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Research Article

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Properties of multiplication operators on the space of functions of bounded φ-variation

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Abstract: In this paper, the functions $u \in BV_\varphi[0,1]$ which define compact and Fredholm multiplication operators M_u acting on the space of functions of bounded φ-variation are studied. All the functions $u \in BV_\varphi[0,1]$ which define multiplication operators $M_u : BV_\varphi[0,1] \to BV_\varphi[0,1]$ with closed range are characterized.

Keywords: multiplication operator, bounded variation functions, compact operators, Fredholm operators

MSC 2020: 47B38, 26A45, 26B30, 46E40

1 Introduction

In 1881, Camille Jordan [1] introduced the notion of functions of bounded variation and established the relation between those functions and monotonic ones when he was studying convergence of Fourier series. Later on the concept of bounded variation was generalized in various directions by many mathematicians such as F. Riesz, N. Wiener, R. E. Love, H. Ursell, L. C. Young, W. Orlicz, J. Musielak, L. Tonelli, L. Cesari, R. Caccioppoli, E. de Giorgi, O. Olcinik, E. Conway, J. Smoller, A. Volpert, S. Hudjacv, L. Ambrosio, G. Dal Maso, among many others. It is noteworthy to mention that many of these generalizations were motivated by problems in such areas as calculus of variation, convergence of Fourier series, geometric measure theory, mathematical physics, etc. For many applications of functions of bounded variation see, e.g., the monograph [2].

On the other side, the space of functions of bounded p-variation, introduced by Wiener in 1924 (see [3]), led in 1937 to Young [4] to give a generalization of the concept of function of bounded variation by introducing the notion of φ-variation of a function; this concept, in turn, was generalized by Chistyakov, [5], for functions which take values in a linear normed vector space. We recommend the excellent book of Appell et al. [6] for the study of the properties of functions of bounded φ-variation.

The aim of this article is to study the properties of multiplication operator acting on $BV_\varphi[0,1]$ spaces, which we define in Section 2. Each function $u \in BV_\varphi[0,1]$ defines a linear and continuous operator $M_u : BV_\varphi[0,1] \to BV_\varphi[0,1]$ given by $M_u(f) = u \cdot f$ which is known as *multiplication operator* with symbol u. This operator has been widely studied in the context of spaces of measurable functions. We mention here the pioneering work by Singh and Kumar in [7,8] on properties of multiplication operators acting on spaces of measurable functions. These authors studied the compactness and closedness of the range of multiplication operators on $L_2(\mu)$. Additionally, we note that the work of Arora et al. in [9–11] examined properties of M_u on Lorentz and Lorentz-Bochner spaces. Further significant results regarding multiplication

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operators were obtained by Castillo et al. in [12–14], in which these authors showed that the techniques used by previously mentioned authors can be modified to study multiplication operators on weak L_p spaces, Orlicz-Lorentz spaces, and variable Lebesgue spaces. The most complete study of the properties of multiplication operators acting on measurable function spaces was carried out by Hudzik et al. [15] (see also [16] and [17]). However, in the context of spaces of functions with certain type of bounded variation, it is a recent topic, being the first work due to Astudillo-Villalba and Ramos-Fernández [18] (see also [19]) where the authors made an exhaustive study of the properties of multiplication operator acting on the space of functions of bounded variation.

Consequently, the main objective of this article is to extend the results achieved in [18] to more general setting of space of functions of bounded φ-variation. With the above end, in Section 2, we give some properties of functions of bounded φ-variation. Section 3 is dedicated to characterize all symbols $u \in BV_\varphi[0,1]$ which define multiplication operators having closed range; while in Section 4 we study the compactness of M_u making special emphasis on the properties of the symbol u which define multiplication operators with finite dimensional range. Finally, in Section 5, we use the results obtained in Sections 4 and 5 to characterize the symbols $u \in BV_\varphi[0,1]$ which define Fredholm multiplication operators.

2 Some remarks on functions of bounded φ-variation

For the reader's ease, in this section we collect some properties of φ-functions which will be of great utility in the proof of our results. A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be a Young function (or φ-function) if it is convex, strictly increasing and such that $\varphi(0) = 0$. In particular, according to this definition we also have that $\varphi(t) \to \infty$ as $t \to \infty$. Thus, if φ is a fixed Young function, then we can define the φ-variation (also known as the Young variation) of a real-valued function f defined on $[0,1]$ as

$$\mathrm{Var}_\varphi(f) = \sup \sum_{k=1}^{n} \varphi(|f(t_k) - f(t_{k-1})|),$$

where the supremum is taken over all finite partitions $P : 0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$. With this notation, we say that a real-valued function f defined on $[0,1]$ belongs to the space of bounded φ-variation functions $BV_\varphi[0,1]$ (which was introduced by Young in 1937, see [4]) if there exists a $\lambda > 0$ such that

$$\mathrm{Var}_\varphi\left(\frac{1}{\lambda} f\right) < \infty.$$

For each $f \in BV_\varphi[0,1]$, we can define the quantity

$$\|f\|_\varphi = \inf\left\{\lambda > 0 : \mathrm{Var}_\varphi\left(\frac{1}{\lambda} f\right) \le 1\right\},$$

which is a seminorm for $BV_\varphi[0,1]$ (known as the Luxemburg seminorm) and $BV_\varphi[0,1]$ is a Banach space with the norm

$$\|f\|_{BV_\varphi} = \|f\|_\infty + \|f\|_\varphi,$$

where

$$\|f\|_\infty := \sup_{t \in [0,1]} |f(t)|.$$

In fact, for all $f, g \in BV_\varphi[0,1]$ we have that $f \cdot g \in BV_\varphi[0,1]$ and

$$\|f \cdot g\|_{BV_\varphi} \le \|f\|_{BV_\varphi} \|g\|_{BV_\varphi}, \tag{2.1}$$

which means that $BV_\varphi[0,1]$ is a Banach algebra. When $\varphi(t) = t$ we have the classical space of bounded variation; while when $\varphi(t) = t^p$, the space $BV_\varphi[0,1]$ becomes into $WBV_p[0,1]$, the space of functions of bounded p-variation, introduced by Wiener in 1924 (see [3]).

We fix a Young function φ and consider the space $BV_\varphi[0,1]$ defined by φ. Next, we have the following:

(a) If $f \in BV_\varphi[0,1]$, then there exists $\lambda > 0$ such that

$$\mathrm{Var}_\varphi\left(\frac{1}{\lambda}f\right) < \infty.$$

Let $0 < \beta \le \frac{1}{\lambda}$. Then $0 < \beta\lambda \le 1$. Since φ is convex and $\varphi(0) = 0$,

$$\mathrm{Var}_\varphi(\beta f) = \mathrm{Var}_\varphi\left(\lambda\beta\frac{f}{\lambda} + (1-\lambda\beta)0\right) \le \lambda\beta\mathrm{Var}_\varphi\left(\frac{1}{\lambda}f\right) < \infty.$$

Hence,

$$0 \le \lim_{\beta\to 0}\mathrm{Var}_\varphi(\beta f) \le \lim_{\beta\to 0}\lambda\beta\mathrm{Var}_\varphi\left(\frac{1}{\lambda}f\right) = 0$$

and so

$$\lim_{\beta\to 0}\mathrm{Var}_\varphi(\beta f) = 0.$$

(b) By (a), there exists $\beta > 0$ such that

$$\mathrm{Var}_\varphi(\beta f) \le 1.$$

Let $\lambda = \frac{1}{\beta}$. And so, there exists $\lambda > 0$ such that

$$\mathrm{Var}_\varphi\left(\frac{1}{\lambda}f\right) \le 1.$$

Hence, we have proved that

$$\left\{\lambda > 0 : \mathrm{Var}_\varphi\left(\frac{1}{\lambda}f\right) \le 1\right\} \ne \varnothing.$$

(c) The quantity

$$\|f\|_\varphi = \inf\left\{\lambda > 0 : \mathrm{Var}_\varphi\left(\frac{1}{\lambda}f\right) \le 1\right\} \tag{2.2}$$

satisfies the following property:

$$\mathrm{Var}_\varphi\left(\frac{f}{\|f\|_\varphi}\right) \le 1 \tag{2.3}$$

whenever $f \in BV_\varphi[0,1]$ is not a constant function. In fact, f is a constant function on $[0,1]$ if and only if $\mathrm{Var}_\varphi(f) = 0$ which occurs if and only if $\|f\|_\varphi = 0$. In particular, we have

$$\left\{\lambda > 0 : \mathrm{Var}_\varphi\left(\frac{1}{\lambda}f\right) \le 1\right\} = (\|f\|_\varphi, +\infty).$$

Hence, $\|f\|_\varphi \le k$ if and only if $\mathrm{Var}_\varphi\left(\frac{1}{k}f\right) \le 1$. Furthermore, the relation (2.3) implies that for all $f \in BV_\varphi[0,1]$ and $t \in [0,1]$ we have

$$|f(t)| \le |f(0)| + \varphi^{-1}(1)\|f\|_\varphi$$

and $BV_\varphi[0,1] \subsetneq B[0,1]$, the space of all bounded functions defined on $[0,1]$, and hence we can write

$$\|f\|_\infty \le |f(0)| + \varphi^{-1}(1)\|f\|_\varphi.$$

We also have that if $f, g \in BV_\varphi[0,1]$, then $f\cdot g \in BV_\varphi[0,1]$ and

$$\|f\cdot g\|_\varphi \le \|f\|_\infty\|g\|_\varphi + \|g\|_\infty\|f\|_\varphi. \tag{2.4}$$

We include a proof of the above inequality to illustrate the use of the property (2.3) and the definition of Luxemburg seminorm (2.2). We can suppose that $\|f\|_\varphi > 0$ and $\|g\|_\varphi > 0$. We set

$$M = \|f\|_\infty \|g\|_\varphi + \|g\|_\infty \|f\|_\varphi,$$

then for any partition $P : 0 = t_0 < t_1 < \dots < t_n = 1$ we can write

$$\begin{aligned}\sum_{k=1}^{n}\varphi\left(\frac{1}{M}|f(t_k)g(t_k) - f(t_{k-1})g(t_{k-1})|\right) &= \sum_{k=1}^{n}\varphi\left(\frac{1}{M}\,|(f(t_k) - f(t_{k-1}))g(t_k) + f(t_{k-1})(g(t_k) - g(t_{k-1}))|\right)\\ &\leq \sum_{k=1}^{n}\varphi\left(\frac{\|g\|_\infty}{M}|f(t_k) - f(t_{k-1})| + \frac{\|f\|_\infty}{M}\,|g(t_k) - g(t_{k-1})|\right)\\ &= \sum_{k=1}^{n}\varphi\left(\frac{\|g\|_\infty\|f\|_\varphi}{M}\frac{|f(t_k) - f(t_{k-1})|}{\|f\|_\varphi} + \frac{\|f\|_\infty\|g\|_\varphi}{M}\frac{|g(t_k) - g(t_{k-1})|}{\|g\|_\varphi}\right)\\ &\leq \frac{1}{M}\|g\|_\infty\|f\|_\varphi \operatorname{Var}_\varphi\left(\frac{f}{\|f\|_\varphi}\right) + \frac{1}{M}\|f\|_\infty\|g\|_\varphi \operatorname{Var}_\varphi\left(\frac{g}{\|g\|_\varphi}\right)\\ &\leq \frac{1}{M}\|g\|_\infty\|f\|_\varphi + \frac{1}{M}\|f\|_\infty\|g\|_\varphi = 1,\end{aligned}$$

where we have used the convexity of φ in the second inequality and the property (2.3) in the last inequality. Thus, since the partition P was arbitrary, we obtain that

$$\operatorname{Var}_\varphi\left(\frac{f\cdot g}{M}\right) \leq 1$$

which proves that $\|f\cdot g\|_\varphi \leq M$ as affirmed.

In this kind of spaces, the following functions will play an important role. For a subset B of $[0,1]$, the characteristic function of B is denoted by $\mathbf{1}_B$ and it is defined by

$$\mathbf{1}_B(t) = \begin{cases} 0, & \text{if } t \notin B,\\ 1, & \text{if } t \in B.\end{cases}$$

This is known as an *indicatrix function*. In particular, we have the following very useful property:

Proposition 2.1. *For each $a \in [0,1]$, the function $\mathbf{1}_{\{a\}}$ belongs to $BV_\varphi[0,1]$ and*

$$1 + \frac{1}{\varphi^{-1}(1)} \leq \|\mathbf{1}_{\{a\}}\|_{BV_\varphi} \leq 1 + \frac{1}{\varphi^{-1}\left(\frac{1}{2}\right)}. \tag{2.5}$$

Proof. Indeed, clearly $\|\mathbf{1}_{\{a\}}\|_\infty = 1$, hence, we shall estimate $\|\mathbf{1}_{\{a\}}\|_\varphi$. To this end, let $P : 0 = t_0 < t_1 < \dots < t_n = 1$ be any partition of $[0,1]$. If $a \notin P$, then

$$\sum_{k=1}^{n}\varphi\left(\frac{1}{\lambda}|\mathbf{1}_{\{a\}}(t_k) - \mathbf{1}_{\{a\}}(t_{k-1})|\right) = 0$$

for all $\lambda > 0$. If $a \in P$, then there exists $k_0 \in \{0,1,2,\dots,n\}$ such that $t_{k_0} = a$. If $k_0 = 0$ or $k_0 = n$, then

$$\sum_{k=1}^{n}\varphi\left(\frac{1}{\lambda}|\mathbf{1}_{\{a\}}(t_k) - \mathbf{1}_{\{a\}}(t_{k-1})|\right) = \varphi\left(\frac{1}{\lambda}|\mathbf{1}_{\{a\}}\left(t_{k_0}\right)|\right) = \varphi\left(\frac{1}{\lambda}\right) \leq 1$$

whenever $\lambda \geq \frac{1}{\varphi^{-1}(1)}$; while if $k_0 \in \{1,2,\dots,n-1\}$, then we have

$$\sum_{k=1}^{n}\varphi\left(\frac{1}{\lambda}|\mathbf{1}_{\{a\}}(t_k) - \mathbf{1}_{\{a\}}(t_{k-1})|\right) = 2\varphi\left(\frac{1}{\lambda}|\mathbf{1}_{\{a\}}(t_{k_0})|\right) = 2\varphi\left(\frac{1}{\lambda}\right) \leq 1$$

whenever $\lambda \geq \frac{1}{\varphi^{-1}\left(\frac{1}{2}\right)}$. Thus, since φ is an increasing function, we conclude that

$$\|\mathbf{1}_{\{a\}}\|_\varphi \leq \frac{1}{\varphi^{-1}\left(\frac{1}{2}\right)}$$

and we have the upper bound in (2.5). Finally, if we consider the partition $P_0 = \{0, a, 1\}$, then we have

$$\mathrm{Var}_\varphi\left(\frac{1}{\frac{1}{\varphi^{-1}(1)}}\mathbf{1}_{\{a\}}\right) = \mathrm{Var}_\varphi(\varphi^{-1}(1)\mathbf{1}_{\{a\}}) \geq \varphi(\varphi^{-1}(1)\mathbf{1}_{\{a\}}(a)) = 1$$

and by definition of Luxemburg seminorm, we conclude that

$$\|\mathbf{1}_{\{a\}}\|_\varphi \geq \frac{1}{\varphi^{-1}(1)}.$$

And so the proposition is proved. □

Remark. It is important to remark that from the proof of the above proposition, we can see that

$$\|\mathbf{1}_{\{0\}}\|_{BV_\varphi} = \|\mathbf{1}_{\{1\}}\|_{BV_\varphi} = 1 + \frac{1}{\varphi^{-1}(1)},$$

while if $a \in (0, 1)$, then

$$\|\mathbf{1}_{\{a\}}\|_{BV_\varphi} = 1 + \frac{1}{\varphi^{-1}\left(\frac{1}{2}\right)}$$

which gives us a more precise information.

3 Multiplication operators with closed range on $BV_\varphi[0, 1]$

In this section, we characterize all functions $u \in BV_\varphi[0, 1]$ which define multiplication operators $M_u : BV_\varphi[0, 1] \to BV_\varphi[0, 1]$ with closed range. It is convenient, from now, to define the following set:

$$\mathbf{Z}_u = \{t \in [0, 1] : u(t) = 0\};$$

hence, $\mathbf{Z}_u^c = [0, 1] \setminus \mathbf{Z}_u = \{t \in [0, 1] : u(t) \neq 0\}$. Also, for $\varepsilon > 0$ fixed, we set

$$E_\varepsilon = \{t \in [0, 1] : |u(t)| \geq \varepsilon\}, \tag{3.1}$$

with these notations, we have the following result.

Theorem 3.1. *Suppose that $u \in BV_\varphi[0, 1]$. The operator $M_u : BV_\varphi[0, 1] \to BV_\varphi[0, 1]$ has closed range if and only if there exists an $\varepsilon > 0$ such that $Z_u^c = E_\varepsilon$.*

Proof. The result is trivial if u is the null function. Let us suppose first that there exists an $\varepsilon > 0$ such that $Z_u^c \subseteq E_\varepsilon$. Then $|u(t)| \geq \varepsilon$ for all $t \in Z_u^c$. We shall prove that $\mathrm{Ran}(M_u)$ is a closed subset of $BV_\varphi[0, 1]$. To see this, suppose that $h \in \overline{\mathrm{Ran}(M_u)}$. Then there exists a sequence $\{h_n\} \subseteq \mathrm{Ran}(M_u)$ such that

$$\lim_{n\to\infty} \|h_n - h\|_{BV_\varphi} = 0.$$

Then, for each $n \in \mathbb{N}$ we can find $f_n \in BV_\varphi[0, 1]$ such that $h_n = u \cdot f_n$. In particular, $h_n(t) = 0$ for all $t \in Z_u$. This last fact implies that $h(t) = 0$ for all $t \in Z_u$ since

$$|h_n(t) - h(t)| \leq \|h_n - h\|_\infty \leq \|h_n - h\|_{BV_\varphi}$$

for all $t \in [0, 1]$.

Next, we consider the function f_u defined by

$$f_u(t) = \begin{cases} \dfrac{1}{u(t)}, & \text{if } t \in Z_u^c = [0,1]\backslash \mathbf{Z}_u, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

We shall prove that $f_u \in BV_\varphi[0,1]$. To this end, let $P: 0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ be any partition of $[0,1]$; then we have the following cases:

Case I. If $t_k, t_{k-1} \in \mathbf{Z}_u$, then $f_u(t_k) = f_u(t_{k-1}) = 0$ and in this case we have

$$\varphi\left(\frac{1}{\lambda}|f_u(t_k) - f_u(t_{k-1})|\right) = 0$$

for all $\lambda > 0$.

Case II. If $t_k, t_{k-1} \in \mathbf{Z}_u^c$, then

$$\varphi\left(\frac{1}{\lambda}|f_u(t_k) - f_u(t_{k-1})|\right) = \varphi\left(\frac{1}{\lambda}\frac{|u(t_k) - u(t_{k-1})|}{|u(t_k)||u(t_{k-1})|}\right) \le \varphi\left(\frac{1}{\lambda\varepsilon^2}|u(t_k) - u(t_{k-1})|\right) \le \varphi\left(\frac{1}{\|u\|_{BV_\varphi}}|u(t_k) - u(t_{k-1})|\right)$$

whenever $\lambda > \frac{1}{\varepsilon^2}\|u\|_{BV_\varphi}$, where we have used that φ is increasing function and $\mathbf{Z}_u^c = E_\varepsilon$.

Case III. If $t_k \in \mathbf{Z}_u^c$ and $t_{k-1} \in \mathbf{Z}_u$, then

$$\varphi\left(\frac{1}{\lambda}|f_u(t_k) - f_u(t_{k-1})|\right) = \varphi\left(\frac{1}{\lambda}\frac{1}{|u(t_k)|}\right) \le \varphi\left(\frac{1}{\lambda\varepsilon^2}|u(t_k) - u(t_{k-1})|\right) \le \varphi\left(\frac{1}{\|u\|_{BV_\varphi}}|u(t_k) - u(t_{k-1})|\right)$$

whenever $\lambda > \frac{1}{\varepsilon^2}\|u\|_{BV_\varphi}$ where we have used that φ is an increasing function. The above estimation is also valid for the case $t_k \in \mathbf{Z}_u$ and $t_{k-1} \in \mathbf{Z}_u^c$.

Next, taking into account the sum in each of the four cases, we obtain

$$\sum_{k=1}^{m}\varphi\left(\frac{1}{\lambda}|f_u(t_k) - f_u(t_{k-1})|\right) \le \sum_{k=1}^{m}\varphi\left(\frac{1}{\|u\|_{BV_\varphi}}|u(t_k) - u(t_{k-1})|\right) \le \mathrm{Var}_\varphi\left(\frac{1}{\|u\|_{BV_\varphi}}u\right) \le 1$$

whenever $\lambda > \frac{1}{\varepsilon^2}\|u\|_{BV_\varphi}$ and since the partition $P: 0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ was arbitrary, we obtain that

$$\mathrm{Var}_\varphi\left(\frac{1}{\lambda}f_u\right) \le 1$$

and $f_u \in BV_\varphi[0,1]$.

Finally, if we set $f = h \cdot f_u$, we have $f \in BV_\varphi[0,1]$ since this space is an algebra and $h = u \cdot f$ which means that $h \in \mathrm{Ran}(M_u)$. This proves that $\mathrm{Ran}(M_u)$ is a closed subspace of $BV_\varphi[0,1]$.

Next, we suppose that $M_u: BV_\varphi[0,1] \to BV_\varphi[0,1]$ has closed range and that the conclusion is false. Then, for each $n \in \mathbb{N}$ we can find $t_n \in Z_u^c$ such that

$$0 < |u(t_n)| < \frac{1}{n^2}. \tag{3.3}$$

In particular, the sequence $\{t_n\}$ is an infinite set, it is contained in $[0,1]$ and hence it has a convergent subsequence. Thus, we can suppose $\{t_n\}$ is a convergent sequence. Furthermore, by passing to another subsequence we may assume that $\{t_n\}$ is an ordered and convergent set (recall that from a convergent sequence we can build a monotone subsequence) such that $t_n \ne t_m$ for $n \ne m$. Also, from (3.3) we can see that the series $\sum_{k=1}^{\infty} u(t_k)$ is absolutely convergent and the order of the summation does not affect the convergence.

Next, for each $n \in \mathbb{N}$ we consider the set $A_{2n+1} = \{t_1, t_3, t_5, \dots, t_{2n+1}\}$ and we define the function

$$\mathbf{1}_{A_{2n+1}} = \sum_{k=0}^{n} \mathbf{1}_{\{t_{2k+1}\}}.$$

Then clearly $\mathbf{1}_{A_{2n+1}} \in BV_\varphi[0,1]$ because it is a finite sum of functions in this space and hence $h_n = u \cdot \mathbf{1}_{A_{2n+1}} \in \operatorname{Ran}(M_u)$. Furthermore, if $n, m \in \mathbb{N}$ and $n > m$, then

$$\begin{aligned}\|h_n - h_m\|_{BV_\varphi} &= \left\| u \cdot \sum_{k=m+1}^{n} \mathbf{1}_{\{t_{2k+1}\}} \right\|_{BV_\varphi} = \left\| \sum_{k=m+1}^{n} u(t_{2k+1}) \mathbf{1}_{\{t_{2k+1}\}} \right\|_{BV_\varphi} \\ &\leq \sum_{k=m+1}^{n} |u(t_{2k+1})| \|\mathbf{1}_{\{t_{2k+1}\}}\|_{BV_\varphi} \leq \left(1 + \frac{1}{\varphi^{-1}\left(\frac{1}{2}\right)}\right) \sum_{k=m+1}^{n} |u(t_{2k+1})| \to 0 \text{ as } m \to \infty.\end{aligned}$$

Thus, $\{h_n\}$ is a Cauchy sequence in the closed set $\operatorname{Ran}(M_u)$ and because of this fact, there exists a function $f \in BV_\varphi[0,1]$ such that $\|h_n - u \cdot f\|_{BV_\varphi} \to 0$ as $n \to \infty$. In particular, $\|u \cdot \mathbf{1}_{A_{2n+1}} - u \cdot f\|_\infty \to 0$ as $n \to \infty$. Thus, if we fix $m \in \mathbb{N}$ and we consider $t \in A_{2m+1}$, then for all $n \geq m$ we have that $t \in A_{2n+1}$ and therefore

$$|u(t)\mathbf{1}_{A_{2m+1}}(t) - u(t)f(t)| \leq \|u \cdot \mathbf{1}_{A_{2n+1}} - u \cdot f\|_\infty$$

for all $n \geq m$ and hence $|u(t)\mathbf{1}_{A_{2m+1}}(t) - u(t)f(t)| = 0$. This last fact implies that $f(t) = 1$ for all $t \in A_\infty := \{t_1, t_3, t_5, \dots\}$ because $u(t) \neq 0$ for all $t \in A_\infty$ and $f(t_{2n}) = 0$ for all $n \in \mathbb{N}$.

We will see that a function f with this property cannot belong to $BV_\varphi[0,1]$. Indeed, if $f \in BV_\varphi[0,1]$, then we can find a constant $\lambda_f > 0$ such that

$$\operatorname{Var}_\varphi\left(\frac{1}{\lambda_f} f\right) < \infty.$$

Thus for each $n \in \mathbb{N}$, we consider the partition $P = \{0, t_1, t_2, \dots, t_{n-1}, t_n, 1\}$ (remember that $\{t_n\}$ is an ordered set) and we have

$$\operatorname{Var}_\varphi\left(\frac{1}{\lambda_f} f\right) \geq \sum_{k=2}^{n} \varphi\left(\frac{1}{\lambda_f} |f(t_k) - f(t_{k-1})|\right) = (n-1)\varphi\left(\frac{1}{\lambda_f}\right) \to \infty,$$

as $n \to \infty$. Where, in the last equality, we have used the fact that $f(t_{2k-1}) = 1$ and $f(t_{2k}) = 0$. This implies that f does not belong to $BV_\varphi[0,1]$ and we have a contradiction. Therefore, there exists a $\delta > 0$ such that $|u(t)| \geq \delta$ for all $t \in Z_u^c$. □

Remark. As an important application of Theorem 3.1 we can see that when $\varphi(t) = t$, we have Theorem 8 in [18], while if $\varphi(t) = t^p$ with $p > 1$ we have Theorem 7 in [19].

4 On the compactness

In this section, we analyze when a function $u \in BV_\varphi[0,1]$ defines compact multiplication operators M_u acting on $BV_\varphi[0,1]$. We recall that an operator $T : X \to X$ is said to be *compact* if $\{T(x_n)\}$ has a convergent subsequence in X for all bounded sequences $\{x_n\} \subset X$. It is known that the limit of compact operators is also a compact operator and that the identity operator $I : X \to X$ defined by $If = f$ is compact if and only if $\dim(X) < \infty$. We recall that for $\varepsilon > 0$ we set

$$E_\varepsilon = \{t \in [0,1] : |u(t)| \geq \varepsilon\}, \tag{4.1}$$

then we have the following result.

Theorem 4.1. *Suppose that $u \in BV_\varphi[0,1]$. If the operator M_u is compact on $BV_\varphi[0,1]$, then for each $\varepsilon > 0$ the set E_ε is finite.*

Proof. Suppose that $M_u : BV_\varphi[0,1] \to BV_\varphi[0,1]$ is a compact operator and that there exists an $\varepsilon > 0$ such that E_ε is an infinite set. Then E_ε has a sequence $\{t_n\}$ such that $t_n \neq t_m$ for $n \neq m$ and the sequence of functions $\{f_n\}$ defined by

$$f_n(t) = \mathbf{1}_{t_n}(t) = \begin{cases} 1, & \text{if } t = t_n, \\ 0, & \text{if } t \neq t_n, \end{cases} \tag{4.2}$$

are linearly independent and belong to the space

$$X_\varepsilon = \{f \in BV_\varphi[0,1] : f(t) = 0 \quad \forall t \in [0,1] \backslash E_\varepsilon\},$$

hence we can conclude that $\dim(X_\varepsilon) = \infty$. Furthermore, if $\{g_n\}$ is a sequence in X_ε such that $\|g_n - g\|_{BV_\varphi}$ goes to zero as $n \to \infty$ for some $g \in BV_\varphi[0,1]$, then in particular $\|g_n - g\|_\infty \to 0$ as $n \to \infty$ and $h(t) = 0$ for all $t \in [0,1] \backslash E_\varepsilon$ which means that $g \in X_\varepsilon$ and X_ε is a closed subspace of $BV_\varphi[0,1]$.

The inclusion operator $i_{X_\varepsilon} : X_\varepsilon \to BV_\varphi[0,1]$ defined by $i_{X_\varepsilon} f = f$ is continuous, and therefore, the composition $M_u \circ i_{X_\varepsilon} : X_\varepsilon \to BV_\varphi[0,1]$ is a compact operator. We shall show that $\operatorname{Ran}(M_u \circ i_{X_\varepsilon}) = X_{E_\varepsilon}$. It is clear that $\operatorname{Ran}(M_u \circ i_{X_\varepsilon}) \subseteq X_{E_\varepsilon}$ and if $f \in X_\varepsilon$, then we consider the function $h : [0,1] \to \mathbb{R}$ defined by

$$h(t) = \begin{cases} \dfrac{f(t)}{u(t)}, & \text{if } t \in E_\varepsilon, \\ 0, & \text{otherwise}. \end{cases}$$

We have that $f(t) = u(t) \cdot h(t)$ for all $t \in [0,1]$, hence we need to show that $h \in BV_\varphi[0,1]$. To see that, we consider, as in the proof of Theorem 3.1, four cases for the function

$$h_u(t) = \begin{cases} \dfrac{1}{u(t)}, & \text{if } t \in E_\varepsilon, \\ 0, & \text{otherwise}, \end{cases}$$

depending on $t_k, t_{k-1} \in E_\varepsilon^c$, $t_k, t_{k-1} \in E_\varepsilon$, $t_k \in E_\varepsilon^c$ and $t_{k-1} \in E_\varepsilon$ or $t_k \in E_\varepsilon$ and $t_{k-1} \in E_\varepsilon^c$ and we conclude that $h_u \in BV_\varphi[0,1]$, which implies that $h = f \cdot h_u \in BV_\varphi[0,1]$ and $f \in \operatorname{Ran}(M_u \circ i_{X_\varepsilon})$. This proves what we claimed.

Next, we have that the operator $M_u \circ i_{X_\varepsilon} : X_\varepsilon \to X_\varepsilon$ is a surjective and compact operator and since $M_u \circ i_{X_\varepsilon} : X_\varepsilon \to X_\varepsilon$ is injective, we conclude that $M_u \circ i_{X_\varepsilon} : X_\varepsilon \to X_\varepsilon$ is a bijective and compact operator. Hence, we have arrived to the following conclusion: X_ε is a finite dimensional space, which leads a contradiction with the conclusion of the first part of the proof. □

For the converse, we need to impose a reasonable condition to the Young function φ. The Young function $\varphi : [0,\infty) \to [0,\infty)$ satisfies the global Δ_2 condition if there exists a constant $K_\Delta > 0$ such that

$$\varphi(2t) \leq K_\Delta \varphi(t)$$

for all $t \geq 0$. For instance, the Young function $\varphi_1(t) = t\log(1+t)$ satisfies the global Δ_2 condition but the Young function $\varphi_2(t) = e^t - 1$ does not satisfy the global Δ_2 condition. We also recall that a continuous operator $S : X \to X$ has *finite range* if $\dim(\operatorname{Ran}(S)) < \infty$. It is well known that each operator having finite range is a compact operator. With this notation and definition, we have the following result:

Theorem 4.2. *Suppose that the Young function $\varphi : [0,\infty) \to [0,\infty)$ satisfies the global Δ_2 condition and that $u \in BV_\varphi[0,1]$. If for each $\varepsilon > 0$ the set E_ε is finite, then the operator M_u is compact on $BV_\varphi[0,1]$.*

Proof. Suppose that for each $\varepsilon > 0$ the set E_ε is finite. Then, since

$$Z_u^c = \bigcup_{n=1}^{\infty} E_{\frac{1}{n}} = \bigcup_{n=1}^{\infty} \left\{ t \in [0,1] : |u(t)| \geq \frac{1}{n} \right\},$$

we conclude that Z_u^c is a finite set or a numerable set.

If Z_u^c is a finite set, then we can write $Z_u^c = \{t_1, \dots, t_m\}$ for some $m \in \mathbb{N}$. Thus, the functions $\{h_1, h_2, \dots, h_m\}$ defined by $h_k = u \cdot f_k$, being f_k as in (4.2) are linearly independent and belong to $\operatorname{Ran}(M_u)$. Furthermore, if $h \in \operatorname{Ran}(M_u)$, then $h = u \cdot f$ for some $f \in BV_\varphi[0,1]$. Thus, if we consider the scalars $\alpha_k = f(t_k)$, with $k = 1, 2, \dots, m$, we have

$$h = \sum_{k=1}^{m} \alpha_k h_k$$

and $\dim(\operatorname{Ran}(M_u)) = m < \infty$, that is, M_u has finite range. This last fact implies that the operator M_u is compact on $BV_\varphi[0,1]$.

If Z_u^c is an infinite numerable set, then we can write

$$Z_u^c = \{t_1, t_2, \dots, t_n, \dots\} \subset [0,1]$$

with $t_n \neq t_m$ for $n \neq m$. We claim that the series

$$\sum_{k=1}^{\infty} \varphi(|u(t_k)|) \tag{4.3}$$

is convergent. Indeed, since $u \in BV_\varphi[0,1]$, then there exists $\lambda_u > 0$ such that

$$\operatorname{Var}_\varphi\left(\frac{u}{\lambda_u}\right) < \infty.$$

We fix $m \in \mathbb{N}$, then we can suppose that the set $\{t_1, t_2, \dots, t_m\}$ is ordered and $t_k < t_{k+1}$ for all $k = 1, 2, \dots, m-1$. Thus, for each $k \in \{1, 2, \dots, m\}$ we can find $x_k \in Z_u$ such that $t_k < x_k < t_{k+1}$ and $t_m < x_m < 1$. Hence, we can build the partition

$$P_m : 0 < t_1 < x_1 < t_2 < x_2 < \cdots < t_m < x_m < 1.$$

Then we have

$$\sum_{k=1}^{m} \varphi\left(\frac{|u(t_k)|}{\lambda_u}\right) = \sum_{k=1}^{m} \varphi\left(\frac{|u(x_k) - u(t_k)|}{\lambda_u}\right) \le \operatorname{Var}_\varphi\left(\frac{u}{\lambda_u}\right) < \infty$$

and the series in (4.3) converges absolutely since φ satisfies a Δ_2 condition and the claim is proved. In particular,

$$\lim_{k\to\infty} |u(t_k)| = 0$$

because $\varphi(t) = 0$ if and only if $t = 0$.

Next, for each $n \in \mathbb{N}$, we set the symbol u_n by

$$u_n = \sum_{k=1}^{n} u(t_k)\mathbf{1}_{\{t_k\}}.$$

Then $u_n \in BV_\varphi[0,1]$ since it is a finite sum of functions in $BV_\varphi[0,1]$, and $Z_{u_n}^c = \{t_1, t_2, \dots, t_n\}$ is a finite set. Thus by the first part of the proof we can conclude that the operator M_{u_n} is compact on $BV_\varphi[0,1]$ for each $n \in \mathbb{N}$.

Also, for any $t \in [0,1]$ we have

$$u(t) - u_n(t) = \sum_{k=n+1}^{\infty} u(t_k)\mathbf{1}_{\{t_k\}}(t)$$

and hence $\|u - u_n\|_\infty = \sup_{k\ge n+1} |u(t_k)|$. This last fact implies that

$$\lim_{n\to\infty} \|u - u_n\|_\infty = \lim_{n\to\infty} \sup_{k\ge n+1} |u(t_k)| = \lim_{k\to\infty} \sup |u(t_k)| = 0. \tag{4.4}$$

Next, we shall prove that

$$\lim_{n\to\infty}\|u-u_n\|_\varphi = \inf\left\{\lambda>0:\ \mathrm{Var}_\varphi\left(\frac{u-u_n}{\lambda}\right)\le 1\right\}=0.$$

With this end, we fix $\varepsilon>0$. There exists $H\in\mathbb{N}$, depending on ε, such that

$$\frac{2}{\varepsilon}\le 2^H.$$

Thus, since φ is an increasing function and it satisfies a global Δ_2 condition, we can find a constant $K_\Delta(\varepsilon)>0$, depending only on φ and ε, such that

$$\varphi\left(\frac{2}{\varepsilon}t\right)\le K_\Delta(\varepsilon)\varphi(t) \tag{4.5}$$

for all $t\ge 0$. Next, since the series $\sum_{k=1}^{\infty}\varphi(|u(t_k)|)$ converges, then we can find $N\in\mathbb{N}$ such that

$$2K_\Delta(\varepsilon)\sum_{k=n+1}^{\infty}\varphi(|u(t_k)|)\le 1$$

for all $n\ge N$. For such $n\ge N$ fixed, we consider the set $A_n=\{t_{n+1},t_{n+2},t_{n+3},\dots\}$ and we shall estimate

$$\sum_{k=1}^{m}\varphi\left(\frac{1}{\varepsilon}|h_n(x_k)-h_n(x_{k-1})|\right),$$

where $h_n=u-u_n$ and $P:0=x_0<x_1<x_2<\dots<x_m=1$ is any partition of $[0,1]$. As in the proof of Theorem 3.1, we consider four cases but now we will use the Δ_2-condition:

Case I. If $x_k, x_{k-1}\notin A_n$, then $h_n(x_k)=h_n(x_{k-1})=0$ and in this case we have

$$\varphi\left(\frac{1}{\varepsilon}|h_n(x_k)-h_n(x_{k-1})|\right)=0.$$

Case II. If $x_k, x_{k-1}\in A_n$, then there are $i,j\ge n+1$ such that $x_k=t_j$ and $x_{k-1}=t_i$. In this case $h_n(x_k)=u(t_j)$, $h_n(x_{k-1})=u(t_i)$ and we can write

$$\begin{aligned}\varphi\left(\frac{1}{\varepsilon}|h_n(x_k)-h_n(x_{k-1})|\right)&\le\varphi\left(\frac{1}{\varepsilon}|u(t_j)|+\frac{1}{\varepsilon}|u(t_i)|\right)\\&\le\frac12\varphi\left(\frac{2}{\varepsilon}|u(t_j)|\right)+\frac12\varphi\left(\frac{2}{\varepsilon}|u(t_i)|\right)\\&\le\frac12K_\Delta(\varepsilon)\varphi(|u(t_j)|)+\frac12K_\Delta(\varepsilon)\varphi(|u(t_i)|),\end{aligned}$$

where we have used that φ is increasing, convex and satisfies the condition (4.5).

Case III. If $x_k\in A_n$ and $x_{k-1}\notin A_n$, then there exists $j\ge n+1$ such that $x_k=t_j$. In this case, $h_n(x_{k-1})=0$, $h_n(x_k)=u(t_j)$ and we have

$$\varphi\left(\frac{1}{\varepsilon}|h_n(x_k)-h_n(x_{k-1})|\right)=\varphi\left(\frac{1}{\varepsilon}|u(t_j)|\right)\le\frac12\varphi\left(\frac{2}{\varepsilon}|u(t_j)|\right)\le\frac12K_\Delta(\varepsilon)\varphi(|u(t_j)|),$$

where we have used that φ is a convex function with $\varphi(0)=0$. The above estimation is also valid for the case $x_k\notin A_n$ and $x_{k-1}\in A_n$.

Next, taking into account the sum in each of the four cases, we obtain

$$\sum_{k=1}^{m}\varphi\left(\frac{1}{\varepsilon}|h_n(x_k)-h_n(x_{k-1})|\right)\le 2K_\Delta(\varepsilon)\sum_{i=n+1}^{\infty}\varphi(|u(t_i)|)<1$$

and since the partition $P : 0 = x_0 < x_1 < x_2 < \cdots < x_m = 1$ was arbitrary, we obtain that

$$\mathrm{Var}_\varphi\left(\frac{1}{\varepsilon}h_n\right) \le 1$$

whenever $n \ge N$, which means that $\|u - u_n\|_\varphi < \varepsilon$ whenever $n \ge N$, that is,

$$\lim_{n\to\infty} \|u - u_n\|_\varphi = 0. \tag{4.6}$$

Finally, from (4.4) and (4.6) we can conclude that

$$\lim_{n\to\infty} \|M_{u_n} - M_u\| = \lim_{n\to\infty} \|M_{u_n - u}\| = \lim_{n\to\infty} \|u_n - u\|_{BV_\varphi} = \lim_{n\to\infty} (\|u - u_n\|_\infty + \|u - u_n\|_\varphi) = 0.$$

This proves that $M_u : BV_\varphi[0,1] \to BV_\varphi[0,1]$ is the uniform limit of compact operators and therefore it is a compact operator. □

Remark. As an application of Theorems 4.1 and 4.2 we can see that when $\varphi(t) = t$, we have Theorem 10 in [18], while if $\varphi(t) = t^p$ with $p > 1$ we have Theorem 5 in [19].

Remark. We finish this section remarking that in the first part of the proof of the above theorem we have proven that if the set Z_u^c is finite, then $\dim(\mathrm{Ran}(M_u)) < \infty$. We shall prove that the converse is also true.

Proposition 4.3. *Suppose that* $u \in BV_\varphi[0,1]$. *Then* $\dim(\mathrm{Ran}(M_u)) < \infty$ *if and only if* Z_u^c *is a finite set.*

Proof. Indeed, if Z_u^c is an infinite set, then we can find a sequence $\{t_n\} \subseteq Z_u^c$ such that $t_n \ne t_m$ whenever $n \ne m$ and the functions $h_n := u \cdot \mathbf{1}_{t_n}$ are linearly independent in $\mathrm{Ran}(M_u)$. This means that $\dim(\mathrm{Ran}(M_u)) = \infty$. □

5 Fredholm multiplication operators on $BV_\varphi[0,1]$

In this section, we shall use the technical and the results of the above sections to characterize all functions $u \in BV_\varphi[0,1]$ which define Fredholm multiplication operators on $BV_\varphi[0,1]$. First, we can observe that if $t_1, t_2 \in \mathbf{Z}_u$ and $t_1 \ne t_2$, then $\{\mathbf{1}_{\{t_1\}}, \mathbf{1}_{\{t_2\}}\}$ is a linearly independent set in $\mathrm{Ker}(M_u)$. This simple fact allows us to write the following proposition:

Proposition 5.1. *Suppose that* $u \in BV_\varphi[0,1]$. *Then* $\dim(\mathrm{Ker}(M_u)) < \infty$ *if and only if* $\mathbf{Z}_u$ *is a finite set.*

Proof. Indeed, if $\mathbf{Z}_u$ is an infinite set, then we can find a sequence $\{t_n\} \subseteq \mathbf{Z}_u^c$ such that $t_n \ne t_m$ whenever $n \ne m$ and the functions $\{\mathbf{1}_{\{t_n\}}\}$ are linearly independent in $\mathrm{Ker}(M_u)$. This means that $\dim(\mathrm{Ker}(M_u)) = \infty$.

Conversely, if $\mathbf{Z}_u = \{t_1, t_2, \dots, t_m\}$ and $f \in \mathrm{Ker}(M_u)$, then $u(t)\cdot f(t) = 0$ for all $t \in [0,1]$ and hence $\{t \in [0,1] : f(t) \ne 0\} \subseteq \mathbf{Z}_u$. This last fact implies that

$$f = \sum_{k=1}^{m} \alpha_k \mathbf{1}_{\{t_k\}},$$

which proves that $\mathrm{Ker}(M_u) = \mathrm{span}\{\mathbf{1}_{\{t_1\}}, \mathbf{1}_{\{t_2\}}, \dots, \mathbf{1}_{\{t_m\}}\}$ and $\dim(\mathrm{Ker}(M_u)) = m < \infty$. □

We recall that for a Banach space X, a bounded operator $T : X \to X$ is called upper semi-Fredholm if $\dim(\mathrm{Ker}(T)) < \infty$ and $\mathrm{Ran}(T)$ is a closed set of X, while T is lower semi-Fredholm if $\mathrm{codim}(\mathrm{Ran}(T)) < \infty$. It is well known that the condition $\mathrm{codim}(\mathrm{Ran}(T)) < \infty$ implies that $\mathrm{Ran}(M_u)$ is a closed set of X. The operator $T : X \to X$ is Fredholm if and only if $T : X \to X$ is lower and upper semi-Fredholm. The following is the main result of this section:

Theorem 5.2. *Suppose that $u \in BV_\varphi[0,1]$. The following statements are equivalent:*
(1) $M_u : BV_\varphi[0,1] \to BV_\varphi[0,1]$ *is upper semi-Fredholm;*
(2) $M_u : BV_\varphi[0,1] \to BV_\varphi[0,1]$ *is lower semi-Fredholm;*
(3) $M_u : BV_\varphi[0,1] \to BV_\varphi[0,1]$ *is Fredholm;*
(4) $\mathbf{Z}_u$ *is a finite set and there exists a* $\delta > 0$ *such that* $E_\delta \subseteq \mathbf{Z}_u^c$.

Proof. It is enough to show that if $\mathrm{Ran}(M_u)$ is a closed subset of $BV_\varphi[0,1]$, then

$$\mathrm{codim}(\mathrm{Ran}(M_u)) < \infty \iff \mathbf{Z}_u \text{ is a finite set}. \tag{5.1}$$

Indeed, as before, if $\mathbf{Z}_u = \{t_1, t_2, \dots, t_m\}$ and $\hat{h} \in BV_\varphi[0,1]/\mathrm{Ran}(M_u)$ is arbitrary, then by definition of quotient space we can write

$$\hat{h} = h + \mathrm{Ran}(M_u) = \{h + g : g \in \mathrm{Ran}(M_u)\}$$

for some $h \in BV_\varphi[0,1]$. Thus, since $\mathrm{Ran}(M_u)$ is a closed subset of $BV_\varphi[0,1]$, Theorem 3.1 and the three cases in the proof of Theorem 3.1 imply that the function

$$f_u(t) = \begin{cases} \dfrac{1}{u(t)}, & \text{if } t \in \mathbf{Z}_u^c, \\ 0, & \text{otherwise}, \end{cases}$$

belongs to $BV_\varphi[0,1]$ and hence the function $g = h \cdot f_u$ also belongs to $BV_\varphi[0,1]$. Thus, if for $k = 1, 2, \dots, m$ we set $\alpha_k = h(t_k)$, then for $t \in \mathbf{Z}_u^c$, we have

$$h(t) - \sum_{k=1}^{m} \alpha_k \mathbf{1}_{\{t_k\}}(t) = h(t) = u(t) \cdot g(t);$$

while if $t \in \mathbf{Z}_u$, then

$$h(t) - \sum_{k=1}^{m} \alpha_k \mathbf{1}_{\{t_k\}}(t) = 0 = u(t) \cdot g(t).$$

This means that $h - \sum_{k=1}^{m} \alpha_k \mathbf{1}_{\{t_k\}} \in \mathrm{Ran}(M_u)$ and

$$\hat{h} = \sum_{k=1}^{m} \alpha_k \mathbf{1}_{\{t_k\}} + \mathrm{Ran}(M_u).$$

Therefore, $BV_\varphi[0,1]/\mathrm{Ran}(M_u) \subseteq \mathrm{span}\{\mathbf{1}_{\{t_k\}} + \mathrm{Ran}(M_u) : k = 1, 2, \dots, m\}$ and

$$\mathrm{codim}(\mathrm{Ran}(M_u)) = \dim(BV_\varphi[0,1]/\mathrm{Ran}(M_u)) \le m < \infty.$$

If $\mathbf{Z}_u$ is an infinite set, then we can find a sequence $\{t_1, t_2, \dots\} \subseteq \mathbf{Z}_u$ such that $t_n \neq t_m$ whenever $n \neq m$ and the set $\{\mathbf{1}_{\{t_k\}} + \mathrm{Ran}(M_u) : k \in \mathbb{N}\}$ is linearly independent in $BV_\varphi[0,1]/\mathrm{Ran}(M_u)$ which means that $\mathrm{codim}(\mathrm{Ran}(M_u)) = \infty$.

Finally, by definition, Proposition 5.1 and Theorem 3.1 we have that statements (1) and (4) are equivalent. While that if (2) holds, then $\mathrm{Ran}(M_u)$ is a closed set of $BV_\varphi[0,1]$ and by the equivalence (5.1) we have that $\mathbf{Z}_u$ is a finite set. Therefore, $\dim(\mathrm{Ker}(M_u)) < \infty$ and $M_u : BV_\varphi[0,1] \to BV_\varphi[0,1]$ is upper semi-Fredholm. That is, (2) and (4) also are equivalent. This proves the result. □

Remark. As an application of Theorem 5.2 we can see that when $\varphi(t) = t$, we have Theorem 11 in [18], while if $\varphi(t) = t^p$ with $p > 1$ we have Theorem 8 in [19].

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References

[1] C. Jordan, *Sur la série de Fourier*, Comptes Rendus de l'Académie des Sciences **2** (1881), 228–230.

[2] A. I. Vol'pert and S. I. Hudjacv, *Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics, Mechanics: Analysis*, vol. 8, Martinus Nijhoff Publishers, Dordrecht, Netherlands, 1985.

[3] N. Wiener, *The quadratic variation of function and its Fourier coefficients*, Massachusetts J. Math. **3** (1924), 72–94, DOI: https://doi.org/10.1002/sapm19243272.

[4] L. C. Young, *Sur une généralisation de la notion de variation de puissance piéme bornée au sens de M. Wiener, et sur la convergence des séries de Fourier*, C.R. Acad. Sci. Paris **204** (1937), 470–472.

[5] V. V. Chistyakov, *Mappings of generalized variation and composition operators*, J. Math. Sci. **11** (2002), 2455–2466, DOI: https://doi.org/10.1023/A:1015018310969.

[6] J. Appell, J. Banaś, and N. Merentes, *Bounded Variation and Around*, vol. 17, De Gruyter Ser. Nonlinear Anal. Appl., Berlin, 2014.

[7] R. K. Singh and A. Kumar, *Multiplication operators and composition operators with closed ranges*, Bull. Austral. Math. Soc. **16** (1977), no. 2, 247–252, DOI: https://doi.org/10.1017/S0004972700023261.

[8] R. K. Singh and A. Kumar, *Compact composition operators*, J. Austral. Math. Soc. Ser. A **28** (1979), no. 3, 309–314, DOI: https://doi.org/10.1017/S1446788700012258.

[9] S. C. Arora, G. Datt, and S. Verma, *Multiplication operators on Lorentz spaces*, Indian J. Math. **48** (2006), no. 3, 317–329.

[10] S. C. Arora, G. Datt, and S. Verma, *Multiplication and composition operators on Lorentz-Bochner spaces*, Osaka J. Math. **45** (2008), no. 3, 629–641.

[11] S. C. Arora, G. Datt, and S. Verma, *Operators on Lorentz sequences spaces*, Math. Bohem. **134** (2009), no. 1, 87–98.

[12] R. E. Castillo, F. A. Vallejo Narvaez, and J. C. Ramos-Fernández, *Multiplication and composition operators on weak L_p spaces*, Bull. Malays. Math. Sci. Soc. **38** (2015), no. 3, 927–973, DOI: https://doi.org/10.1007/s40840-014-0081-1.

[13] R. E. Castillo, H. C. Chaparro, and J. C. Ramos-Fernández, *Orlicz-Lorentz spaces and their multiplication operators*, Hacet. J. Math. Stat. **44** (2015), no. 5, 991–1009.

[14] R. E. Castillo, J. C. Ramos-Fernández, and H. Rafeiro, *Multiplication operators in variable Lebesgue spaces*, Rev. Colombiana Mat. **49** (2015), no. 2, 293–305, DOI: https://doi.org/10.15446/recolma.v49n2.60447.

[15] H. Hudzik, R. Kumar, and R. Kumar, *Matrix multiplication operators on Banach function spaces*, Proc. Indian Acad. Sci. Math. Sci. **116** (2006), 71–81, DOI: https://doi.org/10.1007/BF02829740.

[16] R. E. Castillo, H. Rafeiro, J. C. Ramos-Fernández, and M. Salas-Brown, *Multiplication operator on Köthe spaces: measure of non-compactness and closed range*, Bull. Malays. Math. Sci. Soc. **42** (2019), no. 4, 1523–1534, DOI: https://doi.org/10.1007/s40840-017-0562-0.

[17] J. C. Ramos-Fernández and M. Salas-Brown, *On multiplication operators acting on Köthe sequence spaces*. Afr. Mat. **28** (2017), no. 3–4, 661–667, DOI: https://doi.org/10.1007/s13370-016-0475-3.

[18] F. R. Astudillo-Villalba and J. C. Ramos-Fernández, *Multiplication operators on the space of functions of bounded variation*, Demonstr. Math. **50** (2017), no. 1, 105–115, DOI: https://doi.org/10.1515/dema-2017-0012.

[19] F. R. Astudillo-Villalba, R. E. Castillo, and J. C. Ramos-Fernández, *Multiplication operators on the spaces of functions of bounded p-variation in Wieneras sense*, Real Anal. Exchange **42** (2017), no. 2, 329–344, DOI: https://doi.org/10.14321/realanalexch.42.2.0329.