horadam number and the set  $\{1, i, j, k\}$  forms a basis of split quaternion algebra. Of course, split Horadam quaternions satisfy the following recurrence:

$$QH_n = pQH_{n-1} + qQH_{n-2}, \quad n \geq 2$$

where  $QH_0 = a + bi + (pb + qa)j + (p^2b + pqa + qb)k$  and

$$QH_1 = b + (pb + qa)i + (p^2b + pqa + qb)j + (p^3b + p^2qa + 2pqb + q^2a)k.$$

For special values of  $a, b, p, q$ , we obtain some known sequences of split quaternions:

1. For  $a = 0, b = p = q = 1$ , we obtain split Fibonacci quaternions and on the other hand for  $a = 2, b = p = q = 1$  we obtain split Lucas quaternions (Akyiğit et al. [23]).
2. For  $a = 0, b = q = 1$  and  $p = k, k \in \mathbb{R}_{>0}$ , we obtain split  $k$ -Fibonacci quaternions, whereas for  $a = 2, b = p = k (k \in \mathbb{R}_{>0})$  and  $q = 1$ , we obtain split  $k$ -Lucas quaternions (Polatli et al. [24]).
3. For  $a = 0, b = q = 1$  and  $p = 2$ , we obtain split Pell quaternions, whereas for  $a = b = q = 1$  and  $p = 2$ , we obtain split Pell-Lucas quaternions (Tokeşer et al. [25]).
4. For  $a = 0, b = p = 1$  and  $q = 2$  we obtain split Jacobsthal quaternions, while for  $a = q = 2$  and  $b = p = 1$  we obtain split Jacobsthal-Lucas quaternions (Yağmur [26]).

We have the following connection between Split Quaternaccis and split Horadam quaternions:

$$\begin{aligned} (QH_m)^n &= (H_m + H_{m+1}i + H_{m+2}j + H_{m+3}k)^n \\ &= (H_m)^n \left( 1 + \frac{H_{m+1}}{H_m}i + \frac{H_{m+2}}{H_m}j + \frac{H_{m+3}}{H_m}k \right)^n \\ &= (H_m)^n \left( \mathbf{A}_n \left( \frac{H_{m+1}}{H_m}, \frac{H_{m+2}}{H_m}, \frac{H_{m+3}}{H_m} \right) + \mathbf{B}_n \left( \frac{H_{m+1}}{H_m}, \frac{H_{m+2}}{H_m}, \frac{H_{m+3}}{H_m} \right) i \right. \\ &\quad \left. + \mathbf{C}_n \left( \frac{H_{m+1}}{H_m}, \frac{H_{m+2}}{H_m}, \frac{H_{m+3}}{H_m} \right) j + \mathbf{D}_n \left( \frac{H_{m+1}}{H_m}, \frac{H_{m+2}}{H_m}, \frac{H_{m+3}}{H_m} \right) k \right). \end{aligned}$$

From Proposition 4 and the aforementioned formula we get

$$\sum_{n=0}^{N-1} \frac{QH_m^n}{H_m^n} = \frac{H_m}{H_{m+1}} \mathbf{B}_n \left( \frac{H_{m+1}}{H_m}, \frac{H_{m+2}}{H_m}, \frac{H_{m+3}}{H_m} \right) + \frac{\mathbf{A}_n \left( \frac{H_{m+1}}{H_m}, \frac{H_{m+2}}{H_m}, \frac{H_{m+3}}{H_m} \right) - 1}{\lambda^2} (\beta i + \gamma j + \delta k).$$

**Remark 6.** Some Split Quaternaccis can be found in The On-Line Encyclopedia of Integer Sequences (shortly OEIS), but not all of them!). Below we list a few:

- $\mathbf{A}_n(1, 1, 1) = \mathbf{B}_n(1, 1, 1) = \mathbf{C}_n(1, 1, 1) = \mathbf{D}_n(1, 1, 1) = A011782(n)$  describes the number of permutations in  $S_n$  avoiding patterns<sup>1</sup> 213 and 312 (see [27]). Also some other pairs of patterns can be considered here – like 123 and 132; 123 and 213; 132 and 213; 132 and 231; 213 and 312; 231 and 312; 231 and 321; 312 and 321.
- $\mathbf{A}_n(1, 0, 1) = A001333(n)$  is the sequence of numerators of continued fraction convergents of  $\sqrt{2}$ , whereas  $\mathbf{B}_n(1, 0, 1) = \mathbf{C}_n(1, 0, 1) = A000129(n) = P_n$  are their denominators, where  $P_n$  is the  $n$ th Pell number.<sup>2</sup>
- $\mathbf{D}_n(0, 1, -2) = A028860(n + 2) = A293007(n + 1)$  – a number of associative, quasitrivial and order-preserving binary operations on the  $n + 1$ -element set  $\{1, 2, \dots, m + 1\}$  that have neutral and annihilator<sup>3</sup> elements.
- $\mathbf{A}_n(4, 5, 1) = A025172(n) = 3^n \cos \left( n \arccos \left( \frac{1}{3} \right) \right)$ . This sequence was used for justifying the negative answer for the third Hilbert Problem (see e.g. [28]). Moreover,  $\mathbf{A}_n(4, 5, 1) = A_n(2, 2, 0)$  (see [1]).

<sup>1</sup> More generally it is about permutations of  $\{a_1, a_2, \dots, a_n\}$  avoiding patterns  $a_{k+1}a_k a_{k+2}$  and  $a_{k+2}a_k a_{k+1}$ .

<sup>2</sup> Pell sequence  $(P_n)_0^\infty$  can be defined recursively as follows:  $P_n = 2P_{n-1} + P_{n-2}, n \geq 2, P_0 = 0$  and  $P_1 = 1$ .

<sup>3</sup> That is, elements in the set  $\{r \in \{1, 2, \dots, n + 1\} \mid (\forall s \in \{1, 2, \dots, n + 1\}) (sr = 0)\}$ .

Also the following sequences were found:

$$\begin{aligned}
 \mathbf{B}_n(2, 1, 2) &= \mathbf{D}_n(2, 1, 2) = A274520(n), \\
 \mathbf{B}_n(1, -1, 2) &= -\mathbf{C}_n(1, -1, 2) = A015518(n), \\
 \mathbf{A}_n(1, -1, 2) &= A046717(n), & \mathbf{C}_n(0, 1, -2) &= A002605(n), \\
 \mathbf{A}_n(2, 1, 2) &= A083098(n), & \mathbf{C}_n(2, 1, 2) &= A083099(n), \\
 \mathbf{A}_n(3, 1, -2) &= A090042(n), & \mathbf{C}_n(3, 1, -2) &= A015520(n), \\
 \mathbf{A}_n(4, 3, -2) &= A133294(n), & \mathbf{D}_n(4, 3, -2) &= A274526(n), \\
 \mathbf{A}_n(4, 5, 3) &= A000012(n) \equiv 1, & \mathbf{C}_n(4, 5, 3) &= A008587(n) = 5n, \\
 \mathbf{B}_n(4, 5, 3) &= A008586(n) = 4n, & \mathbf{D}_n(4, 5, 3) &= A008585(n) = 3n.
 \end{aligned}$$

### 4 Bridges between some integer sequences

In this part of the paper, we present new identity connecting Horadam sequence with different known sequences which generalizes results from [29]. In particular, we obtain some new identity for Split Quaternaccis or Quaternaccis. It is worth pointing out that each member of Split Quaternaccis and Quaternaccis is actually a polynomial in three variables. Therefore, the results below give us identities for polynomials. Additionally, we give analogous identities for Horadam quaternions and Horadam split quaternions.

**Notation.** Since all results in this section are true both for Quaternaccis and Split Quaternaccis, we shall not distinguish the notation and use  $\{1, i, j, k\}$  for basis and  $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$  for sequences in both cases. In this section, we will use the following notation:

$$(x, X) \in \{(\beta, B), (\gamma, C), (\delta, D)\}, \quad \eta \in \{1 - \lambda, 1 + \lambda\}.$$

Also recall that  $\lambda = \sqrt{(\beta i)^2 + (\gamma j)^2 + (\delta k)^2}$  (where the set  $\{1, i, j, k\}$  forms the basis of quaternion algebra in the case of Quaternaccis and of split quaternion algebra for Split Quaternaccis).

We start by proving the following identity which generalizes identities for Fibonacci numbers considered in [30].

**Theorem 7.** *The following identity for Horadam sequence holds:*

$$H_{n+1} + (\varphi - p)H_n = A\varphi^n, \tag{32}$$

where  $\varphi \in \left\{ \frac{p - \sqrt{p^2 + 4q}}{2}, \frac{p + \sqrt{p^2 + 4q}}{2} \right\}$  and  $A = (a\varphi + b - pa)$ .

**Proof.** We will prove it by induction.

For  $n = 0$  from definition of Horadam sequence, we have

$$b + (\varphi - p)a = a\varphi + b - pa = A = A\varphi^0.$$

Assume now that (32) is satisfied for all  $n \leq N \in \mathbb{N}$ . Then:

$$A\varphi^{N+1} = \varphi \cdot A\varphi^N = \varphi H_{N+1} + (\varphi^2 - p\varphi)H_N = p H_{N+1} + q H_N + (\varphi - p)H_{N+1} \stackrel{(31)}{=} H_{N+2} + (\varphi - p)H_{N+1}.$$

So (32) is valid for all  $n \in \mathbb{N}_0$  by the induction principle. □

**Corollary 3.** *The following identities hold:*

$$x\eta^n = \mathbf{X}_{n+1} + (\eta - 2)\mathbf{X}_n, \tag{33}$$

$$(\eta - 1)\eta^n = \mathbf{A}_{n+1} + (\eta - 2)\mathbf{A}_n. \tag{34}$$

**Theorem 8.** Let  $(k_n)_{n=1}^\infty$  be any nonnegative integer sequence,  $K_N := \sum_{n=1}^N k_n$ ,  $N \in \mathbb{N}$  and  $A$  defined as in Theorem 7. Then we get

$$\prod_{n=1}^N (H_{k_{n+1}} + (\varphi - p)H_{k_n}) = A^N \varphi^{K_N}, \tag{35}$$

and in particular

$$\prod_{n=1}^N (\mathbf{X}_{k_{n+1}} + (\eta - 2)\mathbf{X}_{k_n}) = x^N \eta^{K_N}, \tag{36}$$

$$\prod_{n=1}^N (\mathbf{A}_{k_{n+1}} + (\eta - 2)\mathbf{A}_{k_n}) = (\eta - 1)^N \eta^{K_N}. \tag{37}$$

**Proof.** By induction, based on (32). □

The identities given in the aforementioned theorem constitute the bridges between Split Quaternaccis or Quaternaccis and other known number sequences. In the next corollaries, we will specify some of them.

**Corollary 4.** Under assumptions of Theorem 8 the following identities hold

$$\prod_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \left( \mathbf{X}_{\binom{n-k}{k-1}+1} + (\eta - 2)\mathbf{X}_{\binom{n-k}{k-1}} \right) = x^{\lfloor \frac{n+1}{2} \rfloor} \eta^{F_n}, \tag{38}$$

$$\prod_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \left( \mathbf{A}_{\binom{n-k}{k-1}+1} + (\eta - 2)\mathbf{A}_{\binom{n-k}{k-1}} \right) = (\eta - 1)^{\lfloor \frac{n+1}{2} \rfloor} \eta^{F_n}, \tag{39}$$

and in particular, for  $\beta = 1$

$$\prod_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \left( U_{\binom{n-k}{k-1}+1}(2, 1 - \lambda_0^2) + (\eta_0 - 2)U_{\binom{n-k}{k-1}}(2, 1 - \lambda_0^2) \right) = x^{\lfloor \frac{n+1}{2} \rfloor} \eta_0^{F_n}, \tag{40}$$

$$\prod_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \left( V_{\binom{n-k}{k-1}+1}(2, 1 - \lambda^2) + (\eta - 2)V_{\binom{n-k}{k-1}}(2, 1 - \lambda^2) \right) = \left( \frac{\eta - 1}{2} \right)^{\lfloor \frac{n+1}{2} \rfloor} \eta^{F_n}, \tag{41}$$

where  $\lambda_0 = \sqrt{i^2 + (y_j)^2 + (\delta k)^2}$ ,  $(U_n)$  and  $(V_n)$  are Lucas sequences of the first and second kind, respectively, and  $F_n$  is the  $n$ th Fibonacci number.

**Proof.** Identities (38) and (39) follow from  $F_n = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k}{k-1}$  (see e.g. [31]). Identities (40) and (41) follow from Remark 3. □

**Corollary 5.** Under assumptions of Theorem 8 the following identities hold:

$$\prod_{k=1}^n (\mathbf{X}_{2 \cdot 4^{n-k} C_{k-1}+1} + (\eta - 2)\mathbf{X}_{2 \cdot 4^{n-k} C_{k-1}}) = x^n \eta^{4^n - (n+1)C_n}, \tag{42}$$

$$\prod_{k=1}^n (\mathbf{A}_{2 \cdot 4^{n-k} C_{k-1}+1} + (\eta - 2)\mathbf{A}_{2 \cdot 4^{n-k} C_{k-1}}) = (\eta - 1)^n \eta^{4^n - (n+1)C_n}, \tag{43}$$

where  $C_n$  is the  $n$ th Catalan number.

**Proof.** Identities (42) and (43) follow from a new (as we suppose) identity for Catalan numbers [32]

$$\frac{n+1}{4^n} C_n = 1 - \frac{1}{2} \sum_{k=0}^{n-1} \frac{C_k}{4^k}, \quad n \in \mathbb{N}. \tag{44}$$

**Corollary 6.** Under assumptions of Theorem 8 the following identities hold:

$$\prod_{k=1}^n \left( X_{\frac{n!}{k} H_{k-1} + 1} + (\eta - 2) X_{\frac{n!}{k} H_{k-1}} \right) = x^n \eta^{\lfloor \frac{n+1}{3} \rfloor}, \tag{45}$$

$$\prod_{k=1}^n \left( A_{\frac{n!}{k} H_{k-1} + 1} + (\eta - 2) \lambda A_{\frac{n!}{k} H_{k-1}} \right) = (\eta - 1)^n \eta^{\lfloor \frac{n+1}{3} \rfloor}, \tag{46}$$

where  $H_n$  is the  $n$ th harmonic number and  $\lfloor \frac{n}{k} \rfloor$  is the Stirling number of the first kind.

**Proof.** It follows from the following known relations (see e.g. [33]):

$$\left[ \begin{matrix} n \\ 2 \end{matrix} \right] = (n-1)! H_{n-1} \quad \text{and} \quad \left[ \begin{matrix} n+1 \\ m+1 \end{matrix} \right] = \sum_{k=0}^n \left[ \begin{matrix} k \\ m \end{matrix} \right] \frac{n!}{k!},$$

assuming that  $\left[ \begin{matrix} k \\ m \end{matrix} \right] = 0$  for  $k < m$ . □

In the case of Horadam quaternions and split Horadam quaternions we have analogous identities.

**Theorem 9.** The following identity holds

$$QH_{n+1} + (\varphi - p)QH_n = A \underline{\varphi} \varphi^n,$$

where  $\varphi \in \left\{ \frac{p - \sqrt{p^2 + 4q}}{2}, \frac{p + \sqrt{p^2 + 4q}}{2} \right\}$ ,  $A = a\varphi + b - pa$  and  $\underline{\varphi} = 1 + \varphi i + \varphi^2 j + \varphi^3 k$ .

**Proof.** By induction, analogously to the proof of Theorem 7. □

By the above theorem, we obtain a “bridge” identity for Horadam quaternions and split Horadam quaternions. This result is interesting since although multiplication in quaternion and split quaternion algebras is noncommutative, the following identity does not depend on left or right multiplication.

**Theorem 10.** Let  $(k_n)_{n=1}^{\infty}$  be any nonnegative integer sequence,  $K_N = \sum_{n=1}^N k_n$  and  $N \in \mathbb{N}$ . Then we get

$$\begin{aligned} \prod_{n=1}^N (QH_{k_{n+1}} + (\varphi - p)QH_{k_n}) &= A^N \underline{\varphi}^N \varphi^{K_N} = A^N (A_N(\varphi, \varphi^2, \varphi^3) + B_N(\varphi, \varphi^2, \varphi^3)i + C_N(\varphi, \varphi^2, \varphi^3)j \\ &\quad + D_N(\varphi, \varphi^2, \varphi^3)k) \varphi^{K_N}. \end{aligned} \tag{47}$$

**Proof.** The proof is by induction on  $N$ . Let  $(k_n)_{n=1}^{\infty}$  be any nonnegative integer sequence. For  $N = 1$  we have

$$QH_{k_1+1} + (\varphi - p)QH_{k_1} = A \underline{\varphi} \varphi^{k_1},$$

which is valid by Theorem 9. Assuming that the formula (47) holds for  $N \in \mathbb{N}$ ,  $N > 1$ , we will prove it for  $N + 1$ . We have

$$\begin{aligned} \prod_{n=1}^{N+1} (QH_{k_{n+1}} + (\varphi - p)QH_{k_n}) &= (QH_{k_{N+1}+1} + (\varphi - p)QH_{k_{N+1}}) \prod_{n=1}^N (QH_{k_{n+1}} + (\varphi - p)QH_{k_n}) \\ &= (QH_{k_{N+1}+1} + (\varphi - p)QH_{k_{N+1}}) A^N \underline{\varphi}^N \varphi^{K_N} \\ &\stackrel{\text{Th. 9}}{=} A \underline{\varphi} \varphi^{k_{N+1}} A^N \underline{\varphi}^N \varphi^{K_N} = A^{N+1} \underline{\varphi}^{N+1} \varphi^{K_{N+1}}. \end{aligned}$$

Note that since multiplication in quaternion and split quaternion algebras is associative, we have  $q^{N+1} = qq^N = q^Nq$ , so if we take

$$\prod_{n=1}^{N+1} (QH_{k_{n+1}} + (\varphi - p)QH_{k_n}) = \left( \prod_{n=1}^N (QH_{k_{n+1}} + (\varphi - p)QH_{k_n}) \right) (QH_{k_{N+1+1}} + (\varphi - p)QH_{k_{N+1}})$$

we obtain the same result. Therefore, (47) is valid for all  $N \in \mathbb{N}$ . □

## 5 Some remarks on the other algebras

### 5.1 Octonions

It is well known that there is only one (up to isomorphism) real finite-dimensional division algebra except for algebras  $\mathbb{R}, \mathbb{C}, \mathbb{H}$ , namely, the Cayley-Graves algebra  $\mathbb{O}$  of octonions (it is in fact a deep topological theorem [34]; no purely algebraic proof of this fact is known, which is similar to – but at a much higher level – the situation with the Fundamental Theorem of Algebra). Algebra  $\mathbb{O}$  was discovered independently by Graves (1843) and Cayley (1845); it can be constructed from  $\mathbb{H}$  in a similar manner to the algebra  $\mathbb{H}$  from  $\mathbb{C}$ , namely, via the Cayley-Dickson construction [35]. More precisely, if  $(a, b)$  and  $(c, d)$  are pairs of quaternions, then we define their addition pairwise, whereas their product is defined by the formula:

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}),$$

where a bar denotes the quaternionic conjugation. From the computational point of view the description of octonionic multiplication can be given by the following formula for eight unit octonions  $e_i, i = 0, 1, 2, \dots, 7$ , comprising a basis for  $\mathbb{R}^8$ :

$$e_i e_j = \begin{cases} e_j & \text{if } i = 0, \\ e_i & \text{if } j = 0, \\ -\delta_{ij}e_0 + \varepsilon_{ijk}e_k & \text{otherwise,} \end{cases}$$

where  $\delta_{ij}$  is the Kronecker delta and  $\varepsilon_{ijk}$  is the Levi-Civita symbol. It is also well known [36] that the Fano plane (which is in fact the projective plane over the two-element Galois field) gives a convenient mnemonic for remembering these products of unit octonions.

Algebra  $\mathbb{O}$  plays a crucial role in the classification of Lie and Jordan algebras [36]. Octonions are also connected with Bott periodicity and the parallelizability of spheres in Euclidean spaces [37]. The octonion algebra is also related to Clifford algebras (in a non-direct way, which is obvious because of the associativity of Clifford algebras); for example, the algebra of endomorphism of a complexification of the algebra  $\mathbb{O}$  can be identified with a Clifford algebra [36]. There were also attempts to apply octonionic methods to problems of particle physics, but with little success until the 1980s, when it was realized that octonions explain some of the features of string theory [36,38].

We can define octonion equivalents for Quaternaccis.

**Definition 4.** Octonion algebra equivalents for Quaternaccis are defined by the following relation:

$$(1 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7)^n = \mathbf{A}_n^{\mathbb{O}} + \mathbf{B}_n^{\mathbb{O}}e_1 + \mathbf{C}_n^{\mathbb{O}}e_2 + \mathbf{D}_n^{\mathbb{O}}e_3 + \mathbf{E}_n^{\mathbb{O}}e_4 + \mathbf{F}_n^{\mathbb{O}}e_5 + \mathbf{G}_n^{\mathbb{O}}e_6 + \mathbf{H}_n^{\mathbb{O}}e_7,$$

where  $n \in \mathbb{N}_0, a_1, a_2, \dots, a_7 \in \mathbb{R}$  and  $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$  is a basis of  $\mathbb{O}$ .

In the aforementioned definition, all sequences  $\mathbf{A}_n^{\mathbb{O}}, \dots, \mathbf{H}_n^{\mathbb{O}}$  depend on parameters  $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ . We omit them in the above formulation for the sake of clarity of definition.

We have the following recurrence relation for Octonion equivalents for Quaternaccis.

**Theorem 11.** Octonion algebra equivalents for Quaternaccis  $\mathbf{A}_n^{\circ}, \dots, \mathbf{H}_n^{\circ}$  satisfy the following recurrence relation:

$$\begin{bmatrix} \mathbf{A}_{n+1}^{\circ} \\ \mathbf{B}_{n+1}^{\circ} \\ \mathbf{C}_{n+1}^{\circ} \\ \mathbf{D}_{n+1}^{\circ} \\ \mathbf{E}_{n+1}^{\circ} \\ \mathbf{F}_{n+1}^{\circ} \\ \mathbf{G}_{n+1}^{\circ} \\ \mathbf{H}_{n+1}^{\circ} \end{bmatrix} = \begin{bmatrix} 1 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & 1 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & 1 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & 1 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & 1 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & 1 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & 1 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}_n^{\circ} \\ \mathbf{B}_n^{\circ} \\ \mathbf{C}_n^{\circ} \\ \mathbf{D}_n^{\circ} \\ \mathbf{E}_n^{\circ} \\ \mathbf{F}_n^{\circ} \\ \mathbf{G}_n^{\circ} \\ \mathbf{H}_n^{\circ} \end{bmatrix} \tag{48}$$

for every  $n \in \mathbb{N}$  with  $\mathbf{A}_0^{\circ} = 1$ , and the remaining initial values equal to 0.

Octonions are defined similar to quaternions (and they form an alternative algebra), so we can also obtain an analogue of Proposition 2, providing two more recurrence relations. Indeed, all three transition matrices defining the relations have analogous eigenvalues to the respective matrices in case of quaternions. The first two have two eigenvalues  $1 \pm \lambda_0$ , where  $\lambda_0 = \sqrt{-a_1^2 - a_2^2 - a_3^2 - a_4^2 - a_5^2 - a_6^2 - a_7^2}$ , whereas the third matrix has one extra eigenvalue equal to 1.

Therefore, all octonion algebra equivalents for Quaternaccis satisfy (the same) recurrence relation:

$$\mathbb{X}_n = 2\mathbb{X}_{n-1} - (1 - \lambda_0^2)\mathbb{X}_{n-2}, \quad n > 2, \tag{49}$$

with the following initial conditions:

$$\begin{aligned} \mathbf{A}_0^{\circ} = 1, \quad \mathbf{B}_0^{\circ} = \mathbf{C}_0^{\circ} = \dots = \mathbf{H}_0^{\circ} = 0, \quad \mathbf{A}_1^{\circ} = 1, \quad \mathbf{B}_1^{\circ} = a_1, \\ \mathbf{C}_1^{\circ} = a_2, \quad \mathbf{D}_1 = a_3, \quad \mathbf{E}_1 = a_4, \quad \mathbf{F}_1 = a_5, \quad \mathbf{G}_1 = a_6, \quad \mathbf{H}_1 = a_7. \end{aligned}$$

Note that because of (49), most of the formulae given in Section 4 are also true for octonion algebra equivalents for Quaternaccis.

### 5.2 Clifford algebras

Let  $V$  be a real vector space endowed with a quadratic form  $Q : V \rightarrow \mathbb{R}$  and let  $\mathcal{A}$  be an associative algebra with unity  $1_{\mathcal{A}}$  and let  $\gamma : V \rightarrow \mathcal{A}$  be a linear mapping. A pair  $(\mathcal{A}, \gamma)$  is called a Clifford algebra for the quadratic space  $(V, Q)$  if and only if  $\mathcal{A}$  is generated by a subset  $\{\gamma(v) : v \in V\} \cup \{a1_{\mathcal{A}} : a \in \mathbb{R}\}$  and the following condition is satisfied for all  $v \in V$ :

$$(\gamma(v))^2 = Q(v)1_{\mathcal{A}}.$$

Considering an orthonormal basis of  $V$  one can show (see e.g. [40]) that the dimension of a Clifford algebra for the quadratic space  $(V, Q)$  is not greater than  $2^{\dim(V)}$ . Although there exist Clifford algebras of dimension less than  $2^{\dim(V)}$  (see e.g. [39]), the prominent role is played by the Clifford algebras of maximal possible dimension. This is a consequence of so-called universality (or universal property, see [40]) and the uniqueness (up to the unique canonical isomorphism) of a universal Clifford algebra for the quadratic space  $(V, Q)$  (i.e. Clifford algebra satisfying the universal property) which in the following will be denoted by  $Cl(V, Q)$ . A standard construction of the universal Clifford algebra (as a quotient of the tensor algebra) was proposed e.g. by Chevalley in 1954 [41] and can be readily found in the literature (see e.g. [39,40]).

Since Clifford mapping  $\gamma$  is a linear injection, in the following we shall dispense with the symbol  $\gamma$ , thus identifying a vector  $v \in V$  with its image  $\gamma(v) \in Cl(V, Q)$ .

It is obvious from the purely computational point of view that the most important thing is to know a basis of the Clifford algebra and relations between elements of this basis with respect to the multiplicative structure of the considered algebra. It turns out that if  $\{e_1, e_2, \dots, e_n\}$  is an orthonormal basis of  $V$ , then the elements of the form  $e_1^{\mu_1} e_2^{\mu_2} \dots e_n^{\mu_n}$ , where  $\mu_k = 0, 1$  for  $k = 0, 1, \dots, n$ , constitute a basis of the Clifford algebra  $Cl(V, Q)$  and we have the following relations  $e_k e_l = -e_l e_k$  and  $e_k^2 = Q(e_k)$ , for all  $k, l = 1, 2, \dots, n$  and  $k \neq l$ .

In what follows, we shall abbreviate  $e_{kl} := e_k e_l$  and the like. Moreover, by  $Cl_{p,q}(\mathbb{R})$ , where  $p, q \in \mathbb{N}_0$ , we shall denote the Clifford algebra for the quadratic space  $(\mathbb{R}^{p+q}, Q)$ , where  $Q$  is given by

$$Q(v_1, \dots, v_{p+q}) = v_1^2 + \dots + v_p^2 - v_{p+1}^2 - \dots - v_{p+q}^2.$$

Let us note that for  $p + q = 2$  our concept of Split Quaternaccis and Quaternaccis can be transferred without any problems to Clifford algebra equivalents. In fact, split quaternion and quaternion algebras can be considered as special cases of Clifford algebras, for example,  $Cl_{0,2}(\mathbb{R})$  (or  $Cl_{2,0}(\mathbb{R})$ ) and  $Cl_{1,1}(\mathbb{R})$ , respectively.

So suppose now that  $p + q = 3$ . Then Definition 3 turns to

**Definition 5.** Clifford algebra equivalents for Quaternaccis for  $Cl_{p,q}(\mathbb{R})$  (shortly Cliffordaccis), where  $p + q = 3$  and  $p, q, \in \mathbb{N}_0$  are defined by the following relation:

$$(1 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_{12} + a_5 e_{13} + a_6 e_{23} + a_7 e_{123})^n = \mathbf{A}_n^{p,q} + \mathbf{B}_n^{p,q} e_1 + \mathbf{C}_n^{p,q} e_2 + \mathbf{D}_n^{p,q} e_3 + \mathbf{E}_n^{p,q} e_{12} + \mathbf{F}_n^{p,q} e_{13} + \mathbf{G}_n^{p,q} e_{23} + \mathbf{H}_n^{p,q} e_{123},$$

where  $n \in \mathbb{N}_0$ ,  $a_1, a_2, \dots, a_7 \in \mathbb{R}$  and  $\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}$  is a basis of  $Cl_{p,q}(\mathbb{R})$ .

Then we can get an analogue of Theorem 1:

**Theorem 12.** Cliffordaccis  $\mathbf{A}_n^{p,q}, \dots, \mathbf{H}_n^{p,q}$  satisfy the following recurrence relation:

$$\begin{bmatrix} \mathbf{A}_{n+1}^{p,q} \\ \mathbf{B}_{n+1}^{p,q} \\ \mathbf{C}_{n+1}^{p,q} \\ \mathbf{D}_{n+1}^{p,q} \\ \mathbf{E}_{n+1}^{p,q} \\ \mathbf{F}_{n+1}^{p,q} \\ \mathbf{G}_{n+1}^{p,q} \\ \mathbf{H}_{n+1}^{p,q} \end{bmatrix} = \begin{bmatrix} 1 & a_1 e_1^2 & a_2 e_2^2 & a_3 e_3^2 & a_4 e_{12}^2 & a_5 e_{13}^2 & a_6 e_{23}^2 & a_7 e_{123}^2 \\ a_1 & 1 & -a_4 e_2^2 & -a_5 e_3^2 & a_2 e_2^2 & a_3 e_3^2 & a_7 e_{23}^2 & a_6 e_{23}^2 \\ a_2 & a_4 e_1^2 & 1 & -a_6 e_3^2 & -a_1 e_1^2 & -a_7 e_3^2 & a_7 e_3^2 & -a_5 e_{13}^2 \\ a_3 & a_5 e_1^2 & a_6 e_2^2 & 1 & a_7 e_{12}^2 & -a_1 e_1^2 & -a_2 e_2^2 & a_4 e_{12}^2 \\ a_4 & a_2 & -a_1 & a_7 e_3^2 & 1 & -a_6 e_3^2 & a_5 e_3^2 & a_3 e_3^2 \\ a_5 & a_3 & -a_7 e_2^2 & -a_1 & a_6 e_2^2 & 1 & -a_4 e_2^2 & -a_2 e_2^2 \\ a_6 & a_7 e_1^2 & a_3 & -a_2 & -a_5 e_1^2 & a_4 e_1^2 & 1 & a_1 e_1^2 \\ a_7 & a_6 & -a_5 & a_4 & a_3 & -a_2 & a_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}_n^{p,q} \\ \mathbf{B}_n^{p,q} \\ \mathbf{C}_n^{p,q} \\ \mathbf{D}_n^{p,q} \\ \mathbf{E}_n^{p,q} \\ \mathbf{F}_n^{p,q} \\ \mathbf{G}_n^{p,q} \\ \mathbf{H}_n^{p,q} \end{bmatrix} \tag{50}$$

for every  $n \in \mathbb{N}$  with  $\mathbf{A}_0^{p,q} = 1$ , and the remaining initial values equal to 0.

We can also get an analogue of Proposition 2. The first form is similar to the above and the second, an analogue for the arrow-head matrix in (7), looks in the following way:

$$\begin{bmatrix} \mathbf{A}_{n+1}^{p,q} \\ \mathbf{B}_{n+1}^{p,q} \\ \mathbf{C}_{n+1}^{p,q} \\ \mathbf{D}_{n+1}^{p,q} \\ \mathbf{E}_{n+1}^{p,q} \\ \mathbf{F}_{n+1}^{p,q} \\ \mathbf{G}_{n+1}^{p,q} \\ \mathbf{H}_{n+1}^{p,q} \end{bmatrix} = \begin{bmatrix} 1 & a_1 e_1^2 & a_2 e_2^2 & a_3 e_3^2 & a_4 e_{12}^2 & a_5 e_{13}^2 & a_6 e_{23}^2 & a_7 e_{123}^2 \\ a_1 & 1 & 0 & 0 & 0 & 0 & a_7 e_{23}^2 & a_6 e_{23}^2 \\ a_2 & 0 & 1 & 0 & 0 & -a_7 e_3^2 & 0 & -a_5 e_{13}^2 \\ a_3 & 0 & 0 & 1 & a_7 e_{12}^2 & 0 & 0 & a_4 e_{12}^2 \\ a_4 & 0 & 0 & a_7 e_3^2 & 1 & 0 & 0 & a_3 e_3^2 \\ a_5 & 0 & -a_7 e_2^2 & 0 & 0 & 1 & 0 & -a_2 e_2^2 \\ a_6 & a_7 e_1^2 & 0 & 0 & 0 & 0 & 1 & a_1 e_1^2 \\ a_7 & a_6 & -a_5 & a_4 & a_3 & -a_2 & a_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A}_n^{p,q} \\ \mathbf{B}_n^{p,q} \\ \mathbf{C}_n^{p,q} \\ \mathbf{D}_n^{p,q} \\ \mathbf{E}_n^{p,q} \\ \mathbf{F}_n^{p,q} \\ \mathbf{G}_n^{p,q} \\ \mathbf{H}_n^{p,q} \end{bmatrix} \tag{51}$$

**Remark 7.** Under certain assumptions the transition matrix in (50) has four different eigenvalues and the one in (51) has six different eigenvalues. They have pretty complicated form in general, but for example for  $C\ell_{2,1}(\mathbb{R})$  they are

$$1 \pm \mu_1 + a_7, 1 \pm \mu_2 + a_7 \quad \text{and} \quad 1 \pm \mu_1 + a_7, 1 \pm \mu_2 + a_7, 1 \pm a_7,$$

respectively, under additional assumption that  $\mu_1 \neq 0$ ,  $\mu_2 \neq 0$ ,  $a_7 \neq 0$ , where

$$\mu_1 = \sqrt{(a_2 + a_5)^2 + (a_1 - a_6)^2 - (a_3 + a_4)^2} \quad \text{and} \quad \mu_2 = \sqrt{(a_2 - a_5)^2 + (a_1 + a_6)^2 - (a_3 - a_4)^2}.$$

It is also worth pointing out that for case  $p + q = 4$  an analogue of matrices in (7) and (51) does not have such simple form (we will not present those matrices here because of their order).

**Conjecture.** We checked numerically that Cliffordaccis for  $C\ell_{2,1}(\mathbb{R})$  and  $C\ell_{0,3}(\mathbb{R})$  satisfy some recurrence relation of order 4. We suspect that all Cliffordaccis for  $p + q = 3$  satisfy a recurrence relation of order 4.

## 6 Final remark

An interesting area of research is a binomial transformation of given sequence. In [42], the authors considered binomial transformations of scaled Fibonacci numbers. We plan to investigate such transformations in the case of Quaternaccis and Split Quaternaccis.

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