



Research Article

Qinghua Zhang* and Zhizhong Tan

Solvability of the abstract evolution equations in L^s -spaces with critical temporal weights

<https://doi.org/10.1515/math-2021-0027>

received June 27, 2020; accepted January 21, 2021

Abstract: This paper deals with the abstract evolution equations in L^s -spaces with critical temporal weights. First, embedding and interpolation properties of the critical L^s -spaces with different exponents s are investigated, then solvability of the linear evolution equation, attached to which the inhomogeneous term f and its average Φf both lie in an $L_{1/s}^s$ -space, is established. Based on these results, Cauchy problem of the semi-linear evolution equation is treated, where the nonlinear operator $F(t, u)$ has a growth number $\rho \geq s + 1$, and its asymptotic behavior acts like $\alpha(t)/t$ as $t \rightarrow 0$ for some bounded function $\alpha(t)$ like $(-\log t)^{-p}$ with $2 \leq p < \infty$.

Keywords: critical L^s -space, sectorial operator, maximal regularity, evolution equation, solvability

MSC 2020: 34K30, 47D06

1 Introduction

This paper deals with the abstract evolution equations in L^s -spaces with critical temporal weights or equivalently critical L^s -spaces. Given a Banach space X and a number $0 < T < \infty$, let $L^0(0, T; X)$ be the collection of strongly measurable X -valued functions. Given two indices $0 < \mu \leq 1$ and $1 < s < \infty$, the weighted abstract-valued L^s -space and $W^{k,s}$ -space are defined as follows:

$$L_\mu^s(0, T; X) = \{f \in L^0(0, T; X) : t^{1-\mu}\|f(t)\|_X \in L^s(0, T)\}$$

with the norm

$$\|f\|_{L_\mu^s(0, T; X)} = \left(\int_0^T (t^{1-\mu}\|f(t)\|_X)^s dt \right)^{1/s},$$

and

$$W_\mu^{k,s}(0, T; X) = \{f \in W_{loc}^{1,1}(0, T; X) : f^{(j)}(t) \in L_\mu^s(0, T; X), j = 0, 1, \dots, k\}$$

with the norm

$$\|f\|_{W_\mu^{k,s}(0, T; X)} = \sum_{j=0}^k \|f^{(j)}\|_{L_\mu^s(0, T; X)},$$

* **Corresponding author: Qinghua Zhang**, School of Sciences, Nantong University, Nantong 226019, Jiangsu Province, P. R. China, e-mail: zhangqh1971@126.com, tel: +81-0513-55003312

Zhizhong Tan: School of Sciences, Nantong University, Nantong 226019, Jiangsu Province, P. R. China

where $f^{(j)}$ denotes the j th derivative of f in the sense of distribution. Evidently, if $\mu = 1$ in the aforementioned definitions, then we obtain the classical Lebesgue-Bochner space $L^s(0, T; X)$ and Sobolev-Bochner space $W^{k,s}(0, T; X)$, respectively.

By means of Hölder’s inequality, Prüss-Simonett [1] showed that under the restriction $\mu > 1/s$, $L^s_\mu(0, T; X)$ can be embedded into $L^1(0, T; X)$, hence the trace of $W^{1,s}_\mu(0, T; X)$ makes sense. Moreover, the average operator

$$\Phi f(t) := \frac{1}{t} \int_0^t \|f(\tau)\|_X d\tau$$

is bounded from $L^s_\mu(0, T; X)$ to $L^s_\mu(0, T)$. More precisely,

$$\|\Phi f\|_{L^s_\mu(0, T)} \leq \frac{1}{\mu - 1/s} \|f\|_{L^s_\mu(0, T; X)}. \tag{1.1}$$

This is a direct corollary of Hardy’s inequality (cf. [1–3]).

Note that under the condition $1/s < \mu \leq 1$, $t^{(1-\mu)s}$ is a one-dimensional \mathcal{A}_s -weight of Muckenhoupt class, so $L^s_\mu(0, T; X)$ can be used to investigate the maximal regularity of the abstract evolution equations. Suppose that A is a closed linear operator defined in X with the dense domain $D(A)$, and $-A$ generates an analytic C_0 -semigroup e^{-tA} . Under this assumption, A is called a sectorial operator (refer to [4, §2.5] or [5, p. 130]).

Consider the linear evolution equation:

$$u' + Au = f(t), \quad t > 0. \tag{1.2}$$

We say (1.2) has the maximal L^s -regularity on the interval $[0, T]$, or symbolically $A \in MR_s(0, T; X)$, if for every $f \in L^s(0, T; X)$, equation (1.2) has a unique strong solution $u \in W^{1,s}(0, T; X)$ such that $u(0) = 0$, $Au \in L^s(0, T; X)$, and (1.2) is satisfied for a.e. $t \in (0, T]$. Evidently, this strong solution u has the integral expression:

$$u(t) = \int_0^t e^{-(t-\tau)A} f(\tau) d\tau. \tag{1.3}$$

Moreover, by the closed graph theorem, there is a constant $C(s, T) > 0$ such that

$$\|u\|_{W^{1,s}(0, T; X)} + \|Au\|_{L^s(0, T; X)} \leq C(s, T) \|f\|_{L^s(0, T; X)}. \tag{1.4}$$

From [6,7], we know that maximal L^s -regularity is an essential property of A , it does not depend on the choice of s and T . In other words, if $A \in MR_{s'}(0, T'; X)$ for some $1 < s' < \infty$ and $0 < T' < \infty$, then $A \in MR_s(0, T; X)$ for all $1 < s < \infty$ and all $0 < T < \infty$. Moreover, the constant $C(s, T)$ in (1.4) is uniform on $[s_0, s_1] \times (0, T_0]$ for $1 < s_0 < s_1 < \infty$ and $0 < T_0 < \infty$. For this reason, throughout this paper, we will omit the indices s, T , only use C to denote the regularity constant in (1.4), without specifying the bounds of s and T . For the sake of convenience, we also assume that semigroup e^{-tA} is uniformly bounded, i.e., there is a constant $M_0 \geq 1$ such that

$$\max\{\|e^{-tA}\|, t\|Ae^{-tA}\|\} \leq M_0, \quad \text{for all } t > 0. \tag{1.5}$$

Equation (1.2) is said to have the maximal L^s_μ -regularity on $[0, T]$, if for every $f \in L^s_\mu(0, T; X)$, (1.2) has a unique strong solution $u \in W^{1,s}_\mu(0, T; X)$ satisfying $u(0) = 0$ and $Au \in L^s_\mu(0, T; X)$. With the aid of the boundedness of Φ on $L^s_\mu(0, T; X)$, [1] proved the equivalence of the maximal regularity of (1.2) on $L^s(0, T; X)$ and on $L^s_\mu(0, T; X)$. This was also shown in [8] as a special case in the framework of singular operator theory.

There is a natural question arising up, that is whether or not the maximal regularity of (1.2) keeps in the critical case $\mu = 1/s$. To our best knowledge, there is not an affirmative answer to this question. In the method used in [1], average operator and Hardy’s inequality are employed, where the constant $(\mu - 1/s)^{-1}$ appears definitely. Since $(\mu - 1/s)^{-1}$ blows up as μ tends to $1/s$, this method fails here. Moreover, since t^{s-1} is not a Muckenhoupt A_s weight anymore, boundedness of the singular integral operator employed in [8]

disappears in the space $L_{1/s}^s(0, T; X)$. For these reasons, the question posed above seems much interesting, and it deserves full attention.

This paper makes a short investigation on the solvability of (1.2) in critical L^s -spaces. Using decomposition of the integral solution (1.3), together with the weak boundedness of one-dimensional fractional integral operator on $L^1(0, T; X)$, we will show that if the function f belongs to $L^1(0, T; X) \cap L_{1/s}^s(0, T; X)$, then equation (1.2) has a strong solution in $W_{\text{loc}}^{1,1}((0, T]; X)$ with $Au, u' \in L_{1/s, w}^s(0, T; X)$. Furthermore, under the additional assumption $\Phi f \in L_{1/s}^s(0, T; X)$, then we also have $Au, u' \in L_{1/s}^s(0, T; X)$, and $u \in L_*^s(0, T; X)$, an L^s -space with the critical weight t^{-1} . This gives a partial answer to the above question.

In the second part of the paper, solvability of (1.2) in critical L^s -spaces, and estimates (1.5) are applied to the Cauchy problem of the semi-linear evolution equation:

$$u' + Au = F(t, u), \quad t > 0, \quad (1.6)$$

where $f: [0, T] \times X \rightarrow X$ is a Caratheodory map, locally Lipschitzian about u , and subject to a growth of the exponent -1 w.r.t. t , i.e.,

$$\|F(t, u) - F(t, v)\| \leq \frac{C}{t} \|u - v\| (\alpha(t) + \|u\|^{\rho-1} + \|v\|^{\rho-1}),$$

and

$$\|F(t, 0)\| \leq C \frac{\alpha(t)}{t}$$

for some positive function $\alpha(t)$ satisfying $\alpha(t)/t \in L^1(0, T)$ and $\Phi(\alpha(\cdot)/\cdot) \in L_{1/s}^s(0, T)$. Using Banach's contraction principle, with the aid of the function space $L_*^{\rho, s}(0, T; X)$ (see Section 3), existence of the strong solution of equation (1.6) with $u(0) = 0$ is established. Since $\alpha(t)$ can be selected such that $\alpha(t)/t$ belongs to not $L^q(0, T)$ for any $q > 1$, but $L^1(0, T)$, results obtained here are useful supplements to the literature in dealing with nonlinear evolution equations despite that the initial value in our model is only 0.

For other investigations on the maximal L^s -regularity of the evolution equations with different focuses, please refer to [9–12] and [13, §3.4] etc.

2 Properties of critical L^s -spaces and maximal $L_{1/s}^s$ -regularity of evolution equations

Let $0 < T < \infty$ and $1 < s < \infty$. Define the critical L^s -space

$$L_{1/s}^s(0, T; X) = \{f \in L^0(0, T; X) : t^{1-1/s}f \in L^s(0, T; X)\}.$$

Evidently, endowed with the norm

$$\|f\|_{L_{1/s}^s(0, T; X)} := \|t^{1-1/s}f\|_{L^s(0, T; X)},$$

$L_{1/s}^s$ becomes a Banach space. Furthermore, in the case $s = 1$, $L_{1/s}^s(0, T; X)$ is exactly $L^1(0, T; X)$. As for $s = \infty$, we define

$$L_{1/\infty}^\infty(0, T; X) = \{f \in L^0(0, T; X) : tf \in L^\infty(0, T; X)\}$$

with the norm

$$\|f\|_{L_{1/\infty}^\infty(0, T; X)} := \|tf\|_{L^\infty(0, T; X)}.$$

Unlike the subcritical case $1/s < \mu \leq 1$, here $L_{1/s}^s(0, T; X)$ is not contained in $L^1(0, T; X)$ for $s > 1$ anymore. For example, the function $f(t) = (t \log t)^{-1}x$ ($x \in X$) belongs to $L_{1/s}^s(0, T; X)$, but it does not belong to $L^1(0, T; X)$. On the other hand, direct calculation shows that

$$L^1(0, T; X) \cap L^\infty(0, T; X) \hookrightarrow L^s_{1/s}(0, T; X) \tag{2.1}$$

for all $1 < s < \infty$, and

$$\|f\|_{L^s_{1/s}(0, T; X)} \leq \|f\|_{L^1(0, T; X)}^{1/s} \|f\|_{L^\infty(0, T; X)}^{1-1/s} \tag{2.2}$$

for all $f \in L^1(0, T; X) \cap L^\infty(0, T; X)$. Furthermore, suppose that $1 \leq s_1 < s < s_2 \leq \infty$, and

$$\frac{1}{s} = \frac{1-\theta}{s_1} + \frac{\theta}{s_2}$$

for some $0 < \theta < 1$, then we have

$$\|f\|_{L^s_{1/s}(0, T; X)} \leq \|f\|_{L^{s_1}_{1/s_1}(0, T; X)}^{1-\theta} \|f\|_{L^{s_2}_{1/s_2}(0, T; X)}^\theta.$$

This shows the interpolation property of the critical L^s -spaces. As a matter of fact, if we regard $L^s_{1/s}(0, T; X)$ as the space containing all the strongly measurable functions f satisfying

$$\int_0^T \|tf(t)\|_X^s \frac{dt}{t} < \infty,$$

it is not hard to show that (cf. [14, §7.24] or [15, §5.2]), for every $1 < s < \infty$,

$$L^s_{1/s}(0, T; X) = (L^1(0, T; X), L^\infty(0, T; X))_{1/s, s}$$

in the sense of isomorphism. Here $(X, Y)_{\theta, s}$ stands for the real interpolation space associated with the indices θ and s between the two Banach spaces X and Y .

Next we make a short investigation on the maximal regularity of evolution equation (1.2) in critical L^s -spaces. Suppose that $1 < s < \infty$ and $f \in L^s_{1/s}(0, T; X)$, consider the decomposition of (1.3),

$$u(t) = t^{1/s-1} \int_0^t e^{-(t-\tau)A} \tau^{1-1/s} f(\tau) d\tau + t^{1/s-1} \int_0^t e^{-(t-\tau)A} (t^{1-1/s} - \tau^{1-1/s}) f(\tau) d\tau.$$

Let

$$J_1(t) = t^{1/s-1} \int_0^t A e^{-(t-\tau)A} \tau^{1-1/s} f(\tau) d\tau,$$

$$J_2(t) = t^{1/s-1} \int_0^t A e^{-(t-\tau)A} (t^{1-1/s} - \tau^{1-1/s}) f(\tau) d\tau.$$

Suppose that $A \in \mathcal{MR}(0, T)$, then we have $J_1 \in L^s_{1/s}(0, T; X)$. As for J_2 , by the estimates (1.5), we have

$$\|J_2(t)\|_X \leq M_0 t^{1/s-1} \int_0^t \frac{\|f(\tau)\|_X}{(t-\tau)^{1/s}} d\tau. \tag{2.3}$$

Since $f \in L^1(0, T; X)$, by the mapping property of the fractional integral operator (cf. [16, §6.2]), we can conclude that $t^{1-1/s} J_2 \in L^s_w(0, T; X)$, or $J_2 \in L^s_{1/s, w}(0, T; X)$ in symbol. Thus, $J_1 + J_2 \in L^s_{1/s, w}(0, T; X)$, which implies that $u(t) \in D(A)$ a.e. on $(0, T]$, and $Au \in L^s_{1/s, w}(0, T; X)$. Consequently, $u'(t)$ exists a.e. on $(0, T]$ and $u' \in L^s_{1/s, w}(0, T; X)$. Summing up, we have

Proposition 2.1. *Under the hypothesis $A \in \mathcal{MR}(0, T)$, for all $f \in L^1(0, T; X) \cap L^s_{1/s}(0, T; X)$, integral (1.3) is the strong solution of equation (1.2) associated with the initial condition $u(0) = 0$, and satisfies $u \in C([0, T], X) \cap W^{1,1}_{loc}(0, T; X)$ and $u', Au \in L^s_{1/s, w}(0, T; X)$.*

Remark 2.2. Inequality (2.3) has a different version as follows:

$$\|J_2(t)\|_X \leq \frac{M_0}{t} \int_0^t \|f(\tau)\|_X d\tau,$$

from which, it is easy to see that, in addition to the condition $f \in L^1(0, T; X) \cap L^{1/s}(0, T; X)$, if $\Phi f \in L^{1/s}(0, T)$, then $J_2 \in L^{1/s}(0, T; X)$, consequently $Au, u' \in L^{1/s}(0, T; X)$, and

$$\|u'\|_{L^{1/s}(0, T; X)} + \|Au\|_{L^{1/s}(0, T; X)} \leq (C + M_0)(\|f\|_{L^{1/s}(0, T; X)} + \|\Phi f\|_{L^{1/s}(0, T)}). \tag{2.4}$$

For $1 < s < \infty$, define another type of weighted L^s -space

$$L_*^s(0, T; X) = \{u \in L^0(0, T; X) : t^{-1/s}u \in L^s(0, T; X)\}$$

with the norm

$$\|u\|_{L_*^s(0, T; X)} = \|t^{-1/s}u\|_{L^s(0, T; X)}.$$

Here the temporal weight t^{-1} is also critical, it does not belong to the Muckenhoupt class.

Let us derive some estimates for the solution of (1.6) for further arguments. First, using (1.3) and (1.5), we have

$$\|u(t)\|_X \leq M_0 \int_0^t \|f(\tau)\|_X d\tau = M_0 t \Phi f(t), \quad t > 0.$$

Thus, under the assumption $\Phi f \in L^{1/s}(0, T; X)$, we can conclude that $t^{-1/s}u \in L^s(0, T; X)$, and

$$\|u\|_{L_*^s(0, T; X)} \leq M_0 \|\Phi f\|_{L^{1/s}(0, T)}. \tag{2.5}$$

Moreover, if $\rho \geq s + 1$, then we have

$$\begin{aligned} \int_0^T \frac{1}{t} \left(\int_0^t \frac{\|u(\tau)\|_X^\rho}{\tau} d\tau \right)^s dt &\leq M_0^{\rho s} \left(\int_0^T \|f(t)\|_X dt \right)^{(q-1-s)s} \int_0^T \frac{1}{t} \left(\int_0^t \|f(\tau)\|_X d\tau \right)^s \left(\int_0^t \frac{1}{\tau} \left(\int_0^\tau \|f(\xi)\|_X d\xi \right)^s d\tau \right)^s dt \\ &\leq M_0^{\rho s} \|f\|_{L^1(0, T; X)}^{(q-1-s)s} \int_0^T t^{s-1} (\Phi f(t))^s \left(\int_0^t \tau^{s-1} (\Phi f(\tau))^s d\tau \right)^s dt \\ &\leq M_0^{\rho s} \|f\|_{L^1(0, T; X)}^{(q-1-s)s} \|\Phi f\|_{L^{1/s}(0, T)}^{(1+s)s}. \end{aligned} \tag{2.6}$$

It is easy to check that

$$\varphi_{\rho, s, T}(u) := \int_0^T \frac{1}{t} \left(\int_0^t \frac{\|u(\tau)\|_X^\rho}{\tau} d\tau \right)^s dt$$

defines a convex functional on $L^0(0, T; X)$. This functional also satisfies the following three properties

- $\lambda \mapsto \varphi_{\rho, s, T}(\lambda u)$ is continuous on $[0, \infty)$,
- $\varphi_{\rho, s, T}(\lambda u) = \varphi_{\rho, s, T}(u)$ provided $|\lambda| = 1$, and
- $\varphi_{\rho, s, T}(u) = 0$ if and only if $u = 0$.

In a word, $\varphi_{\rho, s, T}$ is a continuous modular, so it generates a normed space from $L^0(0, T; X)$ (cf. [17]), denoted by $L_*^{\rho, s}(0, T; X)$. One can also verify that the norm generated by $\varphi_{\rho, s, T}$ is exactly $\varphi_{\rho, s, T}^{1/\rho s}$, according to which, $L_*^{\rho, s}(0, T; X)$ is a Banach space embedded into $L^1(0, T; X)$.

According to the definition of $L_t^{\rho,s}(0, T; X)$ and inequality (2.6), we then obtain the second estimate of u ,

$$\|u\|_{L_t^{\rho,s}(0, T; X)} \leq M_0 \|f\|_{L^1(0, T; X)}^{1-(1+s)/\rho} \|\Phi f\|_{L_{1/s}^s(0, T)}^{(1+s)/\rho}. \quad (2.7)$$

3 Solvability of semilinear evolution equations with temporal growth exponent –1

Now, we are ready to deal with the semi-linear evolution equation (1.6). Here A is a sectorial operator having the maximal regularity on $[0, T]$ with $0 < T < \infty$, $F : [0, T] \times X \rightarrow X$ is a Caratheodory operator satisfying $H(F)$:

- (1) For all $u \in X$, $t \mapsto F(t, u)$ is strongly measurable on $[0, T]$,
- (2) for a.e. $t \in [0, T]$, $u \mapsto F(t, u)$ is locally Lipschitz such that

$$\|F(t, u) - F(t, v)\|_X \leq \frac{C}{t} \|u - v\|_X (\alpha(t) + \|u\|_X^{\rho-1} + \|v\|_X^{\rho-1}) \quad (3.1)$$

for some $\rho \geq s + 1$.

- (3) The one-dimensional Lebesgue measure of the set $\{t \in [0, T] : \|F(t, 0)\|_X > 0\}$ is larger than 0, and

$$\|F(t, 0)\|_X \leq C \frac{\alpha(t)}{t} \quad \text{a.e. on } [0, T]. \quad (3.2)$$

Here $\alpha(t) \geq 0$ is a bounded function verifying

$$\int_0^T \frac{\alpha(t)}{t} dt < \infty \quad \text{and} \quad \Phi\left(\frac{\alpha(\cdot)}{\cdot}\right) \in L_{1/s}^s(0, T). \quad (3.3)$$

Remark 3.1. The function $\alpha(t) = |\log t|^{-p}$ with $2 \leq p < \infty$ satisfies hypotheses (3.3) provided $0 < T < 1$, but $\alpha(\cdot)/\cdot \notin L^q(0, T)$ for all $1 < q < \infty$.

Theorem 3.2. *Under the assumptions $A \in \mathcal{MR}(0, T)$ and $H(F)$, there exists a number $0 < r < \infty$ such that if*

$$h(T) := \int_0^T \frac{\alpha(t)}{t} dt + \left(\int_0^T \frac{\alpha(t)}{t} dt \right)^{1/s} \|\alpha\|_{L_{1/s}^{\infty}(0, T)}^{1-1/s} + \left\| \Phi\left(\frac{\alpha(\cdot)}{\cdot}\right) \right\|_{L_{1/s}^s(0, T)} \leq r,$$

then equation (1.6) admits a unique nontrivial strong solution

$$u \in C([0, T]; X) \cap L_*^s(0, T; X) \cap L_*^{\rho,s}(0, T; X)$$

such that $u(0) = 0$, and both u' and Au belong to $L_{1/s}^s(0, T; X)$.

Proof. Introduce two intersection spaces

$$\begin{aligned} \mathbb{X}_T &= C([0, T]; X) \cap L_*^s(0, T; X) \cap L_*^{\rho,s}(0, T; X), \\ \mathbb{Y}_T &= \{f \in L^1(0, T; X) : f \in L_{1/s}^s(0, T; X), \text{ and } \Phi f \in L_{1/s}^s(0, T)\}. \end{aligned}$$

Evidently, endowed with the norms

$$\begin{aligned} \|u\|_{\mathbb{X}_T} &= \|u\|_{C([0, T]; X)} + \|u\|_{L_*^s(0, T; X)} + \|u\|_{L_*^{\rho,s}(0, T; X)}, \\ \|f\|_{\mathbb{Y}_T} &= \|f\|_{L^1(0, T; X)} + \|f\|_{L_{1/s}^s(0, T; X)} + \|\Phi f\|_{L_{1/s}^s(0, T)}, \end{aligned}$$

respectively, \mathbb{X}_T and \mathbb{Y}_T are both Banach spaces.

Take any $u \in \mathbb{X}_T$, we assert that the composite function $Gu = F(\cdot, u(\cdot))$ belongs to \mathbb{Y}_T . As a matter of fact, by (3.1) and (3.2), we have

$$\begin{aligned} \|Gu(t)\|_X &\leq C \frac{\alpha(t)}{t} (1 + \|u(t)\|_X) + \frac{C}{t} \|u(t)\|_X^\rho \\ &\leq C \frac{\alpha(t)}{t} (1 + \|u\|_{C([0,T];X)}) + C \|u\|_{C([0,T];X)}^{\rho-s} \frac{\|u(t)\|_X^s}{t} \\ &\leq \frac{C}{t} \{ \|\alpha\|_{L^\infty(0,T)} (1 + \|u\|_{C([0,T];X)}) + \|u\|_{C([0,T];X)}^\rho \} \end{aligned} \quad (3.4)$$

a.e. on $[0, T]$. From the second line of (3.4), using the assumption $u \in L_*^s(0, T; X)$ and (3.3), we have $Gu \in L^1(0, T; X)$, and

$$\|Gu\|_{L^1(0,T;X)} \leq C \int_0^T \frac{\alpha(t)}{t} dt \cdot (1 + \|u\|_{C([0,T];X)}) + C \|u\|_{C([0,T];X)}^{\rho-s} \|u\|_{L_*^s(0,T;X)}^s.$$

From the third line of (3.4), we also have $Gu \in L_{1/\infty}^\infty(0, T; X)$. Hence by the imbedding (2.1) and inequality (2.2), we get $Gu \in L_{1/s}^s(0, T; X)$ and

$$\|Gu\|_{L_{1/s}^s(0,T;X)} \leq C \left(\int_0^T \frac{\alpha(t)}{t} dt \right)^{1/s} \|\alpha\|_{L_{1/\infty}^\infty(0,T)}^{1-1/s} (1 + \|u\|_{C([0,T];X)}) + C \|u\|_{C([0,T];X)}^{\rho-1} \|u\|_{L_*^s(0,T;X)}.$$

Moreover, using the first line of (3.4), together with (3.3) and the assumption $u \in L_*^{\rho,s}(0, T; X)$, we assert that $\Phi(Gu) \in L_{1/s}^s(0, T)$, and

$$\|\Phi(Gu)\|_{L_{1/s}^s(0,T;X)} \leq C \left\| \Phi \left(\frac{\alpha(\cdot)}{\cdot} \right) \right\|_{L_{1/s}^s(0,T)} (1 + \|u\|_{C([0,T];X)}) + C \|u\|_{L_*^{\rho,s}(0,T;X)}^\rho.$$

Therefore, $\Phi(Gu) \in \mathbb{Y}_T$, and

$$\|Gu\|_{\mathbb{Y}_T} \leq Ch(T) (1 + \|u\|_{\mathbb{X}_T}) + C \|u\|_{\mathbb{X}_T}^\rho. \quad (3.5)$$

Similarly, for any two points $u, v \in \mathbb{X}_T$, by assumption (3.1), we have

$$\|Gu - Gv\|_{L^1(0,T;X)} \leq C \|u - v\|_{C([0,T];X)} \left(\int_0^T \frac{\alpha(t)}{t} dt + \|u\|_{C([0,T];X)}^{\rho-1-s} \|u\|_{L_*^s(0,T;X)}^s + \|v\|_{C([0,T];X)}^{\rho-1-s} \|v\|_{L_*^s(0,T;X)}^s \right), \quad (3.6)$$

and

$$\begin{aligned} &\|Gu - Gv\|_{L_{1/s}^s(0,T;X)} \\ &\leq C \|u - v\|_{C([0,T];X)} \left\{ \left(\int_0^T \frac{\alpha(t)}{t} dt \right)^{1/s} \|\alpha\|_{L_{1/\infty}^\infty(0,T)}^{1-1/s} + \|u\|_{C([0,T];X)}^{\rho-2} \|u\|_{L_*^s(0,T;X)} + \|v\|_{C([0,T];X)}^{\rho-2} \|v\|_{L_*^s(0,T;X)} \right\}. \end{aligned} \quad (3.7)$$

Note that

$$\begin{aligned} \left\{ \int_0^T \frac{1}{t} \left(\int_0^t \frac{\|w(\tau)\|_X \|u(\tau)\|_X^{\rho-1}}{\tau} d\tau \right)^s dt \right\}^{1/s} &\leq \left\{ \int_0^T \frac{1}{t} \left(\int_0^t \frac{\|w(\tau)\|_X^\rho}{\tau} d\tau \right)^{s/\rho} \left(\int_0^t \frac{\|u(\tau)\|_X^\rho}{\tau} d\tau \right)^{s(1-1/\rho)} dt \right\}^{1/s} \\ &\leq \|w\|_{L_*^{\rho,s}(0,T;X)} \|u\|_{L_*^{\rho,s}(0,T;X)}^{\rho-1}, \end{aligned}$$

we obtain

$$\begin{aligned} \|\Phi(Gu - Gv)\|_{L_{1/s}^s(0, T; X)} &\leq C \left\| \Phi\left(\frac{\alpha(\cdot)}{\cdot}\right) \right\|_{L_{1/s}^s(0, T)} \|u - v\|_{C([0, T]; X)} \\ &\quad + \|u - v\|_{L_*^{\rho, s}(0, T; X)} (\|u\|_{L_*^{\rho, s}(0, T; X)}^{\rho-1} + \|v\|_{L_*^{\rho, s}(0, T; X)}^{\rho-1}). \end{aligned} \quad (3.8)$$

Putting the aforementioned three inequalities (3.6), (3.7), and (3.8) together, we have

$$\|Gu - Gv\|_{\mathbb{Y}_T} \leq C \|u - v\|_{\mathbb{X}_T} (h(T) + \|u\|_{\mathbb{X}_T}^{\rho-1} + \|v\|_{\mathbb{X}_T}^{\rho-1}). \quad (3.9)$$

Define the solution operator

$$Su(t) = \int_0^t e^{-(t-\tau)A} F(\tau, u(\tau)) d\tau, \quad u \in \mathbb{X}_T.$$

Estimates (3.5) and (3.9), jointly with (2.5) and (2.6), yield

$$\|Su\|_{\mathbb{X}_T} \leq CM_0 [h(T)(1 + \|u\|_{\mathbb{X}_T}) + \|u\|_{\mathbb{X}_T}^\rho] \quad (3.10)$$

and

$$\|Su - Sv\|_{\mathbb{X}_T} \leq CM_0 \|u - v\|_{\mathbb{X}_T} (h(T) + \|u\|_{\mathbb{X}_T}^{\rho-1} + \|v\|_{\mathbb{X}_T}^{\rho-1}). \quad (3.11)$$

Now take

$$r = \frac{1}{4CM_0(1 + 4CM_0)}.$$

Direct calculation shows that if $h(T) \leq r$, then for all $u, v \in \mathbb{X}_T$ with the norms no more than $1/(1 + 4CM_0)$, we have

$$\|Su\|_{\mathbb{X}_T} \leq \frac{1}{2(1 + 4CM_0)} + \frac{CM_0}{(1 + 4CM_0)^\rho} \leq \frac{1}{1 + 4CM_0}$$

and

$$\|Su - Sv\|_{\mathbb{X}_T} \leq \|u - v\|_{\mathbb{X}_T} \left(\frac{1}{4(1 + 4CM_0)} + \frac{2CM_0}{(1 + 4CM_0)^{\rho-1}} \right) \leq \frac{1}{2} \|u - v\|_{\mathbb{X}_T}.$$

Therefore, S is a contractive map from the ball of \mathbb{X}_T with the center 0 and the radius $1/(1 + 4CM_0)$ to itself. In terms of Banach's contraction principle, S has a unique fixed point u within the ball, which is exactly the integral solution of (1.6) with the initial value $u(0) = 0$. Furthermore, since $u \in \mathbb{X}_T$, we have $\Phi(Gu) \in \mathbb{Y}_T$. Thus, by invoking Remark 2.2, we conclude that u' and Au are both in $L_{1/s}^s(0, T; X)$, and u solves (1.6) in the strong sense. In addition, by $H(F)$ (3), it follows $u \neq 0$. Finally, uniqueness of the strong solution can be derived from (3.11) directly. Thus, the proof has been completed. \square

Remark 3.3. Note that the function $\alpha(t)/t$ can be selected not belonging to any $L^q(0, T)$ with $q > 1$, so the classical function space $L^s(0, T; X_1) \cap W_0^{1, s}(0, T; X)$ could not be applied in dealing with equation (1.6) under present hypotheses. In this sense, results obtained here can be viewed as useful supplements to the literature in dealing with nonlinear evolution equations.

Remark 3.4. Following the same process, we can also establish solvability of (1.6) provided $F(t, u)$ is replaced by $F(t, u) + f(t)$ with $f \in \mathbb{Y}_T$. It says that under hypotheses $H(F)$ and $F(\cdot, 0) + f \neq 0$, if $h(T) + \|f\|_{\mathbb{Y}_T}$ is sufficiently small, then equation (1.6) has a unique nontrivial solution $u = u(t, f) \in \mathbb{X}_T$. Furthermore, from (3.1) and (2.4), we can also derive that

$$\|u_1 - u_2\|_{\mathbb{X}_T} + \|u_1' - u_2'\|_{L_{1/s}^s(0, T; X)} + \|Au_1 - Au_2\|_{L_{1/s}^s(0, T; X)} \leq C \|f_1 - f_2\|_{\mathbb{Y}_T},$$

where $u_i = u(t, f_i)$, $i = 1, 2$. This shows the structural stability of system (1.6).

Remark 3.5. It should be mentioned that if hypothesis (3.1) is revised to be

$$\|F(t, u) - F(t, v)\|_X \leq \frac{\alpha(t)}{t} \|u - v\|_X (1 + \|u\|_X^{\rho-1} + \|v\|_X^{\rho-1}), \quad (3.12)$$

with $\alpha(\cdot) \in L^1(0, T)$, then for every $u_0 \in X$, equation (1.6) has a local strong solution $u \in C([0, T_1]; X) \cap W_{loc}^{1,s}(0, T_1; X)$ for some $0 < T_1 \leq T$ and all $1 < s < \infty$ such that $u(0) = u_0$. Proof of this conclusion is simple. First by means of Banach's contraction principle, we can show existence and uniqueness of the solution of the integral equation:

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}F(\tau, u(\tau))d\tau$$

in the space $C([0, T_1]; X)$ for some $0 < T_1 \leq T$. Then we can verify that u is exactly the strong solution of (1.6) by checking the inclusion $F(\cdot, u(\cdot)) \in L^1(0, T; X) \cap L_{loc}^{\infty}(0, T; X)$.

Acknowledgments: The authors express their appreciation for the referee's careful reading and comments.

Funding information: This work was supported by the National Natural Science Foundation of China (11771223).

Conflicts of interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

- [1] Jan Prüss and Gieri Simonett, *Maximal regularity for evolution equations in weighted L^p -spaces*, Arch. Math. **82** (2004), 415–431, DOI: <https://doi.org/10.1007/s00013-004-0585-2>.
- [2] Martin Meyries, *Maximal regularity in weighted spaces, nonlinear boundary conditions, and global attractors*, Doctoral Thesis, Karlsruhe Institute of Technology, 2010.
- [3] Martin Meyries and Roland Schnaubelt, *Interpolation, embeddings and traces of anisotropic fractional Sobolev spaces with temporal weights*, J. Func. Anal. **262** (2012), no. 3, 1200–1229, DOI: <https://doi.org/10.1016/j.jfa.2011.11.001>.
- [4] Amnon Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
- [5] Alessandra Lunardi, *Interpolation Theory*, Edizioni Della Normale, Pisa, 2009.
- [6] Giovanni Dore, *L^p regularity for abstract differential equations*, in: H. Komatsu (ed.), *Functional Analysis and Related Topics*, Lecture Notes in Mathematics, vol. 1540, Springer, Berlin, Heidelberg, 1991, pp. 25–38, DOI: <https://doi.org/10.1007/BFb0085472>.
- [7] Jan Prüss, *Maximal regularity for evolution equations in L^p -spaces*, Conf. Semin. Mat. Univ. Bari **285** (2002), 1–39.
- [8] Ralph Chill and Alberto Fiorenza, *Singular integral operators with operator-valued kernels, and extrapolation of maximal regularity into arrangement invariant Banach function*, J. Evol. Equ. **14** (2014), 795–828, DOI: <https://doi.org/10.1007/s00028-014-0239-1>.
- [9] Giovanni Dore and Alberto Venni, *On the closedness of the sum of two closed operators*, Math. Z. **196** (1987), 189–201, DOI: <https://doi.org/10.1007/BF01163654>.
- [10] Matthias Köhne, Jan Prüss, and Mathias Wilke, *On quasilinear parabolic evolution equations in weighted L^p -spaces*, J. Evol. Equ. **10** (2010), 443–463, DOI: <https://doi.org/10.1007/s00028-010-0056-0>.
- [11] Jeremy LeCrone, Jan Prüss, and Mathias Wilke, *On quasilinear parabolic evolution equations in weighted L^p -spaces II*, J. Evol. Equ. **14** (2014), 509–533, DOI: <https://doi.org/10.1007/s00028-014-0226-6>.
- [12] Lutz Weis, *Operator-valued Fourier multiplier theorems and maximal L^p regularity*, Math. Ann. **319** (2001), no. 4, 735–758, DOI: <https://doi.org/10.1007/PL00004457>.
- [13] Herbert Amann, *Linear and Quasilinear Parabolic Problems, Volume I: Abstract Linear Theory*, Birkhäuser, Basel, 1995.
- [14] Robert A. Adams and John J. F. Fournier, *Sobolev Spaces*, 2nd ed., Pure and Applied Mathematics, vol. 140, Academic Press, New York-London, 2003.

- [15] Jöran Bergh and Jörgen Löfström, *Interpolation spaces: An introduction*, Grundlehren der Mathematischen Wissenschaften, no. 223, Springer-Verlag, Berlin-New York, 1976.
- [16] Alberto Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, London, 1986.
- [17] Lars Diening, Petteri Harjulehto, Peter Hästö, and Michael Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Berlin, 2011.