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Research Article

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The B-topology on S^*-doubly quasicontinuous posets

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Abstract: The notions of o_S-convergence and S^*-doubly quasicontinuous posets are introduced, which can be viewed as common generalizations of Birkhoff's order-convergence and S^*-doubly continuous posets, respectively. We first consider the relationship between o_S-convergence and B-topology and show that the topology induced by o_S-convergence according to the standard topological approach is the B-topology precisely. Then, the topological characterization for the S^*-doubly quasicontinuity is presented. It is proved that a poset is S^*-doubly quasicontinuous iff the poset equipped with the B-topology is locally hyperclosed iff the lattice of all B-open subsets of the poset is hypercontinuous. Finally, the order theoretical condition for the o_S-convergence being topological is given and the complete regularity of B-topology on S^*-doubly quasicontinuous posets is explored.

Keywords: o_S-convergence, B-topology, S^*-doubly quasicontinuous poset

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1 Introduction and preliminaries

The concept of order-convergence (o-convergence, for short) in partially ordered sets was introduced by Birkhoff [1], Frink [2] and Mcshane [3] and studied by Mathews and Anderson [4], Wolk [5] and Olejček [6]. It is defined as follows: a net $(x_i)_{i\in I}$ in a poset P is said to *o-converge to* $x \in P$ (we write $(x_i)_{i\in I} \xrightarrow{o} x$ in this paper) if there exist a directed subset D and a filtered subset F of P such that

(1) $\sup D = x = \inf F$;
(2) For every $d \in D$ and $e \in F$, $d \leqslant x_i \leqslant e$ holds eventually, i.e., there exists $i_0 \in I$ such that $d \leqslant x_i \leqslant e$ for all $i \geq i_0$.

Based on the o-convergence, the order topology on posets has also been defined by Birkhoff [1] (note: in [7], the order topology is also called the B-topology) according to the standard topological approach (the reader can refer to [8, p. 133] for details on this approach). This topology played an important role in searching the order-theoretical condition for the o-convergence being topological. This fact can be demonstrated in the work of Zhao and Wang in [9]. They considered the relationship between the B-topology and Bi-Scott topology and showed that a poset P, which satisfies condition (*), is doubly continuous if and only if o-convergence in P is topological with respect to the B-topology on P. This means that, for a special class

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of posets, a sufficient and necessary condition for o-convergence being topological is obtained. Following this ideal, we further proposed the notion of S^*-doubly continuous posets in [7]. By establishing the order-theoretical characterization of B-topology on S^*-doubly continuous posets, it is proved that a poset P is S^*-doubly continuous if and only if the o-convergence in P is topological with respect to the B-topology on P. This reveals that the S^*-double continuity is the equivalent condition for the o-convergence being topological and has close relationship with the B-topology on posets.

The main goal of this paper is to explore the relationship between B-topology and S^*-double quasicontinuity, a generalized form of S^*-double continuity. The most related work to ours is of Gierz et al. [10] who proposed the concept of quasicontinuous domains as a generalization of continuous domains (see [11]) and generalized continuous lattices (see [12]). The core ideal is to generalize the way-below relation between points to the case of sets. The quasicontinuity can be characterized by Scott topology, and it has been shown that quasicontinuous domains equipped with the Scott topologies are precisely the spectra of distributive hypercontinuous lattices. Recently, the notion of s_2-quasicontinuous posets, a generalized form of s_2-continuous posets (see [13]) and quasicontinuous domains, was introduced by Zhang and Xu [14]. Based on this, some topological characterizations by the σ_2-topology (see [14]) to s_2-quasicontinuity are given and the complete regularity of σ_2-topology on s_2-quasicontinuous posets is investigated.

In this paper, as a generalization of o-convergence, the concept of o_s-convergence in posets is proposed. It is shown that the topology induced by o_s-convergence according to the standard topological approach (see [8, p. 133]) is the B-topology precisely. Also, the S^*-double quasicontinuity, a generalized form of S^*-double continuity, is defined for posets. Using the structural characterization for the B-topology, the topological characterization to S^*-double quasicontinuity is obtained. That is, a poset is S^*-doubly quasicontinuous if and only if the poset equipped with the B-topology is locally hyperclosed if and only if the lattice of all B-open subets of the poset is a hypercontinuous lattice. Based on this characterization, we give a necessary and sufficient condition for the o_s-convergence being topological. That is, the o_s-convergence in a poset is topological if and only if the poset is S^*-doubly quasicontinuous. Finally, we consider the complete regularity of B-topology and show that the B-topology on an S^*-doubly quasicontinuous poset satisfying Condition ($\diamond$) is Tychonoff.

Some conventional notions will be used in the sequel. Throughout this paper, given a set X, $F \sqsubseteq X$ means that F is a finite subset of X. And we simply denote
- $\mathcal{P}(X) = \{Y : Y \subseteq X\}$, $\mathcal{P}_0(X) = \mathcal{P}(X)\backslash\{\varnothing\}$;
- $\mathcal{L}(X) = \{F : F \sqsubseteq X\}$, $\mathcal{L}_0(X) = \mathcal{L}(X)\backslash\{\varnothing\}$.

For any subfamily $\mathcal{A} \subseteq \mathcal{P}(X)$ and $Y \subseteq X$, the restriction of $\mathcal{A}$ on Y is written as $\mathcal{A}|_Y = \{A \cap Y : A \in \mathcal{A}\}$. Given a topological space $(X, \mathcal{T})$ and $Y \subseteq X$. We take $\mathrm{int}_{\mathcal{T}}Y$ and $\mathrm{cl}_{\mathcal{T}}Y$ to mean the interior and the closure of Y with respect to the topology $\mathcal{T}$, respectively.

Let P be a poset and $x \in P$, $\uparrow x$ and $\downarrow x$ are always used to denote the *principal filter* $\{y \in P : y \geqslant x\}$ and the *principal ideal* $\{z \in P : z \leqslant x\}$ of P, respectively. For any $A \subseteq P$, we let $\uparrow A = \bigcup\{\uparrow a : a \in A\}$ and $\downarrow A = \bigcup\{\downarrow a : a \in A\}$. By writing $\sup A$ we mean that the least upper bound of A in P exists and equals to $\sup A \in P$. The symbol $\inf A$ has the dual meaning.

Given a poset P. The topology generated by the subbase $\{P\backslash\downarrow x : x \in P\}$ is called the *upper topology on* P, and denoted by $v(P)$. We can define the preorder $\leq$ on $\mathcal{P}_0(P)$ by $A \leq B$ if $B \subseteq \uparrow A$ and the preorder $\leq_{op}$ on $\mathcal{P}_0(P)$ by $B \leq_{op} A$ if $B \subseteq \downarrow A$. A subfamily $\mathcal{M} \subseteq \mathcal{P}_0(P)$ is said to be *directed* if for any $M_1, M_2 \in \mathcal{M}$, there exists $M \in \mathcal{M}$ such that $M_1, M_2 \leq M$, i.e., $M \subseteq \uparrow M_1 \cap \uparrow M_2$; dually, a subfamily $\mathcal{N} \subseteq \mathcal{P}_0(P)$ is said to be *filtered* if for any $N_1, N_2 \in \mathcal{N}$, there exists $N \in \mathcal{N}$ such that $N \leq_{op} N_1, N_2$, i.e., $N \subseteq \downarrow N_1 \cap \downarrow N_2$.

To make this paper self-contained, we briefly review the following notions and propositions:

Definition 1.1. [1] Let P be a poset and $U \subseteq P$. U is said to be *B-open* if and only if for every net $(x_i)_{i\in I} \xrightarrow{o} x \in U$, we have that $x_i \in U$ holds eventually.

It can be formally verified that the collection of all B-open subsets of P forms a topology on P, which is called the *B-topology* and denoted by $\mathcal{T}_P$. And the following is the order-theoretical characterization for B-topology:

Proposition 1.2. [7] *Let P be a poset and $U \subseteq P$. Then $U \in \mathcal{T}_P$ if and only if for any directed set D and filtered set F with $\sup D = \inf F = x \in U$, $\uparrow d_0 \cap \downarrow e_0 \subseteq U$ for some $d_0 \in D$ and $e_0 \in F$.*

Definition 1.3. [7] Let P be a poset and $x, y, z \in P$. We define $y \ll_S x$ if for every directed subset D of P with $\sup D = x$, there exists $d \in D$ such that $y \leqslant d$; dually, we define $z \triangleright_S x$ if for every filtered subset F of P with $\inf F = x$, there exists $e \in F$ such that $z \geqslant e$.

In what follows, for a poset P and $x \in P$, we denote

- $\Downarrow_S x = \{a \in P : a \ll_S x\}$, $\Uparrow_S x = \{b \in P : x \ll_S b\}$;
- $\downdownarrows_S x = \{c \in P : x \triangleright_S c\}$, $\upuparrows_S x = \{d \in P : d \triangleright_S x\}$.

Definition 1.4. [7] A poset P is said to be *S-doubly continuous* if for every $x \in P$, the sets $\Downarrow_S x$ and $\upuparrows_S x$ are directed and filtered, respectively, and $\sup \Downarrow_S x = x = \inf \upuparrows_S x$.

Definition 1.5. [7] An S-doubly continuous poset P is said to be *S^*-doubly continuous* if for every $x \in P$, $y \in \Downarrow_S x$ and $z \in \upuparrows_S x$, there exist $y_0 \in \Downarrow_S x$ and $z_0 \in \upuparrows_S x$ such that $\uparrow y_0 \cap \downarrow z_0 \subseteq \Uparrow_S y \cap \downdownarrows_S z$.

Proposition 1.6. [7] *If P is a doubly continuous poset (see* [15, Definition 2.5]*), then P is S^*-doubly continuous.*

For some natural examples of doubly continuous posets and S^*-doubly continuous posets, one can refer to [15, Example 2.1]. The following is the well-known Rudin Lemma.

Lemma 1.7. [8] *Let P be a poset.*

(1) *If $\mathcal{M}$ is a directed subfamily of $\mathcal{L}_0(P)$, then there exists a directed set $D \subseteq \bigcup\{M : M \in \mathcal{M}\}$ such that $D \cap M \neq \varnothing$ for all $M \in \mathcal{M}$.*

(2) *If $\mathcal{N}$ is a filtered subfamily of $\mathcal{L}_0(P)$, then there exists a filtered set $F \subseteq \bigcup\{N : N \in \mathcal{N}\}$ such that $F \cap N \neq \varnothing$ for all $N \in \mathcal{N}$.*

2 o_S-convergence in posets

In this section, as a generalization of o-convergence, the concept of o_S-convergence in posets is proposed and some basic properties of o_S-convergence are presented.

Definition 2.1. Let P be a poset. A net $(x_i)_{i\in I}$ in P is said to *o_S-converge* to $x \in P$ if there exists a directed family $\mathcal{M}_x \subseteq \mathcal{L}_0(\downarrow x)$ and a filtered family $\mathcal{N}_x \subseteq \mathcal{L}_0(\uparrow x)$ such that

(Q1) $\bigcap\{\uparrow M : M \in \mathcal{M}_x\} = \uparrow x$ and $\bigcap\{\downarrow N : N \in \mathcal{N}_x\} = \downarrow x$.

(Q2) For every $M \in \mathcal{M}_x$ and $N \in \mathcal{N}_x$, $x_i \in \uparrow M \cap \downarrow N$ holds eventually, i.e., there exists $i_0 \in I$ such that $x_i \in \uparrow M \cap \downarrow N$ for all $i \geq i_0$.

In this case, we write $(x_i)_{i\in I} \xrightarrow{o_S} x$. By saying that $(x_j)_{j\in J}$ is an o_S-convergent net in P, we mean $(x_j)_{j\in J} \xrightarrow{o_S} y$ for some $y \in P$.

Proposition 2.2. *Let P be a poset and $(x_i)_{i\in I}$ a net in P. Then the o_S-convergent point of $(x_i)_{i\in I}$ is unique. That is, if $(x_i)_{i\in I} \xrightarrow{o_S} x \in P$ and $(x_i)_{i\in I} \xrightarrow{o_S} y \in P$, then $x = y$.*

Proof. Suppose that $(x_i)_{i\in I} \xrightarrow{o_s} x$ and $(x_i)_{i\in I} \xrightarrow{o_s} y$. Then, by Definition 2.1, there exist directed families $\mathcal{M}_x \subseteq \mathcal{L}_0(\downarrow x)$, $\mathcal{M}_y \subseteq \mathcal{L}_0(\downarrow y)$ and filtered families $\mathcal{N}_x \subseteq \mathcal{L}_0(\uparrow x)$, $\mathcal{N}_y \subseteq \mathcal{L}_0(\uparrow y)$ such that

(1) $\bigcap\{\uparrow M_x : M_x \in \mathcal{M}_x\} = \uparrow x$ and $\bigcap\{\downarrow N_x : N_x \in \mathcal{N}_x\} = \downarrow x$;
(2) $\bigcap\{\uparrow M_y : M_y \in \mathcal{M}_y\} = \uparrow y$ and $\bigcap\{\downarrow N_y : N_y \in \mathcal{N}_y\} = \downarrow y$;
(3) For any $M_x \in \mathcal{M}_x$ and $N_x \in \mathcal{N}_x$, $x_i \in \uparrow M_x \cap \downarrow N_x$ holds eventually;
(4) For any $M_y \in \mathcal{M}_y$ and $N_y \in \mathcal{N}_y$, $x_i \in \uparrow M_y \cap \downarrow N_y$ holds eventually.

For every $M_x \in \mathcal{M}_x$, one can trivially check, by (3) and the finiteness of M_x, that there exists $m_{M_x} \in M_x$ such that $x_i \in \uparrow m_{M_x}$ holds frequently, that is, for every $i \in I$ there exists $j_i \geq i$ such that $x_{j_i} \in \uparrow m_{M_x}$. Since $x_i \in \downarrow N_y$ holds eventually for every $N_y \in \mathcal{N}_y$, there exists $i_{N_y} \in I$ such that $x_i \in \downarrow N_y$ for all $i \geq i_{N_y}$. Let $M_0 = \{m_{M_x} : M_x \in \mathcal{M}_x\}$. Then we have $x_{j_{i_{N_y}}} \in \uparrow m_{M_x} \cap \downarrow N_y$ for every $m_{M_x} \in M_0$ and $N_y \in \mathcal{N}_y$. This means $M_0 \subseteq \bigcap\{\downarrow N_y : N_y \in \mathcal{N}_y\} = \downarrow y$. Since $x \in \bigcap\{\uparrow m_{M_x} : M_x \in \mathcal{N}_x\} \subseteq \bigcap\{\uparrow M_x : M_x \in \mathcal{N}_x\} = \uparrow x$, we have $\sup M_0 = x$. This implies $x \leqslant y$. Similarly, it can be proved that $y \leqslant x$. Thus, we conclude $x = y$. □

Proposition 2.3. *Let P be a poset and $(x_i)_{i\in I}$ a net in P. If $(x_i)_{i\in I} \xrightarrow{o} x \in P$, then $(x_i)_{i\in I} \xrightarrow{o_s} x$.*

Proof. Straightforward by the definitions of o-convergence and o_s-convergence. □

The following example shows that the converse of Proposition 2.3 is not true.

Example 2.4. Let $P = \{x\} \cup \{a_0, a_1, a_2, a_3, \dots, a_n, \dots\} \cup \{b_0, b_1, b_2, b_3, \dots, b_n, \dots\}$. The partially order $\leqslant$ on P is defined by setting

(1) $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant \dots, a_n \leqslant \dots \leqslant x$;
(2) $b_0 \leqslant b_1 \leqslant b_2 \leqslant b_3 \leqslant \dots, b_n \leqslant \dots \leqslant x$.

Let $x_{2k} = a_k$ and $x_{2k+1} = b_k$ for every $k \in \mathbf{N}$, where $\mathbf{N}$ denotes the set of all natural numbers. Then it is easy to verify that the net $(x_i)_{i\in\mathbf{N}}$ o_s-converges to x but not o-converge to x.

Proposition 2.5. *Given a poset P and $x, y, z \in P$. Let a directed family $\mathcal{M}_x \subseteq \mathcal{L}_0(\downarrow x)$ and a filtered family $\mathcal{N}_x \subseteq \mathcal{L}_0(\uparrow x)$ satisfy $\bigcap\{\uparrow M : M \in \mathcal{M}_x\} = \uparrow x$ and $\bigcap\{\uparrow N : N \in \mathcal{N}_x\} = \downarrow x$.*
(1) If $y \ll_S x$, then $M_0 \subseteq \uparrow y$ for some $M_0 \in \mathcal{M}_x$.
(2) If $z \triangleright_S x$, then $N_0 \subseteq \downarrow z$ for some $N_0 \in \mathcal{N}_x$.

Proof. (1): Suppose that $M \nsubseteq \uparrow y$ for every $M \in \mathcal{M}_x$. Then it is easy to verify that $\{M \backslash \uparrow y : M \in \mathcal{M}_x\} \subseteq \mathcal{L}_0(\downarrow x)$ is a directed family such that $\bigcap\{\uparrow(M\backslash\uparrow y) : M \in \mathcal{M}_x\} = \uparrow x$. By Rudin lemma, there exists a directed $D \subseteq \bigcup\{M\backslash\uparrow y : M \in \mathcal{M}_x\}$ such that $D \cap (M\backslash\uparrow y) \neq \varnothing$ for all $M \in \mathcal{M}_x$. This implies

$$x \in \bigcap\{\uparrow d : d \in D\} \subseteq \bigcap\{\uparrow(M\backslash\uparrow y) : M \in \mathcal{M}_x\} = \uparrow x.$$

Thus, we have $\sup D = x$. It follows from the definition of $\ll_S$ that there exists $d_0 \in D$ such that $d_0 \in \uparrow y$. This contradicts that $d_0 \in D \subseteq \bigcup\{M\backslash\uparrow y : M \in \mathcal{M}_x\}$.

The proof of (2) can be completed similarly. □

Proposition 2.6. *Let P be a S-doubly continuous poset and $x \in P$. Then a net*

$$(x_i)_{i\in I} \xrightarrow{o_s} x \iff (x_i)_{i\in I} \xrightarrow{o} x.$$

Proof. Straightforward by Proposition 2.5, the definitions of o-convergence and o_s-convergence. □

In the following proposition, we provide an approach to construct a special kind of o_s-convergent net in a given poset.

Proposition 2.7. *Let P be a poset and $x \in P$. Suppose that $\mathcal{M}_x \subseteq \mathcal{L}_0(\downarrow x)$ is a directed family and $\mathcal{N}_x \subseteq \mathcal{L}_0(\uparrow x)$ a filtered family with $\bigcap\{\uparrow M : M \in \mathcal{M}_x\} = \uparrow x$ and $\bigcap\{\downarrow N : N \in \mathcal{N}_x\} = \downarrow x$. If we*

(1) *set $\mathcal{I}_0 = \{\uparrow M \cap \downarrow N : M \in \mathcal{M}_x \ \& \ N \in \mathcal{N}_x\}$ and $\mathcal{I}_{\mathcal{M}_x}^{\mathcal{N}_x} = \{(i, I) \in P \times \mathcal{I}_0 : i \in I\}$;*

(2) *define the directed preorder $\leqq$ on $\mathcal{I}_{\mathcal{M}_x}^{\mathcal{N}_x}$ by*

$$\left(\forall (i_1, I_1), (i_2, I_2) \in \mathcal{I}_{\mathcal{M}_x}^{\mathcal{N}_x}\right)(i_1, I_1) \leqq (i_2, I_2) \ \Leftrightarrow \ I_2 \subseteq I_1;$$

(3) *let $x_{(i,I)} = i$ for all $(i, I) \in \mathcal{I}_{\mathcal{M}_x}^{\mathcal{N}_x}$,*

then the net $(x_{(i,I)})_{(i,I)\in \mathcal{I}_{\mathcal{M}_x}^{\mathcal{N}_x}} \xrightarrow{o_s} x$.

Proof. The proof can be easily completed by checking that the directed family $\mathcal{M}_x$ and the filtered family $\mathcal{N}_x$ satisfy Conditions (Q1) and (Q2) of Definition 2.1. □

3 The connection between B-topology and o_s-convergence

In this section, we consider the close relationship between o_s-convergence and B-topology. That is, the topology induced by o_s-convergence is precisely the B-topology. To show this fact, we first give the following lemma:

Lemma 3.1. *Let P be a poset and $U \in \mathcal{T}_P$. Then for every $x \in U$, every directed subfamily $\mathcal{M}_x \subseteq \mathcal{L}_0(\downarrow x)$ and every filtered subfamily $\mathcal{N}_x \subseteq \mathcal{L}_0(\uparrow x)$ such that $\bigcap\{\uparrow M : M \in \mathcal{M}_x\} = \uparrow x$ and $\bigcap\{\downarrow N : N \in \mathcal{N}_x\} = \downarrow x$, we have $x \in \uparrow M_0 \cap \downarrow N_0 \subseteq U$ for some $M_0 \in \mathcal{M}_x$ and $N_0 \in \mathcal{N}_x$.*

Proof. Suppose not, that is, for every $M \in \mathcal{M}_x$ and every $N \in \mathcal{N}_x$, we have $x \in \uparrow M \cap \downarrow N \nsubseteq U$. Now, we show

(i) For every $M = \{m_1, m_2, \ldots, m_{k_M}\} \in \mathcal{M}_x$, the set

$$M^K = \{m \in M : (\forall N \in \mathcal{N}_x) \uparrow m \cap \downarrow N \nsubseteq U\} \neq \varnothing.$$

Suppose $M^K = \varnothing$. Then, for every $i \in \{1, 2, \ldots, k_M\}$, there exists $N_i \in \mathcal{N}_x$ such that $\uparrow m_i \cap \downarrow N_i \subseteq U$. Since $\mathcal{N}_x$ is a filtered subfamily of $\mathcal{L}_0(\downarrow x)$, we can take $N_{00} \in \mathcal{N}_x$ such that $N_{00} \subseteq \downarrow N_1 \cap \downarrow N_2 \cap \ldots \cap \downarrow N_{k_M}$. This implies $\uparrow M \cap \downarrow N_{00} \subseteq U$, a contradiction to the hypothesis. Thus, we have $M^K \neq \varnothing$.

(ii) The family $\mathcal{M}_x^K = \{M^K : M \in \mathcal{M}_x\}$ is also directed.

Let $(M_1)^K, (M_2)^K \in \mathcal{M}_x^K$. According to the directness of $\mathcal{M}_x$, it follows that there exists $M_{00} \in \mathcal{M}_x$ such that $M_{00} \subseteq \uparrow M_1 \cap \uparrow M_2$. To show the directness of $\mathcal{M}_x^K$, it suffices to prove $(M_{00})^K \subseteq \uparrow (M_1)^K \cap \uparrow (M_2)^K$. Suppose $(M_{00})^K \nsubseteq \uparrow (M_1)^K$. Then, we have $m_0 \in \uparrow m_1'$ for some $m_0 \in (M_{00})^K$ and $m_1' \in M_1 \backslash (M_1)^K$. Since $m_1' \in M_1 \backslash (M_1)^K$, there exists $N_1 \in \mathcal{N}_x$ such that $\uparrow m_1' \cap \downarrow N_1 \subseteq U$, which implies

$$\uparrow m_0 \cap \downarrow N_1 \subseteq \uparrow m_1' \cap \downarrow N_1 \subseteq U.$$

This contradicts the fact that $m_0 \in (M_{00})^K$. Thus, we have $(M_{00})^K \subseteq \uparrow (M_1)^K$. A similar verification can show $(M_{00})^K \subseteq \uparrow (M_2)^K$. Therefore, we have $(M_{00})^K \subseteq \uparrow (M_1)^K \cap \uparrow (M_2)^K$.

(iii) The existence of directed set D satisfying
  (1) $(\forall M \in \mathcal{M}_x)\ D \cap M^K \neq \varnothing$;
  (2) $D \subseteq \bigcup\{M^K : M \in \mathcal{M}_x\}$;
  (3) $\sup D = x$;
  (4) $(\forall d \in D, \forall N \in \mathcal{N}_x) \uparrow d \cap \downarrow N \nsubseteq U$.

By Rudin lemma and the directness of $\mathcal{M}_x^K$, we can conclude the existence of a directed set D satisfying (1) and (2). Since

$$\uparrow x \subseteq \bigcap \{\uparrow d : d \in D\} \text{ (by (2) and } \mathcal{M}_x^K \subseteq \mathcal{L}_0(\downarrow x))$$

$$\subseteq \bigcap \{\uparrow M^K : M \in \mathcal{M}_x\} \text{ (by (1) and (2))}$$
$$\subseteq \bigcap \{\uparrow M : M \in \mathcal{M}_x\} \text{ (by } (\forall M \in \mathcal{M}_x)\ M^K \subseteq M) = \uparrow x,$$

x is the supremum of the directed set D, i.e., $\sup D = x$. Now, it is easy to verify, by (2) and the definition of $\mathcal{M}_x^K$, that the directed set D satisfies (4).

(iv) For every $N \in \mathcal{N}_x$, the set

$$N^K = \{n \in N : (\forall d \in D) \uparrow d \cap \downarrow n \nsubseteq U\} \neq \emptyset.$$

The verification is similar to that of (i).

(v) The family $\mathcal{N}_x^K = \{N^K : N \in \mathcal{N}_x\}$ is filtered.
The verification is similar to that of (ii).

(vi) The existence of a filtered set F satisfying
(5) $(\forall N \in \mathcal{N}_x)\ F \cap N^K \neq \emptyset$;
(6) $F \subseteq \bigcup\{N^K : N \in \mathcal{N}_x\}$;
(7) $\inf F = x$;
(8) $(\forall d \in D, \forall e \in F) \uparrow d \cap \downarrow e \nsubseteq U$.

The verification is similar to that of (iii).

To summarize what we have proved, we picked a directed set D and a filtered set F such that $\sup D = \inf F = x \in U$, and $\uparrow d \cap \downarrow e \nsubseteq U$ for every $d \in D$ and $e \in F$. By Proposition 1.2, U is not a B-open set. This contradicts to the assumption $U \in \mathcal{T}_P$. Therefore, the proof is complete. □

Theorem 3.2. *Let P be a poset. Then $U \in \mathcal{T}_P$ if and only if for every net $(x_i)_{i\in I} \xrightarrow{o_s} x \in U$, $x_i \in U$ holds eventually.*

Proof. (⇒): Let $U \in \mathcal{T}_P$ and a net $(x_i)_{i\in I} \xrightarrow{o_s} x \in U$. Then, by Definition 2.1, there exist a directed subfamily $\mathcal{M}_x \subseteq \mathcal{L}_0(\downarrow x)$ and a filtered subfamily $\mathcal{N}_x \subseteq \mathcal{L}_0(\uparrow x)$ such that
(1) $\bigcap\{\uparrow M : M \in \mathcal{M}_x\} = \uparrow x$ and $\bigcap\{\downarrow N : N \in \mathcal{N}_x\} = \downarrow x$;
(2) For every $M \in \mathcal{M}_x$ and $N \in \mathcal{N}_x$, $x_i \in \uparrow M \cap \downarrow N$ eventually.

By Lemma 3.1, we have $x \in \uparrow M_0 \cap \downarrow N_0 \subseteq U$ for some $M_0 \in \mathcal{M}_x$ and $N_0 \in \mathcal{N}_x$. This implies that $x_i \in \uparrow M_0 \cap \downarrow N_0 \subseteq U$ holds eventually.

(⇐): By Proposition 2.3 and Definition 1.1. □

Theorem 3.2 clarifies the fact that the topology induced by o-convergence is the same as that induced by o_s-convergence, namely, the B-topology.

4 The B-topology on S^*-doubly quasicontinuous posets

In this section, we first introduce the concept of S^*-doubly quasicontinuous posets as a generalization of S^*-doubly continuous posets. Then the fundamental properties of B-topology on S^*-doubly quasicontinuous posets are presented and the topological characterization for the S^*-double quasicontinuity is investigated.

Definition 4.1. Let P be a poset and $A, M, N \subseteq P$.
(1) We say that M *S_0-approximates A below*, in symbol $M \ll_{S_0} A$, if for every directed subset D of P, $\sup D \in A$ implies $d \in \uparrow M$ for some $d \in D$. We simply write $M \ll_{S_0} x$ for $M \ll_{S_0} \{x\}$ and $y \ll_{S_0} A$ for $\{y\} \ll_{S_0} A$.
(2) Dually, we say that N *S_0-approximates A above*, in symbol $N \triangleright_{S_0} A$, if for every filtered subset F of P, $\inf F \in A$ implies $e \in \downarrow N$ for some $e \in F$. We simply write $x \triangleright_{S_0} A$ for $\{x\} \triangleright_{S_0} A$ and $N \triangleright_{S_0} x$ for $N \triangleright_{S_0} \{x\}$.

For convenience, given a subset A of a poset P and $x \in P$, we simply denote
- $\Uparrow_{S_0} A = \{p \in P : A \ll_{S_0} p\}$, $\Downarrow_{S_0} A = \{p \in P : p \ll_{S_0} A\}$;
- $\downdownarrows_{S_0} A = \{p \in P : A \triangleright_{S_0} p\}$, $\upuparrows_{S_0} A = \{p \in P : p \triangleright_{S_0} A\}$;
- $w(x) = \{M \in \mathcal{L}_0(P) : M \ll_{S_0} x\}$, $v(x) = \{N \in \mathcal{L}_0(P) : N \triangleright_{S_0} x\}$.

The following proposition is basic and the proof is straightforward by Definitions 1.3 and 4.1.

Proposition 4.2. *Let P be a poset and $A, B, M, N \subseteq P$. Then*
(1) $M \ll_{S_0} A \Leftrightarrow (\forall a \in A)\ M \ll_{S_0} a$, *and* $N \triangleright_{S_0} A \Leftrightarrow (\forall a \in A)\ N \triangleright_{S_0} a$;
(2) $B \le M \ll_{S_0} A \Rightarrow B \ll_{S_0} A$, *and* $B \ge_{op} N \triangleright_{S_0} A \Rightarrow B \triangleright_{S_0} A$;
(3) $M \ll_{S_0} A \Rightarrow M \le A$, *and* $N \triangleright_{S_0} A \Rightarrow N \ge_{op} A$;
(4) $\{x\} \ll_{S_0} \{y\}$ *and* $\{z\} \triangleright_{S_0} \{y\}$ *in the sense of* Definition 4.1 *are equivalent to* $x \ll_S y$ *and* $z \triangleright_S y$ *in the sense of* Definition 1.3, *respectively*.

Proposition 4.3. *Let P be a poset, $x \in P$ and $A, B \subseteq P$. For every directed family $\mathcal{M}_x \subseteq \mathcal{L}_0(\downarrow x)$ and every filtered family $\mathcal{N}_x \subseteq \mathcal{L}_0(\uparrow x)$,*
(1) *if* $A \ll_{S_0} x$ *and* $\bigcap\{\uparrow M : M \in \mathcal{M}_x\} = \uparrow x$, *then* $M_0 \subseteq \uparrow A$ *for some* $M_0 \in \mathcal{M}_x$;
(2) *if* $B \triangleright_{S_0} x$ *and* $\bigcap\{\downarrow N : N \in \mathcal{N}_x\} = \downarrow x$, *then* $N_0 \subseteq \downarrow B$ *for some* $N_0 \in \mathcal{N}_x$.

Proof.
(1): Suppose $M \nsubseteq \uparrow A$ for every $M \in \mathcal{M}_x$. Then $\{M \backslash \uparrow A : M \in \mathcal{M}_x\}$ is a directed subfamily of $\mathcal{L}_0(\downarrow x)$ and $\bigcap\{\uparrow(M \backslash \uparrow A) : M \in \mathcal{M}_x\} = \uparrow x$. By Rudin Lemma, there exists a directed set D such that $D \subseteq \bigcup\{M \backslash \uparrow A : M \in \mathcal{M}_x\}$ and $D \cap (M \backslash \uparrow A) \neq \emptyset$ for every $M \in \mathcal{M}_x$. Thus, we have

$$\uparrow x \subseteq \bigcap\{\uparrow d : d \in D\} \subseteq \bigcap\{\uparrow(M \backslash \uparrow A) : M \in \mathcal{M}_x\} \subseteq \bigcap\{\uparrow M : M \in \mathcal{M}_x\} = \uparrow x.$$

This means $\sup D = x$ and $d \notin \uparrow A$ for all $d \in D$. By the definition of $\ll_{S_0}$, we conclude that $A \not\ll_{S_0} x$, which contradicts the assumption $A \ll_{S_0} x$.
(2) It can be similarly proved. □

The S_0-approximate relations $\ll_{S_0}$ and $\triangleright_{S_0}$ on a given poset P can be equivalently characterized by the o_s-convergence in P in the following proposition.

Proposition 4.4. *Let P be a poset and $A, B, H \subseteq P$. Then*
(1) $A \ll_{S_0} H$, *if and only if for every net* $(x_i)_{i \in I} \xrightarrow{o_s} x \in H$, $x_i \in \uparrow A$ *holds eventually*;
(2) $B \triangleright_{S_0} H$, *if and only if for every net* $(x_i)_{i \in I} \xrightarrow{o_s} x \in H$, $x_i \in \downarrow B$ *holds eventually*.

Proof.
(1) ($\Rightarrow$): Straightforward by the definition of o_s-convergence and Proposition 4.3.
($\Leftarrow$): Suppose $x_i \in \uparrow A$ holds eventually for every net $(x_i)_{i \in I} \xrightarrow{o_s} x \in H$. Let D be a directed subset of P with $\sup D \in H$. Consider the net $(x_d)_{d \in D}$ defined by $x_d = d$ for every $d \in D$. Then we have $(x_i)_{i \in I} \xrightarrow{o} \sup D \in H$, and thus $(x_i)_{i \in I} \xrightarrow{o_s} \sup D \in H$. This implies that $x_{d_0} = d_0 \in \uparrow A$ for some $d_0 \in D$, which shows $A \ll_{S_0} H$.
(2) It can be similarly proved. □

Based on the S_0-approximate relations $\ll_{S_0}$ and $\triangleright_{S_0}$ defined on posets, the concept of S^*-doubly quasicontinuous posets can be introduced as a generalization of S^*-doubly continuous posets.

Definition 4.5. A poset P is called an *S-doubly quasicontinuous poset*, if for every $x \in P$
(S1) $w(x)$ is directed and $v(x)$ filtered;
(S2) $\bigcap\{\uparrow M : M \in w(x)\} = \uparrow x$ and $\bigcap\{\downarrow N : N \in v(x)\} = \downarrow x$.

Definition 4.6. An S-doubly quasicontinuous poset P is said to be *S^*-doubly quasicontinuous*, if for every $x \in P$, every $M \in w(x)$ and every $N \in v(x)$, there exist $M_0 \in w(x)$ and $N_0 \in v(x)$ such that

$$\uparrow M_0 \cap \downarrow N_0 \subseteq \Uparrow_{S_0} M \cap \Downarrow_{S_0} N.$$

In some sense, every S^*-doubly continuous poset is a special kind of S^*-doubly quasicontinuous poset in which the relations $\ll_S$ and $\triangleright_S$ between singleton subsets are replaced by the S_0-approximate relations $\ll_{S_0}$ and $\triangleright_{S_0}$ between finite subsets, respectively.

Remark 4.7. Let P be an S-doubly quasicontinuous poset. Then
(1) $w(x)|_{\downarrow x} = \{M \cap \downarrow x : M \in w(x)\}$ is a directed subfamily of $\mathcal{L}_0(\downarrow x)$ and $\bigcap\{\uparrow M' : M' \in w(x)|_{\downarrow x}\} = \uparrow x$;
(2) $w(x)|_{\downarrow x} \subseteq w(x)$;
(3) $v(x)|_{\uparrow x} = \{N \cap \uparrow x : N \in v(x)\}$ is a filtered subfamily of $\mathcal{L}_0(\uparrow x)$ and $\bigcap\{\downarrow N' : N' \in v(x)|_{\uparrow x}\} = \downarrow x$;
(4) $v(x)|_{\uparrow x} \subseteq v(x)$.

Remark 4.8. Every S^*-doubly continuous poset is an S^*-doubly quasicontinuous poset.

However, the converse implication of Remark 4.8 may not be true. This fact can be illustrated by the following example:

Example 4.9. Let $P = \{a_1, a_2, \dots, a_n, \dots\} \cup \{b_1, b_2, \dots, b_n, \dots\} \cup \{x\}$, and the partial order $\leqslant$ on P is defined by setting
(1) $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant \ldots \leqslant x$;
(2) $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n \leqslant \ldots \leqslant x$.

Then a trivial verification according to Definitions 1.5 and 4.11 can show that P is an S^*-doubly quasicontinuous poset but not an S^*-doubly continuous poset.

In the following, we provide a basis for the topological space $(P, \mathcal{T}_P)$ consisting of an S^*-doubly quasicontinuous poset P and the B-topology on P.

Lemma 4.10. *Let P be an S^*-doubly quasicontinuous poset and $A, B \in \mathcal{P}_0(P)$. Then*
(1) $\mathrm{int}_{\mathcal{T}_P} \uparrow A = \Uparrow_{S_0} A$, *where* $\mathrm{int}_{\mathcal{T}_P} \uparrow A$ *represents the interior of the set* $\uparrow A$ *in the B-topology* $\mathcal{T}_P$;
(2) $\mathrm{int}_{\mathcal{T}_P} \downarrow B = \Downarrow_{S_0} B$.

Proof. (1) We first show $\mathrm{int}_{\mathcal{T}_P} \uparrow A \subseteq \Uparrow_{S_0} A$. Let D be a directed set with $\sup D = x \in \mathrm{int}_{\mathcal{T}_P} \uparrow A$. Then, by Proposition 1.2, there exists $d_0 \in D$ such that $\uparrow d_0 \cap \downarrow x \subseteq \mathrm{int}_{\mathcal{T}_P} \uparrow A \subseteq \uparrow A$. This means $d_0 \in \uparrow A$, and thus we have $x \in \Uparrow_{S_0} A$. This shows $\mathrm{int}_{\mathcal{T}_P} \uparrow A \subseteq \Uparrow_{S_0} A$. Conversely, we prove $\Uparrow_{S_0} A \subseteq \mathrm{int}_{\mathcal{T}_P} \uparrow A$. To prove this , it suffices to check $\Uparrow_{S_0} A \in \mathcal{T}_P$. Let D be a directed set and F a filtered set with $\sup D = \inf F = x \in \Uparrow_{S_0} A$. Then, there exists $M \in w(x)$ such that $M \cap \downarrow x \subseteq \uparrow A$ and $M \cap \downarrow x \in w(x)$ by Proposition 4.3 and Remark 4.7. Since P is an S^*-doubly quasicontinuous poset, there exist $M_0 \in w(x)$ and $N_0 \in v(x)$ such that

$$x \in \uparrow M_0 \cap \downarrow N_0 \subseteq \Uparrow_{S_0}(M \cap \downarrow x) \subseteq \Uparrow_{S_0} A.$$

Thus, we have $d_0 \in \uparrow M_0$ and $e_0 \in \downarrow N_0$ for some $d_0 \in D$ and $e_0 \in F$. This implies

$$x \in \uparrow d_0 \cap \downarrow e_0 \subseteq \uparrow M_0 \cap \downarrow N_0 \subseteq \Uparrow_{S_0} A.$$

Therefore, we conclude $\Uparrow_{S_0} A \in \mathcal{T}_P$ by Proposition 1.2.

(2) The proof is similar to that of (1). □

Theorem 4.11. *Let P be an S^*-doubly quasicontinuous poset. Then the family $O_P = \{\Uparrow_{S_0} M \cap \Downarrow_{S_0} N : M, N \in \mathcal{L}_0(P)\}$ is a basis for the B-topology $\mathcal{T}_P$.*

Proof. Obviously, we have $O_P \subseteq \mathcal{T}_P$ by Lemma 4.10. Let $U \in \mathcal{T}_P$ and $x \in U$. Then, by Lemma 3.1 and Remark 4.7, there exist $M_0 \in w(x)$ and $N_0 \in v(x)$ such that

$$x \in \Uparrow_{S_0}(M_0 \cap \downarrow x) \cap \Downarrow_{S_0}(N_0 \cap \uparrow x) \subseteq \uparrow(M_0 \cap \downarrow x) \cap \downarrow(N_0 \cap \uparrow x) \subseteq U.$$

Set $M = M_0 \cap \downarrow x$ and $N = N_0 \cap \uparrow x$. Then we have $x \in \Uparrow_{S_0} M \cap \Downarrow_{S_0} N \subseteq U$. This shows $O_P = \{\Uparrow_{S_0} M \cap \Downarrow_{S_0} N : M, N \in \mathcal{L}_0(P)\}$ is a basis for the B-topology $\mathcal{T}_P$. □

We present some special B-closed set in general posets in the following proposition:

Proposition 4.12. *Let P be a poset and $M, N \in \mathcal{L}_0(P)$. Then*
(1) *$\downarrow M$ is a B-closed set, i.e., $P \backslash \downarrow M \in \mathcal{T}_P$;*
(2) *$\uparrow N$ is a B-closed set.*

Proof. (1) To show $\downarrow M$ is B-closed, it suffices to prove $P \backslash \downarrow x \in \mathcal{T}_P$ for every $x \in P$. Suppose $P \backslash \downarrow x \notin \mathcal{T}_P$. Then, by Proposition 1.2, there exist a directed set D_0 and a filtered set F_0 such that $\sup D_0 = \inf F_0 = y \in P \backslash \downarrow x$ and $\uparrow d \cap \downarrow e \nsubseteq P \backslash \downarrow x$ for every $d \in D_0$ and $e \in F_0$. This follows that $d \in \downarrow x$ for all $d \in D_0$. Thus, we have $D_0 \subseteq \downarrow x$, which implies $\sup D_0 = y \in \downarrow x$. A contradiction to $\sup D_0 = y \in P \backslash \downarrow x$. Therefore, $P \backslash \downarrow x \in \mathcal{T}_P$.

(2) Similar to (1). □

Based on the fundamental discussion about the B-topology on S^*-doubly quasicontinuous posets, we consider the topological characterization to the S^*-double quasicontinuity.

Let $(P, \leqslant)$ be a poset equipped with a topology $\mathcal{T}$. We simply use the pair $(P, \mathcal{T})$ to denote the poset $(P, \leqslant)$ and the topology $\mathcal{T}$ on the underlying set P.

Definition 4.13. Let P be a poset equipped with a topology $\mathcal{T}$. $(P, \mathcal{T})$ is said to be *locally hyperclosed* if for every $U \in \mathcal{T}$ and $x \in U$, there exist $V \in \mathcal{T}$ and $M, N \in \mathcal{L}_0(P)$ such that $x \in V \subseteq \uparrow M \cap \downarrow N \subseteq U$.

Proposition 4.14. *Let P be a poset. Then $(P, \mathcal{T}_P)$ is locally hyperclosed if and only if for every $U \in \mathcal{T}_P$ and $x \in U$, there exist $M_0 \in \mathcal{L}_0(\downarrow x)$, $N_0 \in \mathcal{L}_0(\uparrow x)$ and $V_0 \in \mathcal{T}_P$ such that*

$$x \in V_0 \subseteq \uparrow M_0 \cap \downarrow N_0 \subseteq U.$$

Proof. ($\Rightarrow$): By Definition 4.13, there exist $V \in \mathcal{T}_P$ and $M, N \in \mathcal{L}_0(P)$ such that $x \in V \subseteq \uparrow M \cap \downarrow N \subseteq U$. Let $V_0 = V \backslash (\uparrow(M \backslash \downarrow x) \cup \downarrow(N \backslash \uparrow x))$, $M_0 = M \backslash \downarrow x$ and $N_0 = N \backslash \uparrow x$. Then, by Proposition 4.12, we have $V_0 \in \mathcal{T}_P$, $M_0 \in \mathcal{L}_0(\downarrow x)$, $N_0 \in \mathcal{L}_0(\uparrow x)$ and

$$x \in V_0 \subseteq \uparrow M_0 \cap \downarrow N_0 \subseteq U.$$

($\Leftarrow$): By Definition 4.13. □

Lemma 4.15. *Let P be a poset. Then the following conditions are equivalent:*
(1) *P is an S^*-doubly quasicontinuous poset;*
(2) *$(P, \mathcal{T}_P)$ is a locally hyperclosed space.*

Proof. (1) $\Rightarrow$ (2): Let $U \in \mathcal{T}_P$ and $x \in U$. Since P is S^*-doubly quasicontinuous, there exist $M, N \in \mathcal{L}_0(P)$ with $x \in \Uparrow_{S_0} M \cap \Downarrow_{S_0} N \subseteq U$ by Theorem 4.11. By the S^*-double quasicontinuity of P, we have that

$$x \in \uparrow M_0 \cap \downarrow N_0 \subseteq \Uparrow_{S_0} M \cap \Downarrow_{S_0} N \subseteq U,$$

for some $M_0 \in w(x)$ and $N_0 \in v(x)$. This implies that

$$x \in \Uparrow_{S_0} M_0 \cap \Downarrow_{S_0} N_0 \subseteq \uparrow M_0 \cap \downarrow N_0 \subseteq \Uparrow_{S_0} M \cap \Downarrow_{S_0} N \subseteq U.$$

This shows that there exist $M_0, N_0 \in \mathcal{L}_0(P)$ and $V = \Uparrow_{S_0} M_0 \cap \Downarrow_{S_0} N_0 \in \mathcal{T}_P$ such that $x \in V \subseteq \uparrow M_0 \cap \downarrow N_0 \subseteq U$. Therefore, $(P, \mathcal{T}_P)$ is a locally hyperclosed space.

(2) ⇒ (1): Let $\mathcal{M}_x = \{M \in \mathcal{L}_0(P) : (\exists N \in \mathcal{L}_0(P))\, x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M \cap \downarrow N)\}$ and $\mathcal{N}_x = \{N \in \mathcal{L}_0(P) : (\exists M \in \mathcal{L}_0(P))\, x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M \cap \downarrow N)\}$ for every $x \in P$. Now, we show

(i) $\mathcal{M}_x \neq \varnothing$ and $\mathcal{N}_x \neq \varnothing$.
Since $(P, \mathcal{T}_P)$ is locally hyperclosed and $P \in \mathcal{T}_P$, there exist $V \in \mathcal{T}_P$ and $M_0, N_0 \in \mathcal{L}_0(P)$ such that $x \in V \subseteq \uparrow M_0 \cap \downarrow N_0 \subseteq P$. This means that $x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M_0 \cap \downarrow N_0) \subseteq P$. Thus, we have $\mathcal{M}_x \neq \varnothing$ and $\mathcal{N}_x \neq \varnothing$.

(ii) $\bigcap\{\uparrow M : M \in \mathcal{M}_x\} = \uparrow x$ and $\bigcap\{\downarrow N : N \in \mathcal{N}_x\} = \downarrow x$.
For every $y \notin \uparrow x$, we have $x \in P \backslash \downarrow y \in \mathcal{T}_P$ by Proposition 4.12. It follows from Proposition 4.14 that there exist $M_1 \in \mathcal{L}_0(\downarrow x)$, $N_1 \in \mathcal{L}_0(\uparrow x)$ and $V_1 \in \mathcal{T}_P$ such that

$$x \in V_1 \subseteq \uparrow M_1 \cap \downarrow N_1 \subseteq P \backslash \downarrow y.$$

This implies that $M_1 \in \mathcal{M}_x$ and $y \notin \uparrow M_1$. Thus, we have $\bigcap\{\uparrow M : M \in \mathcal{M}_x\} = \uparrow x$. And the fact that $\bigcap\{\downarrow N : N \in \mathcal{N}_x\} = \downarrow x$ can be similarly proved.

(iii) $\mathcal{M}_x$ is directed and $\mathcal{N}_x$ is filtered.
Let $M_2, M_3 \in \mathcal{M}_x$. Then, by the definition of $\mathcal{M}_x$, there exist $N_2, N_3 \in \mathcal{L}_0(P)$ such that $x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M_2 \cap \downarrow N_2)$ and $x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M_3 \cap \downarrow N_3)$. This implies $x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M_2 \cap \downarrow N_2 \cap \uparrow M_3 \cap \downarrow N_3)$. By Proposition 4.14, we have $x \in V_4 \subseteq \uparrow M_4 \cap \downarrow N_4 \subseteq \mathrm{int}_{\mathcal{T}_P}(\uparrow M_2 \cap \downarrow N_2 \cap \uparrow M_3 \cap \downarrow N_3)$ for some $M_4 \in \mathcal{L}_0(\downarrow x)$, $N_4 \in \mathcal{L}_0(\uparrow x)$ and $V_4 \in \mathcal{T}_P$. This implies

$$x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M_4 \cap \downarrow N_4) \subseteq \uparrow M_4 \cap \downarrow N_4 \subseteq \mathrm{int}_{\mathcal{T}_P}(\uparrow M_2 \cap \uparrow M_3).$$

Thus, we show $M_4 \in \mathcal{M}_x$ and $M_4 \subseteq \uparrow M_2 \cap \uparrow M_3$, which means $\mathcal{M}_x$ is directed. A similar verification can show that $\mathcal{N}_x$ is filtered.

(iv) $\mathcal{M}_x \subseteq w(x)$ and $\mathcal{N}_x \subseteq v(x)$.
For every $M \in \mathcal{M}_x$, there exists $N \in \mathcal{L}_0(P)$ such that $x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M \cap \downarrow N)$ by the definition of $\mathcal{M}_x$. Let D be a directed set with $\sup D = x$. Then, by Proposition 1.2, there exists $d_0 \in D$ such that $x \in \uparrow d_0 \cap \downarrow x \subseteq \mathrm{int}_{\mathcal{T}_P}(\uparrow M \cap \downarrow N)$. This means $d_0 \in \uparrow M$. So, we have $M \ll_{S_0} x$, which implies $\mathcal{M}_x \subseteq w(x)$. It can be similarly proved that $\mathcal{N}_x \subseteq v(x)$.

(v) $\bigcap\{\uparrow A : A \in w(x)\} = \uparrow x$ and $\bigcap\{\downarrow B : B \in v(x)\} = \downarrow x$.
Straightforward by (ii) and (iv).

(vi) $\mathcal{M}_x|_{\downarrow x}$ is directed, $\bigcap\{\uparrow M' : M' \in \mathcal{M}_x|_{\downarrow x}\} = \uparrow x$ and $\mathcal{M}_x|_{\downarrow x} \subseteq \mathcal{M}_x \subseteq w(x)$.
This is straightforward by (ii), (iii), (iv) and the proof of Proposition 4.14.

(vii) $\mathcal{N}_x|_{\uparrow x}$ is filtered, $\bigcap\{\downarrow N' : N' \in \mathcal{N}_x|_{\uparrow x}\} = \downarrow x$ and $\mathcal{N}_x|_{\uparrow x} \subseteq \mathcal{N}_x \subseteq v(x)$.
Similar to (vi).

(viii) $w(x)$ is directed and $v(x)$ is filtered. Let $A_1, A_2 \in w(x)$.
Then, by (vi) and Proposition 4.3, there exist $M_5, M_6 \in \mathcal{M}_x|_{\downarrow x} \subseteq w(x)$ such that $M_5 \subseteq \uparrow A_1$ and $M_6 \subseteq \uparrow A_2$. Since $\mathcal{M}_x|_{\downarrow x}$ is directed, there exists $M_7 \in \mathcal{M}_x|_{\downarrow x} \subseteq w(x)$ such that $M_7 \subseteq \uparrow M_5 \cap \uparrow M_6$. This means $M_7 \subseteq \uparrow A_1 \cap \uparrow A_2$. So, $w(x)$ is directed. It can be similarly shown that $v(x)$ is filtered.

(ix) For every $A \in w(x)$ and $B \in v(x)$, there exist $A_0 \in w(x)$ and $B_0 \in v(x)$ such that $x \in \uparrow A_0 \cap \downarrow B_0 \subseteq \Uparrow_{S_0} A \cap \Downarrow_{S_0} B$.
Let $A \in w(x)$ and $B \in v(x)$. Then, by (vi), (vii) and Proposition 4.3, there exist $M_A \in \mathcal{M}_x|_{\downarrow x} \subseteq \mathcal{M}_x$ and $N_B \in \mathcal{N}_x|_{\uparrow x} \subseteq \mathcal{N}_x$ such that $M_A \subseteq \uparrow A$ and $N_B \subseteq \downarrow B$. This implies, from the definitions of $\mathcal{M}_x$ and $\mathcal{N}_x$, that there exist $M_8, N_8 \in \mathcal{L}_0(P)$ such that $x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M_A \cap \downarrow N_8)$ and $x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M_8 \cap \downarrow N_B)$. Thus, we have

$$x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow M_A \cap \downarrow N_B) \subseteq \uparrow M_A \cap \downarrow N_B \subseteq \uparrow A \cap \downarrow B,$$

which means $x \in \mathrm{int}_{\mathcal{T}_P}(\uparrow A \cap \downarrow B)$. It follows from Lemma 3.1 that $x \in \uparrow M_9 \cap \downarrow N_9 \subseteq \mathrm{int}_{\mathcal{T}_P}(\uparrow A \cap \downarrow B)$ for some $M_9 \in \mathcal{M}_x|_{\downarrow x} \subseteq w(x)$ and $N_9 \in \mathcal{N}_x|_{\uparrow x} \subseteq v(x)$. In the following, we show $x \in \uparrow M_9 \cap \downarrow N_9 \subseteq \Uparrow_{S_0} A \cap \Downarrow_{S_0} B$. For every directed set D with $\sup D \in \uparrow M_9 \cap \downarrow N_9$, there exists $d_1 \in D$ such that $\uparrow d_1 \cap \downarrow \sup D \subseteq \mathrm{int}_{\mathcal{T}_P}(\uparrow A \cap \downarrow B)$ by Proposition 1.2, which implies $d_1 \in \uparrow A$. Thus, we have $x \in \uparrow M_9 \cap \downarrow N_9 \subseteq \Uparrow_{S_0} A$. It can be similarly checked that $x \in \uparrow M_9 \cap \downarrow N_9 \subseteq \Downarrow_{S_0} B$. So, we conclude $x \in \uparrow M_9 \cap \downarrow N_9 \subseteq \Uparrow_{S_0} A \cap \Downarrow_{S_0} B$. Set $A_0 = M_9$ and $B_0 = N_9$. Then, we have $x \in \uparrow A_0 \cap \downarrow B_0 \subseteq \Uparrow_{S_0} A \cap \Downarrow_{S_0} B$.

The combination of (v), (viii) and (ix) shows that P is an S^*-doubly quasicontinuous poset. □

For a topological space $(X, \mathcal{T})$, the complete lattice $(\mathcal{T}, \subseteq)$ consisting of the topology $\mathcal{T}$ and the set inclusion order $\subseteq$ on $\mathcal{T}$ is called the *lattice of the topology $\mathcal{T}$ on X*. And one can easily verify that $\sup \mathcal{T}_0 = \bigcup\{U : U \in \mathcal{T}_0\}$ and $\inf \mathcal{T}_0 = \mathrm{int}_{\mathcal{T}}(\bigcap\{U : U \in \mathcal{T}_0\})$ for every subfamily $\mathcal{T}_0$ of the topology $\mathcal{T}$. From this point of view, given an S^*-doubly quasicontinuous poset P, the lattice $(\mathcal{T}_P, \subseteq)$ of the the B-topology $\mathcal{T}_P$ on P is a hypercontinuous lattice.

Definition 4.16. [16] For a complete lattice L, define a relation $\prec$ on L by

$$(\forall\, x, y \in L)\ x \prec y \iff y \in \mathrm{int}_{\nu(L)} \uparrow x.$$

The complete lattice L is said to be *hypercontinuous* if $p = \sup\{u \in L : u \prec p\}$ for every $p \in P$.

Lemma 4.17. *Let $(\mathcal{T}, \subseteq)$ be the lattice of a topology $\mathcal{T}$ on X. Then $(\mathcal{T}, \subseteq)$ is hypercontinuous if and only if for every $U \in \mathcal{T}$ and $x \in U$, there exist $V, U_1, U_2, \ldots, U_n \in \mathcal{T}$ such that $x \in V \subseteq U$ and*

$$U \in \mathcal{T} \setminus {\downarrow_{\mathcal{T}}}\{U_1, U_2, \ldots, U_n\} \subseteq {\uparrow_{\mathcal{T}}}\{V\},$$

where ${\downarrow_{\mathcal{T}}}\{U_1, U_2, \ldots, U_n\} = \{W \in \mathcal{T} : (\exists i \in \{1, 2, \ldots, n\})\ W \subseteq U_i\}$ and ${\uparrow_{\mathcal{T}}}\{V\} = \{W \in \mathcal{T} : V \subseteq W\}$.

Proof. This follows straightforwardly from Definition 4.16. □

Lemma 4.18. *Let P be a poset. Then the following statements are equivalent*:
(1) *$(P, \mathcal{T}_P)$ is a locally hyperclosed space*;
(2) *$(\mathcal{T}_P, \subseteq)$ is a hypercontinuous lattice.*

Proof. (1) $\Rightarrow$ (2): Suppose that $(P, \mathcal{T}_P)$ is a locally hyperclosed space. Let $U \in \mathcal{T}_P$ and $x \in U$. Then, by Proposition 4.14, there exist $M \in \mathcal{L}_0(\downarrow x)$, $N \in \mathcal{L}_0(\uparrow x)$ and $V \in \mathcal{T}_P$ such that $x \in V \subseteq {\uparrow M} \cap {\downarrow N} \subseteq U$. Let $U_m = P \setminus \mathrm{cl}_{\mathcal{T}_P}\{m\}$ for every $m \in M$ and $V_n = P \setminus \mathrm{cl}_{\mathcal{T}_P}\{n\}$ for every $n \in N$. Then one can trivially verify that

$$U \in \mathcal{T}_P \setminus {\downarrow_{\mathcal{T}_P}}(\{U_m : m \in M\} \cup \{V_n : n \in N\}) \subseteq {\uparrow_{\mathcal{T}_P}}\{V\}.$$

This shows, by Lemma 4.17, that $(\mathcal{T}_P, \subseteq)$ is a hypercontinuous lattice.

(2) $\Rightarrow$ (1): To show this implication, it suffices to prove that for every $U' \in \mathcal{T}_P$ and $x \in U'$, there exist $V' \in \mathcal{T}_P$ and $M_0, N_0 \in \mathcal{L}_0(P)$ such that $x \in V' \subseteq {\uparrow M_0} \cap {\downarrow N_0} \subseteq U'$. We divide the proof in the following three steps:

(i) For every $U \in \mathcal{T}_P$ and $x \in U$, there exist $M_U \in \mathcal{L}_0(\downarrow x)$, $N_U \in \mathcal{L}_0(\uparrow x)$ and $V_U \in \mathcal{T}_P$ such that $M_U, N_U \subseteq U$ and $x \in V_U \subseteq {\uparrow M_U} \cap {\downarrow N_U}$.
Since $(\mathcal{T}_P, \subseteq)$ is a hypercontinuous lattice, by Lemma 4.17, there exist $V, U_1, U_2, \ldots, U_k \in \mathcal{T}_P$ such that $x \in V \subseteq U$ and $U \in \mathcal{T}_P \setminus {\downarrow_{\mathcal{T}_P}}\{U_1, U_2, \ldots, U_k\} \subseteq {\uparrow_{\mathcal{T}_P}}\{V\}$. It follows that $U \nsubseteq U_i$ for every $i \in \{1, 2, \ldots, k\}$, which implies that there exists $x_i \in U \setminus U_i$ for every $i \in \{1, 2, \ldots, k\}$. Set $X_0 = \{x_1, x_2, \ldots, x_k\} \in \mathcal{L}_0(U)$. We show next that $x \in V \subseteq {\uparrow X_0} \cap {\downarrow X_0}$. Suppose $v_0 \notin {\uparrow X_0}$ for some $v_0 \in V$. Then by Proposition 4.12 we have $U \setminus {\downarrow v_0} \in \mathcal{T}_P$ and $X_0 \subseteq U \setminus {\downarrow v_0}$. This implies that $U \setminus {\downarrow v_0} \nsubseteq U_i$ for every $i \in \{1, 2, \ldots, k\}$. This follows, from Lemma 4.17, that $V \subseteq U \setminus {\downarrow v_0}$, which contradicts the fact that $v_0 \in V$. Thus, we have $V \subseteq {\uparrow X_0}$. It can be similarly shown that $V \subseteq {\downarrow X_0}$. Therefore, we conclude $x \in V \subseteq {\uparrow X_0} \cap {\downarrow X_0}$. Now, let $M_U = X_0 \cap {\downarrow x}$, $N_U = X_0 \cap {\uparrow x}$ and $V_U = V \setminus [{\uparrow}(X_0 {\downarrow x}) \cup {\downarrow}(X_0 \setminus {\uparrow x})]$. Then a trivial verification can show that $M_U \in \mathcal{L}_0(\downarrow x)$, $N_U \in \mathcal{L}_0(\uparrow x)$, $M_U, N_U \subseteq U$, $V_U \in \mathcal{T}_P$ and

$$x \in V_U \subseteq {\uparrow M_U} \cap {\downarrow N_U}.$$

(ii) Let $\mathcal{M}_x = \{M_U \in \mathcal{L}_0(\downarrow x) : U \in \mathcal{T}_P\ \&\ x \in U\}$ and $\mathcal{N}_x = \{N_U \in \mathcal{L}_0(\uparrow x) : U \in \mathcal{T}_P\ \&\ x \in U\}$. Then $\mathcal{M}_x$ is directed and $\mathcal{N}_x$ is filtered.
For any $M_{U_1}, M_{U_2} \in \mathcal{M}_x$, by (i), we have $V_{U_1}, V_{U_2} \in \mathcal{T}_P$, $x \in V_{U_1} \subseteq {\uparrow M_{U_1}}$ and $x \in V_{U_2} \subseteq {\uparrow M_{U_2}}$. Let $U_3 = V_{U_1} \cap V_{U_2}$. Then, by (i), we further have $M_{U_3} \subseteq U_3$ and $x \in V_{U_3} \subseteq {\uparrow M_{U_3}}$. This implies $M_{U_3} \in \mathcal{M}_x$ and

$$M_{U_3} \subseteq U_3 = V_{U_1} \cap V_{U_2} \subseteq {\uparrow M_{U_1}} \cap {\uparrow M_{U_2}}.$$

Thus, $\mathcal{M}_x$ is directed. Similarly, we can show that $\mathcal{N}_x$ is filtered.

(iii) $\bigcap\{\uparrow M_U : M_U \in \mathcal{M}_x\} = \uparrow x$ and $\bigcap\{\downarrow N_U : N_U \in \mathcal{N}_x\} = \downarrow x$.
Let $y \in P$ with $y \notin \uparrow x$.
If we take $U_y = P\backslash\downarrow y$, then $x \in U_y = P\backslash\downarrow y \in \mathcal{T}_P$. It follows from (i) that $M_{U_y} \subseteq U_y = P\backslash\downarrow y$ and $M_{U_y} \in \mathcal{M}_x$. This means $y \notin \uparrow M_{U_y}$. Thus, we conclude $\bigcap\{\uparrow M_U : M_U \in \mathcal{M}_x\} = \uparrow x$. And $\bigcap\{\downarrow N_U : N_U \in \mathcal{N}_x\} = \downarrow x$ can be similarly proved.

It follows from (ii), (iii) and Lemma 3.1 that $x \in V_{U_0} \subseteq \uparrow M_{U_0} \cap \downarrow N_{U_0} \subseteq U'$ for some $M_{U_0} \in \mathcal{M}_x$ and $N_{U_0} \in \mathcal{N}_x$. Set $M_0 = M_{U_0}$, $N_0 = N_{U_0}$ and $V' = V_{U_0}$. Then we have

$$x \in V' \subseteq \uparrow M_0 \cap \downarrow N_0 \subseteq U'.$$

This completes the proof. □

Lemma 4.19. *Let P be a poset. Then the following statements are equivalent*:
(1) *$(P, \mathcal{T}_P)$ is a locally hyperclosed space*;
(2) *$\Uparrow_{S_0} M \cap \Downarrow_{S_0} N \in \mathcal{T}_P$ for any $M, N \in \mathcal{L}_0(P)$, and if $U \in \mathcal{T}_P$* and *$x \in U$, then there exist $M_0, N_0 \in \mathcal{L}_0(U)$ such that $x \in \Uparrow_{S_0} M_0 \cap \Downarrow_{S_0} N_0$.*

Proof. (1) $\Rightarrow$ (2): By Theorem 4.11 and Lemma 4.15, Remark 4.7 and Lemma 3.1.
(2) $\Rightarrow$ (1): Similar to the proof of (2) $\Rightarrow$ (1) in Lemma 4.18. □

Theorem 4.20. *Let P be a poset. Then the following statements are equivalent*:
(1) *P is an S^*-doubly quasicontinuous poset*;
(2) *$(P, \mathcal{T}_P)$ is a locally hyperclosed space*;
(3) *$(\mathcal{T}_P, \subseteq)$ is a hypercontinuous lattice*;
(4) *$\Uparrow_{S_0} M \cap \Downarrow_{S_0} N \in \mathcal{T}_P$ for any $M, N \in \mathcal{L}_0(P)$, and if $U \in \mathcal{T}_P$ and $x \in U$, then there exist $M_0, N_0 \in \mathcal{L}_0(U)$ such that $x \in \Uparrow_{S_0} M_0 \cap \Downarrow_{S_0} N_0$.*

Proof. Straightforward by Lemmas 4.15, 4.18 and 4.19. □

5 The order-theoretical characterization to o_s-convergence being topological

In this section, we explore the order-theoretical characterization to o_s-convergence being topological. We prove that the o_s-convergence in a poset is topological if and only if the poset is S^*-doubly quasicontinuous.

Definition 5.1. [17] Given a topological space $(X, \mathcal{T})$ and $x \in X$. A net $(x_i)_{i\in I}$ in X is said to *converge to x with respect to the topology $\mathcal{T}$* if for every $U \in \mathcal{T}$ with $x \in U$, there exists $i_0 \in I$ such that $x_i \in U$ for all $i \geq i_0$. In this case, we write $(x_i)_{i\in I} \xrightarrow{\mathcal{T}} x$.

Proposition 5.2. [17] *Let $(X, \mathcal{T})$ be a topological space. If we*
(1) *set $\mathcal{I}_x = \{U \in \mathcal{T} : x \in U\}$ and $\mathcal{I}_x^{\mathcal{T}} = \{(a, U) \in X \times \mathcal{I}_x : a \in U\}$*;
(2) *define the directed preorder $\preccurlyeq$ on $\mathcal{I}_x^{\mathcal{T}}$ by*

$$(\forall (a_1, U_1), (a_2, U_2) \in \mathcal{I}_x^{\mathcal{T}})\ (a_1, U_1) \preccurlyeq (a_2, U_2) \iff U_2 \subseteq U_1;$$

(3) *let $x_{(a,U)} = a$ for all $(a, U) \in \mathcal{I}_x^{\mathcal{T}}$*;

then the net $(x_{(a,U)})_{(a,U)\in\mathcal{I}_x^{\mathcal{T}}} \xrightarrow{\mathcal{T}} x$.

Definition 5.3. Let P be a poset. The o_s-convergence in P is said to be *topological with respect to a topology $\mathcal{T}$ on P* if for every net $(x_i)_{i\in I}$ in P, we have

$$(x_i)_{i\in I} \xrightarrow{\mathcal{T}} x \in P \;\Leftrightarrow\; (x_i)_{i\in I} \xrightarrow{o_s} x.$$

Lemma 5.4. *Let P be an S^*-doubly quasicontinuous poset. Then the o_s-convergence in P is topological with respect to the B-topology $\mathcal{T}_P$.*

Proof. To prove this lemma, it suffices to show

$$(x_i)_{i\in I} \xrightarrow{\mathcal{T}_P} x \in P \;\Leftrightarrow\; (x_i)_{i\in I} \xrightarrow{o_s} x.$$

($\Rightarrow$): Let P be an S^*-doubly quasicontinuous poset. By Remark 4.7, we have
(1) $w(x)|_{\downarrow x}$ is directed and $v(x)|_{\uparrow x}$ is filtered;
(2) $\bigcap\{\uparrow M : M \in w(x)|_{\downarrow x}\} = \uparrow x$ and $\bigcap\{\downarrow N : N \in v(x)|_{\uparrow x}\} = \downarrow x$.
Suppose $(x_i)_{i\in I} \xrightarrow{\mathcal{T}_P} x$. By Theorem 4.11, we have $x \in \Uparrow_{S_0} M \cap \Downarrow_{S_0} N \in \mathcal{T}_P$ for every $M \in w(x)_{\downarrow x}$ and $N \in v(x)_{\uparrow x}$. This implies that $x_i \in \Uparrow_{S_0} M \cap \Downarrow_{S_0} N \subseteq \uparrow M \cap \downarrow N$ holds eventually. By the definition of o_s-convergence, we conclude $(x_i)_{i\in I} \xrightarrow{o_s} x$.
($\Leftarrow$): By Theorem 3.2. □

Lemma 5.5. *Let P be a poset. If the o_s-convergence is topological with respect to a topology $\mathcal{T}$ on P, then $\mathcal{T} = \mathcal{T}_P$.*

Proof. Suppose that the o_s-convergence is topological with respect to a topology $\mathcal{T}$ on P. Then we have

$$(x_i)_{i\in I} \xrightarrow{\mathcal{T}} x \in P \;\Leftrightarrow\; (x_i)_{i\in I} \xrightarrow{o_s} x.$$

Let $U \in \mathcal{T}$ and $(x_i)_{i\in I} \xrightarrow{o_s} x \in U$. Then we have $(x_i)_{i\in I} \xrightarrow{\mathcal{T}} x \in U$. This implies that $x_i \in U$ holds eventually. By Theorem 3.2, it follows that $U \in \mathcal{T}_P$, which means $\mathcal{T} \subseteq \mathcal{T}_P$. Conversely, let $V \in \mathcal{T}_P$ and $x \in V$. Then the net $(x_{(a,U)})_{(a,U)\in \mathcal{I}_x^{\mathcal{T}}} \xrightarrow{\mathcal{T}} x \in V$ by Proposition 5.2. It follows from the hypothesis that the net $(x_{(a,U)})_{(a,U)\in \mathcal{I}_x^{\mathcal{T}}} \xrightarrow{o_s} x$. By Theorem 3.2, we have that $x_{(a,U)} = a \in V$ holds eventually. This means that there exists $(a_0, U_0) \in \mathcal{I}_x^{\mathcal{T}}$ (definitely, $x \in U_0 \in \mathcal{T}$) such that $x_{(a,U)} = a \in V$ for all $(a, U) \succcurlyeq (a_0, U_0)$. Since $(a, U_0) \succcurlyeq (a_0, U_0)$ for all $a \in U_0$, $x_{(a,U_0)} = a \in V$ or all $a \in U_0$. This implies $x \in U_0 \subseteq V$. Thus, we conclude $\mathcal{T}_P \subseteq \mathcal{T}$. Therefore, we have $\mathcal{T}_P = \mathcal{T}$. □

In fact, the converse implication of Lemma 5.4 is also true, that is:

Lemma 5.6. *Let P be a poset. If the o_s-convergence in P is topological with respect to a topology $\mathcal{T}$ on P, then P is S^*-doubly quasicontinuous.*

Proof. Suppose that the o_s-convergence is topological with respect to a topology $\mathcal{T}$ on P. Then, by Lemma 5.5, we have

$$(x_i)_{i\in I} \xrightarrow{\mathcal{T}_P} x \in P \;\Leftrightarrow\; (x_i)_{i\in I} \xrightarrow{o_s} x.$$

Let $V \in \mathcal{T}_P$ and $x \in V$. By Proposition 5.2, it follows the net $(x_{(a,U)})_{(a,U)\in \mathcal{I}_x^{\mathcal{T}_P}} \xrightarrow{\mathcal{T}_P} x$. This implies that $(x_{(a,U)})_{(a,U)\in \mathcal{I}_x^{\mathcal{T}_P}} \xrightarrow{o_s} x$, which lets one to conclude, from Definition 2.1, that there exist $\mathcal{M}_x \subseteq \mathcal{L}_0(\downarrow x)$ and $\mathcal{N}_x \subseteq \mathcal{L}_0(\uparrow x)$ such that
(1) $\mathcal{M}_x$ is directed and $\mathcal{N}_x$ is filtered;
(2) $\bigcap\{\uparrow M : M \in \mathcal{M}_x\} = \uparrow x$ and $\bigcap\{\downarrow N : N \in \mathcal{N}_x\} = \downarrow x$;
(3) For any $M \in \mathcal{M}_x$ and $N \in \mathcal{N}_x$, $x_{(a,U)} \in \uparrow M \cap \downarrow N$ holds eventually.

By Lemma 3.1, it follows that there exist $M_0 \in \mathcal{M}_x$, $N_0 \in \mathcal{N}_x$ and $(a_0, U_0) \in \mathcal{I}_x^{\mathcal{T}_P}$ (definitely, $x \in U_0 \in \mathcal{T}_P$) such that $x \in \uparrow M_0 \cap \downarrow N_0 \subseteq V$ and $x_{(a,U)} = a \in \uparrow M_0 \cap \downarrow N_0$ for all $(a, U) \succcurlyeq (a_0, U_0)$. Since $(a, U_0) \succcurlyeq (a_0, U_0)$ for all $a \in U_0$, $x_{(a,U_0)} = a \in \uparrow M_0 \cap \downarrow N_0$ for all $a \in U_0$, which means

$$x \in U_0 \subseteq \uparrow M_0 \cap \downarrow N_0 \subseteq V.$$

This shows that $(P, \mathcal{T}_P)$ is a locally hyperclosed space. By Lemma 4.15, the poset P is S^*-doubly quasicontinuous. □

Now, we arrive at the order-theoretical characterization to the o_s-convergence being topological:

Theorem 5.7. *Let P be a poset. Then the following statements are equivalent*:
(1) *The poset P is S^*-doubly quasicontinuous*;
(2) *The o_s-convergence is topological with respect to a topology $\mathcal{T}$ on P, i.e., for every $(x_i)_{i\in I}$ in P, we have*

$$(x_i)_{i\in I} \xrightarrow{\mathcal{T}} x \in P \iff (x_i)_{i\in I} \xrightarrow{o_s} x;$$

(3) *The o_s-convergence is topological with respect to the B-topology $\mathcal{T}_P$, i.e., for every $(x_i)_{i\in I}$ in P, we have*

$$(x_i)_{i\in I} \xrightarrow{\mathcal{T}_P} x \in P \iff (x_i)_{i\in I} \xrightarrow{o_s} x.$$

Proof. (1) $\Rightarrow$ (3): By Lemma 5.4.
(3) $\Rightarrow$ (2): By Lemma 5.5.
(2) $\Rightarrow$ (1): By Lemma 5.6. □

6 The complete regularity of B-topology

In this section, we consider the complete regularity of B-topology, and show that a topological space $(P, \mathcal{T}_P)$ consisting of an S^*-doubly quasicontinuous poset P that satisfies Condition (◇) and the B-topology on P is Tychonoff.

Given a topological space $(X, \mathcal{T})$. $(X, \mathcal{T})$ is called a T_1 *space* if for every pair of distinct points $x_1, x_2 \in X$, there exists $U \in \mathcal{T}$ such that $x_1 \in U$ and $x_2 \notin U$. We call that $(X, \mathcal{T})$ is *regular* if for every $x \in X$ and every closed set $C \subseteq X$ with $x \notin C$, there exist $U_1, U_2 \in \mathcal{T}$ such that $x \in U_1$, $C \subseteq U_2$ and $U_1 \cap U_2 = \varnothing$. We say that a topological space $(X, \mathcal{T})$ is T_3 if it is regular and T_1.

Proposition 6.1. *Let P be a poset. Then $(P, \mathcal{T}_P)$ is T_1.*

Proof. Let $x, y \in P$ and $x \neq y$. Then we have $x \notin \uparrow y$ or $x \notin \downarrow y$. This implies, by Proposition 4.12, that $x \in P\backslash\uparrow y \in \mathcal{T}_P$ or $x \in P\backslash\downarrow y \in \mathcal{T}_P$. It is obvious that $y \notin P\backslash\uparrow y \in \mathcal{T}_P$ and $y \notin P\backslash\downarrow y \in \mathcal{T}_P$. Thus, we conclude that $(P, \mathcal{T}_P)$ is a T_1 space. □

Theorem 6.2. *Let P be an S^*-doubly quasicontinuous poset. Then $(P, \mathcal{T}_P)$ is T_3.*

Proof. To prove $(P, \mathcal{T}_P)$ is T_3, we only need to show that $(P, \mathcal{T}_P)$ is regular by Proposition 6.1. Let $x \in P$ and $P\backslash C \in \mathcal{T}_P$ with $x \notin C$. Then, by Lemma 4.15 and Proposition 4.14, there exist $V \in \mathcal{T}_P$, $M \in \mathcal{L}_0(\downarrow x)$ and $N \in \mathcal{L}_0(\uparrow x)$ such that $x \in V \subseteq \uparrow M \cap \downarrow N \subseteq P\backslash C$. It follows from Proposition 4.12 that $C \subseteq P\backslash(\uparrow M \cap \downarrow N) \in \mathcal{T}_P$, $x \in V \in \mathcal{T}_P$ and $V \cap P\backslash(\uparrow M \cap \downarrow N) = \varnothing$. This shows that $(P, \mathcal{T}_P)$ is T_3. □

Recall that the pair $(\mathbb{R}, \mathcal{T}_0)$ consisting the set $\mathbb{R}$ of all real numbers and the ordinal topology $\mathcal{T}_0$ on $\mathbb{R}$ is a topological space. And the inherited topology $\mathcal{T}_0|_{[0,1]}$ on the unit interval $[0, 1]$ has the basis $\{[0, a) : a \in (0, 1]\} \cup \{(b, c) : b, c \in [0, 1] \text{ and } b < c\} \cap \{(d, 1] : d \in [0, 1)\}$ and subbasis $\{[0, a) : a \in (0, 1]\} \cup \{(b, 1] : b \in [0, 1)\}$.

Definition 6.3. [17] Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is said to be *completely regular* if for every $X\backslash C \in \mathcal{T}$ and $x \in X\backslash C$, there exists a continuous function $f : (X, \mathcal{T}) \to ([0,1], \mathcal{T}_0|_{[0,1]})$ such that $f(x) = 0$ and $f(C) \subseteq \{1\}$.

Definition 6.4. [17] A topological space $(X, \mathcal{T})$ is called a *Tychonoff space* if it is T_1 and completely regular. A Tychonoff space is also said to be $T_{3\frac{1}{2}}$.

Condition $(\diamond)$: A poset P is said to satisfy Condition $(\diamond)$ if
(1) for any $x, y \in P$ and $M \in \mathcal{L}_0(P)$, $M \ll_{S_0} x \leqslant y$ implies $M \ll_{S_0} y$;
(2) for any $a, b \in P$ and $N \in \mathcal{L}_0(P)$, $N \triangleright_{S_0} a \geqslant b$ implies $N \triangleright_{S_0} b$.

Example 6.5.
(1) Every doubly quasicontinuous poset P, i.e., both P and P^{op} are quasicontinuous (see [18, Exercise 5.1.34]), satisfies Condition $(\diamond)$;
(2) Every doubly continuous poset satisfies Condition $(\diamond)$;
(3) Chains, antichains, finite posets all satisfy Condition $(\diamond)$.

Proposition 6.6. *Let P be an S^*-doubly quasicontinuous poset satisfying Condition $(\diamond)$. Then for any $x \in P$, $M_1, M_2 \in \mathcal{L}_0(\downarrow x)$ and $N_1, N_2 \in \mathcal{L}_0(\uparrow x)$ with $x \in \Uparrow_{S_0} M_1 \cap \Downarrow_{S_0} N_1 \subseteq \uparrow M_1 \cap \downarrow N_1 \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2$, there exist $M_0 \in \mathcal{L}_0(\downarrow x)$ and $N_0 \in \mathcal{L}_0(\uparrow x)$ such that*

$$x \in \Uparrow_{S_0} M_1 \cap \Downarrow_{S_0} N_1 \subseteq \uparrow M_1 \cap \downarrow N_1 \subseteq \Uparrow_{S_0} M_0 \cap \Downarrow_{S_0} N_0 \subseteq \uparrow M_0 \cap \downarrow N_0 \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2.$$

Proof. Let $x \in P$, $M_1, M_2 \in \mathcal{L}_0(\downarrow x)$ and $N_1, N_2 \in \mathcal{L}_0(\uparrow x)$ with $x \in \Uparrow_{S_0} M_1 \cap \Downarrow_{S_0} N_1 \subseteq \uparrow M_1 \cap \downarrow N_1 \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2$. Since $m \in \uparrow M_1 \cap \downarrow N_1 \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2$ for every $m \in M_1$, by Theorem 4.20, Lemma 4.10 and Proposition 4.14, there exist $M_0^m \in \mathcal{L}_0(\downarrow m)$ and $A_0^m \in \mathcal{L}_0(\uparrow m)$ such that $m \in \Uparrow_{S_0} M_0^m \cap \Downarrow_{S_0} A_0^m \subseteq \uparrow M_0^m \cap \downarrow A_0^m \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2$ for every $m \in M_1$. Similarly, there exist $B_0^n \in \mathcal{L}_0(\downarrow n)$ and $N_0^n \in \mathcal{L}_0(\uparrow n)$ such that $n \in \Uparrow_{S_0} B_0^n \cap \Downarrow_{S_0} N_0^n \subseteq \uparrow B_0^n \cap \downarrow N_0^n \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2$ for every $n \in N_1$. Let $M_0 = \bigcup\{M_0^m : m \in M_1\}$ and $N_0 = \bigcup\{N_0^n : n \in N_1\}$. Then one can easily see that $M_0 \in \mathcal{L}_0(\downarrow x)$ and $N_0 \in \mathcal{L}_0(\uparrow x)$. Next we show
(1) $\uparrow M_1 \cap \downarrow N_1 \subseteq \Uparrow_{S_0} M_0 \cap \Downarrow_{S_0} N_0$.
Since $M_0^m \subseteq M_0$ and $m \in \Uparrow_{S_0} M_0^m$ for every $m \in M_1$, we have $m \in \Uparrow_{S_0} M_0$ for every $m \in M_1$. Let $y \in \uparrow M_1 \cap \downarrow N_1$. Then $y \in \uparrow m'$ for some $m' \in M_1$. This implies by Condition $(\diamond)$ that $y \in \Uparrow_{S_0} M_0$. It can be similarly shown that $y \in \Downarrow_{S_0} N_0$. Thus, we have $\uparrow M_1 \cap \downarrow N_1 \subseteq \Uparrow_{S_0} M_0 \cap \Downarrow_{S_0} N_0$.
(2) $\uparrow M_0 \cap \downarrow N_0 \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2$.
Since $M_0^m \in \mathcal{L}_0(\downarrow m)$, $A_0^m \in \mathcal{L}_0(\uparrow m)$ and $m \in \uparrow M_0^m \cap \downarrow A_0^m \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2$ for every $m \in M_1$, $M_0^m \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2$ for every $m \in M_1$. This means that $M_0 \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2$. Let $z \in \uparrow M_0 \cap \downarrow N_0$. Then, by Condition $(\diamond)$, we have $z \in \Uparrow_{S_0} M_2$. It can be similarly proved that $z \in \Downarrow_{S_0} N_2$. Thus, we conclude $\uparrow M_0 \cap \downarrow N_0 \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2$.

The combination of (1) and (2) shows

$$x \in \Uparrow_{S_0} M_1 \cap \Downarrow_{S_0} N_1 \subseteq \uparrow M_1 \cap \downarrow N_1 \subseteq \Uparrow_{S_0} M_0 \cap \Downarrow_{S_0} N_0 \subseteq \uparrow M_0 \cap \downarrow N_0 \subseteq \Uparrow_{S_0} M_2 \cap \Downarrow_{S_0} N_2.$$ □

Recall that the set B of all dyadic rational numbers in $[0,1]$ is the union of sets $B_n = \left\{\frac{0}{2^n}, \frac{1}{2^n}, \ldots, \frac{2^n}{2^n}\right\}$ $(n = 0, 1, 2, \ldots, k, \ldots)$, i.e., $B = \bigcup_{n=0}^{\infty} B_n$. It is easy to observe that B is dense in the unit interval $[0,1]$, with respect to the inherited topology $\mathcal{T}_0|_{[0,1]}$.

Proposition 6.7. *Let P be an S^*-doubly quasicontinuous poset satisfying Condition $(\diamond)$ and $x \in P$. If $M \in \mathcal{L}_0(\downarrow x)$ and $N \in \mathcal{L}_0(\uparrow x)$ with $x \in \Uparrow_{S_0} M \cap \Downarrow_{S_0} N$, then there exist families $\{M_b \in \mathcal{L}_0(\downarrow x) : b \in B\}$ and $\{N_b \in \mathcal{L}_0(\downarrow x) : b \in B\}$ such that*
(1) *$M_0 = M$, $N_0 = N$ and $M_1 = N_1 = x$;*
(2) *$\uparrow M_{b_2} \cap \downarrow N_{b_2} \subseteq \Uparrow_{S_0} M_{b_1} \cap \Downarrow_{S_0} N_{b_1}$ whenever $b_1 < b_2$.*

Proof. Straightforward by Theorem 4.11, Remark 4.7, Lemma 3.1, Proposition 6.6 and the induction approach. □

Lemma 6.8. *Let P be an S^*-doubly quasicontinuous poset satisfying Condition ($\diamond$), $x \in P$, $M \in \mathcal{L}_0(\downarrow x)$ and $N \in \mathcal{L}_0(\uparrow x)$ with $x \in \Uparrow_{S_0} M \cap \Downarrow_{S_0} N$. Then there exists a continuous function $f : (P, \mathcal{T}_P) \to ([0,1], \mathcal{T}_0|_{[0,1]})$ such that $f(x) = 1$ and $f(P \backslash (\uparrow M \cap \downarrow N)) \subseteq \{0\}$.*

Proof. By Proposition 6.7, there exist families $\{M_b \in \mathcal{L}_0(\downarrow x) : b \in B\}$ and $\{N_b \in \mathcal{L}_0(\downarrow x) : b \in B\}$ satisfying conditions (1) and (2) of Proposition 6.7. Define $f : P \to [0,1]$ by

$$(\forall p \in P)\ f(p) = \sup\{b \in B : p \in \uparrow M_b \cap \downarrow N_b\},$$

where we stipulate that $\sup \varnothing = 0$. In the following, we present some fundamental properties of f:

(1) If $p \in \uparrow M_b \cap \downarrow N_b$, then $f(p) \geqslant b$.
This is straightforward by Proposition 6.7 and the definition of f.

(2) If $p \notin \Uparrow_{S_0} M_b \cap \Downarrow_{S_0} N_b$, then $f(p) \leqslant b$.
Let $p \notin \Uparrow_{S_0} M_b \cap \Downarrow_{S_0} N_b$. Then, by Proposition 6.7, we have that $p \notin \uparrow M_c \cap \downarrow N_c$ for every $c > b$. This implies

$$f(p) = \sup\Big\{b' \in B : p \in \uparrow M_{b'} \cap \downarrow N_{b'}\Big\} \leqslant \sup(B \cap [0,b]) = b.$$

(3) If $f(p) > b$, then $p \in \Uparrow_{S_0} M_b \cap \Downarrow_{S_0} N_b$.
Since B is dense in $([0,1], \mathcal{T}_0|_{[0,1]})$, there exists $c \in B$ such that $b < c < f(p)$. It follows from the definition of f that $p \in \uparrow M_c \cap \downarrow N_c$. By Proposition 6.7, we have $p \in \Uparrow_{S_0} M_b \cap \Downarrow_{S_0} N_b$.

(4) If $f(p) < b$, then $p \notin \uparrow M_b \cap \downarrow N_b$.
Straightforward by Proposition 6.7 and the definition of f.

(5) $f(x) = 1$ and $f(P \backslash (\uparrow M \cap \downarrow N)) \subseteq \{0\}$.
Straightforward by (1), (2) and Proposition 6.7.

(6) $f^{-1}([0,\alpha)) \in \mathcal{T}_P$ for every $\alpha \in (0,1]$.
To show $f^{-1}([0,\alpha)) \in \mathcal{T}_P$ for every $\alpha \in (0,1]$, it suffices to prove that for every $p \in f^{-1}([0,\alpha))$, there exists $U \in \mathcal{T}_P$ such that $p \in U$ and $f(U) \subseteq [0,\alpha)$. Let $p \in f^{-1}([0,\alpha))$. Then we have $f(p) < b < \alpha$ for some $b \in B$. It follows from (4) and Proposition 4.12 that $p \in P \backslash \uparrow M_b \cap \downarrow N_b \in \mathcal{T}_P$. Since $\Uparrow_{S_0} M_b \cap \Downarrow_{S_0} N_b \subseteq \uparrow M_b \cap \downarrow N_b$, we have $f(P \backslash \uparrow M_b \cap \downarrow N_b) \subseteq f(P \backslash \Uparrow_{S_0} M_b \cap \Downarrow_{S_0} N_b) \subseteq [0,b] \subseteq [0,\alpha)$ by (2). This shows $f^{-1}([0,\alpha)) \in \mathcal{T}_P$.

(7) $f^{-1}((\beta,1]) \in \mathcal{T}_P$ for every $\beta \in [0,1)$.
To prove $f^{-1}((\beta,1]) \in \mathcal{T}_P$ for every $\beta \in [0,1)$, we only need to show that for every $p \in f^{-1}((\beta,1])$, there exists $V \in \mathcal{T}_P$ such that $p \in V$ and $f(V) \subseteq (\beta,1]$. Let $p \in f^{-1}((\beta,1])$. Then we have $\beta < c < f(p)$ for some $c \in B$. This follows from (1) and (3) that $p \in \Uparrow_{S_0} M_c \cap \Downarrow_{S_0} N_c \in \mathcal{T}_P$ and $f(\Uparrow_{S_0} M_c \cap \Downarrow_{S_0} N_c) \subseteq f(\uparrow M_c \cap \downarrow N_c) \subseteq [c,1] \subseteq (\beta,1]$.

By (5), (6) and (7), we conclude that $f : (P, \mathcal{T}_P) \to ([0,1], \mathcal{T}_0|_{[0,1]})$ is a continuous function such that $f(x) = 1$ and $f(P \backslash (\uparrow M \cap \downarrow N)) \subseteq \{0\}$. □

Theorem 6.9. *Let P be an S^*-doubly quasicontinuous poset satisfying Condition ($\diamond$). Then $(P, \mathcal{T}_P)$ is Tychonoff.*

Proof. Let $P \backslash C \in \mathcal{T}_P$ and $x \in P \backslash C$. Then, by Lemma 4.15, Proposition 4.14 and Lemma 4.10, there exists $M \in \mathcal{L}_0(\downarrow x)$ and $N \in \mathcal{L}_0(\uparrow x)$ such that $x \in \Uparrow_{S_0} M \cap \Downarrow_{S_0} N \subseteq \uparrow M \cap \downarrow N \subseteq P \backslash C$. By Lemma 6.8, there exists a continuous function $f : (P, \mathcal{T}_P) \to ([0,1], \mathcal{T}_0|_{[0,1]})$ such that $f(x) = 1$ and $f(C) \subseteq f(P \backslash (\uparrow M \cap \downarrow N)) \subseteq \{0\}$. Define $f : (P, \mathcal{T}_P) \to ([0,1], \mathcal{T}_0|_{[0,1]})$ by

$$(\forall p \in P)\ g(x) = 1 - f(p).$$

Then one can easily check that $g(x) = 0$, $g(C) \subseteq \{1\}$ and g is continuous. This shows $(P, \mathcal{T}_P)$ is completely regular. By Theorem 6.2, we have that $(P, \mathcal{T}_P)$ is T_1. Thus, $(P, \mathcal{T}_P)$ is Tychonoff. □

Corollary 6.10. *Let P be a poset.*
(1) *If P is a doubly quasicontinuous poset, then $(P, \mathcal{T}_P)$ is Tychonoff;*
(2) *If P is a doubly continuous poset, then $(P, \mathcal{T}_P)$ is Tychonoff;*
(3) *If P is a completely distributive lattice, then $(P, \mathcal{T}_P)$ is Tychonoff.*

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