Research Article

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Existence and asymptotical behavior of solutions for a quasilinear Choquard equation with singularity

Abstract: In this study, we consider the following quasilinear Choquard equation with singularity

$$\begin{cases} -\Delta u + V(x)u - u\Delta u^2 + \lambda(I_\alpha * |u|^p)|u|^{p-2}u = K(x)u^{-\gamma}, & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N, \end{cases}$$

where I_α is a Riesz potential, $0 < \alpha < N$, and $\dfrac{N+\alpha}{N} < p < \dfrac{N+\alpha}{N-2}$, with $\lambda > 0$. Under suitable assumption on V and K, we research the existence of positive solutions of the equations. Furthermore, we obtain the asymptotic behavior of solutions as $\lambda \to 0$.

Keywords: quasilinear Schrödinger equation, singularity, Choquard type, variational methods

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1 Introduction and main results

In this study, we investigate the following quasilinear Choquard equation with singularity

$$\begin{cases} -\Delta u + V(x)u - u\Delta u^2 + \lambda(I_\alpha * |u|^p)|u|^{p-2}u = K(x)u^{-\gamma}, & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where I_α is a Riesz potential, $0 < \alpha < N$, and $\dfrac{N+\alpha}{N} < p < \dfrac{N+\alpha}{N-2}$, with $\lambda > 0$. Quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + V(x)z - f(|z|^2)z - \Delta h(|z|^2)h'(|z|^2)z \tag{1.2}$$

have been derived as models of several physical phenomena. Here, $V = V(x)$, $x \in \mathbb{R}^N$, is a given potential, and K, V are real functions. For instance, in the case $h(s) = s$, we obtain

$$i\partial_t z = -\Delta z + V(x)z - f(|z|^2)z - (\Delta|z|^2)z, \tag{1.3}$$

which has been called the superfluid film equation in plasma physics by Kurihara in [1] (cf. [2,3]). In the case $h(s) = (1 + s)^{\frac{1}{2}}$, equation (1.2) models the self-channeling of a high-power ultra-short laser in mater, see [4,5] and the references in [6]. Equation (1.2) also appears in plasma physics and fluid mechanics [7,8],

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in the theory of Heisenberg ferromagnets and magnons [9], in dissipative quantum mechanics, and in condensed matter theory [10,11]. For more details, we refer the readers to [12,13] and the references therein.

In recent years, the study on the quasilinear Schrödinger equation (1.2) is always a topic of great interest. Mathematicians have established several methods to treat equation (1.3), for example, the dual approach, the perturbation method, and the Nehari method, see for instance [14–29], and the references therein. However, the system (1.1) with Choquard type nonlinearity has only been studied in [30,31].

It is remarkable that there are few papers investigating quasilinear equation with singularity. To the best of our knowledge, it only appears in [32], J. Marcos do Ó and A. Moameni established the singular quasilinear Schrödinger equation

$$-\Delta u - \frac{1}{2}\Delta(u^2)u = \lambda u^3 - u - u^{-\alpha}, \quad u > 0, \quad x \in \Omega,$$

where Ω is a ball in $\mathbb{R}^N$ ($N \geq 2$) centered at the origin, $0 < \alpha < 1$. Furthermore, they obtained the existence of radially symmetric positive solutions by taking advantage of Nehari manifold and some techniques about implicit function theorem when λ belongs to a certain neighborhood of the first eigenvalue λ_1 of the eigenvalue problem

$$-\Delta u - \frac{1}{2}\Delta(u^2)u = \lambda u^3$$

In [33], they studied the following Choquard-type quasilinear Schrödinger equation:

$$-\Delta u + V(x)u - \Delta(u^2)u = (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N$$

where $N \geq 3$, $0 < \alpha < N$, $\dfrac{2(N+\alpha)}{N} < p < \dfrac{2(N+\alpha)}{N-2}$, $V : \mathbb{R}^N \to \mathbb{R}$ is radial potential, and I_α is a Riesz potential. They consider the existence of ground state solutions.

To the best of our knowledge, there seems to be little progress on the existence of a positive solution for quasilinear Choquard equation with singularity. By the motivation of the above work, in our study, we establish the existence of a positive solution for problem (1.1) with singularity. First, the nonlinearity of problem (1.1) is nonlocal, and it is much more difficult to obtain the existence of positive solutions. Second, we investigate the relationships between quasilinear Choquard equation involving and without convolution, which makes our studies more interesting. At last, we obtain the asymptotic behavior of solutions as $\lambda \to 0$.

Before stating our main result, we suppose that the functions $V(x)$ and $K(x)$ satisfy the following assumptions:

(V_1)　$V \in C(\mathbb{R}^N)$ satisfies $\inf_{x \in \mathbb{R}^N} V(x) > V_0 > 0$, where V_0 is a constant.

(V_2)　meas$\{x \in \mathbb{R}^N : -\infty < V(x) \leq \mu\} < +\infty$ for all $\mu \in \mathbb{R}$.

(K_1)　$K \in L^{\frac{22^*}{22^*-1+r}}(\mathbb{R}^N)$ is a nonnegative function.

Now, we state our main results as follows.

Theorem 1.1. *Suppose that* $\gamma \in (0, 1)$, $0 < \alpha < N$, $\dfrac{N+\alpha}{N} < p < \dfrac{N+\alpha}{N-2}$ *and* (V_1), (V_2), (K_1) *hold, then equation* (1.1) *admits a unique solution in* E.

Theorem 1.2. *Suppose that* $\gamma \in (0, 1)$, $0 < \alpha < N$, $\dfrac{N+\alpha}{N} < p < \dfrac{N+\alpha}{N-2}$　(V_1), (V_2), (K_1) *are satisfied for any sequence* $\{\lambda_n\} > 0$ *with* $\lambda_n \to 0$ *as* $n \to \infty$. w_{λ_n} *are the corresponding solutions of problem* (1.1) *obtained in Theorem 1.1 with* $\lambda = \lambda_n$, *then* $w_{\lambda_n} \to w_0$ *in* E *where* w_0 *is the unique positive solution to problem*

$$-\Delta u + V(x)u - \Delta(u^2)u = K(x)u^{-\gamma}.$$

Notation. In this study, we make use of the following notations: C will denote various positive constants; the strong (resp. weak) convergence is denoted by $\to$ (resp. $\rightharpoonup$); $o(1)$ denotes $o(1) \to 0$ as $n \to \infty$, and $B_\rho(0)$ denotes a ball centered at the origin with radius $\rho > 0$.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented, and in Section 3, we give the proof of our main results.

2 Variational setting and preliminaries

To prove our conclusion, we give some basic notations and preliminaries. First, we can rewrite (1.1) as

$$-\Delta u + V(x)u - \Delta(u^2)u + \lambda I_\alpha * |u|^p|u|^{p-2}u - K(x)|u|^{-\gamma} = 0 \quad \text{in} \quad \mathbb{R}^N.$$

It may also be noted that we can not apply directly the variational method to study (1.1), since the natural associated functional I given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1 + 2u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2 dx + \frac{\lambda}{2p}\int_{\mathbb{R}^N}I_\alpha * |u|^p|u|^{p-1}u dx - \frac{1}{1-\gamma}\int_{\mathbb{R}^N}K(x)u^{1-\gamma}dx$$

is not well defined in general. We make the changing of variables $w = f^{-1}(u)$, where f is defined by: $f'(t) = \dfrac{1}{\sqrt{1 + 2f^2(t)}}$ on $[0, \infty)$ and $f(t) = f(-t)$ on $(-\infty, 0]$.

If we make the change of variable $u = f(w)$, we may rewrite equation $I(u)$ in the form

$$\begin{aligned}
\mathcal{J}_\lambda(w) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + V(x)f^2(w))dx + \frac{\lambda}{2p}\int_{\mathbb{R}^N}I_\alpha * |f(w)|^p|f(w)|^{p-1}f(w)dx \\
&\quad - \frac{1}{1-\gamma}\int_{\mathbb{R}^N}K(x)|f(w)|^{1-\gamma}dx.
\end{aligned} \tag{2.1}$$

It can be easily proved that the functional $\mathcal{J}_\lambda(w)$ is of class C^1 (see [34]) in E. Moreover, the critical points of $\mathcal{J}_\lambda$ are weak solutions of the equation

$$\int_{\mathbb{R}^N}|\nabla w\nabla\varphi| + V(x)f(w)f'(w)\varphi + \lambda\int_{\mathbb{R}^N}(I_\alpha * |f(w)|^p|f(w)|^{p-1}f'(w)\varphi - \int_{\mathbb{R}^N}K(x)f^{-\gamma}(w)f'(w)\varphi = 0$$

Be aware that if w is a critical point of $\mathcal{J}_\lambda$, then, $u = f(w)$ is a weak solution of the problem (1.1). For any $w \in E$, let

$$E = \left\{w \in H^1(\mathbb{R}^N)| \int_{\mathbb{R}^N}V(x)f^2(w)dx < \infty\right\}$$

be a Hilbert space endowed with inner product and norm

$$\|w\|_E := \sqrt{\langle w, w\rangle} = \left(\int_{\mathbb{R}^N}(|\nabla w|^2 + V(x)w^2)dx\right)^{\frac{1}{2}}.$$

We denote by $\|\bullet\|_p$ the usual L^p-norm in the sequel for convenience, where $1 \le p \le +\infty$. In this step, we see that (1.1) is variational and its weak solutions are the critical points of the functional given by

$$\mathcal{J}_\lambda(w) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla w|^2 + V(x)f^2(w) + \frac{\lambda}{2p}\int_{\mathbb{R}^N}I_\alpha * |f(w)|^p|f(w)|^{p-1}f(w)dx - \frac{1}{(1-\gamma)}\int_{\mathbb{R}^N}K(x)|f(w)|^{1-\gamma}dx.$$

Lemma 2.1. (*See* [35]) *The function f satisfies the following properties:*
 (A1) *f is uniquely defined C^∞ function and invertible*;
 (A2) $|f'(s)| \le 1$ *and* $|f(s)| \le |s|$ *for all* $s \in \mathbb{R}$;

(A3) $\dfrac{f(s)}{s} \to 1$ *as* $s \to 0$;

(A4) $\dfrac{f(s)}{\sqrt{s}} \to 2^{\frac{1}{4}}$ *as* $s \to \infty$;

(A5) $\dfrac{f(s)}{2} \le sf'(s) \le f(s)$ *for all* $s \ge 0$;

(A6) $|f(s)| \le 2^{\frac{1}{4}}|s|^{\frac{1}{2}}$ *for all* $s \in \mathbb{R}$;

(A7) *the function* $f^2(s)$ *is strictly convex*;

(A8) *there exists a positive constant* C *such that*

$$|f(s)| \ge \begin{cases} C|s|, & |s| \le 1, \\ C|s|^{\frac{1}{2}}, & |s| \ge 1; \end{cases}$$

(A9) *for each* $\lambda > 1$, *we have* $f^2(\lambda s) \le \lambda f^2(s)$ *for all* $t \in \mathbb{R}$;

(A10) *the function* $f^{-q}(s)f'(s)$ *is strictly decreasing for* $s > 0$ *and* $0 < q < 1$;

(A11) *the function* $f^q(s)f'(s)s^{-1}$ *is strictly increasing for* $q \ge 3$ *and* $s > 0$.

3 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following results.

Lemma 3.1. *Suppose that* (V_1), (V_2), (K_1) *are satisfied, then* (1.1) *has the global minimizer in* E. *In other words, there exists* $w_0 \in E$ *such that* $\mathcal{J}_\lambda(w_0) = m_\lambda = \inf_E \mathcal{J}_\lambda < 0$.

Proof. By the Sobolev inequality, Hölder inequality and Lemma 2.1 (A_6) yield

$$\int_{\mathbb{R}^N} K(x)|f(w)|^{1-\gamma} \le C\|K\|_{\frac{22^*}{22^*-1+\gamma}}\|w\|^{\frac{1-\gamma}{2}}. \tag{3.1}$$

For any $w \in E$, using (2.1) and (3.1), for $\lambda > 0$ and $0 < \gamma < 1$

$$
\begin{aligned}
\mathcal{J}_\lambda(w) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + V(x)f^2(w)) + \frac{\lambda}{2p}\int_{\mathbb{R}^N}(I_\alpha * |f(w)|^p)|f(w)|^p - \frac{1}{1-\gamma}\int_{\mathbb{R}^N}K(x)|f(w)|^{1-\gamma} \\
&\ge \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + V(x)f^2(w)) - \frac{1}{1-\gamma}\int_{\mathbb{R}^N}K(x)|f(w)|^{1-\gamma} \\
&\ge \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + V(x)f^2(w)) - \frac{C}{1-\gamma}\|K\|_{\frac{22^*}{22^*-1+\gamma}}\|w\|^{\frac{1-\gamma}{2}}.
\end{aligned}
\tag{3.2}
$$

Since $\gamma \in (0,1)$, $\mathcal{J}_\lambda$ is coercive and bounded from below on E for each $\lambda > 0$. Thus $m_\lambda := \inf_E \mathcal{J}_\lambda$ is well defined. For $t > 0$ and given $w \in E\backslash\{0\}$, by Lemma 2.1(A_3), we have

$$
\begin{aligned}
\mathcal{J}_\lambda(tw) &= t^2\int_{\mathbb{R}^N}|\nabla w|^2 + V(x)f^2(tw) + \frac{\lambda}{2p}(I_\alpha * |f(tw)|^p)|f(tw)|^p - \frac{1}{1-\gamma}\int_{\mathbb{R}^N}K(x)|f(tw)|^{1-\gamma} \\
&\le \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla w|^2 + V(x)w^2 + \frac{\lambda t^{2p}}{2p}\int_{\mathbb{R}^N}(I_\alpha * |w|^p)|w|^p - \frac{1}{1-\gamma}f^{1-\gamma}(t)\int_{\mathbb{R}^N}K(x)|w|^{1-\gamma}.
\end{aligned}
\tag{3.3}
$$

Let

$$g(t) = \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla w|^2 + V(x)w^2 + \frac{\lambda t^{2p}}{2p}\int_{\mathbb{R}^N}(I_\alpha * |w|^p)|w|^p - \frac{1}{1-\gamma}f^{1-\gamma}(t)\int_{\mathbb{R}^N}K(x)|w|^{1-\gamma},$$

$$\lim_{t \to 0^+} \frac{g(t)}{t^{(1-\gamma)}} = \lim_{t \to 0^+} \left\{ t^{(1+\gamma)} \int_{\mathbb{R}^N} |\nabla w|^2 + V(x)w^2 + \frac{\lambda t^{(2p-1+\gamma)}}{2p} \int_{\mathbb{R}^N} (I_\alpha * |w|^p)|w|^p \right\}$$

$$- \frac{1}{1-\gamma} \lim_{t \to 0^+} \frac{f^{1-\gamma}(t)}{t^{(1-\gamma)}} \int_{\mathbb{R}^N} K(x)|w|^{1-\gamma}.$$

Therefore, by (3.3), we obtain $\mathcal{J}_\lambda(tw) < 0$, for all $w \not\equiv 0$ and $t > 0$, and there exists a minimizing sequence $\{w_n\} \subset E$ such that $\lim_{n \to \infty} \mathcal{J}_\lambda(w_n) = m_\lambda < 0$. Since $\mathcal{J}_\lambda(|w_n|) = \mathcal{J}_\lambda(w_n)$, we could suppose that $w_n \geq 0$. The coerciveness of $\mathcal{J}_\lambda$ on E shows that $\{w_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that $w_n \rightharpoonup w_0$ in E, $w_n \to w_0$ in $L^p(\mathbb{R}^N)$, $p \in [2, 2^*)$ and $w_n \to w_0$, *a.e.* in $\mathbb{R}^N$, since $0 < \gamma < 1$, $K \in L^{\frac{2 2^*}{2 2^* - 1 + \gamma}}(\mathbb{R}^N)$ is nonnegative, by Hölder's inequality. Similar to (3.1), we have

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x) f^{1-\gamma}(w_n) = \int_{\mathbb{R}^N} K(x) f^{1-\gamma}(w_0). \tag{3.4}$$

Then, by the weakly lower semi-continuity of the norm, Lemma 2.4 in [36] and (3.4), we obtain

$$\mathcal{J}_\lambda(w_0) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w_0|^2 + V(x)|f(w_0)|^2 + \frac{\lambda}{2p} \int_{\mathbb{R}^N} (I_\alpha * |f(w_0)|^n)|f(w_0)|^p - \frac{1}{1-\gamma} \int_{\mathbb{R}^N} K(x) f^{1-\gamma}(w_0)$$

$$\leq \liminf_{n \to \infty} \left[\frac{1}{2} \int_{\mathbb{R}^N} |\nabla w_n|^2 + V(x)|f(w_n)|^2 + \frac{\lambda}{2p} \int_{\mathbb{R}^N} (I_\alpha * |f(w_n)|^p)|f(w_n)|^p \right] - \frac{1}{1-\gamma} \int_{\mathbb{R}^N} K(x) f^{1-\gamma}(w_n)$$

$$= \liminf_{n \to \infty} \mathcal{J}_\lambda(w_n) = m_\lambda.$$

In addition, $\mathcal{J}_\lambda(w_0) = m_\lambda < 0$. We complete the proof. $\qquad\square$

Proof of Theorem 1.1. We divide the proof into three parts.
(1) We will prove that for any $0 \leq \varphi \in E$

$$\int_{\mathbb{R}^N} |\nabla w_0 \nabla \varphi| + V(x) f(w_0) f'(w_0)\varphi + \lambda \int_{\mathbb{R}^N} (I_\alpha * |f(w_0)|^p)|f(w_0)|^{p-1} f'(w_0)\varphi - \int_{\mathbb{R}^N} K(x) f^{-\gamma}(w_0) f'(w_0)\varphi \geq 0.$$

On the basis of Lemma 3.1, w_0 is bounded in E and $w_0 \geq 0$ with $w_0 \not\equiv 0$. For $0 \leq \varphi \in E$ and $\delta > 0$, one has

$$0 \leq \mathcal{J}_\lambda(w_0 + \delta\varphi) - \mathcal{J}_\lambda(w_0)$$

$$= \frac{1}{2}\|w_0 + \delta\varphi\|^2 - \frac{1}{2}\|w_0\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|f(w_0 + \delta\varphi)|^2 - \frac{1}{2} \int_{\mathbb{R}^N} V(x)|f(w_0)|^2$$

$$+ \lambda \int_{\mathbb{R}^N} (I_\alpha * |f(w_0 + \delta\varphi)|^p)|f(w_0 + \delta\varphi)|^p - (I_\alpha * |f(w_0)|^p)|f(w_0)|^p \tag{3.5}$$

$$- \frac{1}{1-\gamma} \int_{\mathbb{R}^N} K(x) [f^{1-\gamma}(w_0 + \delta\varphi) - f^{1-\gamma}(w_0)],$$

since $\gamma \in (0, 1)$ and $K(x)$ is nonnegative.

Dividing (3.5) by $\delta > 0$ and passing to the liminf as $\delta \to 0^+$, then we can get from Fatou's Lemma that

$$\frac{1}{1-\gamma} \liminf_{\delta \to 0^+} \int_{\mathbb{R}^N} \frac{f^{1-\gamma}(w_0 + \delta\varphi) - f^{1-\gamma}(w_0)}{\delta} \leq \int_{\mathbb{R}^N} \nabla w_0 \nabla\varphi + V(x) f(w_0) f'(w_0)\varphi$$

$$+ \lambda \int_{\mathbb{R}^N} (I_\alpha * |f(w_0)|^p)|f(w_0)|^{p-1} f'(w_0)\varphi. \tag{3.6}$$

Since

$$\int_{\mathbb{R}^N} K(x) \frac{f^{1-\gamma}(w_0 + \delta\varphi) - f^{1-\gamma}(w_0)}{\delta} \le (1-\gamma) \int_{\mathbb{R}^N} K(x) f^{-\gamma}(w_0 + \delta\varphi) f'(w_0 + \delta\varphi) \varphi,$$

by the Beppolevi Monotone convergence Theorem and Lemma 2.1(A_{10}), we have

$$\liminf_{\delta \to 0^+} \frac{1}{1-\gamma} \int_{\mathbb{R}^N} K(x) \frac{f^{1-\gamma}(w_0 + \delta\varphi) - f^{1-\gamma}(w_0)}{\delta} = (1-\gamma) \int_{\mathbb{R}^N} K(x) f^{-\gamma}(w_0 + \delta\theta\varphi) f'(w_0 + \theta\delta\varphi) \varphi,$$

where $0 < \theta < 1$, which together with (3.6) implies that

$$\int_{\mathbb{R}^N} \nabla w_0 \nabla\varphi + V(x) f(w_0) f'(w_0)\varphi + \lambda \int_{\mathbb{R}^N} (I_\alpha * |f(w_0)|^p) |f(w_0)|^{p-1} f'(w_0)\varphi - \int_{\mathbb{R}^N} K(x) f^{-\gamma}(w_0) f'(w_0)\varphi \ge 0 \tag{3.7}$$

(2) We show that $w_0 > 0$ in $\mathbb{R}^N$ and w_0 is a solution of problem (1.1). Given $\varepsilon > 0$, define $g : [-\varepsilon, \varepsilon] \to R$ by $g(t) = \mathcal{J}_\lambda(w_0 + tw_0)$. Then, g attains its minimum at $t = 0$ by Lemma 3.1, which implies that

$$g'(0) = \int_{\mathbb{R}^N} |\nabla w_0|^2 + V(x)|f(w_0)|^2 + \lambda \int_{\mathbb{R}^N} (I_\alpha * |f(w_0)|^p) |f(w_0)|^p - \int_{\mathbb{R}^N} K(x)|f(w_0)|^{1-\gamma} = 0 \tag{3.8}$$

For any $v \in E$ and $\varepsilon > 0$, set $\varphi_\varepsilon = (w_0 + \varepsilon v)^+$ and $\Omega_\varepsilon = \{x \in \mathbb{R}^N : \varphi_\varepsilon \le 0\}$. Then, using (3.7) and (3.8) with $\varphi = \varphi_\varepsilon$ lead to

$$
\begin{aligned}
0 &\le \int_{\mathbb{R}^N} \nabla w_0 \nabla\varphi + V(x) f(w_0) f'(w_0)\varphi + \lambda \int_{\mathbb{R}^N} (I_\alpha * |f(w_0)|^p) |f(w_0)|^{p-1} f'(w_0)\varphi - \int_{\mathbb{R}^N} K(x) f^{-\gamma}(w_0) f'(w_0)\varphi \\
&= \left(\int_{\mathbb{R}^N} - \int_{\Omega_\varepsilon} \right) \nabla w_0 \nabla(w_0 + \varepsilon v) + V(x) f(w_0) f'(w_0)(w_0 + \varepsilon v) \\
&\quad + \lambda \left(\int_{\mathbb{R}^N} - \int_{\Omega_\varepsilon} \right) (I_\alpha * |f(w_0)|^p) |f(w_0)|^{p-1} f'(w_0)(w_0 + \varepsilon v) - \left(\int_{\mathbb{R}^N} - \int_{\Omega_\varepsilon} \right) K(x) f^{-\gamma}(w_0) f'(w_0)(w_0 + \varepsilon v) \\
&= \varepsilon \left\{ \nabla w_0 \nabla v + V(x) f(w_0) f'(w_0) v + \lambda \int_{\mathbb{R}^N} (I_\alpha * |f(w_0)|^p) |f(w_0)|^{p-1} f'(w_0) v - \int_{\mathbb{R}^N} K(x) f^{-\gamma}(w_0) f'(w_0) v \right\} \\
&\quad - \int_{\Omega_\varepsilon} \nabla w_0 \nabla(w_0 + \varepsilon v) + V(x) f(w_0) f'(w_0)(w_0 + \varepsilon v) - \lambda \int_{\Omega_\varepsilon} (I_\alpha * |f(w)|^p) |f(w)|^{p-1} f'(w)(w + \varepsilon v) \\
&\quad + \left(\int_{\mathbb{R}^N} - \int_{\Omega_\varepsilon} \right) K(x) f^{-\gamma}(w_0) f'(w_0)(w_0 + \varepsilon v).
\end{aligned}
$$

Taking $\varepsilon \to 0^+$ to the above inequality and based on the fact that $\Omega_\varepsilon \to 0$ as $\varepsilon \to 0^+$, we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} \nabla w_0 \nabla\varphi &+ V(x) f(w_0) f'(w_0)\varphi + \lambda \int_{\mathbb{R}^N} (I_\alpha * |f(w_0)|^p) |f(w_0)|^{p-1} f'(w_0)\varphi \\
&- \int_{\mathbb{R}^N} K(x) f^{-\gamma}(w_0) f'(w_0)\varphi \ge 0 \quad \forall\varphi \in E
\end{aligned}
$$

The above inequality also holds for $-v$; hence, we have

$$\int_{\mathbb{R}^N} \nabla w_0 \nabla \varphi + V(x)f(w_0)f'(w_0)\varphi + \lambda \int_{\mathbb{R}^N} (I_\alpha * |f(w_0)|^p)|f(w_0)|^{p-1}f'(w_0)\varphi$$

$$- \int_{\mathbb{R}^N} K(x)f^{-\gamma}(w_0)f'(w_0)\varphi = 0 \tag{3.9}$$

Analogous to the proof of [32, Theorem 1], we obtain $w_0 \in C^2_{\text{loc}}(\mathbb{R}^N)$. Since $w_0 \geq 0$, the strong maximum principle implies $w_0 > 0$, and $w_0 \in E$ is a solution of problem (1.1).

(3) We show that the solution w_0 is unique. Assume that $\overline{w} \in E$ is also a solution, then for any $\varphi \in E$

$$\int_{\mathbb{R}^N} \nabla \overline{w} \nabla \varphi + V(x)f(\overline{w})f'(\overline{w})\varphi + \lambda \int_{\mathbb{R}^N} (I_\alpha * |f(\overline{w})|^p)|f(\overline{w})|^{p-1}f'(\overline{w})\varphi - \int_{\mathbb{R}^N} K(x)f^{-\gamma}(\overline{w})f'(\overline{w})\varphi = 0 \tag{3.10}$$

Subtracting (3.9) and (3.10), since $K(x) > 0$, it follows from Lemma 2.4 (see [36]) and $\lambda > 0$ that

$$\|w_0 - \overline{w}\|^2 = \int_{\mathbb{R}^N} K(x)[f^{-\gamma}(w_0)f'(w_0) - f^{-\gamma}(\overline{w})f'(\overline{w})](w_0 - \overline{w}) - \int_{\mathbb{R}^N} V(x)f(w_0)f'(w_0)(w_0 - \overline{w})$$

$$- \lambda \left[\int_{\mathbb{R}^N} (I_\alpha * |f(w_0)|^p)|f(w_0)|^{p-1}f'(w_0) - (I_\alpha * |f(w_0)|^p)|f(w_0)|^{p-1}f'(\overline{w}) \right](w_0 - \overline{w}) \leq 0$$

Hence, $w_0 = \overline{w}$ and w_0 is the unique solution of problem (1.1), which completes the proof. □

Proof of Theorem 1.2. From the proof of Lemma 3.1 and Theorem 1.1, we can get that $\lambda = 0$ is allowed. Therefore, under the conditional assumptions of Theorem 1.2, equation (1.1) has a unique positive solution $w_0 \in E$, i.e., for any $w \in E$, we obtain

$$\int_{\mathbb{R}^N} \nabla w_0 \nabla \varphi + V(x)f(w_0)f'(w_0)\varphi = \int_{\mathbb{R}^N} K(x)|f(w_0)|^{-\gamma}f'(w_0)\varphi.$$

For any sequence $\{\lambda_n\} > 0$ with $\lambda_n \to 0$, as $n \to \infty$, according to Theorem 1.1, we can obtain a positive solution sequence $\{w_{\lambda_n}\} \subset E$ corresponding solution of problem (1.1) with $\lambda = \lambda_n$ for $n \in N$. Thus, we obtain

$$\int_{\mathbb{R}^N} \nabla w_{\lambda_n} \nabla \varphi + V(x)|f(w_{\lambda_n})|f'(w_{\lambda_n})\varphi + \lambda_n \int_{\mathbb{R}^N} (I_\alpha * |f(w_{\lambda_n})|^p)|f(w_{\lambda_n})|^{p-2}f(w)f'(w_{\lambda_n})\varphi$$

$$= \int_{\mathbb{R}^N} K(x)|f(w_{\lambda_n})|^{-\gamma}f'(w_{\lambda_n})\varphi \tag{3.11}$$

for any $w_{\lambda_n} \in E$. From Lemma 2.1 and the proof of Theorem 1.1, we get $\mathcal{J}_{\lambda_n} = m_{\lambda_n} < 0$ and then $\{w_{\lambda_n}\}$ is bounded in E since $\mathcal{J}_{\lambda_n}$ is coercive according to (3.3). As a result, there exist a subsequence of $\{w_{\lambda_n}\}$ (still denoted by $\{w_{\lambda_n}\}$) and a nonnegative function $w_0 \in E$ such that $w_{\lambda_n} \rightharpoonup w_0$ in E, $w_{\lambda_n} \to w_0$ in $L^p(\mathbb{R}^N)$, $p \in [2, 2^*)$ and $w_{\lambda_n} \to w_0$ a.e in $\mathbb{R}^N$. Let us define $w_n = w_{\lambda_n}$ in (3.11) and passing to the liminf as $n \to \infty$, we can obtain from Lemma 2.4 (see [36]), (3.3) and the weakly lower semi-continuity of the norm that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, the support of φ is contained in $B_{R_0(0)}$ for some $R_0 > 0$ since $w_n \rightharpoonup w_0$ in $H^1(\mathbb{R}^N)$, we have

$$\left| \int_{\mathbb{R}^N} \nabla w_n \nabla \varphi - \nabla w_0 \nabla \varphi \right| \to 0. \tag{3.12}$$

By $w_n \to w_0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$,

$$\left| \int_{\mathbb{R}^N} V(x)[f(w_n)f'(w_n)\varphi - f(w_0)f'(w_0)]\varphi \right| \leq \mu \int_{B_{R_0(0)}} |f(w_n)f'(w_n)\varphi - f(w_0)f'(w_0)\varphi|$$

$$\leq \mu \left(\int_{B_{R_0(0)}} |f(w_n)f'(w_n) - f(w_0)f'(w_0)|^2 \right)^{\frac{1}{2}} \left(\int_{B_{R_0(0)}} |\varphi|^2 \right)^{\frac{1}{2}} \to 0. \tag{3.13}$$

Passing to the lim inf as $n \to \infty$ in (3.11), by (3.12), (3.13) and the weakly lower semi-continuity of the definition, we have

$$\int_{\mathbb{R}^N} \nabla w_0 \nabla \varphi + V(x)f(w_0)f'(w_0)\varphi \leq \int_{\mathbb{R}^N} K(x)|f(w_0)|^{-\gamma}f'(w_0)\varphi. \tag{3.14}$$

Furthermore, passing to the lim inf as $n \to \infty$ in (3.11), by (3.12), (3.13) and the Fatou's Lemma, we obtain

$$\int_{\mathbb{R}^N} \nabla w_0 \nabla \varphi + V(x)f(w_0)f'(w_0)\varphi \geq \int_{\mathbb{R}^N} K(x)|f(w_0)|^{-\gamma}f'(w_0)\varphi. \tag{3.15}$$

Using (3.14) and (3.15), we have

$$\int_{\mathbb{R}^N} \nabla w_0 \nabla \varphi + V(x)f(w_0)f'(w_0)\varphi = \int_{\mathbb{R}^N} K(x)|f(w_0)|^{-\gamma}f'(w_0)\varphi.$$

Analogous to step (2) in the proof of Theorem 1.1, we can get that $0 < w_0 \in E$ is also a solution. Therefore, $w_{\lambda_n} \to w_0$ in E and w_0 is the unique positive solution to the equation (1.2). We complete the proof. $\square$

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References

[1] S. Kurihura, *Large-amplitude quasi-solitons in superfluid films*, J. Phys. Soc. Jpn. **50** (1981), no. 10, 3262–3267, DOI: https://doi.org/10.1143/JPSJ.50.3262.

[2] E. Laedke, K. Spatschek, and L. Stenflo, *Evolution theorem for a class of perturbed envelope soliton solutions*, J. Math. Phys. **24** (1983), no. 12, 2764–2769, DOI: https://doi.org/10.1063/1.525675.

[3] H. Lange, M. Poppenberg, and H. Teismann, *Nash-Moser methods for the solution of quasilinear Schrödinger equations*, Commun. Partial Differ. Equ. **24** (1999), no. 7–8, 1399–1418, DOI: https://doi.org/10.1080/03605309908821469.

[4] A. Borovskii and A. Galkin, *Dynamical modulation of an ultrashort high-intensity laser pulse in matter*, J. Exp. Theor. Phys. **77** (1993), 562–573.

[5] E. Gloss, *Existence and concentration of positive solutions for a quasilinear equation in* $\mathbb{R}^N$, J. Math. Anal. Appl. **371** (2010), no. 2, 465–484, DOI: https://doi.org/10.1016/j.jmaa.2010.05.033.

[6] A. de Bouard, N. Hayashi, and J. Saut, *Global existence of small solutions to a relativistic nonlinear Schrödinger equation*, Comm. Math. Phys. **189** (1997), 73–105, DOI: https://doi.org/10.1007/s002200050191.

[7] A. Litvak and A. Sergeev, *One-dimensional collapse of plasma waves*, JETP Lett. **27** (1978), no. 10, 517–520.

[8] A. Nakamura, *Damping and modification of exciton solitary waves*, J. Phys. Soc. Jpn. **42** (1977), no. 6, 1824–1835, DOI: https://doi.org/10.1143/JPSJ.42.1824.

[9] F. G. Bass and N. N. Nasanov, *Nonlinear electromagnetic spin waves*, Phys. Rep. **189** (1990), no. 4, 165–223, DOI: https://doi.org/10.1016/0370-1573(90)90093-H.

[10] R. Hasse, *A general method for the solution of nonlinear soliton and kink Schrödinger equations*, Z. Physik B. **37** (1980), 83–87, DOI: https://doi.org/10.1007/BF01325508.

[11] V. G. Makhankov and V. K. Fedyanin, *Non-linear effects in quasi-one-dimensional models of condensed matter theory*, Physics Reports **104** (1984), no. 1, 1–86, DOI: https://doi.org/10.1016/0370-1573(84)90106-6.

[12] B. Ritchie, *Relativistic self-focusing and channel formation in laser-plasma interactions*, Phys. Rev. E **50** (1994), no. 2, R687–R689, DOI: https://doi.org/10.1103/PhysRevE.50.R687.

[13] H. Lange, B. Toomire, and P. Zweifel, *Time-dependent dissipation in nonlinear Schrödinger systems*, J. Math. Phys. **36** (1995), no. 3, 1274–1283, DOI: https://doi.org/10.1063/1.531120.

[14] C. O. Alves and M. Yang, *Multiplicity and concentration of solutions for a quasilinear Choquarde quation*, J. Math. Phys. **55** (2014), no. 6, 061502, DOI: https://doi.org/10.1063/1.4884301.

[15] E. Silva and G. Vieira, *Quasilenear asymptotically periodic Schrödinger equations with critical growth*, Calc. Var. Partial Differ. Equ. **39** (2010), 1–33, DOI: https://doi.org/10.1007/s00526-009-0299-1.

[16] E. Silva and G. Vieira, *Quasilinear asymptotically periodic Schrödinger equations with subcritical growth*, Nonlinear Anal. **72** (2010), no. 6, 2935–2949, DOI: https://doi.org/10.1016/j.na.2009.11.037.

[17] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations Hardy-Littlewood-Sobolev critical exponent*, Commun. Contemp. Math. **17** (2015), no. 5, 1550005, DOI: https://doi.org/10.1142/S0219199715500054.

[18] M. Millem, *Minimax Theorems*, Birkhöuser, Berlin, 1996.

[19] V. Moroz and J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. **367** (2015), 6557–6579, DOI: https://doi.org/10.1090/S0002-9947-2014-06289-2.

[20] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), no. 2, 153–184, DOI: https://doi.org/10.1016/j.jfa.2013.04.007.

[21] D. Cao and S. Peng, *A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms*, J. Diff. Eqns. **193** (2003), no. 2, 424–434, DOI: https://doi.org/10.1016/S0022-0396(03)00118-9.

[22] S. Cingolani, M. Clapp, and S. Secchi, *Multiple solutions to a magnetic nonlinear Choquard equation*, Z. Angew. Math. Phys. **63** (2012), 233–248, DOI: https://doi.org/10.1007/s00033-011-0166-8.

[23] M. Poppenberg, K. Schmitt, and Z. Wang, *On the existence of soliton solutions to quasilinear Schrödinger equations*, Calc. Var. Partial Differ. Equ. **14** (2002), 329–344, DOI: https://doi.org/10.1007/s005260100105.

[24] H. Brözis and T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl. **9** (1979), 137–151.

[25] N. Hirano, C. Saccon, and N. Shioji, *Brezis-Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem*, J. Diff. Eqns. **245** (2008), no. 8, 1997–2037, DOI: https://doi.org/10.1016/j.jde.2008.06.020.

[26] D. Lü, *Existence and concentration behavior of ground state solutions for magnetic nonlinear Choquard equations*, Commun. Pure. Appl. Anal. **15** (2016), no. 5, 1781–1795, DOI: https://doi.org/10.3934/cpaa.2016014.

[27] Z. Shen, F. Gao, and M. Yang, *Multiple solutions for nonhomogeneous Choquard equation involving Hardy-Littlewood-Sobolev critical exponent*, Z. Angew. Math. Phys. **68** (2017), 61, DOI: https://doi.org/10.1007/s00033-017-0806-8.

[28] M. Colin and L. Jeanjean, *Solutions for a quasilinear Schrödinger equations: a dual approach*, Nonlinear Anal. **56** (2004), no. 2, 213–226, DOI: https://doi.org/10.1016/j.na.2003.09.008.

[29] S. Chen and X. Wu, *Existence of positive solutions for a class of quasilinear Schrödinger equations of Choquard type*, J. Math. Anal. Appl. **475** (2019), no. 2, 1754–1777, DOI: https://doi.org/10.1016/j.jmaa.2019.03.051.

[30] X. Yang, W. Zhang, and F. Zhao, *Existence and muliplicity of solutions for a quasilinear Choquard equation via perturbation method*, J. Math. Phys. **59** (2018), no. 8, 081503, DOI: https://doi.org/10.1063/1.5038762.

[31] C. Alves and M. Yang, *Multiplicity and concentration behavior of solutions for a quasilinear Choquard equation via penalization method*, Proc. Roy. Soc. Edinburgh Sect. A. **146** (2016), no. 6, 23–58.

[32] J. Marcos do Ó and A. Moameni, *Solutions for singular quasilinear Schrödinger quation with one parameter*, Commun. Pure. Appl. Anal. **9** (2010), no. 4, 1011–1023, DOI: https://doi.org/10.3934/cpaa.2010.9.1011.

[33] J. Chen, B. Cheng, and X. Huang, *Ground state solutions for a class of quasilinear Schrödinger equations with Choquard type nonlinearity*, Appl. Math. Lett. **102** (2020), 106141, DOI: https://doi.org/10.1016/j.aml.2019.106141.

[34] J. Liu, Y. Wang, and Z. Wang, *Soliton solutions for quasilinear Schrödinger equations: II*, J. Diff. Eqns. **187** (2003), no. 2, 473–493, DOI: https://doi.org/10.1016/S0022-0396(02)00064-5.

[35] X. Yang, W. Wang, and F. Zhao, *Infinitely many radial and non-radial solutions to a quasilinear Schrödinger equation*, Nonlinear Anal. **114** (2015), no. 11, 158–168, DOI: https://doi.org/10.1016/j.na.2014.11.015.

[36] X. Li, S. Ma, and G. Zhang, *Existence and qualitative properties of solutions for Choquard equations with a local term*, Nonlinear Anal. Real World Appl. **45** (2019), 1–25, DOI: https://doi.org/10.1016/j.nonrwa.2018.06.007.