Research Article

Xiao Yu*, Pu Zhang, and Hongliang Li

Some estimates for the commutators of multilinear maximal function on Morrey-type space

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Abstract: In this paper, we study the equivalent conditions for the boundedness of the commutators generated by the multilinear maximal function and the bounded mean oscillation (BMO) function on Morrey space. Moreover, the endpoint estimate for such operators on generalized Morrey spaces is also given.

Keywords: Morrey-type space, multilinear maximal function, commutator, Orlicz function

MSC 2020: 42B20, 42B25

1 Introduction

Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We say that b belongs to the mean oscillation space $\mathrm{BMO}(\mathbb{R}^n)$ if b satisfies

$$\|b\|_{\mathrm{BMO}} := \sup_{Q} \frac{1}{|Q|} \int_{Q} |b(y) - b_Q| \mathrm{d}y < \infty,$$

where Q denotes any cube of $\mathbb{R}^n$ and $b_Q = \frac{1}{|Q|}\int_Q b(x)\mathrm{d}x$.

The commutator $[b, M](f)(x)$, which is generated by the maximal function and the BMO function, can be stated as follows:

$$[b, M](f)(x) = bM(f)(x) - M(bf)(x).$$

Here, $M(f)(x)$ is the classical Hardy-Littlewood maximal function defined as

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_{Q} |f(y)| \mathrm{d}y,$$

where the supremum is taken over all cubes Q containing x.

For the study of $[b, M]$, one may see [1–4] for more details. Here, we would like to point out that in [2], Bastero, Milman and Ruiz gave the sufficient and necessary conditions for the boundedness of $[b, M]$ on $L^p(\mathbb{R}^n)$. Precisely, they proved the following theorem.

* **Corresponding author: Xiao Yu,** Department of Mathematics, Shangrao Normal University, Shangrao 334001, China, e-mail: yx2000s@163.com
Pu Zhang: Department of Mathematics, Mudanjiang Normal University, Mudanjiang 157011, China, e-mail: puzhang@sohu.com
Hongliang Li: Department of Mathematics, Zhejiang International Studies University, Hangzhou 310023, China, e-mail: hongli@126.com

Theorem A. [2] *Let b be a real valued, locally integrable function in $\mathbb{R}^n$. Then, the following three assertions are equivalent.*

(*i*) *The commutator $[b, M]$ is bounded on L^p for $1 < p < \infty$.*

(*ii*) *b is in BMO and b^- belongs to L^∞ with $b^- = -\min\{b(x), 0\}$.*

(*iii*) *For $p \in (1, \infty)$, there is*

$$\sup_Q \frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^p \mathrm{d}x < \infty,$$

where $M_Q(b)(x) = \sup_{Q_0 \ni x, Q_0 \subset Q} \left(\frac{1}{|Q_0|} \int_{Q_0} |b(t)|^p \mathrm{d}t \right)^{1/p}$.

Later, Xie [4] extended Theorem A to the Morrey space. Here the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda} \int_{B(x,t)} |f(y)|^p \mathrm{d}y \right)^{1/p} < \infty \right\},$$

where $0 \le \lambda < n$ and $1 \le p < \infty$.

Similarly, the weak Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$ can be stated as follows:

$$WL^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{WL^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, t>0, \beta>0} \frac{\beta}{t^{\lambda/p}} |\{t \in Q(x, t) : |f(t)| > \beta\}|^{1/p} < \infty \right\}.$$

In 1938, Morrey [5] obtained the Hölder regularity of the solutions of elliptic equations by applying new technique. Here, the new technique was based on the estimates of the integrals over balls of the gradient of the solutions via the radius of the same ball. The classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, usually attributed to him, were introduced in the 1960s by Campanato, Peetre and Brudneii, independently. Readers may refer to [6] for more details.

Later, the boundedness of Hardy-Littlewood maximal function on Morrey space can be found in [7] and the maximal inequality in generalized Morrey-type spaces can be found in [8].

Obviously, if we choose $\lambda = 0$, then $L^{p,\lambda}(\mathbb{R}^n)$ becomes the classical $L^p(\mathbb{R}^n)$ space.

On the other hand, the multilinear theory was also developed a lot in the past 20 years and readers may refer to [9,10]. for more details. Moreover, Pérez and Torres [11] introduced the commutators of multilinear Calderón-Zygmund operators defined as

$$\mathcal{T}_{\vec{b}}(\vec{f})(x) = \sum_{i=1}^m \mathcal{T}^i_{\vec{b}}(\vec{f})(x) \quad (\vec{b} = (b_1, \dots, b_m)),$$

where

$$\mathcal{T}^i_{\vec{b}}(\vec{f})(x) = b_i(x)\mathcal{T}(\vec{f})(x) - \mathcal{T}(f_1, \dots, f_{i-1}, b_i f_i, f_{i+1}, \dots, f_m)(x),$$

$\vec{b} = (b_1, \dots, b_m)$ and $\mathcal{T}$ is the multilinear Calderón-Zygmund operator which was first studied by Grafakos and Torres [9].

Later, Lerner et al. [10] introduced a new type of multiple weight classes $A_{\vec{P}}$ which is very adopted to the weighted norm estimates for the commutators of multilinear Calderón-Zygmund operators. Moreover, in order to give the characterization of $A_{\vec{P}}$, Lerner et al. [10] introduced the following multilinear maximal function $\mathcal{M}(\vec{f})(x)$ defined as

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)| \mathrm{d}y_i,$$

where $\vec{f} = (f_1, \dots, f_m)$ with $m \in \mathbb{Z}^+ (m > 1)$ and the supremum is taken over all cubes Q containing x.

Obviously, it is easy to see $\mathcal{M}(\vec{f})(x) \le \prod_{i=1}^m M(f_i)(x)$.

For any cube Q, we write $Q^m = \underbrace{Q \times \cdots \times Q}_{m}$. In 2015, Zhang [12] studied the commutator generated by the BMO function and the multilinear maximal functions $[\vec{b}, \mathcal{M}](\vec{f})(x)$ and $\mathcal{M}_{\vec{b}}(\vec{f})(x)$ as follows:

$$[\vec{b}, \mathcal{M}](\vec{f})(x) = \sum_{j=1}^{m} [\vec{b}, M]_j(\vec{f})(x), \quad \mathcal{M}_{\vec{b}}(\vec{f})(x) = \sum_{j=1}^{m} \mathcal{M}^{b_j}(\vec{f})(x).$$

Here

$$[\vec{b}, M]_j(\vec{f})(x) = b_j(x)\mathcal{M}(\vec{f})(x) - \mathcal{M}(f_1, \dots, f_{j-1}, b_j f_j, f_{j+1}, \dots, f_m)(x)$$

and

$$\mathcal{M}^{b_j}(\vec{f})(x) = \sup_{Q \ni x} \frac{1}{|Q|^m} \int_{Q^m} |b_j(x) - b_j(y_j)| \prod_{i=1}^{m} |f_i(y_i)| d\vec{y},$$

where $\vec{y} = (y_1, \dots, y_m)$.

Zhang [12] proved the weighted boundedness of $[\vec{b}, \mathcal{M}](\vec{f})(x)$ and $\mathcal{M}_{\vec{b}}(\vec{f})(x)$ on the product L^p spaces.

Motivated by the above backgrounds, in this paper, we would like to extend Theorem A with $[\vec{b}, \mathcal{M}](\vec{f})(x)$ on Morrey-type spaces. The first result of this paper can be regarded as follows.

Theorem 1.1. *Let b_i be a real valued, locally integrable function in $\mathbb{R}^n$ with $i = 1, \dots, m$. Then, the following three assertions are equivalent.*

(I) *The commutator $[\vec{b}, \mathcal{M}](\vec{f})(x)$ is bounded from $L^{p_1,\lambda}(\mathbb{R}^n) \times \cdots \times L^{p_m,\lambda}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$ for $1 < p_i < \infty$ and $1/p = \sum_{i=1}^{m} 1/p_i$ with $p > 1$ and $0 \le \lambda < n$.*

(II) *b_i is in BMO and b_i^- belongs to L^∞.*

(III) *For $p \in (1, \infty)$, there is*

$$\sup_{Q} \frac{1}{|Q|} \int_{Q} |b_i(x) - M_Q(b_i)(x)|^p \mathrm{d}x < \infty,$$

where $M_Q(b_i)(x) = \sup_{Q_0 \ni x, Q_0 \subset Q} \frac{1}{|Q_0|} \int_{Q_0} |b_i(t)| \mathrm{d}t$.

Remark 1.2. Obviously, our results improve the main results of [2, 4].

On the other hand, in 1995, Pérez [13] gave a counter example that the commutator $[b, T]$ is not of weak type (1, 1) and he proved that $[b, T]$ satisfies the weak $L \log L$-type estimates. For the study of the commutators on the endpoint case, one may see [10,12,14,15] for more details. Particularly in [12], Zhang proved the weak weighted $L \log L$ estimates for $\mathcal{M}_{\vec{b}}(\vec{f})(x)$. Recently, Wang [16] proved that the commutator generated by some integral operators is bounded from $L^{1,\omega}_{L\log L}(\mathbb{R}^n)$ to $WL^{1,\omega}(\mathbb{R}^n)$ in some sense and the definitions of $WL^{1,\omega}(\mathbb{R}^n)$ and $L^{1,\omega}_{L\log L}(\mathbb{R}^n)$ will be given in Sections 1 and 2, respectively.

Thus, it is natural to ask whether we can prove the endpoint estimates of the commutators generated by the multilinear maximal functions and BMO functions on Morrey-type space.

Before giving the second result of this paper, we introduce the generalized Morrey space $L^{p,\omega}(\mathbb{R}^n)$ proposed by Nakai [8].

Definition A. [8] Let $\omega : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^+$, $1 \le p < \infty$ and $Q(a, r)$ be the cube

$$\{x \in \mathbb{R}^n : |x_i - a_i| \le r/2, \ i = 1, 2, \dots, n\}$$

whose edges have length r and are parallel to the coordinate axes. For $Q = Q(a, r)$, we denote $kQ = Q(a, kr)$ and $\omega(Q) = \omega(a, r)$. Moreover, we suppose that ω satisfies

$$\omega(a, 2t) \leq C_0 \omega(a, t) \tag{1.1}$$

with $1 < C_0 < 2^n$.

Then, the generalized Morrey space $L^{p,\omega}(\mathbb{R}^n)$ $(1 \leq p < \infty)$ is defined as

$$L^{p,\omega}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\omega}(\mathbb{R}^n)} = \sup_Q \left(\frac{1}{\omega(Q)} \int_Q |f(x)|^p \mathrm{d}x \right)^{1/p} < \infty \right\}.$$

Similarly, the weak generalized Morrey space $WL^{p,\omega}(\mathbb{R}^n)(1 \leq p < \infty)$ is defined by

$$WL^{p,\omega}(\mathbb{R}^n) = \left\{ f \in WL^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\omega}(\mathbb{R}^n)} = \sup_Q \sup_{\beta>0} \frac{\beta}{\omega(Q)^{1/p}} |\{x \in Q : |f(x)| > \beta\}|^{1/p} < \infty \right\}.$$

Obviously, if we choose $\omega(Q) = |Q|^{\frac{\lambda}{n}}$ with $0 \leq \lambda < n$, there is

$$\omega(2Q) = |2Q|^{\frac{\lambda}{n}} = 2^\lambda |Q|^{\frac{\lambda}{n}} < 2^n \omega(Q). \tag{1.2}$$

In this case, $L^{p,\omega}(\mathbb{R}^n)$ and $WL^{p,\omega}(\mathbb{R}^n)$ become $L^{p,\lambda}(\mathbb{R}^n)$ and $WL^{p,\lambda}(\mathbb{R}^n)$ with $0 \leq \lambda < n$, respectively.

The second result of this paper can be stated as follows.

Theorem 1.3. *Let $b_i \in \mathrm{BMO}(\mathbb{R}^n)$ and ω satisfy (1.1) with $1 < C_0 \leq 2^\gamma$ for any $0 < \gamma < n$. Then, for any $\beta > 0$ and cube $B = B(x, t)$ with center $x \in \mathbb{R}^n$ and radius $t > 0$, there is*

$$\frac{1}{\omega(B)^m} |\{z \in B(x, t) : \mathcal{M}_{\vec{b}}(\vec{f})(z) > \beta^m\}|^m \leq C \|\vec{b}\|_{\mathrm{BMO}} \prod_{i=1}^m \left\| \Phi\left(\frac{|f_i|}{\beta} \right) \right\|_{L^{1,\omega}_{L \log L}(\mathbb{R}^n)},$$

where $\Phi(x) = t(1 + \log^+ t)$, $\|\vec{b}\|_{\mathrm{BMO}} = \sup_{1 \leq i \leq m} \|b_i\|_{\mathrm{BMO}}$ and C is a positive constant depending on the dimension n.

Remark 1.4. By the definition of $WL^{\frac{1}{m},\omega}(\mathbb{R}^n)$, we may roughly say that $\mathcal{M}_{\vec{b}}(\vec{f})(x)$ is bounded from $L^{1,\omega}_{L \log L}(\mathbb{R}^n) \times \cdots \times L^{1,\omega}_{L \log L}(\mathbb{R}^n)$ to $WL^{\frac{1}{m},\omega}(\mathbb{R}^n)$ in some sense.

2 Orlicz space, generalized Hölder inequalities, weak L^p spaces and another kind of multilinear maximal function

First, we introduce some facts and theory related to the Orlicz spaces. For more information about Orlicz spaces, one may refer to [17].

Suppose that $\Phi : [0, \infty) \to [0, \infty)$ is a continuous, convex, increasing function with $\Phi(0) = 0$ and Φ satisfies $\Phi(t) \to \infty$ if $t \to \infty$. Then, we say that Φ is a Young function.

For a function f defined on a cube Q, we define

$$\|f\|_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(y)|}{\lambda} \right) \mathrm{d}y \leq 1 \right\},$$

which is called the mean Luxemburg norm of f on Q.

Obviously, if $\Phi \leq \Psi$, then

$$\|f\|_{\Phi,Q} \leq \|f\|_{\Psi,Q}.$$

For a Young function Φ, we may define its complementary function $\bar{\Phi}(s)$ as

$$\bar{\Phi}(s) = \sup_{t>0}\{st - \Phi(t)\}.$$

Obviously, $\Phi(t) = t(1 + \log^+ t)$ is a Young function and its complementary function $\bar{\Phi}(s) \approx e^s$ (see [17]).

From [18], we know that if E, F, G are Young functions and satisfy

$$E^{-1}(t)G^{-1}(t) \le F^{-1}(t) \quad (\forall t > 0).$$

Then, the following generalized Hölder inequality holds.

$$\|fg\|_{F,Q} \le 2\|f\|_{E,Q}\|g\|_{G,Q}.$$

In this case, we have

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|\mathrm{d}y \le \|f\|_{\Phi,Q}\|g\|_{\bar{\Phi},Q}.$$

When $\Phi(t) = t(1 + \log^+ t)$, we write $\|f\|_{L\log L,Q} = \|f\|_{\Phi,Q}$ and $\|f\|_{\exp L,Q} = \|f\|_{\bar{\Phi},Q}$. Thus, we get

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|\mathrm{d}y \le \|f\|_{L\log L,Q}\|g\|_{\exp L,Q}. \tag{2.1}$$

By the definition of $\|f\|_{\Phi,Q}$, it is easy to see that

$$\frac{1}{|Q|}\int_Q |f(x)|\mathrm{d}x = \|f\|_{L^1,Q} \le \|f\|_{L\log L,Q}. \tag{2.2}$$

Moreover, for $b \in \mathrm{BMO}(\mathbb{R}^n)$, using the John-Nirenberg inequality, we have $\|b - b_Q\|_{\exp L,Q} \le C\|b\|_{\mathrm{BMO}}$. Thus, (2.1) can be written as follows:

$$\frac{1}{|Q|}\int_Q |f(y)(b(y) - b_Q)|\mathrm{d}y \le C\|f\|_{L\log L,Q}\|b\|_{\mathrm{BMO}}. \tag{2.3}$$

Next, following [16], we introduce the generalized Morrey space $L^{1,\omega}_{L\log L}(\mathbb{R}^n)$ related to $L\log L$ type associated with ω, that is,

$$L^{1,\omega}_{L^1\log L}(\mathbb{R}^n) = \left\{f \in L_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{1,\omega}_{L\log L}(\mathbb{R}^n)} = \sup_Q\left(\frac{|Q|}{\omega(Q)}\|f\|_{L\log L,Q}\right) < \infty\right\}.$$

Obviously, it is easy to see that $L^{1,\omega}_{L\log L}(\mathbb{R}^n) \subset L^{1,\omega}(\mathbb{R}^n)$ from (2.1) (see [16]).

Now, we introduce the definition of the weak L^p space (see [19]).

Suppose that X is a measure space and has the normal Lebesgue measure. For $0 < p < \infty$, the weak $L^p(X)$ is defined as the set of all Lebesgue measurable functions f, such that

$$\|f\|_{L^{p,\infty}(X)} = \sup\{\lambda d_f(\lambda)^{1/p} : \lambda > 0\} < \infty,$$

where the definition of $d_f(y)$ is

$$d_f(\lambda) = |\{x \in X : |f(x)|>\lambda\}|.$$

Finally, we introduce another kind of multilinear maximal function $\mathcal{M}_c(\vec{f})(x)$ as follows:

$$\mathcal{M}_c(\vec{f})(x) = \sup_{r>0}\frac{1}{|Q(x,r)|^m}\int_{Q(x,r)^m}\prod_{j=1}^m |f_j(y_j)|d\vec{y}.$$

Moreover, we define the commutator of $\mathcal{M}_c^{b_j}(\vec{f})(x)$ as

$$\mathcal{M}_c^{b_j}(\vec{f})(x) = \sup_{r>0} \frac{1}{|Q(x,r)|^m} \int_{Q(x,r)^m} |b_j(x) - b_j(y_j)| \prod_{i=1}^m |f_i(y_i)| d\vec{y}.$$

Then, it is easy to see [12]

$$\mathcal{M}_c(\vec{f})(x) \sim \mathcal{M}(\vec{f})(x) \quad \text{and} \quad \mathcal{M}_c^{b_j}(\vec{f})(x) \sim \mathcal{M}^{b_j}(\vec{f})(x). \tag{2.4}$$

3 Some useful lemmas

In this section, we would like to give some lemmas, which is very useful throughout this paper. For some techniques to deal with commutators of operators on Morrey-type spaces, one may see [4,12,16,20–22] for more details.

Lemma 3.1. *Suppose that* $f \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $r_1, r_2 \in \mathbb{R}^+$, *then for any* $1 < p < \infty$ *and* $j \in \mathbb{Z}^+$, *we have*

(*i*) $\left(\frac{1}{|Q(x_0,r_1)|} \int_{Q(x_0,r_1)} |f(x) - f_{Q(x_0,r_2)}| \mathrm{d}x\right) \le C\left(1 + |\log \frac{r_1}{r_2}|\right) \|f\|_{\mathrm{BMO}}$.

(*ii*) $\|f\|_{\mathrm{BMO}} \sim \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \mathrm{d}x\right)^{1/p}$.

(*iii*) $|f_Q - f_{2^{j+1}Q}| \le C(j+1)\|f\|_{\mathrm{BMO}}$.

The proof of Lemma 3.1 is very standard and can be found in many papers.

Lemma 3.2. *Suppose that* $1/p = \sum_{i=1}^m 1/p_i$ *with* $p_i > 1$. *If* $b_i \in \mathrm{BMO}$ *with* $i = 1, \ldots, m$, *we obtain that* $\mathcal{M}_{\vec{b}}(\vec{f})(x)$ *is bounded from* $L^{p_1,\lambda}(\mathbb{R}^n) \times \cdots \times L^{p_m,\lambda}(\mathbb{R}^n)$ *to* $L^{p,\lambda}(\mathbb{R}^n)$ *with* $0 \le \lambda < n$. *That is,*

$$\|\mathcal{M}_{\vec{b}}(\vec{f})\|_{L^{p,\lambda}(\mathbb{R}^n)} \le C\|\vec{b}\|_{\mathrm{BMO}} \prod_{i=1}^m \|f_i\|_{L^{p_i,\lambda}(\mathbb{R}^n)}$$

with $\|\vec{b}\|_{\mathrm{BMO}} = \sup_i \|b_i\|_{\mathrm{BMO}}$.

Proof. By the definition of $\mathcal{M}_{\vec{b}}(\vec{f})(x)$ and the fact $\mathcal{M}_c^{b_i}(\vec{f})(x) \sim \mathcal{M}^{b_i}(\vec{f})(x)$, it suffices to consider $\mathcal{M}_c^{b_1}(\vec{f})(x)$. For any cube $B = B(x,t)$, we split each $f_i = f_i^0 + f_i^\infty$ where $f_i^0 = f_i\chi_{2B}$ and $2B = B(x,2t)$. Then, we get

$$\|\mathcal{M}_c^{b_1}(\vec{f})\|_{L^p(B)} \le \|\mathcal{M}_c^{b_1}(\vec{f^0})\chi_B\|_{L^p} + \sum{}' \|\mathcal{M}_c^{b_1}(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})\chi_B\|_{L^p} := I + II.$$

where $\alpha_1, \ldots, \alpha_m \in \{0, \infty\}$ and each term in the sum $\sum'$ contains at least one $\alpha_i = \infty$ and one $\alpha_j = 0$.

For I, by the boundedness of $\mathcal{M}_c^{b_1}(\vec{f})(x)$ on the product L^p spaces (see [12]), it is easy to see

$$I \le C\|b_1\|_{\mathrm{BMO}} \prod_{i=1}^m \|f_i\|_{L^{p_i,\lambda}} t^{\lambda/p}. \tag{3.1}$$

To estimate II, first, we consider the case $\alpha_1 = \cdots = \alpha_m = \infty$. For any cube $Q = Q(z,r)$ with $z \in B(x,t)$ and using the fact $y_1 \in B(x,2t)^c$ and $z \in B(x,t)$, we have

$$\begin{aligned}\sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |b_1(z) - b_1(y_1)||f_1^\infty(y_1)| \mathrm{d}y_1 &= \sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r) \bigcap B(x,2t)^c} |b_1(z) - b_1(y_1)||f_1(y_1)| \mathrm{d}y_1 \\ &\le C \sup_{r>2t} \int_{Q(z,r) \bigcap B(x,2t)^c} \frac{|b_1(z) - b_1(y_1)||f_1(y_1)|}{|y_1 - z|^n} \mathrm{d}y_1\end{aligned}$$

$$
\begin{aligned}
&\leq C \sup_{r>2t} \int\limits_{Q(z,r)\bigcap B(x,2t)^c} \frac{|b_1(z)-b_1(y_1)||f_1(y_1)|}{|y_1-x|^n} \mathrm{d}y_1 \\
&\leq C \sup_{r>2t} \int\limits_{B(x,2t)^c} \frac{|b_1(z)-b_1(y_1)||f_1(y_1)|}{|y_1-x|^n} \mathrm{d}y_1 \\
&\leq \int\limits_{B(x,2t)^c} \frac{|f_1(y_1)||b_1(z)-b_1(y_1)|}{|y_1-x|^n} \mathrm{d}y_1 \\
&\leq \int\limits_{B(x,2t)^c} \frac{|f_1(y_1)||b_1(z)-(b_1)_B|}{|y_1-x|^n} \mathrm{d}y_1 + \int\limits_{B(x,2t)^c} \frac{|f_1(y_1)||b_1(y_1)-(b_1)_B|}{|y_1-x|^n} \mathrm{d}y_1 \\
&=: A_1 + A_2.
\end{aligned}
$$

For A_1, we decompose it as

$$
A_1 \leq \sum_{j=1}^{\infty} \int\limits_{2^{j+1}B\setminus 2^jB} \frac{|f_1(y_1)||b_1(z)-(b_1)_B|}{|y_1-x|^n} \mathrm{d}y_1
$$

with $2^{j+1}B = B(x, 2^{j+1}t)$.

As it is easy to see $\frac{2^jt}{2} \leq |x-y_1| \leq \frac{2^{j+1}t\sqrt{n}}{2}$, which implies

$$
\frac{C_1}{|2^{j+1}B|} \leq \frac{1}{|x-y_1|^n} \leq \frac{C_2}{|2^jB|},
$$

where C_1 and C_2 are positive constants depending on the dimension n and $2^{j+1}B = B(x, 2^{j+1}t)$. Thus, we conclude that there exists a constant C depending on the dimension n, such that

$$
\frac{1}{|x-y_1|^n} \sim \frac{C}{|2^{j+1}B|}. \tag{3.2}
$$

Then, we have

$$
\begin{aligned}
A_1 &\leq |b_1(z)-(b_1)_B| \int\limits_{B(x,2t)^c} \frac{|f_1(y_1)|}{|y_1-x|^n} \mathrm{d}y_1 \\
&\leq |b_1(z)-(b_1)_B| \sum_{j=1}^{\infty} \int\limits_{2^{j+1}B\setminus 2^jB} \frac{|f_1(y_1)|}{|y_1-x|^n} \mathrm{d}y_1 \\
&\leq |b_1(z)-(b_1)_B| \sum_{j=1}^{\infty} \frac{C}{|2^{j+1}B|} \int\limits_{2^{j+1}B} |f_1(y_1)| \mathrm{d}y_1 \\
&\leq |b_1(z)-(b_1)_B| \sum_{j=1}^{\infty} \frac{C}{|2^{j+1}B|} \left(\int\limits_{2^{j+1}B} |f_1(y_1)|^{p_1} \mathrm{d}y_1 \right)^{1/p_1} |2^{j+1}B|^{1-\frac{1}{p_1}} \\
&\leq C|b_1(z)-(b_1)_B| \|f_1\|_{L^{p_1,\lambda}} \sum_{j=1}^{\infty} |2^{j+1}B|^{\frac{1}{p_1}\left(\frac{\lambda}{n}-1\right)} \\
&\leq C|b_1(z)-(b_1)_B| \|f_1\|_{L^{p_1,\lambda}} |B|^{\frac{1}{p_1}\left(\frac{\lambda}{n}-1\right)}
\end{aligned}
$$

with the constant C depending on the dimension n.

For the estimates of A_2, we decompose it as

$$
A_2 \leq \sum_{j=1}^{\infty} \int\limits_{2^{j+1}B\setminus 2^jB} |f_1(y_1)||b_1(y_1)-(b_1)_B||y_1-x|^{-n} \mathrm{d}y_1.
$$

Then, using Lemma 3.1, the Hölder inequality and (3.2) again, there is

$$\begin{aligned} A_2 &\le \sum_{j=1}^{\infty} \frac{C}{|2^{j+1}B|} \int_{2^{j+1}B} |f_1(y_1)||b_1(y_1) - (b_1)_B| \mathrm{d}y_1 \\ &\le \sum_{j=1}^{\infty} \frac{C}{|2^{j+1}B|} \left(\int_{2^{j+1}B} |f_1(y_1)|^{p_1} \mathrm{d}y_1 \right)^{1/p_1} \left(\int_{2^{j+1}B} |b_1(y_1) - (b_1)_B|^{\frac{p_1}{p_1-1}} \mathrm{d}y_1 \right)^{1-\frac{1}{p_1}} \\ &\le C\|f_1\|_{L^{p_1,\lambda}} \sum_{j=1}^{\infty} |2^{j+1}B|^{\frac{1}{p_1}\left(\frac{\lambda}{n}-1\right)} \\ &\le C\|f_1\|_{L^{p_1,\lambda}} \|b_1\|_{\mathrm{BMO}} |B|^{\frac{1}{p_1}\left(\frac{\lambda}{n}-1\right)}, \end{aligned}$$

where the positive constant C is depending on the dimension n.

Thus, we obtain

$$\sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |b_1(z) - b_1(y_1)||f_1^{\infty}(y_1)| \mathrm{d}y_1 \le C\left(\|b_1\|_{\mathrm{BMO}} + |b_1(z) - (b_1)_B|\right)\|f_1\|_{L^{p_1,\lambda}} |B|^{\frac{1}{p_1}\left(\frac{\lambda}{n}-1\right)}.$$

Moreover, for $\int_{B(x,2t)^c} \frac{|f_i(y_i)|}{|y_i - x|^n} \mathrm{d}y_i$ with $i = 2, 3, \ldots, m$, there is

$$\begin{aligned} \int_{B(x,2t)^c} \frac{|f_i(y_i)|}{|y_i - x|^n} \mathrm{d}y_i &\le \sum_{j=1}^{\infty} \int_{2^{j+1}B\setminus 2^jB} \frac{|f_i(y_i)|}{|y_i - x|^n} \mathrm{d}y_i \\ &\le \sum_{j=1}^{\infty} \frac{C}{|2^{j+1}B|} \int_{2^{j+1}B} |f_i(y_i)| \mathrm{d}y_i \\ &\le \sum_{j=1}^{\infty} \frac{C}{|2^{j+1}B|} \left(\int_{2^{j+1}B} |f_i(y_i)|^{p_i} \mathrm{d}y_i \right)^{1/p_i} |2^{j+1}B|^{1-\frac{1}{p_i}} \\ &\le C\|f_i\|_{L^{p_i,\lambda}} \sum_{j=1}^{\infty} |2^{j+1}B|^{\frac{1}{p_i}\left(\frac{\lambda}{n}-1\right)} \\ &\le C\|f_i\|_{L^{p_i,\lambda}} |B|^{\frac{1}{p_i}\left(\frac{\lambda}{n}-1\right)}, \end{aligned}$$

which implies

$$M_c(f_i^{\infty})(z) := \sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |f_i^{\infty}(y_i)| \mathrm{d}y_i \le C\|f_i\|_{L^{p_i,\lambda}} |B|^{\frac{1}{p_i}\left(\frac{\lambda}{n}-1\right)}. \tag{3.3}$$

Combining the aforementioned estimates, we may obtain

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |\mathcal{M}_c^{b_1}(\vec{f}^{\infty})(z)|^p \chi_B(z) \mathrm{d}z \right)^{1/p} &\le C\|b_1\|_{\mathrm{BMO}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\lambda}} |B|^{\frac{1}{p_i}\left(\frac{\lambda}{n}-1\right)} \left(\int_B \mathrm{d}z \right)^{1/p} \\ &\quad + \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\lambda}} |B|^{\frac{\lambda}{np_i}} \left(|B|^{-1} \int_B |b_1(z) - (b_1)_B| \mathrm{d}z \right)^{1/p} \\ &\le C\|b_1\|_{\mathrm{BMO}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\lambda}} t^{\lambda/p}. \end{aligned}$$

Finally, for the case that $\alpha_{j_1} = \cdots = \alpha_{j_l} = 0$ for some $\{j_1, \ldots, j_l\} \subset \{1, \ldots, m\}$ where $1 \le l < m$, we only consider the case $\alpha_1 = \infty$, $\alpha_2 = \infty$ and $\alpha_3 = \cdots = \alpha_m = 0$ since the other cases follow in a similar way. Using (2.4) with $m = 1$, the Hölder inequality and the aforementioned estimates, there is

$$
\begin{aligned}
\|\mathcal{M}^{b_1}(f_1^\infty, f_2^\infty, f_3^0, \ldots, f_m^0)\chi_B\|_{L^p} &\le \|M_{b_1}(f_1^\infty)\chi_B\|_{L^{p_1}} \|M(f_2^\infty)\chi_B\|_{L^{p_2}} \|M(f_3^0)\chi_B\|_{L^{p_3}} \cdots \|M(f_m^0)\chi_B\|_{L^{p_m}} \\
&\le C\|f_1\|_{L^{p_1,\omega}} |B|^{\lambda/np_1} |B|^{-\frac{1}{p_1}} \left(\int_B (\|b_1\|_{\mathrm{BMO}} + |b_1(z) - (b_1)_B|)\,\mathrm{d}z \right)^{1/p_1} \\
&\quad \times |B|^{\lambda/np_2} \|f_2\|_{L^{p_2,\omega}} |B|^{-1/p_2} |B|^{1/p_2} \prod_{i=3}^m \left(\|f_i\|_{L^{p_i,\lambda}} |B|^{\frac{1}{p_i}\frac{\lambda}{n}} \right) \\
&\le C\|b_1\|_{\mathrm{BMO}} \prod_{i=1}^m \|f_i\|_{L^{p_i,\lambda}} t^{\lambda/p},
\end{aligned}
$$

where $M_b(f)(x)$ is the commutator generated by the $M(f)(x)$ and the symbol b, that is,

$$
M_b(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)|\mathrm{d}y.
$$

Combining the estimates of I and II, we finish the proof of Lemma 3.2 from the definition of $L^{p,\lambda}(\mathbb{R}^n)$. □

Using the similar ideas and estimates in the proof of Lemma 3.2, we can easily get the boundedness of $\mathcal{M}(\vec{f})(x)$ on $L^{p,\lambda}(\mathbb{R}^n)$ and we still give a sketch and simple proof of the following lemma for the sake of completeness.

Lemma 3.3. *Let $1/p = \sum_{i=1}^m 1/p_i$ with $p_i > 1$. Then, $\mathcal{M}(\vec{f})(x)$ is bounded from $L^{p_1,\lambda}(\mathbb{R}^n) \times \cdots \times L^{p_m,\lambda}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$ with $0 \le \lambda < n$.*

Proof. We only give some main steps of this proof. It suffices to study $\mathcal{M}_c(\vec{f})(x)$. Following the proof of Lemma 3.2, we split each $f_i = f_i^0 + f_i^\infty$ with $f_i^0 = f_i\chi_{2B}$. Then, we decompose $\mathcal{M}_c(\vec{f})(z)$ as

$$
\mathcal{M}_c(\vec{f})(z) = \mathcal{M}_c(\vec{f^0})(z) + \sum{}' \mathcal{M}_c(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(z) := I_1 + I_2,
$$

where $\alpha_1, \ldots, \alpha_m \in \{0, \infty\}$ and each term in the sum $\sum'$ contains at least one $\alpha_i = \infty$.

Thus, it suffices to show the following three inequalities:

$$
\|\mathcal{M}_c(f_1^0, \ldots, f_m^0)\|_{L^p(B)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\lambda}} t^{\lambda/p}, \tag{3.4}
$$

$$
\|\mathcal{M}_c(f_1^\infty, \ldots, f_m^\infty)\|_{L^p(B)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\lambda}} t^{\lambda/p} \tag{3.5}
$$

and

$$
\|\mathcal{M}_c(f_1^{\alpha_1}, f_2^{\alpha_2}, \ldots, f_m^{\alpha_m})\|_{L^p(B)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\lambda}} t^{\lambda/p}, \tag{3.6}
$$

where each term in (3.6) contains at least one $\alpha_i = \infty$ and one $\alpha_j = 0$.

By checking the proof of Lemma 3.2, we know that (3.4) follows from the boundedness of $\mathcal{M}(\vec{f})(x)$ on product L^p spaces (see [10]) and (3.5) follows from (3.3) and the multilinear Hölder inequality on L^p spaces.

For (3.6), without loss of generality, using (3.3) and the similar estimates of $\|\mathcal{M}^{b_1}(f_1^\infty, f_2^\infty, f_3^0, \ldots, f_m^0)\chi_B\|_{L^p}$ in the proof of Lemma 3.2, we know that (3.6) is also true. □

Lemma 3.4. [12] *Let $\vec{b} = (b_1, \ldots, b_m)$ and $\vec{f} = (f_1, \ldots, f_m)$ be two collections of locally integrable functions, then*

$$
|[\vec{b}, \mathcal{M}](\vec{f})(x)| \le \mathcal{M}_{\vec{b}}(\vec{f})(x) + 2\left(\sum_{i=1}^m b_i^-(x)\right)\mathcal{M}(\vec{f})(x).
$$

Lemma 3.5. [22] *Let $1 \le p < \infty$ and $0 \le \lambda \le n$. Then for any $B(x_0, t_0)$, we have*

$$\|\chi_{B(x_0,t_0)}\|_{L^{p,\lambda}(\mathbb{R}^n)} \le Ct^{\frac{n-\lambda}{p}},$$

where C is a positive constant only depending on the dimension n.

4 Proof of Theorem 1.1.

In this section, we will give the proof of Theorem 1.1. First, using Lemmas 3.2–3.4, we know that (II) implies (I).

Next, we show (I) $\Rightarrow$ (III) and we only prove the case for b_1. Choose $f_i(x) = \chi_Q(x) \in L^{p_i,\lambda}$ with any cube Q. Then, using Lemma 3.5, we have

$$\|[\vec{b}, \mathcal{M}](\vec{f})\|_{L^{p,\lambda}} \le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\lambda}} \le C\prod_{i=1}^{m} t^{\frac{n-\lambda}{p_i}} = Ct^{\frac{n-\lambda}{p}} = C|Q|^{\frac{n-\lambda}{np}},$$

where C is a positive constant depending on the dimension n.

From [12, p. 991], we have

$$\mathcal{M}(b_1\chi_Q, \chi_Q, \dots, \chi_Q)(x) = M_Q(b_1)(x)$$

for $x \in Q$ and

$$M_Q(b_1)(x) = \sup_{Q_0 \ni x, Q_0 \subset Q} \frac{1}{|Q_0|}\int_{Q_0} |b_1(x)|\mathrm{d}x.$$

From [2], we know that $M_Q(b_i) \ge b_i$ with $i = 1, 2, \dots, m$. Then, for $\vec{f} = (\chi_Q(x), \dots, \chi_Q(x))$, we have

$$\left(\frac{1}{|Q|^{\frac{\lambda}{n}}}\int_Q |M_Q(b_1)(x) - b_1(x)|^p\mathrm{d}x\right)^{1/p} \le \left(\frac{1}{|Q|^{\frac{\lambda}{n}}}\int_Q \left|\sum_{i=1}^{m}|M_Q(b_i)(x) - b_i(x)|\right|^p \mathrm{d}x\right)^{1/p} \le C\|[\vec{b}, \mathcal{M}](\vec{f})\|_{L^{p,\lambda}} \le C|Q|^{\frac{n-\lambda}{np}}.$$

Thus, we obtain

$$\left(\frac{1}{|Q|}\int_Q |M_Q(b_1)(x) - b_1(x)|^p\mathrm{d}x\right)^{1/p} \le C,$$

which implies (I) $\Rightarrow$ (III).

Finally, (III) $\rightarrow$ (II) can be found in [2].

5 Proof of Theorem 1.3.

Proof of Theorem 1.3. Obviously, it suffices to consider $\mathcal{M}_c^{b_1}(\vec{f})(x)$. As in the above section, for any cube $B = B(x, t)$, we split each f_i as $f_i = f_i^0 + f_i^\infty$ with $f_i^0 = f_i\chi_{2B}$. Thus, for any $\beta > 0$, there is

$$\begin{aligned}\frac{1}{\omega(B)^m}\,|\{z \in B(x,t) : \mathcal{M}_c^{b_1}(\vec{f})(z) > \beta^m\}|^m &\le \frac{1}{\omega(B)^m}\left|\left\{z \in B(x,t) : \mathcal{M}_c^{b_1}(\vec{f^0})(z) > \frac{\beta^m}{4}\right\}\right|^m \\ &\quad + \frac{1}{\omega(B)^m}\left|\left\{z \in B(x,t) : \mathcal{M}_c^{b_1}(\vec{f^\infty})(z) > \frac{\beta^m}{4}\right\}\right|^m \\ &\quad + \sum{}' \frac{1}{\omega(B)^m}\left|\left\{z \in B(x,t) : \mathcal{M}_c^{b_1}(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(z) > \frac{\beta^m}{2}\right\}\right|^m \\ &:= I + II + III,\end{aligned}$$

where $\alpha_1, \ldots, \alpha_m \in \{0, \infty\}$ and each term in the sum $\sum'$ contains at least one $\alpha_i = \infty$ and $\alpha_j = 0$.

For I, using [12, Theorem 1.9] and (2.2), we have

$$\begin{aligned} I &\le C\|b_1\|_{\mathrm{BMO}} \frac{1}{\omega(B)^m} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \Phi\left(\frac{|f_i^0(y_i)|}{\beta} \right) \mathrm{d}y_i \right) \\ &= C\|b_1\|_{\mathrm{BMO}} \frac{\omega(2B)^m}{\omega(B)^m} \prod_{i=1}^m \left(\frac{|2B|}{\omega(2B)} \cdot \frac{1}{|2B|} \int_{2B} \Phi\left(\frac{|f_i(y_i)|}{\beta} \right) \mathrm{d}y_i \right) \\ &\le C\|b_1\|_{\mathrm{BMO}} \prod_{i=1}^m \left\| \Phi\left(\frac{|f_i|}{\beta} \right) \right\|_{L^{1,\omega}_{L\log L}(\mathbb{R}^n)} . \end{aligned}$$

To give the estimates of II, for any $Q = Q(z, r)$ with $z \in B(x, t)$, we show the following decomposition:

$$\begin{aligned} &\sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |b_1(z) - b_1(y_1)||f_1^\infty(y_1)|\mathrm{d}y_1 \prod_{i=2}^m \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |f_i^\infty(y_i)|\mathrm{d}y_i \\ &\le \sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |b_1(z) - (b_1)_B||f_1^\infty(y_1)|\mathrm{d}y_1 \prod_{i=2}^m \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |f_i^\infty(y_i)|\mathrm{d}y_i \\ &\quad + \sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |(b_1)_B - b_1(y_1)||f_1^\infty(y_1)|\mathrm{d}y_1 \prod_{i=2}^m \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |f_i^\infty(y_i)|\mathrm{d}y_i =: F + H. \end{aligned}$$

First, we have

$$\frac{1}{\omega(B)^m} \left| \left\{ z \in B : F > \frac{\beta^m}{8} \right\} \right|^m \le \left(\frac{1}{\omega(B)} \int_B \prod_{i=1}^m \left(\frac{\sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} |f_i^\infty(y_i)|\mathrm{d}y_i}{\beta} \right)^{1/m} |b_1(z) - (b_1)_B|^{1/m} \mathrm{d}z \right)^m .$$

As $y_i \in (2B)^c \bigcap Q(z, r)$ and $z \in B(x, t)$. Following the discussion in Section 3 and using (3.2), we get

$$\begin{aligned} \sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} \frac{|f_i^\infty(y_i)|}{\beta} \mathrm{d}y_i &\le \frac{C}{\beta} \sup_{r>2t} \int_{(2B)^c \bigcap Q(z,r)} \frac{|f_i(y_i)|}{|y_i - z|^n} \mathrm{d}y_i \\ &\le \frac{C}{\beta} \int_{(2B)^c} \frac{|f_i(y_i)|}{|y_i - x|^n} \mathrm{d}y_i \\ &\le \frac{C}{\beta} \sum_{j=1}^\infty \int_{2^{j+1}B \setminus 2^j B} \frac{|f_i(y_i)|}{|y_i - x|^n} \mathrm{d}y_i \\ &\le \frac{C}{\beta} \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f_i(y_i)|\mathrm{d}y_i \\ &\le C \sum_{j=1}^\infty \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} \Phi\left(\frac{|f_i(y_i)|}{\beta} \right) \mathrm{d}y_i. \end{aligned}$$

Moreover, as ω satisfies (1.1), there is

$$\frac{\omega(2^{j+1}B)}{\omega(B)} = \frac{\omega(2^{j+1}B)}{\omega(2^j B)} \frac{\omega(2^j B)}{\omega(2^{j-1}B)} \cdots \frac{\omega(2B)}{\omega(2B)} \le C_0^{j+1}. \tag{5.1}$$

Recall the condition of Theorem 1.3 and the fact $1 < C_0 \le 2^\gamma$ with $0 < \gamma < n$. Thus, using the Hölder inequality and (2.2), we obtain

$$\begin{aligned}
\frac{1}{\omega(B)^m}\left|\left\{z \in B : F > \frac{\beta^m}{8}\right\}\right|^m &\le \left(\frac{1}{\omega(B)}\int_B |b_1(z) - (b_1)_B|^{1/m}\prod_{i=1}^m\left(\sum_{j=1}^\infty \frac{C}{|2^{j+1}B|}\int_{2^{j+1}B}\Phi\left(\frac{|f_i(y_i)|}{\beta}\right)\mathrm{d}y_i\right)^{1/m}\mathrm{d}z\right)^m \\
&\le C\left(\frac{1}{|B|}\int_B |b_1(z) - (b_1)_B|\mathrm{d}z\right)\left(\prod_{i=1}^m\left(\sum_{j=1}^\infty \frac{|B|}{|2^{j+1}B|\omega(B)}\int_{2^{j+1}B}\Phi\left(\frac{|f_i(y_i)|}{\beta}\right)\mathrm{d}y_i\right)^{1/m}\right)^m \\
&\le C\|b_1\|_{\mathrm{BMO}}\left(\prod_{i=1}^m\left(\sum_{j=1}^\infty \frac{|B|\omega(2^{j+1}B)}{|2^{j+1}B|\omega(B)}\cdot\frac{|2^{j+1}B|}{\omega(2^{j+1}B)|2^{j+1}B|}\int_{2^{j+1}B}\Phi\left(\frac{|f_i(y_i)|}{\beta}\right)\mathrm{d}y_i\right)^{1/m}\right)^m \\
&\le C\|b_1\|_{\mathrm{BMO}}\left(\prod_{i=1}^m\left(\sum_{j=1}^\infty \frac{|B|\omega(2^{j+1}B)}{|2^{j+1}B|\omega(B)}\left\|\Phi\left(\frac{|f_i|}{\beta}\right)\right\|_{L\log L, 2^{j+1}B}\right)^{1/m}\right)^m \\
&\le C\|b_1\|_{\mathrm{BMO}}\prod_{i=1}^m\left\|\Phi\left(\frac{|f_i|}{\beta}\right)\right\|_{L^{1,\omega}_{L\log L}(\mathbb{R}^n)},
\end{aligned}$$

where C is a positive constant depending on γ and the dimension n.

For H, we have

$$\begin{aligned}
\frac{1}{\omega(B)^m}\left|\left\{z \in B : H > \frac{\beta^m}{4}\right\}\right|^m \le C\Bigg(\frac{1}{\omega(B)}\int_B&\left(\sup_{r>0}\frac{1}{|Q(z,r)|}\int_{Q(z,r)}\frac{|(b_1)_B - b_1(y_i)||f_1^\infty(y_1)|}{\beta}\mathrm{d}y_1\right)^{1/m} \\
&\times\prod_{i=2}^m\left(\sup_{r>0}\frac{1}{|Q(z,r)|}\int_{Q(z,r)}\frac{|f_i^\infty(y_i)|}{\beta}\mathrm{d}y_i\right)^{1/m}\mathrm{d}z\Bigg)^m.
\end{aligned}$$

Using (3.2) again, we know that there exists a positive constant C depending on the dimension n, such that

$$\begin{aligned}
\left(\sup_{r>0}\frac{1}{|Q(z,r)|}\int_{Q(z,r)}\frac{|(b_1)_B - b_1(y_i)||f_1^\infty(y_1)|}{\beta}\mathrm{d}y_1\right)^{1/m} &\le \left(\int_{(2B)^c}\frac{|(b_1)_B - b_1(y_1)||f_1(y_1)|}{\beta|x-y_1|^n}\mathrm{d}y_1\right)^{1/m} \\
&\le \left(\sum_{j=1}^m\int_{2^{j+1}B\setminus 2^jB}\frac{|b_1(y_1) - (b_1)_B||f_1(y_1)|}{\beta|x-y_1|^n}\mathrm{d}y_1\right)^{1/m} \\
&\le \left(\sum_{j=1}^m\frac{C}{|2^{j+1}B|}\int_{2^{j+1}B\setminus 2^jB}\frac{|b_1(y_1) - (b_1)_B||f_1(y_1)|}{\beta}\mathrm{d}y_1\right)^{1/m} \\
&\le \left(\sum_{j=1}^m\frac{C}{|2^{j+1}B|}\int_{2^{j+1}B\setminus 2^jB}\frac{|(b_1)_{2^{j+1}B} - (b_1)_B||f_1(y_1)|}{\beta}\mathrm{d}y_1\right)^{1/m} \\
&\quad + \left(\sum_{j=1}^m\frac{C}{|2^{j+1}B|}\int_{2^{j+1}B\setminus 2^jB}\frac{|(b_1)_{2^{j+1}B} - b_1(y_1)||f_1(y_1)|}{\beta}\mathrm{d}y_1\right)^{1/m}.
\end{aligned}$$

Thus, we obtain

$$\frac{1}{\omega(B)^m}\left|\left\{z\in B : H>\frac{\beta^m}{4}\right\}\right|^m \le H_1+H_2,$$

where

$$H_1=\left(\frac{1}{\omega(B)}\int_B\left(\sum_{j=1}^m\frac{C}{|2^{j+1}B|}\int_{2^{j+1}B\setminus 2^jB}\frac{|(b_1)_{2^{j+1}B}-(b_1)_B||f_1(y_1)|}{\beta}\mathrm{d}y_1\right)^{1/m}\prod_{i=2}^m\left(\sup_{r>0}\frac{1}{|Q(z,r)|}\int_{Q(z,r)}\frac{|f_i^\infty(y_i)|}{\beta}\mathrm{d}y_i\right)^{1/m}\mathrm{d}z\right)^m$$

and

$$H_2=\left(\frac{1}{\omega(B)}\int_B\left(\sum_{j=1}^m\frac{C}{|2^{j+1}B|}\int_{2^{j+1}B\setminus 2^jB}\frac{|(b_1)_{2^{j+1}B}-b_1(y_1)||f_1(y_1)|}{\beta}\mathrm{d}y_1\right)^{1/m}\prod_{i=2}^m\left(\sup_{r>0}\frac{1}{|Q(z,r)|}\int_{Q(z,r)}\frac{|f_i^\infty(y_i)|}{\beta}\mathrm{d}y_i\right)^{1/m}\mathrm{d}z\right)^m.$$

For H_1, using Lemma 3.1, we have

$$\begin{aligned}
H_1&\le\left(\frac{1}{\omega(B)}|B|\prod_{i=2}^m\left(\sup_{r>0}\frac{1}{|Q(z,r)|}\int_{Q(z,r)}\frac{|f_i^\infty(y_i)|}{\beta}\mathrm{d}y_i\right)^{1/m}\right.\\
&\quad\left.\times\left(\sum_{j=1}^\infty\frac{C}{|2^{j+1}B|}\int_{2^{j+1}B}|(b_1)_B-(b_1)_{2^{j+1}B}|\frac{|f_1(y_1)|}{\beta}\mathrm{d}y_1\right)^{1/m}\right)^m\\
&\le\left(\frac{C}{\omega(B)}|B|\prod_{i=2}^m\left(\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\Phi\left(\frac{|f_i(y_i)|}{\beta}\right)\mathrm{d}y_i\right)^{1/m}\right.\\
&\quad\left.\times\left(\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|(b_1)_B-(b_1)_{2^{j+1}B}|\frac{|f_1(y_1)|}{\beta}\mathrm{d}y_1\right)^{1/m}\right)^m\\
&\le C\left(\prod_{i=2}^m\left(\sum_{j=1}^\infty\frac{|B|\omega(2^{j+1}B)}{|2^{j+1}B|\omega(B)}\cdot\frac{|2^{j+1}B|}{\omega(2^{j+1}B)}\cdot\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\Phi\left(\frac{|f_i(y_i)|}{\beta}\right)\mathrm{d}y_i\right)^{1/m}\right.\\
&\quad\left.\times\left(\sum_{j=1}^\infty\frac{\omega(2^{j+1}B)(j+1)|B|}{\omega(2^{j+1}B)\omega(B)|2^{j+1}B|}\|b_1\|_{\mathrm{BMO}}\int_{2^{j+1}B}\frac{|f_1(y_1)|}{\beta}\mathrm{d}y_1\right)^{1/m}\right)^m\\
&\le C\|b_1\|_{\mathrm{BMO}}\left(\prod_{i=2}^m\left(\sum_{j=1}^\infty\frac{|B|\omega(2^{j+1}B)}{|2^{j+1}B|\omega(B)}\cdot\frac{|2^{j+1}B|}{\omega(2^{j+1}B)}\left\|\Phi\left(\frac{|f_i|}{\beta}\right)\right\|_{L\log L,2^{j+1}B}\right)^{1/m}\right.\\
&\quad\left.\times\left(\sum_{j=1}^\infty\frac{\omega(2^{j+1}B)(j+1)|B|}{\omega(B)|2^{j+1}B|}\frac{|2^{j+1}B|}{\omega(2^{j+1}B)}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\Phi\left(\frac{|f_1(y_1)|}{\beta}\right)\mathrm{d}y_1\right)^{1/m}\right)^m\\
&\le C\|b_1\|_{\mathrm{BMO}}\prod_{i=1}^m\left\|\Phi\left(\frac{|f_i|}{\beta}\right)\right\|_{L^{1,\omega}_{L\log L}(\mathbb{R}^n)},
\end{aligned}$$

where C is a positive constant depending on γ and the dimension n.

For H_2, there is

$$H_2 \le \left(\frac{|B|}{\omega(B)} \prod_{i=2}^{m} \left(\sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} \frac{|f_i^0(y_i)|}{\beta} \mathrm{d}y_i \right)^{1/m} \left(\sum_{j=1}^{m} \frac{C}{|2^{j+1}B|} \int_{2^{j+1}B} |b_1(y_1) - (b_1)_{2^{j+1}B}| \frac{|f_1(y_1)|}{\beta} \mathrm{d}y_1 \right)^{1/m} \right)^m .$$

Then, using (2.3), we have

$$\begin{aligned} &\sum_{j=1}^{\infty} \frac{C}{|2^{j+1}B|} \int_{2^{j+1}B} |b_1(y_1) - (b_1)_{2^{j+1}B}| \frac{|f_1(y_1)|}{\beta} \mathrm{d}y_1 \\ &\le C \sum_{j=1}^{\infty} \|b_1(\cdot) - (b_1)_{2^{j+1}B}\|_{\exp L, 2^{j+1}B} \left\| \Phi\left(\frac{|f_1|}{\beta} \right) \right\|_{L \log L, 2^{j+1}B} \\ &\le C \|b_1\|_{\mathrm{BMO}} \sum_{j=1}^{\infty} \left\| \Phi\left(\frac{|f_1|}{\beta} \right) \right\|_{L \log L, 2^{j+1}B} . \end{aligned}$$

Thus, we obtain

$$\begin{aligned} H_2 &\le C\|b_1\|_{\mathrm{BMO}} \left(\frac{|B|}{\omega(B)} \prod_{i=2}^{m} \left(\sup_{r>0} \frac{1}{|Q(z,r)|} \int_{Q(z,r)} \frac{|f_i(y_i)|}{\beta} \mathrm{d}y_i \right)^{1/m} \left(\sum_{j=1}^{\infty} \left\| \Phi\left(\frac{|f_1|}{\beta} \right) \right\|_{L \log L, 2^{j+1}B} \right)^{1/m} \right)^m \\ &\le C\|b_1\|_{\mathrm{BMO}} \left(\prod_{i=2}^{m} \left(\frac{|B|}{\omega(B)} \frac{1}{|Q|} \int_{Q} \frac{|f_i(y_i)|}{\beta} \mathrm{d}y_i \right)^{1/m} \left(\sum_{j=1}^{\infty} \frac{\omega(2^{j+1}B)|B|}{|2^{j+1}B||\omega(B)|} \frac{|2^{j+1}B|}{\omega(2^{j+1}B)} \left\| \Phi\left(\frac{|f_1|}{\beta} \right) \right\|_{L \log L, 2^{j+1}B} \right)^{1/m} \right)^m . \end{aligned}$$

Similar to the estimates of H_1, it is easy to see

$$H_2 \le C\|b_1\|_{\mathrm{BMO}} \prod_{i=1}^{m} \left\| \Phi\left(\frac{|f_i|}{\beta} \right) \right\|_{L^{1,\omega}_{L \log L}(\mathbb{R}^n)} ,$$

where C is a positive constant depending on γ and the dimension n.

Combining the estimates of H_1 and H_2, we get

$$\frac{1}{\omega(B)^m} \left| \left\{ z \in B : H > \frac{\beta^m}{4} \right\} \right|^m \le C\|b_1\|_{\mathrm{BMO}} \prod_{i=1}^{m} \left\| \Phi\left(\frac{|f_i|}{\beta} \right) \right\|_{L^{1,\omega}_{L \log L}(\mathbb{R}^n)} .$$

For III, without loss of generality, we only consider $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \alpha_4 = \alpha_5 = \cdots = \alpha_m = \infty$.

Recall the definition of $M_b(f)(x)$ and $M(f)(x)$. Then, using the Hölder inequality for weak L^p space (see [19, p. 16]) and (2.4) with $m = 1$, we obtain

$$\begin{aligned} &\frac{1}{\omega(B)^m} \left| \left\{ z \in B : \sup_{r>0} \frac{1}{|Q|} \int_Q |b_1(y_1) - b_1(z)||f_1^0(y_1)| \mathrm{d}y_1 \frac{1}{|Q|} \int_Q |f_2^0(y_2)| \mathrm{d}y_2 \right.\right. \\ &\quad \left.\left. \times \frac{1}{|Q|} \int_Q |f_3^\infty(y_3)| \mathrm{d}y_3 \frac{1}{|Q|} \int_Q |f_4^\infty(y_4)| \mathrm{d}y_4 \cdots \frac{1}{|Q|} \int_Q |f_m^\infty(y_m)| \mathrm{d}y_m > \beta^m \right\} \right|^m \\ &\le \frac{1}{\omega(B)^m} |\{z \in B : M_{b_1}(f_1^0) M(f_2^0) M(f_3^\infty) \cdots M(f_m^\infty) > \beta^m\}|^m \\ &\le \frac{1}{\omega(B)^m} \frac{1}{\beta^m} \|M_{b_1}(f_1^0) M(f_2^0) M(f_3^\infty) \cdots M(f_m^\infty)\|_{L^{\frac{1}{m},\infty}(B)} \\ &\le C \frac{1}{\omega(B)^m} \frac{1}{\beta^m} \|M_{b_1}(f_1^0)\|_{L^{1,\infty}(B)} \|M(f_2^0)\|_{L^{1,\infty}(B)} \|M(f_3^\infty) \cdots M(f_m^\infty)\|_{L^{\frac{1}{m-2},\infty}(B)} . \end{aligned}$$

According to the endpoint estimates of M_b (see [1]), (2.2) and the definition of $L^{1,\omega}_{L\log L}(\mathbb{R}^n)$, it is easy to get

$$\frac{1}{\omega(B)\beta}\|M_{b_1}(f_1^0)\|_{L^{1,\infty}(B)} \le C\|b_1\|_{\mathrm{BMO}}\frac{1}{\omega(B)\beta}\int_{2B}\Phi(|f_1|)\mathrm{d}y_1 \le C\|b_1\|_{\mathrm{BMO}}\left\|\Phi\left(\frac{|f_1|}{\beta}\right)\right\|_{L^{1,\omega}_{L\log L}(\mathbb{R}^n)}.$$

For $\|M(f_2^0)\|_{L^{1,\infty}(B)}$, noting the fact $M(f)(x)$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, it is easy to see

$$\frac{1}{\omega(B)\beta}\|M(f_2^0)\|_{L^{1,\infty}(B)} \le C\left\|\Phi\left(\frac{|f_2|}{\beta}\right)\right\|_{L^{1,\omega}_{L\log L}(\mathbb{R}^n)}.$$

Finally, using the fact $L^1(B) \hookrightarrow L^{1,\infty}(B)$ and adopting some similar estimates in the proof of II, we can easily obtain

$$\frac{1}{\omega(B)^{m-2}\beta^{m-2}}\|M(f_3^\infty)\cdots M(f_m^\infty)\|_{L^{\frac{1}{m-2},\infty}(B)} \le C\prod_{i=3}^{m}\left\|\Phi\left(\frac{|f_i|}{\beta}\right)\right\|_{L^{1,\omega}_{L\log L}(\mathbb{R}^n)}.$$

Combining the estimates of I, II and III, we finish the proof of Theorem 1.3.

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