Research Article

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Partial sums and inclusion relations for analytic functions involving (p, q)-differential operator

Abstract: Let $f_k(z) = z + \sum_{n=2}^{k} a_n z^n$ be the sequence of partial sums of the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. In this paper, we determine sharp lower bounds for $\text{Re}\{f(z)/f_k(z)\}$, $\text{Re}\{f_k(z)/f(z)\}$, $\text{Re}\{f'(z)/f_k'(z)\}$ and $\text{Re}\{f_k'(z)/f'(z)\}$, where $f(z)$ belongs to the subclass $\mathcal{J}_{p,q}^m(\mu, \alpha, \beta)$ of analytic functions, defined by Sălăgean (p, q)-differential operator. In addition, the inclusion relations involving $N_\delta(e)$ of this generalized function class are considered.

Keywords: analytic, univalent, (p, q)-differential operator, partial sum, inclusion relation

MSC 2020: 30C45, 30C50

1 Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic and univalent in the open disc $\mathbb{D} = \{z : |z| < 1\}$. We also denote $\mathcal{T}$ a subclass of $\mathcal{A}$ introduced and studied by Silverman [1], consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0; z \in \mathbb{D}. \tag{1.2}$$

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$, $z \in \mathbb{D}$.

We briefly recall here the notion of *q-operators* i.e. *q-difference operator* that plays vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of *q-calculus* was initiated by Jackson [2] (also see [3–5]). Kanas and Răducanu [4] have used the fractional *q-calculus operators*

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in investigations of certain classes of functions which are analytic in $\mathbb{D}$. For $p > 0$, $q > 0$ the $(p; q)$-*differential operator* of a function f, analytic in $\mathbb{D}$ is defined for a function $f \in \mathcal{A}$ is, by definition, given as follows [2]:

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p - q)z} \quad (p \neq q). \tag{1.3}$$

From (1.3), we have

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}, \tag{1.4}$$

where

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad [0]_{p,q} = 0. \tag{1.5}$$

The *twin-basic* number is a natural generalization of q-number, that is

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad (q \neq 1), \tag{1.6}$$

which is sometimes called *the basic number n*.

One can easily verify that $D_{p,q}f(z) \to f'(z)$ as $p \to 1^-$ and $q \to 1^-$. It is clear that *q-integers* and $(p; q)$-*integers* differ, that is, we cannot obtain $(p; q)$-*integers* just by replacing q by $\frac{q}{p}$ in the definition of *q-integers*. However, (1.5) reduces to (1.6) for the case $p = 1$. Thus, we can say that $(p; q)$-*calculus* can be taken as a generalization of *q-calculus*. The $(p; q)$-*factorial* is defined by

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}! \quad (n \geq 1), \quad [0]_{p,q}! = 1. \tag{1.7}$$

Note that $p \to 1^-$ the $(p; q)$-*factorial* reduces to the *q-factorial*. Also, clearly

$$\lim_{p\to 1^-}\lim_{q\to 1^-} = [n]_{p,q}! = n! \quad \text{and} \quad \lim_{p\to 1^-}\lim_{a\to 1^-} = [n]_{p,q} = n.$$

For details on *q-calculus* and (p, q)-*calculus*, one can refer to [2,6,7] and also references cited therein. Recently for $f \in \mathcal{A}$, Govindaraj and Sivasubramanian [8] defined Sălăgean (q)-*differential operator* and further Kanas and Răducanu [4] defined and discussed Sălăgean (p, q)-*differential operator* as follows:

$$\begin{aligned}
\mathcal{D}_{p,q}^0 f(z) &= f(z), \\
\mathcal{D}_{p,q}^1 f(z) &= z\mathcal{D}_{p,q}f(z), \\
\mathcal{D}_{p,q}^m f(z) &= z\mathcal{D}_{p,q}^m(\mathcal{D}_{p,q}^{m-1}f(z)), \\
\mathcal{D}_{p,q}^m f(z) &= z + \sum_{n=2}^{\infty} [n]_{p,q}^m a_n z^n \quad (m \in \mathbb{N}_0, z \in \mathbb{D}).
\end{aligned} \tag{1.8}$$

It is interesting that one can observe

$$\lim_{(p,q)\to(1^-,1^-)} \mathcal{D}_{p,q}^m f(z) = \mathcal{D}^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n \quad (m \in \mathbb{N}_0, \ z \in \mathbb{D}), \tag{1.9}$$

the familiar Sălăgean derivative [9].

For $0 \leq \mu \leq 1$, $0 \leq \alpha < 1$, $\beta \geq 0$ and $m \in \mathbb{N}_0$, we let $\mathcal{J}_{p,q}^m(\mu, \alpha, \beta)$ be the subclass of $\mathcal{A}$, consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\mathfrak{R}\left(\frac{\mathcal{D}_{p,q}^{m+1}f(z)}{(1 - \mu)z + \mu\mathcal{D}_{p,q}^m f(z)} - \alpha \right) > \beta \left| \frac{\mathcal{D}_{p,q}^{m+1}f(z)}{(1 - \mu)z + \mu\mathcal{D}_{p,q}^m f(z)} - 1 \right|, \quad z \in \mathbb{D}, \tag{1.10}$$

where $\mathcal{D}_{p,q}^m f(z)$ is given by (1.8). We further let $\mathcal{TJ}_{p,q}^m(\mu, \alpha, \beta) = \mathcal{J}_{p,q}^m(\mu, \alpha, \beta) \cap \mathcal{T}$.

By taking $\mu = 1$ we get $\mathcal{TJ}_{p,q}^m(1, \alpha, \beta) \equiv \mathcal{TSP}_{p,q}^m(\alpha, \beta)$ studied by Kanas and Răducanu [4]. Further by specializing the parameter $\mu = 0$, we define the following new subclass:

Remark 1.1. For $\mu = 0$, $m \in \mathbb{N}_0$, $0 \leq \alpha < 1$ and $\beta \geq 0$, let $\mathcal{TJ}_{p,q}^m(0, \alpha, \beta) \equiv \mathcal{USD}_{p,q}^m(\alpha, \beta)$ be the subclass of $\mathcal{A}$, consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re\left(\frac{\mathcal{D}_{p,q}^{m+1}f(z)}{z} - \alpha\right) > \beta \left|\frac{\mathcal{D}_{p,q}^{m+1}f(z)}{z} - 1\right|, \quad z \in \mathbb{D}, \tag{1.11}$$

where $\mathcal{D}_{p,q}^m f(z)$ is given by (1.8).

Remark 1.2. For $\mu = 0$, $\beta = 0$, $0 \leq \alpha < 1$ and $m \in \mathbb{N}_0$, let $\mathcal{TJ}_{p,q}^m(1, \alpha, 0) \equiv \mathcal{R}_{p,q}^m(\alpha)$ be the subclass of $\mathcal{A}$, consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re\left(\frac{\mathcal{D}_{p,q}^{m+1}f(z)}{z}\right) > \alpha \quad z \in \mathbb{D}, \tag{1.12}$$

where $\mathcal{D}_{p,q}^m f(z)$ is given by (1.8).

Remark 1.3. As $\lim_{(p,q)\to(\Gamma,\Gamma)}$, $\mathcal{D}_{p,q}^m f(z)$ reduces to various interesting subclasses (as mentioned earlier) satisfying the analytic criterion

$$\Re\left(\frac{\mathcal{D}^{m+1}f(z)}{(1-\mu)z + \mu\mathcal{D}^m f(z)} - \alpha\right) > \beta \left|\frac{\mathcal{D}^m f(z)}{(1-\mu)z + \mu\mathcal{D}^m f(z)} - 1\right|, \quad z \in \mathbb{D}, \tag{1.13}$$

where $\mathcal{D}^m f(z)$ is given by (1.9).

2 Basic properties

In this section, we obtain the characterization properties for the classes $\mathcal{TJ}_{p,q}^m(\mu, \alpha, \beta)$.

Theorem 2.1. *A function $f(z)$ of the form* (1.1) *is in* $\mathcal{J}_{p,q}^m(\mu, \alpha, \beta)$ *if*

$$\sum_{n=2}^{\infty}[n]_{p,q}^m([n]_{p,q}(1+\beta) - \mu(\alpha+\beta)) \ |a_n| \leq 1 - \alpha, \tag{2.1}$$

where $0 \leq \mu \leq 1$, $0 \leq \alpha < 1$, $\beta \geq 0$ *and* $m \in \mathbb{N}_0$.

    The result is sharp for the function

$$f_n(z) = z - \frac{1-\alpha}{[n]_{p,q}^m([n]_{p,q}(1+\beta) - \mu(\alpha+\beta))}z^n.$$

Proof. It suffices to show that

$$\beta \left|\frac{\mathcal{D}_{p,q}^{m+1}f(z)}{(1-\mu)z + \mu\mathcal{D}_{p,q}^m f(z)} - 1\right| - \Re\left(\frac{\mathcal{D}_{p,q}^{m+1}f(z)}{(1-\mu)z + \mu\mathcal{D}_{p,q}^m f(z)} - 1\right) \leq 1 - \alpha.$$

We have

$$\beta \left|\frac{\mathcal{D}_{p,q}^{m+1}f(z)}{(1-\mu)z + \mu\mathcal{D}_{p,q}^m f(z)} - 1\right| - \Re\left(\frac{\mathcal{D}_{p,q}^{m+1}f(z)}{(1-\mu)z + \mu\mathcal{D}_{p,q}^m f(z)} - 1\right)$$

$$\leq (1+\beta) \left|\frac{\mathcal{D}_{p,q}^{m+1}f(z)}{(1-\mu)z + \mu\mathcal{D}_{p,q}^m f(z)} - 1\right| \leq \frac{(1+\beta)\sum_{n=2}^{\infty}[n]_{p,q}^m([n]_{p,q} - \mu)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}[n]_{p,q}^m\mu|a_n||z|^{n-1}}.$$

As $|z| \to 1^-$, the last expression is bounded above by $1 - \alpha$ if (2.1) holds. It is obvious that the function f_n satisfies the inequality (2.1), and thus $1 - \alpha$ cannot be replaced by a larger number. Therefore, we need only to prove that $f \in \mathcal{TJ}^m_{p,q}(\mu, \alpha, \beta)$. Since

$$\Re\left(\frac{1 - \sum_{n=2}^{\infty}[n]^{m+1}_{p,q}a_n z^{n-1}}{1 - \sum_{n=2}^{\infty}[n]^m_{p,q}\mu a_n z^{n-1}} - \alpha\right) > \beta \left|\frac{\sum_{n=2}^{\infty}[n]^m_{p,q}([n]_{p,q} - \mu)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty}[n]^m_{p,q}\mu a_n z^{n-1}}\right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality given in (2.1). □

Corollary 2.2. *If $f \in \mathcal{TJ}^m_{p,q}(\mu, \alpha, \beta)$, then*

$$|a_n| \leq \frac{1 - \alpha}{\Phi^m_{p,q}(\mu, \alpha, \beta)}. \tag{2.2}$$

Equality holds for the function $f(z) = z - \frac{1-\alpha}{\Phi^m_{p,q}(\mu,\alpha,\beta)}z^n$, where

$$\Phi^m_{p,q}(\mu, \alpha, \beta) = [n]^m_{p,q}([n]_{p,q}(1 + \beta) - \mu(\alpha + \beta)).$$

Throughout this paper for convenience, unless otherwise stated, we let

$$\Phi^m_n = \Phi^m_{p,q}(\mu, \alpha, \beta) = [n]^m_{p,q}([n]_{p,q}(1 + \beta) - \mu(\alpha + \beta)) \tag{2.3}$$

and

$$\Phi^m_2 = \Phi^m_{p,q}(\mu, \alpha, \beta) = [2]^m_{p,q}([2]_{p,q}(1 + \beta) - \mu(\alpha + \beta)), \tag{2.4}$$

where $0 \leq \mu \leq 1$, $0 \leq \alpha < 1$, $\beta \geq 0$ and $m \in \mathbb{N}_0$.

3 Partial sums

Silverman [10] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. In this section, following the earlier work by Silverman [10] and also the work cited in [11–15] on partial sums of analytic functions, we study the ratio of a function of the form (1.1) to its sequence of partial sums of the form

$$f_k(z) = z + \sum_{n=2}^{k} a_n z^n,$$

when the coefficients of $f(z)$ satisfy the condition (2.1).

Theorem 3.1. *If $f \in \mathcal{A}$ of the form* (1.1) *satisfies the condition* (2.1), *then*

$$\Re\left(\frac{f(z)}{f_k(z)}\right) \geq \frac{\Phi^m_{k+1} - 1 + \alpha}{\Phi^m_{k+1}} \quad (z \in \mathbb{D}), \tag{3.1}$$

where

$$\Phi^m_n = \Phi^m_{p,q}(\mu, \alpha, \beta) \geq \begin{cases} 1 - \alpha, & \text{if } n = 2, 3, \ldots, k, \\ \Phi^m_{k+1}, & \text{if } n = k + 1, k + 2, \ldots. \end{cases} \tag{3.2}$$

The result (3.1) *is sharp with the function given by*

$$f(z) = z + \frac{1 - \alpha}{\Phi^m_{k+1}}z^{k+1}. \tag{3.3}$$

Proof. Define the function $w(z)$ by

$$\frac{1 + w(z)}{1 - w(z)} = \frac{\Phi_{k+1}^m}{1 - \alpha}\left[\frac{f(z)}{f_k(z)} - \frac{\Phi_{k+1}^m - 1 + \alpha}{\Phi_{k+1}^m}\right] = \frac{1 + \sum_{n=2}^k a_n z^{n-1} + \left(\frac{\Phi_{k+1}^m}{1 - \alpha}\right)\sum_{n=k+1}^\infty a_n z^{n-1}}{1 + \sum_{n=2}^k a_n z^{n-1}}. \tag{3.4}$$

It suffices to show that $|w(z)| \leq 1$. Now, from (3.4) we can write

$$w(z) = \frac{\left(\frac{\Phi_{k+1}^m}{1 - \alpha}\right)\sum_{n=k+1}^\infty a_n z^{n-1}}{2 + 2\sum_{n=2}^k a_n z^{n-1} + \left(\frac{\Phi_{k+1}^m}{1 - \alpha}\right)\sum_{n=k+1}^\infty a_n z^{n-1}}.$$

Hence, we obtain

$$|w(z)| \leq \frac{\left(\frac{\Phi_{k+1}^m}{1 - \alpha}\right)\sum_{n=k+1}^\infty |a_n|}{2 - 2\sum_{n=2}^k |a_n| - \left(\frac{\Phi_{k+1}^m}{1 - \alpha}\right)\sum_{n=k+1}^\infty |a_n|}.$$

Now $|w(z)| \leq 1$ if and only if

$$2\left(\frac{\Phi_{k+1}^m}{1 - \alpha}\right)\sum_{n=k+1}^\infty |a_n| \leq 2 - 2\sum_{n=2}^k |a_n|,$$

or, equivalently,

$$\sum_{n=2}^k |a_n| + \sum_{n=k+1}^\infty \frac{\Phi_{k+1}^m}{1 - \alpha}|a_n| \leq 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{n=2}^k |a_n| + \sum_{n=k+1}^\infty \frac{\Phi_{k+1}^m}{1 - \alpha}|a_n| \leq \sum_{n=2}^\infty \frac{\Phi_n^m}{1 - \alpha}|a_n|,$$

which is equivalent to

$$\sum_{n=2}^k \left(\frac{\Phi_k^m - 1 + \alpha}{1 - \alpha}\right)|a_n| + \sum_{n=k+1}^\infty \left(\frac{\Phi_k^m - \Phi_{k+1}^m}{1 - \alpha}\right)|a_n| \geq 0. \tag{3.5}$$

To see that the function given by (3.3) gives the sharp result, we observe that for $z = re^{i\pi/n}$

$$\frac{f(z)}{f_k(z)} = 1 + \frac{1 - \alpha}{\Phi_{k+1}^m}z^n \to 1 - \frac{1 - \alpha}{\Phi_{k+1}^m} = \frac{\Phi_{k+1}^m - 1 + \alpha}{\Phi_{k+1}^m}, \quad \text{when } r \to 1^-. \qquad \square$$

Theorem 3.2. *If f of the form* (1.1) *satisfies the condition* (2.1)*, then*

$$\Re\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{\Phi_{k+1}^m}{\Phi_{k+1}^m + 1 - \alpha} \quad (z \in \mathbb{D}), \tag{3.6}$$

where $\Phi_{n+1}^m \geq 1 - \alpha$ and

$$\Phi_k^m \geq \begin{cases} 1 - \alpha, & \text{if } n = 2, 3, \dots, k, \\ \Phi_{k+1}^m, & \text{if } n = k + 1, k + 2, \dots. \end{cases} \tag{3.7}$$

The result (3.6) *is sharp with the function given by* (3.3)*.*

Proof. The proof follows by defining

$$\frac{1 + w(z)}{1 - w(z)} = \frac{\Phi_{k+1}^m + 1 - \alpha}{1 - \alpha}\left[\frac{f_k(z)}{f(z)} - \frac{\Phi_{k+1}^m}{\Phi_{k+1}^m + 1 - \alpha}\right]$$

and much akin to similar arguments in Theorem 3.1. $\qquad\square$

We next turn to ratios involving derivatives.

Theorem 3.3. *If f of the form* (1.1) *satisfies the condition* (2.1), *then*

$$\Re\left(\frac{f'(z)}{f_n'(z)}\right) \geq \frac{\Phi_{n+1}^m - (n+1)(1-\alpha)}{\Phi_{n+1}^m} \quad (z \in \mathbb{D}) \tag{3.8}$$

and

$$\Re\left(\frac{f_n'(z)}{f'(z)}\right) \geq \frac{\Phi_{n+1}^m}{\Phi_{n+1}^m + (n+1)(1-\gamma)} \quad (z \in \mathbb{D}), \tag{3.9}$$

where $\Phi_{n+1}^m \geq (n+1)(1-\alpha)$ *and*

$$\Phi_k^m \geq \begin{cases} k(1 - \alpha), & \text{if } k = 2, 3, \dots, n, \\ k\left(\dfrac{\Phi_{n+1}^m}{n+1}\right), & \text{if } k = n+1, n+2, \dots. \end{cases} \tag{3.10}$$

The results are sharp with the function given by (3.3).

Proof. We write

$$\frac{1 + w(z)}{1 - w(z)} = \frac{\Phi_{n+1}^m}{(n+1)(1-\alpha)}\left[\frac{f'(z)}{f_n'(z)} - \left(\frac{\Phi_{n+1}^m - (n+1)(1-\alpha)}{\Phi_{n+1}^m}\right)\right],$$

where

$$w(z) = \frac{\left(\frac{\Phi_{n+1}^m}{(n+1)(1-\alpha)}\right)\sum_{k=n+1}^{\infty} k a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} k a_k z^{k-1} + \left(\frac{\Phi_{n+1}^m}{(n+1)(1-\alpha)}\right)\sum_{k=n+1}^{\infty} k a_k z^{k-1}}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{k=2}^{n} k|a_k| + \frac{\Phi_{n+1}^m}{(n+1)(1-\alpha)} \sum_{k=n+1}^{\infty} k|a_k| \leq 1.$$

From the condition (2.1), it is sufficient to show that

$$\sum_{k=2}^{n} k|a_k| + \frac{\Phi_{n+1}^m}{(n+1)(1-\alpha)} \sum_{k=n+1}^{\infty} k|a_k| \leq \sum_{k=2}^{\infty} \frac{\Phi_k^m}{1-\gamma}|a_k|,$$

which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\Phi_k^m - (1-\alpha)k}{1-\alpha}\right)|a_k| + \sum_{k=n+1}^{\infty} \frac{(n+1)\Phi_k^m - k\Phi_{n+1}^m}{(n+1)(1-\alpha)}|a_k| \geq 0.$$

To prove the result (3.9), we define the function $w(z)$ by

$$\frac{1 + w(z)}{1 - w(z)} = \frac{(n+1)(1-\alpha) + \Phi_{n+1}^m}{(1-\alpha)(n+1)}\left[\frac{f_n'(z)}{f'(z)} - \frac{\Phi_{n+1}^m}{(n+1)(1-\alpha) + \Phi_{n+1}^m}\right]$$

and by similar arguments in the first part we get desired result. $\qquad\square$

4 Inclusion relations involving $N_\delta(e)$

In this section following [16–18], we define the n, δ neighborhood of function $f(z) \in \mathcal{T}$ and discuss the inclusion relations involving $N_\delta(e)$.

$$N_\delta(f) = \left\{ g \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad \text{and} \quad \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \right\}. \tag{4.1}$$

Particularly for the identity function $e(z) = z$, we have

$$N_\delta(e) = \left\{ g \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad \text{and} \quad \sum_{n=2}^{\infty} n|b_n| \leq \delta \right\}. \tag{4.2}$$

Theorem 4.1. *Let*

$$\delta = \frac{1 - \alpha}{[2]_{p,q}^m ([2]_{p,q}(1 + \beta) - \mu(\alpha + \beta))}. \tag{4.3}$$

Then $\mathcal{TJ}_{p,q}^m(\mu, \alpha, \beta) \subset N_\delta(e)$.

Proof. For $f \in \mathcal{TJ}_{p,q}^m(\mu, \alpha, \beta)$, Theorem 2.1, yields

$$[2]_{p,q}^m ([2]_{p,q}(1 + \beta) - \mu(\alpha + \beta)) \sum_{n=2}^{\infty} a_n \leq 1 - \alpha,$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1 - \alpha}{[2]_{p,q}^m ([2]_{p,q}(1 + \beta) - \mu(\alpha + \beta))}. \tag{4.4}$$

On the other hand, from (2.1) and (4.4) we have

$$\begin{aligned}
[2]_{p,q}^m(1 + \beta) \sum_{n=2}^{\infty} [n]_{p,q} a_n &\leq 1 - \alpha + [2]_{p,q}^m \mu(\alpha + \beta) \sum_{n=2}^{\infty} a_n \\
&\leq 1 - \alpha + \frac{[2]_{p,q}^m \mu(\alpha + \beta)(1 - \alpha)}{[2]_{p,q}^m ([2]_{p,q}(1 + \beta) - \mu(\alpha + \beta))} \\
&\leq \frac{[2]_{p,q}^m(1 + \beta)(1 - \alpha)}{[2]_{p,q}(1 + \beta) - \mu(\alpha + \beta)} \\
\sum_{n=2}^{\infty} [n]_{p,q} a_n &\leq \frac{1 - \alpha}{[2]_{p,q}(1 + \beta) - \mu(\alpha + \beta)}. \quad \square
\end{aligned} \tag{4.5}$$

Now we determine the neighborhood for each of the class $\mathcal{TJ}_{p,q}^m(\mu, \alpha, \beta)$, which we define as follows.

A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{TJ}_{p,q}^m(\mu, \alpha, \beta)$ if there exists a function $g \in \mathcal{TJ}^m(\mu, \alpha, \beta, \eta)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \eta, \quad (z \in \mathbb{D}, \ 0 \leq \eta < 1). \tag{4.6}$$

Theorem 4.2. *If $g \in \mathcal{TJ}_{p,q}^m(\mu, \alpha, \beta)$ and*

$$\eta = 1 - \frac{\delta[2]_{p,q}^m ([2]_{p,q}(1 + \beta) - \mu(\alpha + \beta))}{2[2]_{p,q}^m ([2]_{p,q}(1 + \beta) - \mu(\alpha + \beta))}. \tag{4.7}$$

Then $N_\delta(g) \subset \mathcal{TJ}_{p,q}^m(\mu, \alpha, \beta, \eta)$.

Proof. Suppose that $f \in N_\delta(g)$, then we find from 4.1 that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \le \delta,$$

which implies that the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \le \frac{\delta}{2}.$$

Next, since $g \in \mathcal{TJ}_{p,q}^m(\mu, \alpha, \beta)$, we have

$$\sum_{n=2}^{\infty} b_n \le \frac{1 - \alpha}{[2]_{p,q}^m([2]_{p,q}(1 + \beta) - \mu(\alpha + \beta))}.$$

So that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} b_n} \le \frac{\delta}{2} \times \frac{[2]_{p,q}^m([2]_{p,q}(1 + \beta) - \mu(\alpha + \beta))}{[2]_{p,q}^m([2]_{p,q}(1 + \beta) - \mu(\alpha + \beta)) - (1 - \alpha)} \le 1 - \eta,$$

provided that η is given precisely by (4.7). Thus by definition, $f \in \mathcal{TJ}_{p,q}^m(\mu, \alpha, \beta, \eta)$ for η given by (4.7), which completes the proof. □

5 Concluding remarks and observations

As a special case of the aforementioned theorems, we can determine new sharp lower bounds for $\Re\left(\frac{f(z)}{f_n(z)}\right)$, $\Re\left(\frac{f_n(z)}{f(z)}\right)$, $\Re\left(\frac{f'(z)}{f_n'(z)}\right)$ and $\Re\left(\frac{f_n'(z)}{f'(z)}\right)$ for various function classes stated in Remarks 1.1 and 1.2 and upon specializing the values of μ and β one can deduce various new subclasses on p, q-difference operator and prove the above partial sums and neighborhood results.

Author contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of interest: Authors state no conflict of interest.

References

[1]   H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

[2]   F. H. Jackson, *On q-functions and a certain difference operator*, Trans. R. Soc. Edinburgh **46** (1908), 253–281.

[3]   A. Aral, V. Gupta, and R. P. Agarwal, *Applications of* q-*Calculus in Operator Theory*, Springer, New York, 2013.

[4] S. Kanas and D. Răducanu, *Some class of analytic functions related to conic domains*, Math. Slovaca **64** (2014), no. 5, 1183–1196.

[5] S. D. Purohit and R. K. Raina, *Fractional q-calculus and certain subclasses of univalent analytic functions*, Mathematica **55(78)** (2013), no. 1, 62–74.

[6] S. Araci, U. Duran, M. Acikgoz, and H. M. Srivastava, *A certain (p, q)-derivative operator and associated divided differences*, J. Inequal. Appl. **2016** (2016), art. 301, DOI: https://doi.org/10.1186/s13660-016-1240-8.

[7] H. M. Srivastava, *Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inf. Sci. **5** (2011), no. 3, 390–444.

[8] M. Govindaraj and S. Sivasubramanian, *On a class of analytic functions related to conic domains involving q-calculus*, Anal. Math. **43** (2017), no. 3, 475–487.

[9] G. Sălăgean, *Subclasses of univalent functions*, in: C. A. Cazacu, N. Boboc, M. Jurchescu, I. Suciu (eds.), Complex Analysis - Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Mathematics, vol. 1013, Springer, Berlin, 1981, pp. 362–372.

[10] H. Silverman, *Partial sums of starlike and convex functions*, J. Math. Anal. Appl. **209** (1997), no. 1, 221–227.

[11] B. A. Frasin, *Partial sums of certain analytic and univalent functions*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **21** (2005), no. 2, 135–145.

[12] B. A. Frasin and G. Murugusundaramoothy, *Partial sums of certain analytic functions*, Mathematica **53(76)** (2011), no. 2, 131–142.

[13] T. Rosy, K. G. Subramanian, and G. Murugusundaramoorthy, *Neighbourhoods and partial sums of starlike functions based on Ruscheweyh derivatives*, JIPAM. J. Inequal. Pure Appl. Math. **4** (2003), no. 4, art. 64.

[14] T. Sheil-Small, *A note on the partial sums of convex schlicht functions*, Bull. London Math. Soc. **2** (1970), 165–168.

[15] E. M. Silvia, *On partial sums of convex functions of order α*, Houston J. Math. **11** (1985), no. 3, 397–404.

[16] A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. **8** (1957), 598–601.

[17] G. Murugusundaramoorthy and H. M. Srivastava, *Neighborhoods of certain classes of analytic functions of complex order*, JIPAM J. Inequal. Pure Appl. Math. **5** (2004), no. 2, art. 24.

[18] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **81** (1981), no. 4, 521–527.