Research Article

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Numerical methods for time-fractional convection-diffusion problems with high-order accuracy

Abstract: In this paper, we consider the numerical method for solving the two-dimensional time-fractional convection-diffusion equation with a fractional derivative of order α ($1 < \alpha < 2$). By combining the compact difference approach for spatial discretization and the alternating direction implicit (ADI) method in the time stepping, a compact ADI scheme is proposed. The unconditional stability and H^1 norm convergence of the scheme are proved rigorously. The convergence order is $O(\tau^{3-\alpha} + h_1^4 + h_2^4)$, where τ is the temporal grid size and h_1, h_2 are spatial grid sizes in the x and y directions, respectively. It is proved that the method can even attain $(1 + \alpha)$ order accuracy in temporal for some special cases. Numerical results are presented to demonstrate the effectiveness of theoretical analysis.

Keywords: 2D time-fractional convection-diffusion equation, ADI scheme, unconditional stability, convergence

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1 Introduction

Fractional differential equations have attracted considerable interest due to their ability to model many phenomena. They are widely applied in various fields of science and engineering, such as signal processing, anomalous diffusion, wave propagation, and turbulence [1–3]. Generally, these equations are of three types that contain derivatives of fractional order in space, time, or space-time [4]. Because of the nonlocal nature of fractional differential operators, analytical solutions of these equations are not available in most cases. Even if these solutions can be given, the part of special functions makes the computation complex. Therefore, a number of authors proposed numerical methods for solving fractional diffusion equations [5–11].

Currently, there are many algorithms designed to solve one-dimensional problems based on the memory effect in fractional derivatives. For example, many finite difference methods are developed for sub-diffusion equations [12–20]. Particularly, Yuste and Acedo [12] proposed an explicit finite difference method and a von Neumann-type stability analysis for a class of anomalous sub-diffusion equations. They pointed out the difficulty of convergence analysis when implicit methods were considered. Zhuang et al. [13] derived implicit numerical methods for an anomalous sub-diffusion equation by using the energy method. For the time-fractional diffusion equation with variable coefficients, difference schemes of second and fourth order

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of approximation in space and second order in time were constructed by Alikhanov in [14], where the stability and convergence were studied by the method of energy inequalities. In [15], Chen et al. proposed a numerical scheme with first-order temporal accuracy and fourth-order spatial accuracy for a variable-order anomalous sub-diffusion equation. The convergence, stability, and solvability of the numerical scheme were shown via Fourier analysis. Karatay et al. [16] extended the idea of the Crank-Nicholson method to the time-fractional heat equation. By the method of Fourier analysis, they proved that the proposed method is stable and the numerical solution converges to the exact solution with the order $O(\tau^{2-\alpha} + h^2)$. There are some other numerical methods applied for solving fractional diffusion equations. Particularly, Atanackovic et al. [17] used the expansion formula for the fractional derivative to reduce the difficulty of solving the non-linear fractional differential equations arising in mechanics. Mohebbi et al. [18] studied a modified anomalous time-fractional sub-diffusion equation with a nonlinear source term and proposed a finite difference scheme which is temporally first-order accurate and spatially fourth-order accurate. In [19], a time-stepping discontinuous Petrov-Galerkin method with a continuous conforming finite element method in space for time-fractional subdiffusion problems was proposed. In [20], two fully discrete schemes were proposed for the time-fractional sub-diffusion equation with space discretized by the finite element method and time discretized by the fractional linear multistep method.

Obviously, solving two-dimensional fractional problems numerically is difficult, which may be viewed from two aspects. In the first place, fractional derivatives are nonlocal operators and have the character of history dependence, which means that the current function value depends on all the previous values. More specifically, the storage will be very expensive if we adopt low-order methods to spatial discretization. In the second place, it spends a large amount of computational complexity and CPU time if the existing implicit schemes are applied, especially for solving multidimensional problems. Therefore, it is valuable to discuss high-order efficient methods for the high-dimensional fractional equations. As far as I know, there are some work about numerical methods of the high-dimensional fractional problem. Abbaszadeh and Mohebbi [21] established a fourth-order compact solution of a two-dimensional modified anomalous fractional sub-diffusion equation. Some other numerical methods based on compact scheme are referred to in [22–26]. In addition, Zeng et al. [27] investigated the two-dimensional Riesz space fractional nonlinear reaction-diffusion equation, a new alternating direction implicit (ADI) Galerkin-Legendre spectral method was proposed. In [28], two numerical methods for solving a two-dimensional anomalous sub-diffusion equation were presented. The stability, convergence, and solvability were analyzed. Besides, combining the matrix transfer technique with finite difference method, Yang et al. [29,30] proposed a new numerical method for solving a space-time-fractional diffusion equation. Huang et al. [31,32] constructed ADI schemes for two-dimensional time-space-fractional nonlinear diffusion-wave equations by equivalently transforming the problem into their partial integro-differential forms.

This paper is devoted to designing and analyzing efficient numerical methods for multi-dimensional time-fractional convection-diffusion equations. For simplicity, we consider the two-dimensional problem. For higher dimensional cases, the convergence order of our method is better than that of the previous work (see [33]).

This paper deals with the following 2D time-fractional convection-diffusion equation:

$$_{0}^{C}D_{t}^{\alpha}u = \Delta u + au_x + bu_y + cu + f, \qquad (x, y, t) \in \Omega \times (0, T], \tag{1}$$

$$u(x, y, 0) = u^0(x, y), \quad \frac{\partial u(x, y, 0)}{\partial t} = \psi(x, y), \quad (x, y) \in \bar{\Omega} = \Omega \cup \partial\Omega, \tag{2}$$

$$u(x, y, t) = \phi(x, y, t), \qquad (x, y, t) \in \partial\Omega \times (0, T], \tag{3}$$

where $_{0}^{C}D_{t}^{\alpha}$ denotes the Caputo derivative operator defined as

$$_{0}^{C}D_{t}^{\alpha}u(x, y, t) = \frac{1}{\Gamma(2 - \alpha)} \int_{0}^{t} \frac{\partial^2 u(x, y, s)}{\partial s^2}(t - s)^{1-\alpha} \mathrm{d}s, \quad 1 < \alpha < 2, \tag{4}$$

where a, b, c are constants with $a^2 + b^2 - 4c \geq 0$, Δ is the two-dimensional Laplacian, $\Omega = (0, L_1) \times (0, L_2)$, $\partial\Omega$ is the boundary, and f, ϕ, u^0, and ψ are known sufficiently smooth functions.

Equation (1) can be viewed as a generalization of the classical convection diffusion equation. Meanwhile, the case of $1 < \alpha < 2$ models a super-diffusive flow, which may occur in chaotic or turbulent processes through enhanced transport of particles [34]. The review paper by Klafter et al. [35] provided numerous references to physical phenomena in which anomalous diffusion occurs. Compared with the considerable work on sub-diffusion models, solving time-fractional convection-diffusion problems is much more difficult, and there is very little research about them. Recently, for the multidimensional time-fractional diffusion equations with the fractional order lies in (1, 2), some high-order compact schemes have been proposed in [36,37].

Recently, the compact difference scheme for fractional diffusion problems has been developed for promoting the spatial accuracy in [14,15,37]. The advantage of the compact difference scheme is high accuracy in the spatial direction. Therefore, we use the compact exponential difference scheme to solve the time-fractional convection-diffusion problem, and we find that this algorithm is very effective indeed.

In this paper, we aim to design effective and fast numerical methods for solving the problem (1)–(3) and establish the corresponding error estimates. As we mentioned before, we adopt a fourth-order compact difference method for spatial approximation, which needs fewer grid points to produce a highly accurate solution. Therefore, the storage requirement is reduced to some extent. On the other hand, we derive an ADI scheme by utilizing the idea of the ADI method for the parabolic problem. At each time level, only two sets of tri-diagonal linear systems need to be solved, while the size of coefficient matrices is equal to that of the one-dimensional problem. As a result, this method reduces storage requirements and computational complexities greatly.

As for the error estimate, we introduce a new norm denoted by $\tilde{H}^2$, which is equivalent to H^1 norm. We prove that this ADI scheme is unconditionally stable to the initial value and the inhomogeneous term. The temporal convergence order of the proposed scheme is $(3 - \alpha)$ and can attain $(1 + \alpha)$ in some cases. In spatial direction, the scheme has fourth order accuracy. The unconditional stability and $\tilde{H}^2$ norm convergence of the scheme are proved by the discrete energy method.

The content is organized as follows. In Section 2, some notations and preliminary lemmas are introduced, and a compact ADI scheme is derived and the truncation error is analyzed. In Section 3, the unique solvability, unconditional stability, and H^1 norm convergence are proved. In Section 4, two numerical examples are given to support the theoretical result. Some comments are presented in Section 5.

2 Construction of the compact ADI scheme and the truncation error

2.1 Notations and preliminary lemmas

For the finite difference approximation, let N be a positive integer, $\tau = \frac{T}{N}$, and $t_n = n\tau (0 \leq n \leq N)$. The time domain $[0, T]$ is covered by $\{t_n | 0 \leq n \leq N\}$. In addition, for a given grid function $w = \{w^n | 0 \leq n \leq N\}$, we denote

$$w^{n-\frac{1}{2}} = \frac{1}{2}(w^n + w^{n-1}), \quad \delta_t w^{n-\frac{1}{2}} = \frac{\omega^n - \omega^{n-1}}{\tau}.$$

For spatial approximation, let $h_1 = \frac{L_1}{M_1}$, $h_2 = \frac{L_2}{M_2}$ with positive integers M_1, M_2, $h = \max\{h_1, h_2\}$, $x_i = ih_1$ $(0 \leq i \leq M_1)$, and $y_j = jh_2 (0 \leq j \leq M_2)$. Let $\bar{\Omega}_h = \{(x_i, y_j) | 0 \leq i \leq M_1, 0 \leq j \leq M_2\}$, $\Omega_h = \bar{\Omega}_h \cap \Omega$, and $\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega$. For any grid function $v = \{v_{ij} | 0 \leq i \leq M_1, 0 \leq j \leq M_2\}$ defined on $\bar{\Omega}_h$, we denote

$$\delta_x v_{i-\frac{1}{2},j} = \frac{v_{i,j} - v_{i-1,j}}{h_1}, \quad \delta_x^2 v_{ij} = \frac{\delta_x v_{i+\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j}}{h_1},$$

$$\delta_y \delta_x v_{i-\frac{1}{2},j-\frac{1}{2}} = \frac{\delta_x v_{i-\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j-1}}{h_2}, \quad \delta_y \delta_x^2 v_{i,j-\frac{1}{2}} = \frac{\delta_x^2 v_{i,j} - \delta_x^2 v_{i,j-1}}{h_2}.$$

Similar notations $\delta_y v_{i,j-\frac{1}{2}}$, and $\delta_y^2 v_{ij}$, $\delta_x \delta_y^2 v_{i-\frac{1}{2},j}$, $\delta_x^2 \delta_y^2 v_{ij}$ can be defined, respectively. The discrete Laplace operator is denoted as $\Delta_h v_{ij} = \delta_x^2 v_{ij} + \delta_y^2 v_{ij}$.

For convenience of writing, we also define

$$
L_x v_{ij} = \begin{cases} \dfrac{1}{12}(v_{i-1,j} + 10v_{ij} + v_{i+1,j}), & 1 \le i \le M_1 - 1, \quad 0 \le j \le M_2, \\ v_{ij}, & i = 0 \text{ or } M_1, \quad 0 \le j \le M_2, \end{cases}
$$

$$
L_y v_{ij} = \begin{cases} \dfrac{1}{12}(v_{i,j-1} + 10v_{ij} + v_{i,j+1}), & 1 \le j \le M_2 - 1, \quad 0 \le i \le M_1, \\ v_{ij}, & j = 0 \text{ or } M_2, \quad 0 \le i \le M_1. \end{cases}
$$

One verifies readily that

$$
L_x v_{ij} = \left(I + \frac{h_1^2}{12}\delta_x^2\right)v_{ij}, \quad L_y v_{ij} = \left(I + \frac{h_2^2}{12}\delta_y^2\right)v_{ij}, \quad (x_i, y_j) \in \Omega_h,
$$

where I denotes the standard identical operator. Moreover, we introduce the following notations:

$$
L v_{ij} = L_x L_y v_{ij}, \quad \Lambda_h v_{ij} = (L_y \delta_x^2 + L_x \delta_y^2)v_{ij}, \quad (x_i, y_j) \in \Omega_h.
$$

The following lemmas will be used in derivation of the difference scheme.

Lemma 2.1. [38] *Suppose* $1 < \alpha < 2$, $y \in C^2[0, t_n]$. *It holds that*

$$
\left| \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{y'(s)\,ds}{(t_n - s)^{\alpha-1}} - \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0 y(t_n) - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})y(t_k) - a_{n-1}y(0) \right] \right|
$$

$$
\le \frac{1}{\Gamma(3-\alpha)}\left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1 + 2^{1-\alpha}) \right] \max_{0 \le t \le t_n} |y''(t)|\, \tau^{3-\alpha},
$$

where

$$
a_k = (k+1)^{2-\alpha} - k^{2-\alpha}, \quad k \ge 0. \tag{5}
$$

Lemma 2.2. [39] *Let* $y \in C^3[t_{k-1}, t_k]$. *It holds that*

$$
\frac{1}{2}[y'(t_k) + y'(t_{k-1})] = \frac{1}{\tau}[y(t_k) - y(t_{k-1})] + \frac{\tau^2}{16}\int_0^1 \left[y^{(3)}\left(t_{k-\frac{1}{2}} + \frac{s\tau}{2}\right) + y^{(3)}\left(t_{k-\frac{1}{2}} - \frac{s\tau}{2}\right) \right](1 - s^2)\,ds,
$$

where $t_{k-\frac{1}{2}} = t_k - \frac{\tau}{2}$.

Lemma 2.3. [39] *Let* $g(x) \in C^6[x_{i-1}, x_{i+1}]$ *and* $\zeta(s) = 5(1-s)^3 - 3(1-s)^5$, *then*

$$
\frac{g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})}{12} = \frac{g(x_{i+1}) - 2g(x_i) + g(x_{i-1})}{h_1^2} + \frac{h_1^4}{360}\int_0^1 [g^{(6)}(x_i - sh_1) + g^{(6)}(x_i + sh_1)]\zeta(s)\,ds.
$$

2.2 Derivation of the compact ADI scheme

In this subsection, we will give a compact ADI scheme for (1)–(3). In order to keep fourth-order accuracy in space and tridiagonal nature of the scheme, we first introduce a transformation, which is similar to that of Liao [40]. To eliminate the convection terms in (1), we let

$$
u(x, y, t) = \lambda_1(x)\lambda_2(y)v(x, y, t), \tag{6}
$$

where $\lambda_1(x) = \exp\left(-\frac{a}{2}x\right)$, $\lambda_2(y) = \exp\left(-\frac{b}{2}y\right)$.

A combination of (1)–(3) and (6) leads to the following fractional diffusion-wave equation satisfied by $v = v(x, y, t)$:

$$\begin{align} {}_0^C D_t^\alpha v &= \Delta v + dv + g, & (x, y, t) \in \Omega \times (0, T], \tag{7} \end{align}$$

$$\begin{align} v(x, y, 0) &= (\lambda_1 \lambda_2)^{-1} u^0(x, y) = v^0(x, y), & (x, y) \in \bar\Omega = \Omega \cup \partial\Omega, \tag{8} \end{align}$$

$$\begin{align} \frac{\partial v(x, y, 0)}{\partial t} &= (\lambda_1 \lambda_2)^{-1} \psi(x, y) = \psi_v(x, y), & (x, y) \in \bar\Omega = \Omega \cup \partial\Omega, \tag{9} \end{align}$$

$$\begin{align} v(x, y, t) &= (\lambda_1 \lambda_2)^{-1} \phi(x, y, t) = \phi_v(x, y, t), & (x, y, t) \in \partial\Omega \times (0, T], \tag{10} \end{align}$$

where $d = c - \frac{a^2 + b^2}{4}$, $(\lambda_1 \lambda_2)^{-1} = \frac{1}{\lambda_1(x)\lambda_2(y)} = \exp((1/2)(ax + by))$, and $g = (\lambda_1\lambda_2)^{-1}f$.

After the transformation, we introduce a high-order compact difference method for (7)–(10). Let

$$p = v_t, \quad q = v_{xx}, \quad r = v_{yy},$$

then (7) can be rewritten as

$$\frac{1}{\Gamma(2 - \alpha)} \int_0^t \frac{\partial p(x, y, s)}{\partial s} (t - s)^{1-\alpha} ds = q(x, y, t) + r(x, y, t) + dv(x, y, t) + g(x, y, t). \tag{11}$$

In addition, we define grid functions

$$V_{ij}^n = v(x_i, y_j, t_n), \quad P_{ij}^n = p(x_i, y_j, t_n), \quad Q_{ij}^n = q(x_i, y_j, t_n), \quad R_{ij}^n = r(x_i, y_j, t_n),$$

and

$$g_{ij}^n = g(x_i, y_j, t_n), \quad (x_i, y_j) \in \bar\Omega_h, \quad 0 \le n \le N.$$

Considering (11) at the point (x_i, y_j, t_n), in view of Lemma 2.1, we derive that

$$\frac{\tau^{1-\alpha}}{\Gamma(3 - \alpha)} \left[a_0 P_{ij}^n - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})P_{ij}^k - a_{n-1}P_{ij}^0 \right] = Q_{ij}^n + R_{ij}^n + g_{ij}^n + dV_{ij}^n + (R_t^\alpha)_{ij}^n, \tag{12}$$

for $(x_i, y_j) \in \Omega_h, 1 \le n \le N$, where

$$|(R_t^\alpha)_{ij}^n| \le \frac{1}{\Gamma(3 - \alpha)} \left[\frac{2 - \alpha}{12} + \frac{2^{3-\alpha}}{3 - \alpha} - (1 + 2^{1-\alpha}) \right] \max_{0 \le t \le t_n} \left| \frac{\partial^2 p}{\partial t^2}(x_i, y_j, t) \right| \tau^{3-\alpha}.$$

For $2 \le n \le N$, it holds that

$$\frac{\tau^{1-\alpha}}{\Gamma(3 - \alpha)} \left[a_0 P_{ij}^{n-1} - \sum_{k=1}^{n-2}(a_{n-k-2} - a_{n-k-1})P_{ij}^k - a_{n-2}P_{ij}^0 \right] = Q_{ij}^{n-1} + R_{ij}^{n-1} + g_{ij}^{n-1} + dV_{ij}^{n-1} + (R_t^\alpha)_{ij}^{n-1}.$$

According to the equality

$$-\sum_{k=1}^{n-2}(a_{n-k-2} - a_{n-k-1})P_{ij}^k - a_{n-2}P_{ij}^0 = -\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})P_{ij}^{k-1} - a_{n-1}P_{ij}^0,$$

we get

$$\begin{align} &\frac{\tau^{1-\alpha}}{\Gamma(3 - \alpha)} \left[a_0 P_{ij}^{n-1} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})P_{ij}^{k-1} - a_{n-1}P_{ij}^0 \right] \\ &= Q_{ij}^{n-1} + R_{ij}^{n-1} + g_{ij}^{n-1} + dV_{ij}^{n-1} + (R_t^\alpha)_{ij}^{n-1}, \quad (x_i, y_j) \in \Omega_h, \ 2 \le n \le N. \end{align} \tag{13}$$

Let $t = 0$ in (11). It holds that

$$0 = Q_{ij}^0 + R_{ij}^0 + dv_{ij}^0 + g_{ij}^0,$$

which means that (13) holds for $n = 1$ and $(R_t^\alpha)_{ij}^0 = 0$. Due to (12)–(13), using the notation $P_{ij}^{k-\frac{1}{2}} = \frac{1}{2}(P_{ij}^k + P_{ij}^{k-1})$ gives

$$\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\left[a_0 P_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})P_{ij}^{k-\frac{1}{2}} - a_{n-1}P_{ij}^0\right]$$

$$= Q_{ij}^{n-\frac{1}{2}} + R_{ij}^{n-\frac{1}{2}} + g_{ij}^{n-\frac{1}{2}} + dV_{ij}^{n-\frac{1}{2}} + (R_t^\alpha)_{ij}^{n-\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N. \tag{14}$$

Operating $L = L_x L_y$ on the above equality, we have

$$\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}L\left[a_0 P_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})P_{ij}^{k-\frac{1}{2}} - a_{n-1}P_{ij}^0\right]$$

$$= L_y L_x Q_{ij}^{n-\frac{1}{2}} + L_x L_y R_{ij}^{n-\frac{1}{2}} + Lg_{ij}^{n-\frac{1}{2}} + dLV_{ij}^{n-\frac{1}{2}} + L(R_t^\alpha)_{ij}^{n-\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N. \tag{15}$$

Lemma 2.2 implies that

$$P_{ij}^{n-\frac{1}{2}} = \delta_t V_{ij}^{n-\frac{1}{2}} + (R_t)_{ij}^{n-\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N, \tag{16}$$

where

$$(R_t)_{ij}^{n-\frac{1}{2}} = \frac{\tau^2}{16}\int_0^1\left[\frac{\partial^3 v}{\partial t^3}\left(x_i, y_j, t_{n-\frac{1}{2}} + \frac{s\tau}{2}\right) + \frac{\partial^3 v}{\partial t^3}\left(x_i, y_j, t_{n-\frac{1}{2}} - \frac{s\tau}{2}\right)\right](1 - s^2)\,ds.$$

Besides, Lemma 2.3 implies that

$$L_x Q_{ij}^n = \delta_x^2 V_{ij}^n + (R_x)_{ij}^n, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N,$$
$$L_y R_{ij}^n = \delta_y^2 V_{ij}^n + (R_y)_{ij}^n, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N,$$

where

$$(R_x)_{ij}^n = \frac{h_1^4}{360}\int_0^1\left[\frac{\partial^6 v}{\partial x^6}(x_i - sh_1, y_j, t_n) + \frac{\partial^6 v}{\partial x^6}(x_i + sh_1, y_j, t_n)\right]\zeta(s)\,ds,$$

$$(R_y)_{ij}^n = \frac{h_2^4}{360}\int_0^1\left[\frac{\partial^6 v}{\partial y^6}(x_i, y_j - sh_2, t_n) + \frac{\partial^6 v}{\partial y^6}(x_i, y_j + sh_2, t_n)\right]\zeta(s)\,ds.$$

Obviously,

$$L_y L_x Q_{ij}^{n-\frac{1}{2}} = L_y \delta_x^2 V_{ij}^{n-\frac{1}{2}} + L_y (R_x)_{ij}^{n-\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N, \tag{17}$$

$$L_x L_y R_{ij}^{n-\frac{1}{2}} = L_x \delta_y^2 V_{ij}^{n-\frac{1}{2}} + L_x (R_y)_{ij}^{n-\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N. \tag{18}$$

Substituting (16)–(18) into (15), we obtain

$$\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}L\left[a_0 \delta_t V_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\delta_t V_{ij}^{k-\frac{1}{2}} - a_{n-1}P_{ij}^0\right]$$

$$= \Lambda_h V_{ij}^{n-\frac{1}{2}} + Lg_{ij}^{n-\frac{1}{2}} + dLV_{ij}^{n-\frac{1}{2}} + (R_1)_{ij}^{n-\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N, \tag{19}$$

where

$$(R_1)_{ij}^{n-\frac{1}{2}} = L(R_t^\alpha)_{ij}^{n-\frac{1}{2}} - \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}L\left[a_0 (R_t)_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(R_t)_{ij}^{k-\frac{1}{2}}\right] + L_y (R_x)_{ij}^{n-\frac{1}{2}} + L_x (R_y)_{ij}^{n-\frac{1}{2}}.$$

In view of the estimate

$$(R_t)_{ij}^{n-\frac{1}{2}} \leq \frac{\tau^2}{12} \max_{0 \leq t \leq T} \left| \frac{\partial^2 v}{\partial t^2}(x_i, y_j, t) \right|,$$

we derive that

$$\left| -\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} L \left[a_0 (R_t)_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k})(R_t)_{ij}^{k-\frac{1}{2}} \right] \right| \leq \frac{\tau^{3-\alpha}}{6\Gamma(3-\alpha)} \max_{0 \leq t \leq T} \left| \frac{\partial^2 v}{\partial t^2}(x_i, y_j, t) \right|.$$

As a consequence of the aforementioned facts, we find that

$$\left| (R_1)_{ij}^{n-\frac{1}{2}} \right| \leq C_1 (\tau^{3-\alpha} + h_1^4 + h_2^4), \quad (x_i, y_j) \in \Omega_h, \ 1 \leq n \leq N, \tag{20}$$

where C_1 is a positive constant, which is dependent on the regularity of solution $v(x, y, t)$ but independent of the time step size τ, grid spacing h_1, h_2, and time level n.

We add small term $-\frac{\Gamma(3-\alpha)\tau^{\alpha+1}}{4 - 2\Gamma(3-\alpha)\mathrm{d}\tau^\alpha} \delta_x^2 \delta_y^2 \delta_t V_{ij}^{n-\frac{1}{2}}$ into (19) in order to rewrite (19) in a familiar ADI form, thus we have

$$\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} L \left[a_0 \delta_t V_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \delta_t V_{ij}^{k-\frac{1}{2}} - a_{n-1} P_{ij}^0 \right]$$
$$= \Lambda_h V_{ij}^{n-\frac{1}{2}} - \frac{\Gamma(3-\alpha)\tau^{\alpha+1}}{4 - 2\Gamma(3-\alpha)\mathrm{d}\tau^\alpha} \delta_x^2 \delta_y^2 \delta_t V_{ij}^{n-\frac{1}{2}} + Lg_{ij}^{n-\frac{1}{2}} + dLV_{ij}^{n-\frac{1}{2}} + (R)_{ij}^{n-\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \leq n \leq N, \tag{21}$$

where

$$(R)_{ij}^{n-\frac{1}{2}} = (R_1)_{ij}^{n-\frac{1}{2}} + \frac{\Gamma(3-\alpha)\tau^{\alpha+1}}{4 - 2\Gamma(3-\alpha)\mathrm{d}\tau^\alpha} \delta_x^2 \delta_y^2 \delta_t V_{ij}^{n-\frac{1}{2}}.$$

Furthermore, from the initial and boundary value conditions, it follows that

$$P_{ij}^0 = \psi_v(x_i, y_j), \qquad (x_i, y_j) \in \Omega_h, \tag{22}$$

$$V_{ij}^0 = v^0(x_i, y_j), \qquad (x_i, y_j) \in \bar{\Omega}_h, \tag{23}$$

$$V_{ij}^n = \phi_v(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h, \ 1 \leq n \leq N. \tag{24}$$

Substituting (22) into (21), omitting the small term $(R)_{ij}^n$, and replacing the function V_{ij}^n with its numerical approximation v_{ij}^n in (21) and (23)–(24), we obtain the following difference scheme:

$$\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} L \left[a_0 \delta_t v_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \delta_t v_{ij}^{k-\frac{1}{2}} - a_{n-1} (\psi_v)_{ij} \right]$$
$$= \Lambda_h v_{ij}^{n-\frac{1}{2}} - \frac{\Gamma(3-\alpha)\tau^{\alpha+1}}{4 - 2\Gamma(3-\alpha)\mathrm{d}\tau^\alpha} \delta_x^2 \delta_y^2 \delta_t v_{ij}^{n-\frac{1}{2}} + Lg_{ij}^{n-\frac{1}{2}} + dLv_{ij}^{n-\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \leq n \leq N, \tag{25}$$

$$v_{ij}^n = \phi_v(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h, \ 1 \leq n \leq N, \tag{26}$$

$$v_{ij}^0 = v^0(x_i, y_j), \quad (x_i, y_j) \in \bar{\Omega}_h, \tag{27}$$

where a_k is defined as in (5), and $(\psi_v)_{ij} = \psi_v(x_i, y_j)$.

For convenience, we denote $\mu = \Gamma(3-\alpha)\tau^{\alpha-1}$ throughout the paper. In order to rewrite (25) in a more familiar ADI form, we multiply (25) by μ and note that $a_0 = 1$. Hence,

$$L\delta_t v_{ij}^{n-\frac{1}{2}} - \mu\Lambda_h v_{ij}^{n-\frac{1}{2}} + \frac{\mu^2\tau^2}{4 - 2d\mu\tau} \delta_x^2 \delta_y^2 \delta_t v_{ij}^{n-\frac{1}{2}}$$
$$= \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) L\delta_t v_{ij}^{k-\frac{1}{2}} + a_{n-1} L(\psi_v)_{ij} + L\mu g_{ij}^{n-\frac{1}{2}} + d\mu L v_{ij}^{n-\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \leq n \leq N,$$

i.e.,

$$
\left(L_x - \frac{\tau\mu}{2 - \mathrm{d}\mu\tau}\delta_x^2\right)\left(L_y - \frac{\tau\mu}{2 - \mathrm{d}\mu\tau}\delta_y^2\right)v_{ij}^n = \frac{2\tau}{2 - \mathrm{d}\mu\tau}\left[\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})L\delta_t v_{ij}^{k-\frac{1}{2}} + a_{n-1}L(\psi_v)_{ij} + L\mu g_{ij}^{n-\frac{1}{2}}\right]
$$

$$
+ \left[\frac{2 + \mathrm{d}\mu\tau}{2 - \mathrm{d}\mu\tau}L + \frac{\mu\tau}{2 - \mathrm{d}\mu\tau}\Lambda_h + \frac{\mu^2\tau^2}{(2 - \mathrm{d}\mu\tau)^2}\delta_x^2\delta_y^2\right]v_{ij}^{n-1},
$$

$$
(x_i, y_j) \in \Omega_h, \ \ 1 \le n \le N.
$$

We calculate $\{v_{ij}^n\}$ by solving two sets of independent one-dimensional problems. By introducing intermediate variables

$$
v_{ij}^* = \left(L_y - \frac{\tau\mu}{2 - \mathrm{d}\mu\tau}\delta_y^2\right)v_{ij}^n, \quad 0 \le i \le M_1, \ 1 \le j \le M_2 - 1,
$$

we firstly solve the following one-dimensional problem for fixed $j \in \{1, 2, \ldots, M_2 - 1\}$ to compute $\{v_{ij}^*\}$:

$$
\begin{cases}
\left(L_x - \dfrac{\tau\mu}{2 - \mathrm{d}\mu\tau}\delta_x^2\right)v_{ij}^* = \left[\dfrac{2 + \mathrm{d}\mu\tau}{2 - \mathrm{d}\mu\tau}L + \dfrac{\mu\tau}{2 - \mathrm{d}\mu\tau}\Lambda_h + \dfrac{\mu^2\tau^2}{(2 - \mathrm{d}\mu\tau)^2}\delta_x^2\delta_y^2\right]v_{ij}^{n-1} \\
\quad + \dfrac{2\tau}{2 - \mathrm{d}\mu\tau}\left[\displaystyle\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})L\delta_t v_{ij}^{k-\frac{1}{2}} + a_{n-1}L(\psi_v)_{ij} + L\mu g_{ij}^{n-\frac{1}{2}}\right], \quad 1 \le i \le M_1 - 1, \\
v_{0j}^* = \left(L_y - \dfrac{\tau\mu}{2 - \mathrm{d}\mu\tau}\delta_y^2\right)v_{0j}^n, \quad v_{M_1 j}^* = \left(L_y - \dfrac{\tau\mu}{2 - \mathrm{d}\mu\tau}\delta_y^2\right)v_{M_1 j}^n.
\end{cases}
$$

Once $\{v_{ij}^*\}$ is available, we solve the following system for fixed $i \in \{1, 2, \ldots, M_1 - 1\}$:

$$
\begin{cases}
\left(L_y - \dfrac{\tau\mu}{2 - \mathrm{d}\mu\tau}\delta_y^2\right)v_{ij}^n = v_{ij}^*, \quad 1 \le j \le M_2 - 1, \\
v_{i0}^n = \phi_v(x_i, y_0, t_n), \quad v_{iM_2}^n = \phi_v\left(x_i, y_{M_2}, t_n\right).
\end{cases}
$$

Hence, we obtain the wanted solution $\{v_{ij}^n\}$.

2.3 Solvability and truncation error

As we analyzed before, there are two sets of one-dimensional linear systems that need to be solved at each time level. It is clear to see that the coefficient matrices are strictly diagonally dominant; therefore, we have the following result.

Theorem 2.1. *The difference scheme* (25)–(27) *is uniquely solvable.*

It remains to give the estimate of the truncation error, which will be used in the convergence analysis.

Lemma 2.4. *Suppose that* $v(x, y, t) \in C_{x,y,t}^{6,6,3}(\Omega \times [0, T])$ *is the solution of the problem* (1)–(3). *Then the truncation error of the scheme* (25)–(27) *satisfies*

$$
\left|(R)_{ij}^{n-\frac{1}{2}}\right| \le C_R(\tau^{3-\alpha} + h_1^4 + h_2^4), \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N,
$$

where C_R *is a positive constant independent of* τ, h_1, *and* h_2.

Proof. Since the estimate of $(R_1)_{ij}^{n-\frac{1}{2}}$ has been given in (20), it remains to estimate the small term $\frac{\Gamma(3-\alpha)\tau^{\alpha+1}}{4 - 2\Gamma(3-\alpha)\mathrm{d}\tau^\alpha}\delta_x^2\delta_y^2\delta_t V_{ij}^{n-\frac{1}{2}}$.

A direct calculation by using the Taylor expansion with integral remainder leads to

$$\delta_x^2 \delta_y^2 V_{ij}^n = D(x_i, y_j, t_n),$$

where

$$D(x, y, t) = \int\limits_0^1\int\limits_0^1 (1 - \lambda)(1 - \zeta)\left[\frac{\partial^4 v}{\partial^2 x \partial^2 y}(x - \lambda h_1, y - \zeta h_2, t) + \frac{\partial^4 v}{\partial^2 x \partial^2 y}(x + \lambda h_1, y - \zeta h_2, t) \right.$$
$$\left. + \frac{\partial^4 v}{\partial^2 x \partial^2 y}(x - \lambda h_1, y + \zeta h_2, t) + \frac{\partial^4 v}{\partial^2 x \partial^2 y}(x + \lambda h_1, y + \zeta h_2, t) \right] \mathrm{d}\lambda \mathrm{d}\zeta.$$

Hence, we have

$$\delta_x^2 \delta_y^2 \delta_t V_{ij}^{n-\frac{1}{2}} = \int\limits_0^1 \frac{\partial D}{\partial s}(x_i, y_j, t_{n-1} + s\tau)\mathrm{d}s, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N$$

and

$$\frac{\Gamma(3 - \alpha)\tau^{\alpha+1}}{4 - 2\Gamma(3 - \alpha)\mathrm{d}\tau^\alpha}\delta_x^2 \delta_y^2 \delta_t V_{ij}^{n-\frac{1}{2}} = \frac{\Gamma(3 - \alpha)\tau^{\alpha+1}}{4 - 2\Gamma(3 - \alpha)\mathrm{d}\tau^\alpha} \int\limits_0^1 \frac{\partial D}{\partial s}(x_i, y_j, t_{n-1} + s\tau)\mathrm{d}s = O(\tau^{1+\alpha}).$$

In view of the fact that $1 < \alpha < 2$ and $1 + \alpha > 2 > 3 - \alpha$, we complete the proof.　　　□

Remark 2.1. From Lemmas 2.1, 2.2, and 2.4, it follows that if the solution $v(x, y, t) \in C_{x,y,t}^{6,6,3}(\Omega \times [0, T])$ of (7)–(10) satisfies $\frac{\partial^3 v(x, y, t)}{\partial t^3} = 0$ for $(x, y, t) \in \Omega \times [0, T]$, then the truncation error of the scheme (25)–(27) satisfies

$$\left| (R)_{ij}^{n-\frac{1}{2}} \right| \le \tilde{C}_R \left(\tau^{1+\alpha} + h_1^4 + h_2^4 \right), \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N,$$

where $\tilde{C}_R$ is a positive constant independent of τ, h_1, and h_2.

3 Stability and convergence of the compact ADI scheme

3.1 Stability

In order to analyze the stability of the proposed scheme, we introduce the space of grid functions on $\bar{\Omega}_h$,

$$V_h = \{v|v = \{v_{ij}|(x_i, y_j) \in \bar{\Omega}_h\} \quad \text{and} \quad v_{ij} = 0 \quad \text{if} \quad (x_i, y_j) \in \partial\Omega_h\}.$$

In addition, for any grid function $w, v \in V_h$, we define the discrete inner product by

$$\langle w, v \rangle_h = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} w_{ij} \cdot v_{ij}$$

and denote $\|v\| = \sqrt{\langle v, v \rangle_h}$. Similar notations $\|\delta_x^2 v\|$, $\|\delta_y^2 v\|$, $\|L_x v\|$, $\|L_y v\|$, $\|Lv\|$, and $\|\delta_x^2 \delta_y^2 v\|$ can be defined immediately. Furthermore, we also denote

$$\|\delta_x v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2-1} \left| \delta_x v_{i-\frac{1}{2},j} \right|^2}, \quad \|\delta_x \delta_y v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \left| \delta_x \delta_y v_{i-\frac{1}{2}, j-\frac{1}{2}} \right|^2}$$

and

$$\|\delta_y \delta_x^2 v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2} \left| \delta_y \delta_x^2 v_{i,j-\frac{1}{2}} \right|^2}.$$

Then we denote $\|\delta_y v\|$, $\|\delta_x \delta_y^2 v\|$ in a similar way. The discrete H^1 seminorm and H^1 norm of the grid function $v \in V_h$ are defined as

$$\|\nabla_h v\| = \sqrt{\|\delta_x v\|^2 + \|\delta_y v\|^2}, \quad \|v\|_{H^1} = \sqrt{\|v\|^2 + \|\nabla_h v\|^2}.$$

For any grid function $v \in V_h$, we denote

$$\|v\|_A = \sqrt{\|\delta_x v\|^2 - \frac{h_1^2}{12} \|\delta_x^2 v\|^2}, \quad \|v\|_B = \sqrt{\|\delta_y v\|^2 - \frac{h_2^2}{12} \|\delta_y^2 v\|^2}$$

and

$$\|v\|_{\tilde{H}^1} = \sqrt{\|L_y v\|_A^2 + \|L_x v\|_B^2}, \quad \|v\|_{\tilde{H}^2} = \sqrt{\|L_y v\|_A^2 + \|L_x v\|_B^2 - d\|Lv\|^2}.$$

As we will see later, the norm $\|\cdot\|_{\tilde{H}^2}$ is equivalent to the standard H^1 norm and is more convenient than the standard H^1 norm for the stability and convergence analysis.

Lemma 3.1. [41] *For any grid function $v \in V_h$, it holds that*

$$\sqrt{\frac{48(L_1^2 + L_2^2)}{27[6(L_1^2 + L_2^2) + L_1^2 L_2^2]}} \|v\|_{H^1} \leq \|v\|_{\tilde{H}^1} \leq \frac{4}{3} \|v\|_{H^1}.$$

Lemma 3.2. *For any grid function $v \in V_h$, it holds that*

$$\sqrt{\frac{48(L_1^2 + L_2^2)}{27[6(L_1^2 + L_2^2) + L_1^2 L_2^2]}} \|v\|_{H^1} \leq \|v\|_{\tilde{H}^2} \leq \sqrt{\frac{16}{9} - \frac{485d}{162}} \|v\|_{H^1}.$$

Proof. As a consequence of $\|v\|_{\tilde{H}^2} \geq \|v\|_{\tilde{H}^1}$ and Lemma 3.1, we find that

$$\sqrt{\frac{48(L_1^2 + L_2^2)}{27[6(L_1^2 + L_2^2) + L_1^2 L_2^2]}} \|v\|_{H^1} \leq \|v\|_{\tilde{H}^2}.$$

The rest is to verify the inequality on the right side. Noticing that $\|v\|_{\tilde{H}^2}^2 = \|v\|_{\tilde{H}^1}^2 - d\|Lv\|^2$ and applying the discrete Green formula, we get

$$\|Lv\|^2 = \left\langle v + \frac{h_1^2}{12} \delta_x^2 v + \frac{h_2^2}{12} \delta_y^2 v + \frac{h_1^2 h_2^2}{144} \delta_x^2 \delta_y^2 v, v + \frac{h_1^2}{12} \delta_x^2 v + \frac{h_2^2}{12} \delta_y^2 v + \frac{h_1^2 h_2^2}{144} \delta_x^2 \delta_y^2 v \right\rangle_h$$

$$= \|v\|^2 + \frac{h_1^2}{12} \|\delta_x v\|^2 + \frac{h_2^2}{12} \|\delta_y v\|^2 + \frac{h_1^2 h_2^2}{144} \|\delta_x \delta_y v\|^2 + \frac{h_1^2}{12} \|\delta_x v\|^2 + \frac{h_1^4}{144} \|\delta_x^2 v\|^2 + \frac{h_1^2 h_2^2}{144} \|\delta_x \delta_y v\|^2$$

$$+ \frac{h_1^4 h_2^2}{1,728} \langle \delta_x^2 v, \delta_x \delta_y v \rangle_h + \frac{h_2^2}{12} \|\delta_y v\|^2 + \frac{h_2^4}{144} \|\delta_y^2 v\|^2 + \frac{h_1^2 h_2^2}{144} \|\delta_x \delta_y v\|^2 + \frac{h_1^2 h_2^4}{1,728} \langle \delta_y^2 v, \delta_x \delta_y v \rangle_h$$

$$+ \frac{h_1^2 h_2^2}{144} \langle v, \delta_x^2 \delta_y^2 v \rangle_h + \frac{h_1^4 h_2^2}{1,728} \|\delta_x^2 \delta_y v\|^2 + \frac{h_1^2 h_2^4}{1,728} \|\delta_x \delta_y^2 v\|^2 + \frac{h_1^4 h_2^4}{20,736} \|\delta_x^2 \delta_y^2 v\|^2$$

$$= \|v\|^2 + \frac{h_1^2}{6} \|\delta_x v\|^2 + \frac{h_2^2}{6} \|\delta_y v\|^2 + \frac{h_1^4}{144} \|\delta_x^2 v\|^2 + \frac{h_2^4}{144} \|\delta_y^2 v\|^2$$

$$+ \frac{h_1^2 h_2^2}{36} \|\delta_x \delta_y v\|^2 + \frac{h_1^4 h_2^2}{864} \|\delta_x^2 \delta_y v\|^2 + \frac{h_1^2 h_2^4}{864} \|\delta_x \delta_y^2 v\|^2 + \frac{h_1^4 h_2^4}{20,736} \|\delta_x^2 \delta_y^2 v\|^2.$$

In view of inverse estimates $\|\delta_x^2 v\| \leq \frac{4}{h_1^2}\|v\|$ and $\|\delta_y^2 v\| \leq \frac{4}{h_2^2}\|v\|$, we derive that

$$\|Lv\|^2 \leq \frac{485}{162}\|v\|^2 \leq \frac{485}{162}\|v\|_{H^1}^2.$$

Moreover, as a direct consequence of Lemma 3.1, we have

$$\|v\|_{\tilde{H}_2}^2 \leq \left(\frac{16}{9} - \frac{485d}{162}\right)\|v\|_{H^1}^2.$$

This completes the proof.   □

Lemma 3.3. *For any grid function $v^n \in V_h$, it holds that*

$$-\left\langle L_x \delta_t v^{n-\frac{1}{2}}, \delta_x^2 v^{n-\frac{1}{2}} \right\rangle_h = \frac{1}{2\tau}\left(\|v^n\|_A^2 - \|v^{n-1}\|_A^2\right), \tag{28}$$

$$-\left\langle L_y \delta_t v^{n-\frac{1}{2}}, \delta_y^2 v^{n-\frac{1}{2}} \right\rangle_h = \frac{1}{2\tau}\left(\|v^n\|_B^2 - \|v^{n-1}\|_B^2\right). \tag{29}$$

Proof. A combination of the definition of operator L_x and the discrete Green formula leads to

$$-\left\langle L\delta_t v^{n-\frac{1}{2}}, \delta_x^2 v^{n-\frac{1}{2}} \right\rangle_h = -\left\langle \delta_t v^{n-\frac{1}{2}}, \delta_x^2 v^{n-\frac{1}{2}} \right\rangle_h - \frac{h_1^2}{12}\left\langle \delta_t \delta_x^2 v^{n-\frac{1}{2}}, \delta_x^2 v^{n-\frac{1}{2}} \right\rangle_h$$

$$= \frac{1}{2\tau}\left[(\|\delta_x v^n\|^2 - \|\delta_x v^{n-1}\|^2) - \frac{h_1^2}{12}(\|\delta_x^2 v^n\|^2 - \|\delta_x^2 v^{n-1}\|^2)\right].$$

Thus, the proof of equality (28) is complete. Obviously, we can prove equality (29) similarly.   □

Lemma 3.4. *For any grid function $v \in V_h$, it holds that $\langle \delta_x^2 \delta_y^2 v, Lv \rangle_h \geq \frac{1}{3}\|\delta_x \delta_y v\|^2$.*

Proof. From the definition of operator L, it follows that

$$Lv = v + \frac{h_1^2}{12}\delta_x^2 v + \frac{h_2^2}{12}\delta_y^2 v + \frac{h_1^2 h_2^2}{144}\delta_x^2 \delta_y^2 v.$$

In view of the discrete Green formula, we derive that

$$\langle \delta_x^2 \delta_y^2 v, Lv \rangle_h = \left\langle \delta_x^2 \delta_y^2 v, v + \frac{h_1^2}{12}\delta_x^2 v + \frac{h_2^2}{12}\delta_y^2 v + \frac{h_1^2 h_2^2}{144}\delta_x^2 \delta_y^2 v \right\rangle_h$$

$$= \|\delta_x \delta_y v\|^2 - \frac{h_1^2}{12}\|\delta_x^2 \delta_y v\|^2 - \frac{h_2^2}{12}\|\delta_x \delta_y^2 v\|^2 + \frac{h_1^2 h_2^2}{144}\|\delta_x^2 \delta_y^2 v\|^2$$

$$\geq \frac{1}{3}\|\delta_x \delta_y v\|^2,$$

where inverse estimates $\|\delta_x^2 \delta_y v\| \leq \frac{2}{h_1}\|\delta_x \delta_y v\|$ and $\|\delta_x \delta_y^2 v\| \leq \frac{2}{h_2}\|\delta_x \delta_y v\|$ are used.   □

Lemma 3.5. *For any grid function $v^n \in V_h$, it holds that*

$$\langle \Lambda_h v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \rangle_h = -\frac{1}{2\tau}\left(\|v^n\|_{\tilde{H}^1}^2 - \|v^{n-1}\|_{\tilde{H}^1}^2\right). \tag{30}$$

Proof. From the definition of difference operators Λ_h and L, it follows that

$$\langle \Lambda_h v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \rangle_h = \langle L_y \delta_x^2 v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \rangle_h + \langle L_x \delta_y^2 v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \rangle_h. \tag{31}$$

Since operators L_x, L_y, δ_t, and δ_x^2 are linear, we obtain

$$\left\langle L_y \delta_x^2 v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h = \left\langle L_y \delta_x^2 v^{n-\frac{1}{2}}, L_x L_y \delta_t v^{n-\frac{1}{2}} \right\rangle_h = \left\langle L_x \delta_t \left(L_y v^{n-\frac{1}{2}} \right), \delta_x^2 \left(L_y v^{n-\frac{1}{2}} \right) \right\rangle_h,$$

which, together with Lemma 3.3, leads to

$$\left\langle L_y \delta_x^2 v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h = -\frac{1}{2\tau} \left(\|L_y v^n\|_A^2 - \|L_y v^{n-1}\|_A^2 \right). \tag{32}$$

Similarly, we have

$$\left\langle L_x \delta_y^2 v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h = -\frac{1}{2\tau} \left(\|L_x v^n\|_B^2 - \|L_x v^{n-1}\|_B^2 \right). \tag{33}$$

Substituting (32) and (33) into (31), we complete the proof. $\qquad\square$

We now prove that the difference scheme (25)–(27) is stable to the initial values v^0, ψ_v and the forcing term g.

Noting that $a_k = (2 - \alpha) \int_k^{k+1} x^{1-\alpha} dx$, and $x^{1-\alpha}$ is a monotone decreasing function, one verifies readily that

$$1 = a_0 > a_1 > a_2 > \cdots > a_n > \cdots \;\; \to \;\; 0, \tag{34}$$

$$(2 - \alpha)(k + 1)^{1-\alpha} < a_k < (2 - \alpha)k^{1-\alpha}. \tag{35}$$

Theorem 3.1. *Suppose $v^n \in V_h$ is the solution of the difference scheme (25)–(27), then it holds that*

$$\|v^n\|_{\tilde{H}^2}^2 \leq \|v^0\|_{\tilde{H}^2}^2 + \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)} \|L\psi_v\|^2 + \Gamma(2-\alpha) t_n^{\alpha-1} \tau \sum_{k=1}^n \left\| Lg^{k-\frac{1}{2}} \right\|^2, \quad 1 \leq n \leq N.$$

Proof. Multiplying (25) by $2\mu h_1 h_2 L\delta_t v_{ij}^{n-\frac{1}{2}}$ and summing over i, j for $(x_i, y_j) \in \Omega_h$, we have the following equation for $1 \leq n \leq N$:

$$2 \left\| L\delta_t v^{n-\frac{1}{2}} \right\|^2 - 2 \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \left\langle L\delta_t v^{k-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h$$
$$= 2a_{n-1} \left\langle L\psi_v, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h + 2\mu \left\langle \Lambda_h v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h + 2\mu d \left\langle Lv^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h \tag{36}$$
$$+ 2\mu \left\langle Lg^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h - \frac{(\tau\mu)^2}{2 - d\mu\tau} \left\langle \delta_x^2 \delta_y^2 \delta_t v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h.$$

In view of Lemmas 3.4 and 3.5, we find that

$$-\frac{(\tau\mu)^2}{2 - d\mu\tau} \left\langle \delta_x^2 \delta_y^2 \delta_t v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h \leq -\frac{(\tau\mu)^2}{6 - 3d\mu\tau} \|\delta_x \delta_y \delta_t v^{n-\frac{1}{2}}\|^2 \leq 0 \tag{37}$$

and

$$2\mu \left\langle \Lambda_h v^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h = -\frac{\mu}{\tau} \left(\|v^n\|_{\tilde{H}^1}^2 - \|v^{n-1}\|_{\tilde{H}^1}^2 \right). \tag{38}$$

Substituting (37)–(38) into (36) and noticing that both a_{n-1} and $a_{n-k-1} - a_{n-k}$ are positive, we obtain

$$2 \left\| L\delta_t v^{n-\frac{1}{2}} \right\|^2 + \frac{\mu}{\tau} \left(\|v^n\|_{\tilde{H}^1}^2 - \|v^{n-1}\|_{\tilde{H}^1}^2 \right)$$
$$\leq 2 \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \left| \left\langle L\delta_t v^{k-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h \right| + 2a_{n-1} \left| \left\langle L\psi_v, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h \right|$$
$$+ 2\mu \left| \left\langle Lg^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h \right| + 2\mu d \left\langle Lv^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}} \right\rangle_h$$

$$
\begin{aligned}
\leq &\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\left(\left\|L\delta_t v^{k-\frac{1}{2}}\right\|^2 + \left\|L\delta_t v^{n-\frac{1}{2}}\right\|^2\right) + a_{n-1}\left(\left\|L\delta_t v^{n-\frac{1}{2}}\right\|^2 + \|L\psi_v\|^2\right) \\
&+2\mu\left|\left\langle Lg^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}}\right\rangle_h\right| + \frac{\mu d}{\tau}(\|Lv^n\|^2 - \|Lv^{n-1}\|^2) \\
= &\left\|L\delta_t v^{n-\frac{1}{2}}\right\|^2 + \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\left\|L\delta_t v^{k-\frac{1}{2}}\right\|^2 + a_{n-1}\|L\psi_v\|^2 + 2\mu\left|\left\langle Lg^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}}\right\rangle_h\right| \\
&+\frac{\mu d}{\tau}(\|Lv^n\|^2 - \|Lv^{n-1}\|^2),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\left\|L\delta_t v^{n-\frac{1}{2}}\right\|^2 &+ \frac{\mu}{\tau}(\|v^n\|_{\tilde{H}^1}^2 - \|v^{n-1}\|_{\tilde{H}^1}^2) \\
&\leq \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\left\|L\delta_t v^{k-\frac{1}{2}}\right\|^2 + a_{n-1}\|L\psi_v\|^2 + 2\mu\left|\left\langle Lg^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}}\right\rangle_h\right| \\
&+ \frac{\mu d}{\tau}(\|Lv^n\|^2 - \|Lv^{n-1}\|^2), \quad 1 \leq n \leq N.
\end{aligned} \tag{39}
$$

For convenience, we denote $F^0 = \|v^0\|_{\tilde{H}^1}^2 - d\|Lv^0\|^2$ and

$$
F^n = \|v^n\|_{\tilde{H}^1}^2 + \frac{\tau}{\mu}\sum_{k=1}^{n}a_{n-k}\|L\delta_t v^{k-\frac{1}{2}}\|^2 - d\|Lv^n\|^2, \quad 1 \leq n \leq N.
$$

Multiplying inequality (39) by $\frac{\tau}{\mu}$, we have

$$
\begin{aligned}
F^n &\leq F^{n-1} + \frac{a_{n-1}\tau}{\mu}\|L\psi_v\|^2 + 2\tau\left|\left\langle Lg^{n-\frac{1}{2}}, L\delta_t v^{n-\frac{1}{2}}\right\rangle_h\right| \\
&\leq F^0 + \frac{\tau}{\mu}\sum_{k=1}^{n}a_{n-k}\|L\psi_v\|^2 + \tau\sum_{k=1}^{n}\left(\frac{\mu}{a_{n-k}}\left\|Lg^{k-\frac{1}{2}}\right\|^2 + \frac{a_{n-k}}{\mu}\left\|L\delta_t v^{k-\frac{1}{2}}\right\|^2\right), \quad 1 \leq n \leq N.
\end{aligned}
$$

Hence, for $1 \leq n \leq N$, it holds that

$$
\|v^n\|_{\tilde{H}^1}^2 - d\|Lv^n\|^2 \leq \|v^0\|_{\tilde{H}^1}^2 - d\|Lv^0\|^2 + \frac{\tau}{\mu}\sum_{k=1}^{n}a_{n-k}\|L\psi_v\|^2 + \tau\sum_{k=1}^{n}\frac{\mu}{a_{n-k}}\left\|Lg^{k-\frac{1}{2}}\right\|^2. \tag{40}
$$

By a direct calculation, we have $\sum_{k=1}^{n}a_{k-1} = n^{2-\alpha}$, which means

$$
\frac{\tau}{\mu}\sum_{k=1}^{n}a_{n-k}\|L\psi_v\|^2 = \frac{t_n^{2-\alpha}}{\Gamma(3-\alpha)}\|L\psi_v\|^2. \tag{41}
$$

Noting that $a_{n-k} \geq (2-\alpha)(n-k+1)^{1-\alpha} \geq (2-\alpha)n^{1-\alpha}$, we obtain

$$
\frac{\mu}{a_{n-k}} = \frac{\Gamma(3-\alpha)\tau^{\alpha-1}}{a_{n-k}} \leq \frac{\Gamma(3-\alpha)\tau^{\alpha-1}}{(2-\alpha)n^{1-\alpha}} = \Gamma(2-\alpha)t_n^{\alpha-1}.
$$

Furthermore, we have

$$
\tau\sum_{k=1}^{n}\frac{\mu}{a_{n-k}}\left\|Lg^{k-\frac{1}{2}}\right\|^2 \leq \Gamma(2-\alpha)t_n^{\alpha-1}\tau\sum_{k=1}^{n}\left\|Lg^{k-\frac{1}{2}}\right\|^2. \tag{42}
$$

Substituting estimates (41) and (42) into (40), we complete the proof. $\qquad\square$

3.2 Convergence

On the basis of the analysis of stability, the convergence result of the difference scheme (25)–(27) is proposed in this subsection. Let

$$e_{ij}^n = V_{ij}^n - v_{ij}^n, \quad (x_i, y_j) \in \bar{\Omega}_h, \ 0 \le n \le N.$$

Subtracting (25)–(27) from (21)–(23), we get the error system

$$\frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} L \left[a_0 \delta_t e_{ij}^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \delta_t e_{ij}^{k-\frac{1}{2}} \right]$$

$$= \Lambda_h e_{ij}^{n-\frac{1}{2}} - \frac{\Gamma(3-\alpha)\tau^{\alpha+1}}{4 - 2\Gamma(3-\alpha)d\tau^\alpha} \delta_x^2 \delta_y^2 \delta_t e_{ij}^{n-\frac{1}{2}} + dL e_{ij}^{n-\frac{1}{2}} + (R)_{ij}^{n-\frac{1}{2}}, \quad (x_i, y_j) \in \Omega_h, \ 1 \le n \le N, \tag{43}$$

$$e_{ij}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h, \ 1 \le n \le N, \tag{44}$$

$$e_{ij}^0 = 0, \quad (x_i, y_j) \in \bar{\Omega}_h. \tag{45}$$

Applying Theorem 3.1, we have that

$$\|e^n\|_{\tilde{H}^2}^2 \le \Gamma(2-\alpha) t_n^{\alpha-1} \tau \sum_{k=1}^n \left\| (R)^{k-\frac{1}{2}} \right\|^2, \quad 1 \le n \le N.$$

By utilizing Lemma 2.4, we get the following convergence result.

Theorem 3.2. *Assume that the problem* (7)–(10) *has a smooth solution* $v(x, y, t) \in C_{x,y,t}^{6,6,3}(\bar{\Omega} \times [0, T])$, *and let* $\{v_{ij}^n | (x_i, y_j) \in \Omega_h, \ 1 \le n \le N\}$ *be the solution of the difference scheme* (25)–(27). *Then it holds that*

$$\|V^n - v^n\|_{\tilde{H}^2} \le C(\tau^{3-\alpha} + h_1^4 + h_2^4),$$

where $C = C_R \sqrt{\Gamma(2-\alpha) T^\alpha L_1 L_2}$.

Based on Remark 2.1, we get the following corollary:

Corollary 3.1. *Assume that the problem* (7)–(10) *has a smooth solution* $v(x, y, t) \in C_{x,y,t}^{6,6,3}(\bar{\Omega} \times [0, T])$ *and* $\frac{\partial^3 v(x,y,t)}{\partial t^3} = 0$ *for* $(x, y, t) \in \Omega \times [0, T]$, *let* $\{v_{ij}^n | (x_i, y_j) \in \Omega_h, \ 1 \le n \le N\}$ *be the solution of the difference scheme* (25)–(27). *Then it holds that*

$$\|V^n - v^n\|_{\tilde{H}^2} \le \tilde{C}(\tau^{1+\alpha} + h_1^4 + h_2^4),$$

where $\tilde{C} = \tilde{C}_R \sqrt{\Gamma(2-\alpha) T^\alpha L_1 L_2}$.

4 Numerical experiments and implementation

In the following, we present a few results to numerically validate the analysis. In this section, we present implementation of the compact ADI scheme briefly and give some numerical examples to support our theory. All experiments are conducted on a Macbook Pro with a 2.9 GHz Intel i5 CPU and 8 GB RAM.

In the runs, we use the same spacing h in each direction, $h_1 = h_2 = h$, and compute the maximum norm errors of the numerical solution

$$e(\tau, h) = \max_{1 \le n \le N} \max_{(x_i, y_j) \in \Omega_h} |v(x_i, y_j, t_n) - v_{ij}^n|,$$

and we denote

$$\text{rate}_1 = \log_{\frac{\tau_1}{\tau_2}}\left(\frac{e(\tau_1, h)}{e(\tau_2, h)}\right), \quad \text{rate}_2 = \log_{\frac{h_1}{h_2}}\left(\frac{e(\tau, h_1)}{e(\tau, h_2)}\right).$$

4.1 Example 1

As the first example, we consider the fractional convection-diffusion equation:

$$_0^C D_t^\alpha u = \Delta u - u_x - 2u_y + \frac{u}{4} + f, \qquad (x, y, t) \in \Omega \times (0, T], \tag{46}$$

$$u(x, y, 0) = u^0(x, y), \quad \frac{\partial u(x, y, 0)}{\partial t} = 0, \quad (x, y) \in \bar{\Omega} = \Omega \cup \partial\Omega, \tag{47}$$

$$u(x, y, t) = 0, \qquad (x, y, t) \in \partial\Omega \times (0, T], \tag{48}$$

where $\Omega = (0, \pi) \times (0, \pi)$ and $T = 1$. We choose the source term

$$f(x, y, t) = e^{\frac{x+y}{2}} \sin x \sin y \left[\frac{\Gamma(3 + \alpha)t^2}{2} + 3t^{2+\alpha}\right]$$

to obtain an exact solution $u(x, y, t) = e^{\frac{x+y}{2}} t^{2+\alpha} \sin x \sin y$.

We first investigate the temporal errors and convergence orders of the compact ADI scheme. In this test, we fix $h = \pi/16$, a value small enough such that the spatial error is negligible as compared with the temporal error. Table 1 presents the maximum norm errors and temporal convergence orders of the scheme for $\alpha = 1.25, 1.5, 1.75$. It is observed that the scheme generates $(3 - \alpha)$ temporal convergence order. Then, we plot the exact solution (Figure 1) and the numerical result (Figure 2(a)) and the error between them (Figure 2(b)) at $T = 1$, with $\alpha = 1.5$, $M = 20$, and $N = 40$.

Figure 3 shows the contour plot of the exact solution in the $x-y$ plane and numerical solutions using the compact ADI scheme for different M at $T = 1$ with $\alpha = 1.5$, $N = 100$. We can see that when $M = 20$, the numerical solution is a good approximation of the exact solution. Finally, we observe the CPU time of the scheme, and the results are shown in Table 2. In order to illustrate the efficiency of the proposed ADI method, we compare the CPU time of the proposed ADI method with that of the direct method, which is

Table 1: Maximum norm errors and temporal convergence orders of the proposed scheme when $h = \pi/16$ for Example 1

	τ	$e(\tau, h)$	rate_1
$\alpha = 1.25$	1/5	0.0212	
	1/10	0.0066	1.6835
	1/20	0.0020	1.7225
	1/40	6.1280×10^{-4}	1.7065
	1/80	1.8568×10^{-4}	1.7226
$\alpha = 1.5$	1/5	0.0693	
	1/10	0.0254	1.4480
	1/20	0.0091	1.4809
	1/40	0.0033	1.4634
	1/80	0.0012	1.4594
$\alpha = 1.75$	1/5	0.1721	
	1/10	0.0736	1.2255
	1/20	0.0312	1.2382
	1/40	0.0132	1.2410
	1/80	0.0056	1.2370

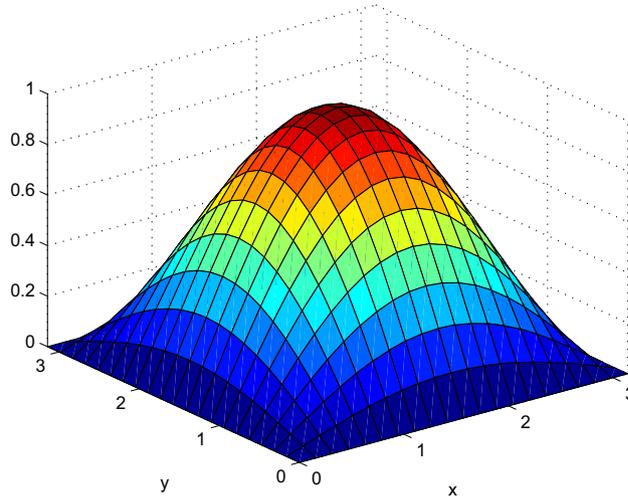**Figure 1:** The exact solution at $T = 1$ with $\alpha = 1.5$, $M = 20$, and $N = 40$ for Example 1.

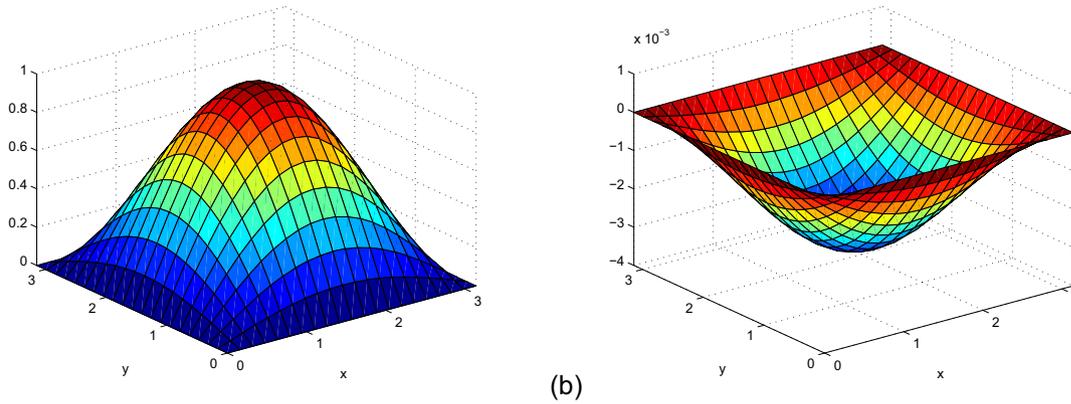(a)   (b)

Figure 2: The plot of the numerical solution and the error between exact solution and the numerical solution computed by the proposed scheme at $T = 1$ with $\alpha = 1.5$, $M = 20$, and $N = 40$ for Example 1. (a) Numerical solution (b) Error.

derived from (19) by neglecting the truncation error. The comparison results are listed in Table 3. It is obvious that the proposed method is more efficient than the direct method, especially for large M and N.

4.2 Example 2

In order to verify Corollary 3.1, the following TF convection-diffusion problem is considered:

$$
\begin{aligned}
&{}_0^C D_t^\alpha u = \Delta u - 2u_x - 2u_y + u + f, &&(x, y, t) \in \Omega \times (0, T], \\
&u(x, y, t) = (t^2 + 1)y^3(1 - y)^2 \sin(\pi x)e^{x+y}, &&(x, y, t) \in \bar{\Omega} \times (0, T], \\
&u(x, y, 0) = y^3(1 - y)^2 \sin(\pi x)e^{x+y}, \quad \frac{\partial u(x, y, 0)}{\partial t} = 0, &&(x, y) \in \bar{\Omega} = \Omega \cup \partial\Omega, \\
&u(x, y, t) = 0, &&(x, y, t) \in \partial\Omega \times (0, T].
\end{aligned}
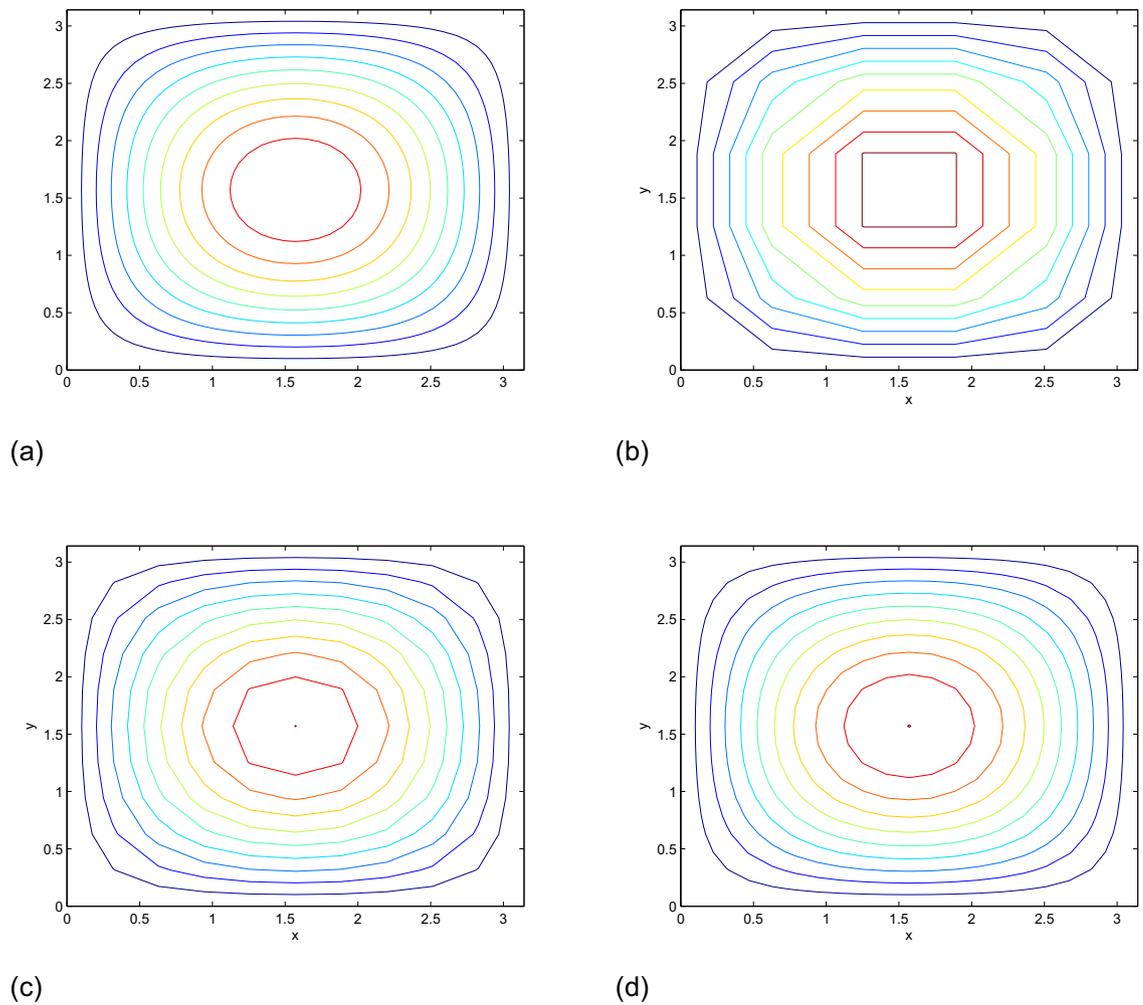$$

(a)

(b)

(c)

(d)

Figure 3: The contour plot of the exact solution and the numerical solutions using the compact ADI scheme for different M at $T = 1$ with $\alpha = 1.5$, $N = 100$ for Example 1. (a) Exact solution, (b) $M = 5$, (c) $M = 10$, (d) $M = 20$.

Table 2: Maximum norm error and CPU time of the proposed scheme for Example 1

α	N	M	$e(\tau, h)$	CPU time (s)
1.25	20	4	0.0018	0.024000
	40	8	6.3446×10^{-4}	0.132590
	80	16	1.8568×10^{-4}	1.783507
	160	32	5.5383×10^{-5}	23.578173
1.5	20	4	0.0094	0.032618
	40	8	0.0033	0.129855
	80	16	0.0012	1.586771
	160	32	4.0983×10^{-4}	26.438435
1.75	20	4	0.0314	0.030852
	40	8	0.0132	0.149223
	80	16	0.0056	1.799616
	160	32	0.0023	23.534627

Table 3: CPU time (s) of the direct method and that of the proposed ADI method for Example 1

α	N	M	Direct method	ADI method
1.25	100	40	7.965835	5.015337
	150	60	71.260503	25.786474
	200	80	367.464825	79.461959
1.5	100	40	8.434888	5.450105
	150	60	66.999567	23.968653
	200	80	360.612136	79.041365

Table 4: Maximum norm errors and temporal convergence orders of the proposed scheme when $h = 1/20$ for Example 2

	τ	$e(\tau, h)$	rate$_1$
$\alpha = 1.25$	1/5	0.0021	
	1/10	4.6123×10^{-4}	2.1868
	1/20	9.8952×10^{-5}	2.2207
	1/40	2.0907×10^{-5}	2.2558
	1/80	4.3548×10^{-6}	2.2633
$\alpha = 1.5$	1/5	0.0014	
	1/10	2.5717×10^{-4}	2.4446
	1/20	4.5920×10^{-5}	2.4855
	1/40	8.0901×10^{-6}	2.5049
	1/80	1.3771×10^{-6}	2.5545
$\alpha = 1.75$	1/5	0.0010	
	1/10	1.5437×10^{-4}	2.6955
	1/20	2.3112×10^{-5}	2.7397
	1/40	3.3932×10^{-6}	2.7679
	1/80	4.5338×10^{-7}	2.9039

Table 5: Maximum norm errors and spatial convergence orders of the proposed scheme when $\tau = 1/1,000$ for Example 2

	h	$e(\tau, h)$	rate$_2$
$\alpha = 1.25$	1/5	1.6698×10^{-5}	
	1/10	1.0664×10^{-6}	3.9689
	1/15	1.9859×10^{-7}	4.1451
	1/20	6.3018×10^{-8}	3.9899
	1/25	2.4538×10^{-8}	4.2269
$\alpha = 1.5$	1/5	1.6447×10^{-5}	
	1/10	1.0618×10^{-6}	3.9532
	1/15	2.0648×10^{-7}	4.0386
	1/20	6.4268×10^{-8}	4.0571
	1/25	2.4801×10^{-8}	4.2671
$\alpha = 1.75$	1/5	1.4937×10^{-5}	
	1/10	9.6587×10^{-7}	3.9509
	1/15	1.8890×10^{-7}	4.0245
	1/20	5.9753×10^{-8}	4.0009
	1/25	2.4200×10^{-8}	4.0506

The domain considered here is $(0, 1) \times (0, 1)$ and $T = 1$. The function f is specified to satisfy the given equation and exact solution. We compute the problem by using the proposed scheme (25)–(27).

In this experiment, we test the temporal errors and convergence orders of the scheme by letting τ vary and fixing the space step h sufficiently small to avoid contamination of the spatial error. Table 4 gives the maximum norm errors and temporal convergence orders of the scheme. In the runs, we let $h = 1/20$ for the compact ADI scheme. It is observed that the temporal order of the scheme is $(\alpha + 1)$, which verifies Corollary 3.1 successfully. Finally, the spatial error and convergence order of the scheme are presented in Table 5. It is shown that the compact ADI scheme has fourth order spatial accuracy.

Figure 3 shows the contour plot of the exact solution in the x–y plane and numerical solutions using the scheme for different M at $T = 1$ with $\alpha = 1.75$, $N = 100$. We can see that the numerical solution is a good approximation of the exact solution when $M = 20$.

5 Conclusion

In this paper, a compact ADI scheme for solving the convection-diffusion equation is proposed. We proved that the ADI scheme is unconditionally stable to the initial values and the inhomogeneous term. Besides, the numerical solution is convergent in the maximum norm. The coefficient matrix of the scheme is tridiagonal at each temporal level, so it can be solved by the Thomas algorithm. It is presented that the compact ADI scheme has fourth order spatial accuracy. In addition, the scheme generates $(3 - \alpha)$ temporal convergence order. We show that when the solution of the problem satisfies certain condition, the temporal convergence order can attain $(1 + \alpha)$. Numerical experiments are carried out to verify the convergence order and show the efficiency of the scheme. Temporal error bounds of the solutions of the ADI scheme are achieved without losing the accuracy.

Conflict of interest: Authors state no conflict of interest.

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