

Research Article

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Global attractors for a class of semilinear degenerate parabolic equations

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Abstract: In this paper, we consider the long-time behavior for a class of semi-linear degenerate parabolic equations with the nonlinearity f satisfying the polynomial growth of arbitrary $p - 1$ order. We establish some new estimates, i.e., asymptotic higher-order integrability for the difference of the solutions near the initial time. As an application, we obtain the $(L^2(\Omega), L^p(\Omega))$ -global attractors immediately; moreover, such an attractor can attract every bounded subset of $L^2(\Omega)$ with the $L^{p+\delta}$ -norm for any $\delta \in [0, +\infty)$.

Keywords: degenerate parabolic equations, polynomial growth of arbitrary order, asymptotic higher-order integrability, global attractors

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain with smooth boundary, we consider the long-time behavior for the solutions of the following semi-linear degenerate parabolic equation:

$$\begin{cases} \partial_t u(x, t) = \Delta_\lambda u(x, t) + f(u(x, t)) + g(x) & \text{in } \Omega \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty), \\ u(x, 0) = u_0, & x \in \Omega, \end{cases} \quad (1.1)$$

where Δ_λ is the degenerate elliptic operator, which will be characterized in Section 2. The external forcing $g(x) = \sum_{i=1}^N D_i g^i + g^0(x) \in H^{-1}(\Omega)$, the nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$ and satisfies the following classical assumptions (e.g., see [1–3]):

$$f'(u) \leq l, \quad (1.2)$$

and

$$-c_0 - c_1|u|^p \leq f(u)u \leq c_0 - c_2|u|^p, \quad p \geq 2 \quad (1.3)$$

for some positive constants c_0, c_1, c_2 and $u \in \mathbb{R}$.

Let $\mathcal{F}(u) = \int_0^u f(r)dr$, then there exist constants $\tilde{c}_i > 0$ ($i = 0, 1, 2$) such that

$$-\tilde{c}_0 - \tilde{c}_1|u|^p \leq \mathcal{F}(u) \leq \tilde{c}_0 - \tilde{c}_2|u|^p, \quad \forall u \in \mathbb{R}. \quad (1.4)$$

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The long-time behavior for the solutions of semi-linear degenerate parabolic equations has been considered by many researchers, e.g., see [4–6] and references therein. For the subcritical growth case, the authors have established the existence of the global attractors in $S_0^1(\Omega)$ (see [4] for details) for equation (1.1) involving the Grushin operator. In [5], the authors extended the result for the Δ_λ -Laplacian. They considered Δ_λ as a self-adjoint operator and showed that $-\Delta_\lambda$ generates an analytic semigroup in $L^2(\Omega)$. They not only proved the existence of the global attractors in $\dot{W}_\lambda^{1,2}(\Omega)$ together with its fractal dimension but also showed the convergence of solutions to an equilibrium solution as the time $t \rightarrow \infty$. In the following, the authors of [6] have considered the case of critical growth nonlinearity and obtained the global attractors in $\dot{W}_\lambda^{1,2}(\Omega)$ by applying the decomposition technique introduced in [7].

In this paper, similar to the reaction–diffusion equations in [8,9], we consider equation (1.1) with the nonlinearity f satisfying the polynomial growth of arbitrary $p - 1$ order, i.e., $f(u) \sim -|u|^{p-2}u$ ($p \geq 2$). For our problem, we will confront two main difficulties when we establish the asymptotic higher-order integrability of solutions and the existence of the $(L^2(\Omega), L^p(\Omega))$ -global attractors. One difficulty is that the external force term $g(x) = \sum_{i=1}^N D_i g^i + g^0(x)$ with $g^i(x)$ ($i = 0, 1, \dots, N$) belongs only to $L^2(\Omega)$, which leads to the fact that the solutions of equation (1.1) are only bounded in $\dot{W}_\lambda^{1,2}(\Omega) \cap L^p(\Omega)$ and do not have any higher regularity than the order $\max\left\{p, \frac{2Q}{Q-2}\right\}$ (where Q is the homogeneous dimension of \mathbb{R}^N with a group of dilations corresponding to the Δ_λ -operator, see [5] for details). The other difficulty is that the Sobolev embedding theorem is not any longer valid since the growth exponent is arbitrary $p - 1$ ($p \geq 2$) order.

Based on the aforementioned difficulties and motivated by the idea of [10–13], we first decompose equation (1.1) as a stationary equation and an evolutionary equation, then establish some asymptotic higher-order integrability results about the difference of the solutions near the initial time by using the bootstrap method (see Theorems 4.1 and 4.3). As an application, we obtain the $(L^2(\Omega), L^p(\Omega))$ -global attractors immediately (see Corollary 4.4). Moreover, the $(L^2(\Omega), L^p(\Omega))$ -global attractors indeed can attract every bounded subset of $L^2(\Omega)$ with the $L^{p+\delta}$ -norm for any $\delta \in [0, +\infty)$. Finally, we also obtain the $(L^2(\Omega), \dot{W}_\lambda^{1,2}(\Omega))$ -global attractors (see Theorem 5.3).

2 Preliminaries

2.1 The Δ_λ -operator

As in [5,14–17], we consider the degenerate elliptic operator of the form

$$\Delta_\lambda := \sum_{i=1}^N \partial_{x_i} (\lambda_i^2 \partial_{x_i}), \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) : \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad (2.1)$$

where $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, 2, \dots, N$. The functions $\lambda_i : \mathbb{R}^N \rightarrow \mathbb{R}$ are continuous, strictly positive and of C^1 outside the coordinate hyperplane¹ and satisfy the following properties (see [5,17] for details):

- (1) $\lambda_1(x) \equiv 1$, $\lambda_i(x) = \lambda_i(x_1, x_2, \dots, x_{i-1})$, $i = 2, 3, \dots, N$.
- (2) For every $x \in \mathbb{R}^N$, the function $\lambda_i(x) = \lambda_i(x^*)$, $i = 1, 2, \dots, N$, where

$$x^* = (|x_1|, |x_2|, \dots, |x_N|) \quad \text{if } x = (x_1, x_2, \dots, x_N).$$

¹ $\lambda_i > 0$ in $\mathbb{R}^N \setminus \Pi$, where $\Pi = \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N : \prod_{i=1}^N x_i = 0\}$.

(3) There exists a constant $\rho \geq 0$ such that

$$0 \leq x_k \partial_{x_k} \lambda_i(x) \leq \rho \lambda_i(x), \quad \forall k \in \{1, 2, \dots, i-1\}, \quad i = 2, 3, \dots, N,$$

and for every $x \in \mathbb{R}_+^N := \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N \mid x_i \geq 0, \forall i = 2, 3, \dots, N\}$.

(4) There exists a group of dilations $(\delta_r)_{r>0}$ which satisfy

$$\delta_r : \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad \delta_r(x) = \delta_r(x_1, x_2, \dots, x_N) = (r^{\varepsilon_1} x_1, r^{\varepsilon_2} x_2, \dots, r^{\varepsilon_N} x_N),$$

where $1 \leq \varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_N$, such that λ_i is δ_r -homogeneous of degree $\varepsilon_i - 1$, i.e.,

$$\lambda_i(\delta_r(x)) = r^{\varepsilon_i-1} \lambda_i(x), \quad \forall x \in \mathbb{R}^N, \quad r > 0, \quad i = 1, 2, \dots, N.$$

This implies that the operator Δ_λ is δ_r -homogeneous of degree two, i.e.,

$$\Delta_\lambda(u(\delta_r(x))) = r^2 (\Delta_\lambda u)(\delta_r(x)), \quad \forall u \in C^\infty(\mathbb{R}^N).$$

We will denote by Q the homogeneous dimension of \mathbb{R}^N with respect to the group of dilations $(\delta_r)_{r>0}$, that is,

$$Q = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_N.$$

We have another expression of operator Δ_λ from the property (1) as (e.g., see [5])

$$\Delta_\lambda = \sum_{i=1}^N (\lambda_i \partial_{x_i})^2.$$

2.2 Functional settings

For a function $u \in C^1$, we define

$$\begin{aligned} \nabla_\lambda u &= (\lambda_1 \partial_{x_1} u, \lambda_2 \partial_{x_2} u, \dots, \lambda_N \partial_{x_N} u), \quad |\nabla_\lambda u|^2 := \sum_{i=1}^N |\lambda_i \partial_{x_i} u|^2, \\ \Delta_\lambda u &= (\lambda_1^2 \partial_{x_1}^2 u, \lambda_2^2 \partial_{x_2}^2 u, \dots, \lambda_N^2 \partial_{x_N}^2 u), \quad |\Delta_\lambda u|^2 := \sum_{i=1}^N |\lambda_i^2 \partial_{x_i}^2 u|^2, \end{aligned}$$

where λ_i ($i = 1, 2, \dots, N$) comes from (2.1). For any $p \in (1, \infty)$, we define that $\dot{W}_\lambda^{1,p}(\Omega)$ is the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_{\dot{W}_\lambda^{1,p}(\Omega)} := \left(\int_\Omega |\nabla_\lambda u|^p dx \right)^{\frac{1}{p}},$$

and $W_\lambda^{2,p}(\Omega)$ with respect to the norm

$$\|u\|_{W_\lambda^{2,p}(\Omega)} := \left(\int_\Omega |\nabla_\lambda u|^p dx + \int_\Omega |\Delta_\lambda u|^p dx \right)^{\frac{1}{p}}.$$

Similar to that in [5,17], we have the following embedding theorem.

Lemma 2.1. *For any $p \in (1, Q)$, $q \in [1, p_\lambda^*]$, the embedding*

$$\dot{W}_\lambda^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is continuous for $q \in [1, p_\lambda^]$ and compact for $q \in [1, p_\lambda^*)$, where $p_\lambda^* := \frac{pQ}{Q-p}$.*

In particular, for any $u \in C_0^1(\Omega)$, there exists a constant C such that

$$\|u\|_{L^2(\Omega)} \leq C \|u\|_{\dot{W}_\lambda^{1,2}(\Omega)}, \quad \forall u \in C_0^1(\Omega), \quad (2.2)$$

and the optimal constant of C is $\frac{1}{\sqrt{\mu_1}}$, where $\mu_1 > 0$ denotes the first eigenvalue of the operator $-\Delta_\lambda$ on Ω with homogeneous Dirichlet boundary conditions.

For later application, we recall the following results (see [12] for details).

Lemma 2.2. For any $\phi \in \dot{W}_\lambda^{1,2}(\Omega) \cap L^\infty(\Omega)$ and any $r > 0$, the following equality holds:

$$\int_{\Omega} \nabla_\lambda \phi \cdot \nabla_\lambda (|\phi|^r \phi) dx = (r+1) \left(\frac{2}{r+2} \right)^2 \int_{\Omega} |\nabla_\lambda |\phi|^{\frac{r+2}{2}}|^2 dx, \quad (2.3)$$

where \cdot stands for the usual inner product in \mathbb{R}^N .

Lemma 2.3. Let X, Y, Z be three Banach spaces satisfying $Z \subset Y \subset X$ with continuous embedding and suppose that there exist $\theta_0 \in (0, 1]$ and a constant C_0 such that $\|\cdot\|_Y \leq C_0 \|\cdot\|_X^{\theta_0} \|\cdot\|_Z^{1-\theta_0}$. Then, for any bounded sets $A, B \subset Z$, we have

$$\text{dist}_Y(A, B) \leq C_0 (\|A\|_Z + \|B\|_Z + 1) \text{dist}_X^{\theta_0}(A, B),$$

where $\|A\|_Z = \sup_{x \in A} \|x\|_Z$, $\|B\|_Z = \sup_{x \in B} \|x\|_Z$ for any $A, B \subset Z$.

Similar to Theorem 3.2 in [12], we have the following theorem.

Theorem 2.4. Let X, Y, Z be three Banach spaces satisfying $Z \subset Y \subset X$ with continuous embedding, $S(\cdot)$ be a semigroup defined on X . Moreover, we assume that

- (a) $S(\cdot)$ has a global attractor \mathcal{A} in X ;
- (b) $v(x)$ is the solution of the stationary equation $\Delta_\lambda v + f(v) + g(x) = 0$ for (1.1);
- (c) there exist a family of operators $S_1(\cdot)$ defined on X satisfying

$$S(t) \cdot = v(x) + S_1(t)(\cdot - v(x)) \quad \text{for any } t \geq 0,$$

and a bounded set B_0 in Z satisfying that for any $t \geq 0$ and any bounded set B in X there exists a $T = T(B)$ such that

$$S_1(t)(B - v(x)) \subset B_0 \quad \text{for any } t \geq T. \quad (2.4)$$

Then the following conclusions hold:

- (i) $B_1 = v(x) + B_0$ is a bounded absorbing set for the semigroup $S(\cdot)$;
- (ii) $\text{dist}_X(\mathcal{A}, B_1) = 0$, i.e.,

$$\text{dist}_X(\mathcal{A}, v(x) + B_0) = \text{dist}_X(\mathcal{A} - v(x), B_0) = 0; \quad (2.5)$$

- (iii) if B_0 is closed in X , then

$$\mathcal{A} - v(x) \subset B_0. \quad (2.6)$$

Furthermore, we assume that the space Y satisfies $\|\cdot\|_Y \leq C_0 \|\cdot\|_X^{\theta_0} \|\cdot\|_Z^{1-\theta_0}$ for constant C_0 and some $\theta_0 \in (0, 1]$. Then,

$$\text{dist}_Y(S(t)B, \mathcal{A}) \rightarrow 0 \quad \text{as } t \rightarrow +\infty. \quad (2.7)$$

Proof. For any $t, T \geq 0$ with $t \geq T$ and any bounded set B in X , from (2.4) we have

$$S(t)B = v(x) + S_1(t)(B - v(x)) \subset v(x) + B_0 \quad \text{for any } t \geq T,$$

which implies (i) immediately.

Given the fact that $\mathcal{A} \in B_1$, (ii) is a direct result of the invariant of \mathcal{A} and (i):

$$\text{dist}_X(\mathcal{A}, v(x) + B_0) = \text{dist}_X(S(t)\mathcal{A}, v(x) + B_0) \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

In the following, we prove (iii).

At first, (2.6) follows directly from (2.5) and the assumption that B_0 is closed in X .

Then, from the assumption (c) we know that

$$S(t)B - v(x) \subset B_0 \quad \text{as } t \geq T,$$

and consequently, by applying Lemma 2.3, we obtain that

$$\begin{aligned} \text{dist}_Y(S(t)B, \mathcal{A}) &= \text{dist}_Y(S(t)B - v(x), \mathcal{A} - v(x)) \\ &\leq C_0(2\|B_0\|_Z + 1) \text{dist}_X^{\theta_0}(S(t)B - v(x), \mathcal{A} - v(x)) \quad \text{as } t \geq T \\ &= C_0(2\|B_0\|_Z + 1) \text{dist}_X^{\theta_0}(S(t)B, \mathcal{A}), \end{aligned}$$

which implies (2.7) immediately. \square

2.3 Solutions for equations (1.1)

In this subsection, being similar to the reaction–diffusion equations, we will give the definition of different solutions about equation (1.1) (see [2] for details).

Definition 2.5. (*Weak solutions*) A function $u = u(x, t)$ is said to be a weak solution of equation (1.1) if

$$u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; \dot{W}_\lambda^{1,2}(\Omega)) \cap L^p(0, T; L^p(\Omega)), \quad u_t \in L^2(0, T; H^{-1}(\Omega))$$

with $u(0) = u_0 \in L^2(\Omega)$, $g \in H^{-1}(\Omega)$ and for all $\varphi \in L^2(0, T; \dot{W}_\lambda^{1,2}(\Omega)) \cap L^p(0, T; L^p(\Omega))$, it satisfies

$$\int_0^T \int_\Omega \left(\frac{\partial u}{\partial t} \varphi + \nabla_\lambda u \nabla_\lambda \varphi \right) dx dt = \int_0^T \int_\Omega (f(u) \varphi + g \varphi) dx dt.$$

Definition 2.6. (*Strong solutions*) A weak solution $u = u(x, t)$ of equation (1.1) is said to be a strong solution if

$$u \in C([0, T]; \dot{W}_\lambda^{1,2}(\Omega)) \cap L^2(0, T; W_\lambda^{2,2}(\Omega)) \cap L^\infty(0, T; L^p(\Omega)), \quad u_t \in L^2(0, T; L^2(\Omega))$$

with $u(0) = u_0 \in \dot{W}_\lambda^{1,2}(\Omega) \cap L^p(\Omega)$ and $g \in L^2(\Omega)$.

3 $(L^2(\Omega), L^2(\Omega))$ -global attractors

3.1 Existence and uniqueness of solutions

In this subsection, we first give the existence and uniqueness of weak solutions, which can be obtained by the Fadeo-Galerkin method (similar to Theorem 8.4, p. 221 in [2]). Here we only state the results.

Lemma 3.1. *Let f satisfy (1.2)–(1.4), $g \in H^{-1}(\Omega)$ and $u_0 \in L^2(\Omega)$. Then, for any $T > 0$, there exists a unique weak solution u to equation (1.1) such that*

$$u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; \dot{W}_\lambda^{1,2}(\Omega)) \cap L^p(0, T; L^p(\Omega)).$$

By Lemma 3.1, we can define the semigroup $\{S(t)\}_{t \geq 0}$ in $L^2(\Omega)$ as follows:

$$S(t) : L^2(\Omega) \rightarrow L^2(\Omega), \quad S(t)u_0 := u(t), \quad \forall t \geq 0, \quad (3.1)$$

and $\{S(t)\}_{t \geq 0}$ is Lipschitz continuous in $L^2(\Omega)$.

Moreover, we also have the following lemma about strong solutions, which can be obtained as that in [2] (similar to Theorem 8.5 in p. 227).

Lemma 3.2. *Let f satisfy (1.2)–(1.4), $g \in L^2(\Omega)$ and $u_0 \in \dot{W}_\lambda^{1,2}(\Omega) \cap L^p(\Omega)$. Then, for any $T > 0$, there exists a unique strong solution u to equation (1.1) such that*

$$u \in C([0, T]; \dot{W}_\lambda^{1,2}(\Omega)) \cap L^2(0, T; W_\lambda^{2,2}(\Omega)) \cap L^\infty(0, T; L^p(\Omega)), \quad u_t \in L^2(0, T; L^2(\Omega)).$$

Similar to [12,13], we need the following lemma about the strong solutions, which guarantees that the test functions that we used in the following are meaningful.

Theorem 3.3. *Let f satisfy (1.2)–(1.4). Then for any $T > 0$ and any initial data $(u_0, g) \in (\dot{W}_\lambda^{1,2}(\Omega) \cap L^\infty(\Omega), L^\infty(0, T; L^\infty(\Omega)))$, the unique strong solution $u(t)$ of equation (1.1) belongs to $L^\infty(0, T; L^\infty(\Omega))$.*

3.2 Some results

In this subsection, we give some results about the solutions of equation (1.1) under conditions (1.2)–(1.4).

At first, similar to that in [2] (see Proposition 11.1 in p. 286 and Theorem 11.4 in p. 290), we have the following lemmas, which will be used to obtain the bounded absorbing sets and the global attractors in $L^2(\Omega)$.

Lemma 3.4. *Let f satisfy (1.2)–(1.3), $g \in H^{-1}(\Omega)$ and $u_0 \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ defined by (3.1) has a $(L^2(\Omega), L^2(\Omega))$ -bounded absorbing set, that is, for any bounded subset B in $L^2(\Omega)$, there exists a $T = T(B)$, depending only on the L^2 -norm of B , such that*

$$\|u(t)\|_2^2 \leq \rho_0^2 \quad \text{for any } t \geq T \text{ and } u_0 \in B,$$

where $\rho_0 > 0$ is a constant independent of B and $u(t)$.

Lemma 3.5. *Let f satisfy (1.2)–(1.4), $g \in H^{-1}(\Omega)$ and $u_0 \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ defined by (3.1) has a $(L^2(\Omega), L^2(\Omega))$ -global attractor, which is nonempty, compact, invariant in $L^2(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ with respect to the L^2 -norm.*

Next, similar to Theorem 3.4 in [11], we have the following lemma, which will be used to obtain the bounded absorbing sets in $L^p(\Omega)$ and $\dot{W}_\lambda^{1,2}(\Omega)$.

Lemma 3.6. *Let f satisfy (1.2)–(1.4), $g \in H^{-1}(\Omega)$ and $u_0 \in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ defined by (3.1) has a $(L^2(\Omega), L^p(\Omega))$ and $(L^2(\Omega), \dot{W}_\lambda^{1,2}(\Omega))$ -bounded absorbing set, that is, for any bounded subset B in $L^2(\Omega)$, there exists a $T = T(B)$, depending only on the L^2 -norm of B , such that*

$$\|u(t)\|_p^p \leq \rho_1^2 \quad \text{for any } t \geq T \text{ and } u_0 \in B,$$

and

$$\|\nabla_\lambda u(t)\|_2^2 \leq \rho_1^2 \quad \text{for any } t \geq T \text{ and } u_0 \in B,$$

where $\rho_1 > 0$ is a constant independent of B and $u(t)$.

4 Asymptotic higher-order integrability

In this section, we will establish some asymptotic higher-order integrability for the solutions of equation (1.1).

4.1 Some *a priori* estimate for the solutions of (4.7)

In this subsection, based on the Alikakos-Moser iteration technique (see [18]), we will establish the following induction estimates about the solutions of (4.7).

For this purpose, we decompose equation (1.1) as a stationary equation

$$\begin{cases} \Delta_\lambda v + f(v) + \sum_{i=1}^N D_i g^i + g^0(x) = 0 & \text{in } \Omega, \\ v(x) = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

and an evolutionary equation

$$\begin{cases} \partial_t w(x, t) = \Delta_\lambda w(x, t) + f(u(x, t)) - f(v(x)) & \text{in } \Omega \times (0, \infty), \\ w(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty), \\ w(x, 0) = u_0(x) - v(x), & x \in \Omega. \end{cases} \quad (4.2)$$

By [19] we know that equation (4.1) has a weak solution $v \in \dot{W}_\lambda^{1,2}(\Omega) \cap L^p(\Omega)$, and then we cannot expect any higher-order integrability of the solutions of (4.1) than the order $\max\left\{p, \frac{2Q}{Q-2}\right\}$.

Then, we consider the asymptotic higher-order integrability of equation (4.2).

Note that $\dot{W}_\lambda^{1,2}(\Omega) \cap L^p(\Omega)$ is dense in $L^2(\Omega)$; then for each $g^i \in L^2(\Omega)$ and every initial data $u_0 \in L^2(\Omega)$, we can find $g_n^i \in \dot{W}_\lambda^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $u_{0n} \in \dot{W}_\lambda^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{aligned} g_n^i &\rightarrow g^i && \text{in } L^2(\Omega), \quad i = 1, 2, \dots, N, \\ u_{0n} &\rightarrow u_0 && \text{in } L^2(\Omega) \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.3)$$

Consider the following approximation equations: $n = 1, 2, \dots$,

$$\begin{cases} \partial_t u_n(x, t) = \Delta_\lambda u_n(x, t) + f(u_n(x, t)) + \sum_{i=1}^N D_i g_n^i + g_n^0(x) & \text{in } \Omega \times (0, \infty), \\ u_n(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty), \\ u_n(x, 0) = u_{0n}, & x \in \Omega, \end{cases} \quad (4.4)$$

and

$$\begin{cases} \Delta_\lambda v_n + f(v_n) + \sum_{i=1}^N D_i g_n^i + g_n^0(x) = 0 & \text{in } \Omega, \\ v_n(x) = 0, & \text{on } \partial\Omega. \end{cases} \quad (4.5)$$

Then, thanks to Lemma 3.2 and Theorem 3.3, we know that for each $n \in \mathbb{N}$, equation (4.4) has a unique strong solution u_n satisfying

$$u_n \in L^\infty(0, T; L^\infty(\Omega)) \cap L^\infty(0, T; \dot{W}_\lambda^{1,2}(\Omega)) \cap L^2(0, T; W_\lambda^{2,2}(\Omega)) \quad \text{for any } T > 0,$$

and by [19] we know that equation (4.5) has a strong solution v_n satisfying

$$v_n \in L^\infty(\Omega) \cap W_\lambda^{2,2}(\Omega). \quad (4.6)$$

Now, for each $n \in \mathbb{N}$, we choose the solutions of (4.5), which satisfies (4.6) and denote it by $v_n(x)$.

Set $w_n = u_n - v_n$, then w_n satisfies the following equation in distribution sense:

$$\begin{cases} \partial_t w_n(x, t) = \Delta_\lambda w_n(x, t) + f(u_n(x, t)) - f(v_n(x, t)) & \text{in } \Omega \times (0, \infty), \\ w_n(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty), \\ w_n(x, 0) = u_{0n}(x) - v_n(x), & x \in \Omega. \end{cases} \quad (4.7)$$

Moreover, we have

$$w_n(\cdot, t) \in \dot{W}_\lambda^{1,2}(\Omega) \cap L^\infty(\Omega) \quad \text{for a.e. } t \in [0, \infty),$$

and so for any $r \geq 0$,

$$|w_n(\cdot, t)|^r \cdot w_n(\cdot, t) \in \dot{W}_\lambda^{1,2}(\Omega) \cap L^\infty(\Omega) \quad \text{for a.e. } t \in [0, \infty). \quad (4.8)$$

Hence, for any $r \geq 0$, we can take $|w_n(\cdot, t)|^r \cdot w_n(\cdot, t)$ as a test function for equation (4.7) (this is the main reason for the approximations above).

With the preparation above, we have the following main result of this subsection.

Theorem 4.1. *For each $k = 0, 1, 2, \dots$, there exist two positive constants T_k and M_k , which depend only on $k, p, Q, \|u_{0n}\|_2$ and $\|g_n^i\|_2$ ($i = 1, 2, \dots, N$), such that for any $n = 1, 2, \dots$, the solutions w_n of (4.7) satisfy*

$$\int_{\Omega} |w_n(t)|^{2\left(\frac{Q}{Q-2}\right)^k} dx \leq M_k \quad \text{for any } t \geq T_k, \quad (A'_k)$$

and

$$\int_t^{t+1} \left(\int_{\Omega} |w_n(s)|^{2\left(\frac{Q}{Q-2}\right)^{k+1}} dx \right)^{\frac{Q-2}{Q}} ds \leq M_k \quad \text{for any } t \geq T_k, \quad (B'_k)$$

where Q is the homogeneous dimension of \mathbb{R}^N with a group of dilations corresponding to the Δ_λ -operator.

Proof. At first, by taking the L^2 -inner product between equation (4.5) and v_n , we know that there exists a positive constant $R = R_{\|g_n^i\|_2}$ which depends only on the L^2 -bounds of $\|g_n^i\|_2$ ($i = 0, 1, 2, \dots, N$) such that for every $n \in \mathbb{N}$, the solution $v_n(x)$ of (4.5) satisfies

$$\|v_n\|_{\dot{W}_\lambda^{1,2}(\Omega) \cap L^p(\Omega)} \leq R_{\|g_n^i\|_2}. \quad (4.9)$$

Second, for the solution of (4.4), similar to the estimate for the solution of equation (1.1) (e.g., Lemma 3.6), we know that for each $n \in \mathbb{N}$, there are positive constants $S_{\|g_n^i\|_2}$ (that depends only on the bounds of $\|g_n^i\|_2$) and $T_{\|u_{0n}\|_2}$ (that depends only on the bounds of $\|u_{0n}\|_2$) such that

$$\|u_n(t)\|_{\dot{W}_\lambda^{1,2}(\Omega) \cap L^p(\Omega)} \leq S_{\|g_n^i\|_2} \quad \text{for any } t \geq T_{\|u_{0n}\|_2}. \quad (4.10)$$

In the following, we will complete our proof.

(1) For the case $k = 0$.

Obviously, (4.9) and (4.10) imply (A'_0) directly.

Then, based on (4.9) and (4.10), by applying the embedding $\dot{W}_\lambda^{1,2}(\Omega) \hookrightarrow L^{\frac{2Q}{Q-2}}(\Omega)$ (see Lemma 2.1) to $w_n(t) = u_n(t) - v_n \in \dot{W}_\lambda^{1,2}(\Omega)$ with $\|w_n(t)\|_{\dot{W}_\lambda^{1,2}(\Omega)} \leq \|u_n(t)\|_{\dot{W}_\lambda^{1,2}(\Omega)} + \|v_n\|_{\dot{W}_\lambda^{1,2}(\Omega)}$, we easily obtain (B'_0) .

(2) Assume that (A'_k) and (B'_k) hold for $k \geq 0$.

(3) We need to prove that (A'_{k+1}) and (B'_{k+1}) hold.

By (4.8), multiplying (4.7) by $|w_n|^{2\left(\frac{Q}{Q-2}\right)^{k+1}-2} \cdot w_n$ and integrating over $x \in \Omega$, then we obtain that

$$\frac{1}{2} \left(\frac{Q-2}{Q} \right)^{k+1} \frac{d}{dt} \int_{\Omega} |w_n|^{2\left(\frac{Q}{Q-2}\right)^{k+1}} dx + \int_{\Omega} \nabla_\lambda w_n \cdot \nabla_\lambda (|w_n|^{2\left(\frac{Q}{Q-2}\right)^{k+1}-2} w_n) dx = \int_{\Omega} (f(u_n) - f(v_n)) |w_n|^{2\left(\frac{Q}{Q-2}\right)^{k+1}-2} w_n dx.$$

Thanks to (1.2) and Lemma 2.2 with $r = 2\left(\frac{Q}{Q-2}\right)^{k+1} - 2 > 0$, we can deduce that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |w_n|^{2\left(\frac{Q}{Q-2}\right)^{k+1}} dx + 2\left(2\left(\frac{Q}{Q-2}\right)^{k+1} - 1\right)\left(\frac{Q-2}{Q}\right)^{k+1} \int_{\Omega} \left| \nabla_{\lambda} |w_n|^{\left(\frac{Q}{Q-2}\right)^{k+1}} \right|^2 dx \\ & \leq 2l\left(\frac{Q}{Q-2}\right)^{k+1} \int_{\Omega} |w_n|^{2\left(\frac{Q}{Q-2}\right)^{k+1}} dx. \end{aligned} \quad (4.11)$$

Applying the uniform Gronwall lemma (see p. 91, Lemma 1.1 in [3]) to (4.11) and (B'_k) , we obtain that

$$\int_{\Omega} |w_n(t)|^{2\left(\frac{Q}{Q-2}\right)^{k+1}} dx \leq C_{M_k, l, Q, k} \quad \text{for any } t \geq T_k + 1,$$

that is, (A'_{k+1}) holds.

Let $t \geq T_k + 1$, we integrate (4.11) over $[t, t+1]$ and obtain that

$$\int_t^{t+1} \int_{\Omega} \left| \nabla_{\lambda} |w_n|^{\left(\frac{Q}{Q-2}\right)^{k+1}} \right|^2 dx ds \leq C_{M_k, l, Q, k}. \quad (4.12)$$

From (4.8) we know that

$$|w_n(\cdot, s)|^{\left(\frac{Q}{Q-2}\right)^{k+1}} \in \dot{W}_{\lambda}^{1,2}(\Omega) \quad \text{for a.e. } s \in [0, \infty). \quad (4.13)$$

By (4.13), then applying the embedding $\dot{W}_{\lambda}^{1,2}(\Omega) \hookrightarrow L^{\frac{2Q}{Q-2}}(\Omega)$ (see Lemma 2.1) to $|w_n(\cdot, s)|^{\left(\frac{Q}{Q-2}\right)^{k+1}}$, we can deduce (B'_{k+1}) immediately from (4.12). \square

Moreover, similar to Lemmas 3.2 and 3.3 in [10], we have the following results.

Lemma 4.2. *Under assumption (4.3), let v_n ($n = 1, 2, \dots$) be the solutions of (4.5). Then the sequence $\{v_n\}_{n=1}^{\infty}$ has a subsequence $\{v_{n_j}\}_{j=1}^{\infty}$ satisfying*

$$v_{n_j}(x) \rightarrow v(x) \quad \text{for a.e. } x \in \Omega \quad \text{as } j \rightarrow \infty.$$

Moreover, $v(x)$ is a solution of equation (4.1).

On the other hand, let u_n ($n = 1, 2, \dots$) be the unique solutions of (4.4) and u be the unique solutions of (1.1). Then we have the following continuity (w.r.t. initial data and forcing term):

$$\|u_n(t) - u(t)\|_2^2 \leq e^{2lt} \left(1 + \frac{1}{2l} \right) \left(\|u_{0n} - u_0\|_2^2 + \sum_{i=1}^N \|g_n^i - g^i\|_2^2 + \frac{1}{\mu_1} \|g_n^0 - g^0\|_2^2 \right)$$

for all $t \in [0, +\infty)$, where $\mu_1 > 0$ is the first eigenvalue of the operator $-\Delta_{\lambda}$ on Ω with homogeneous Dirichlet boundary conditions (see (2.2) in Lemma 2.1).

Moreover, for any $t \in [0, +\infty)$, there is a subsequence $\{u_{n_j}\}_{j=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ such that

$$u_{n_j}(x, t) \rightarrow u(x, t) \quad \text{for a.e. } x \in \Omega \quad \text{as } j \rightarrow \infty.$$

4.2 $(L^2(\Omega), L^p(\Omega))$ -global attractors

In the following, we will establish the main results of this paper as follows.

Theorem 4.3. *Under the assumptions of Lemma 3.4, suppose that $\{S(t)\}_{t \geq 0}$ is the semigroup defined by (3.1). Then, for any $\delta \in [0, \infty)$, there exists a bounded subset B_{δ} in $\dot{W}_{\lambda}^{1,2}(\Omega) \cap L^p(\Omega)$ satisfying the following properties:*

$$B_\delta = \{w \in \dot{W}_\lambda^{1,2}(\Omega) \cap L^{p+\delta}(\Omega) : \|w\|_{\dot{W}_\lambda^{1,2}(\Omega) \cap L^p(\Omega)} + \|w\|_{L^{p+\delta}(\Omega)} \leq \Lambda_{p,Q,\delta} < \infty\},$$

and for any L^2 -bounded subset $B \subset L^2(\Omega)$, there exists a $T = T(B, \delta)$ such that

$$S(t)B \subset v(x) + B_\delta \quad \text{for any } t \geq T. \quad (4.14)$$

Here, the constant $\Lambda_{p,Q,\delta}$ depends only on p, Q, δ , and $v(x)$ is one of the fixed (independent of δ) solutions of (4.1).

Proof. Note that the homogeneous dimension $Q \geq 3$, we have $\frac{Q}{Q-2} > 1$ and then

$$\left(\frac{Q}{Q-2}\right)^k \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Therefore, for any $\delta \in [0, \infty)$, we can take k large enough such that

$$2 \leq p + \delta \leq 2\left(\frac{Q}{Q-2}\right)^k.$$

Now, let u_n, v_n be the strong solution of (4.4) and (4.5), respectively, then for any $t \geq T_k$, by Lemma 4.2, we can find n_j such that $u_{n_j}(x, t) \rightarrow u(x, t), v_{n_j}(x) \rightarrow v(x)$ for a.e. $x \in \Omega$ as $j \rightarrow \infty$. Then, by applying the Fatou lemma to (A_k) in Theorem 4.1, we obtain that

$$\int_{\Omega} |u(x, t) - v(x)|^{2\left(\frac{Q}{Q-2}\right)^k} dx \leq \liminf_{j \rightarrow \infty} \int_{\Omega} |u_{n_j}(x, t) - v_{n_j}(x)|^{2\left(\frac{Q}{Q-2}\right)^k} dx \leq M_k < \infty. \quad (4.15)$$

Therefore, combining with (4.15) and the $\dot{W}_\lambda^{1,2} \cap L^p$ -dissipation (see Lemma 3.6), applying the interpolation inequality, we can define B_δ as follows:

$$B_\delta = \{w \in \dot{W}_\lambda^{1,2} \cap L^{p+\delta} : \|w\|_{\dot{W}_\lambda^{1,2} \cap L^p} + \|w\|_{L^{p+\delta}} \leq C_{M_k, M, \rho_0, \delta, \|v\|_{\dot{W}_\lambda^{1,2} \cap L^p}} < \infty\},$$

and (4.14) follows from (4.15) immediately. \square

As a directly application of Theorem 4.3, we have the following corollary.

Corollary 4.4. Under the assumptions of Lemma 3.4, the semigroup $\{S(t)\}_{t \geq 0}$ defined by (3.1) has a $(L^2(\Omega), L^p(\Omega))$ -global attractor \mathcal{A} . Moreover, \mathcal{A} can attract every bounded subset of $L^2(\Omega)$ with $L^{p+\delta}$ -norm for any $\delta \in [0, \infty)$, and \mathcal{A} allows the decomposition $\mathcal{A} = v(x) + \mathcal{B}$; here \mathcal{B} is bounded in $L^{p+\delta}(\Omega)$ for any $\delta \in [0, \infty)$, $v(x)$ is a fixed solution of (4.1).

Proof. From Lemma 3.5, we know that the semigroup $\{S(t)\}_{t \geq 0}$ is $(L^2(\Omega), L^2(\Omega))$ -asymptotically compact. Then by (4.15) with k large enough such that $p + 1 \leq 2\left(\frac{Q}{Q-2}\right)^k$, using the following interpolation estimates:

$$\begin{aligned} & \|u_1(x, t) - u_2(x, t)\|_{L^p} \\ &= \|u_1(x, t) - v(x) - (u_2(x, t) - v(x))\|_{L^p} \\ &\leq \|u_1(x, t) - v(x) - (u_2(x, t) - v(x))\|_{L^{p+1}}^{1-\theta} \|u_1(x, t) - v(x) - (u_2(x, t) - v(x))\|_{L^2}^\theta \\ &\leq C_{M_k, \theta} \|u_1(x, t) - u_2(x, t)\|_{L^2}^\theta, \end{aligned}$$

where $u_i(t) = S(t)u_{i0}$ ($i = 1, 2$) and $\theta \in (0, 1)$, we can deduce that the semigroup $\{S(t)\}_{t \geq 0}$ is $(L^2(\Omega), L^p(\Omega))$ -asymptotically compact.

Hence, combining with Lemmas 3.5 and 3.6, and by the theory of dynamical system (e.g., see [1–3, 20, 21]), we obtain the existence of the $(L^2(\Omega), L^p(\Omega))$ -global attractors.

Moreover, by applying the interpolation inequality and (4.15), we can get that

$$\begin{aligned} & \|u_1(x, t) - u_2(x, t)\|_{L^{p+\delta}} \\ &= \|u_1(x, t) - v(x) - (u_2(x, t) - v(x))\|_{L^{p+\delta}} \\ &\leq \|u_1(x, t) - v(x) - (u_2(x, t) - v(x))\|_{L^{2\frac{Q}{Q-2}k_1}}^{1-\theta_1} \|u_1(x, t) - v(x) - (u_2(x, t) - v(x))\|_{L^2}^{\theta_1} \\ &\leq C_{M_k, \theta_1} \|u_1(x, t) - u_2(x, t)\|_{L^2}^{\theta_1}, \end{aligned} \quad (4.16)$$

where $\theta_1 \in (0, 1)$, and for any $\delta \in [0, \infty)$, we can take $k_1 = \left\lceil \log_{\frac{Q}{Q-2}} \left(\frac{p+\delta}{2} \right) \right\rceil + 1$ such that

$$2 \left(\frac{Q}{Q-2} \right)^{k_1} > p + \delta.$$

On the other hand, from (A_k') in Theorem 4.1, there are positive constants T_{k_1} and M_{k_1} such that for any $\varepsilon > 0$, we have

$$\text{dist}_{L^2(\Omega)}(S(t)B - v(x), \mathcal{A} - v(x)) = \text{dist}_{L^2(\Omega)}(S(t)B, \mathcal{A}) \leq \left(\frac{\varepsilon}{C_{M_{k_1}, \theta_1}} \right)^{\frac{1}{\theta_1}} \quad (4.17)$$

for all $t \geq T_{k_1}$, where we have used (2.5) in Theorem 2.4 and the attraction of $(L^2(\Omega), L^2(\Omega))$ -global attractors, B is a L^2 -bounded set.

Combining with (4.15)–(4.17), we obtain that

$$\text{dist}_{L^{p+\delta}(\Omega)}(S(t)B, \mathcal{A}) \leq \varepsilon \quad \text{for any } t \geq T_{k_1}.$$

Finally, similar to (4.16) and applying (4.15) again, we can complete the proof of the residual part. \square

5 $(L^2(\Omega), \dot{W}_\lambda^{1,2}(\Omega))$ -global attractors

In this section, we will prove the existence of the $(L^2(\Omega), \dot{W}_\lambda^{1,2}(\Omega))$ -global attractors.

At first, we need the following lemma.

Lemma 5.1. *Under the assumptions of Lemma 3.4, then for any bounded subset B in $L^2(\Omega)$, there exists a $T = T(B)$, depending only on the L^2 -norm of B , such that*

$$\|\partial_t u(s)\|_2^2 \leq \rho_2^2 \quad \text{for any } u_0 \in B \text{ and } s \geq T, \quad (5.1)$$

where $\partial_t u(s) = \frac{d}{dt}(S(t)u_0)|_{t=s}$, $\{S(t)\}_{t \geq 0}$ is the semigroup defined by (3.1) and $\rho_2 > 0$ is a constant independent of B .

Proof. We give below only formal derivation of the estimate (5.1), which can be justified by the Galerkin approximation method.

By differentiating (1.1) with respect to time t and denoting $z = \partial_t u$, we can get

$$\partial_t z(x, t) = \Delta_\lambda z(x, t) + f'(u(x, t))z(x, t). \quad (5.2)$$

Multiplying (5.2) by z and integrating over Ω , then by using (1.2), we obtain that

$$\frac{1}{2} \frac{d}{dt} \|z\|_2^2 + \|\nabla_\lambda z\|_2^2 \leq l \|z\|_2^2.$$

Furthermore,

$$\frac{d}{dt} \|z\|_2^2 \leq 2l \|z\|_2^2. \quad (5.3)$$

On the other hand, multiplying (1.1) by $\partial_t u$, we obtain that

$$\|\partial_t u\|_2^2 + \frac{d}{dt} \left(\frac{1}{2} \|\nabla_\lambda u\|_2^2 - \int_\Omega \mathcal{F}(u) - \int_\Omega gu \right) = 0, \quad (5.4)$$

where $\mathcal{F}(u)$ satisfies (1.4).

Integrating (5.4) with respect to time t from t to $t + 1$, and combining with Lemmas 3.4 and 3.6, we can deduce that

$$\int_t^{t+1} \|\partial_t u\|_2^2 \leq C(\rho_0, \rho_1, |\Omega|, \|\tilde{g}\|_2^2, \|g^0\|_2^2) \quad (5.5)$$

as t is large enough.

Combining with (5.3) and (5.5), and using the uniform Gronwall lemma (see p. 91, Lemma 1.1 in [3]), we have

$$\|\partial_t u\|_2^2 \leq C(\rho_0, \rho_1, |\Omega|, \|\tilde{g}\|_2^2, \|g^0\|_2^2)$$

as t is large enough. \square

Theorem 5.2. *Under the assumptions of Lemma 3.4, the semigroup $\{S(t)\}_{t \geq 0}$ defined by (3.1) is $(L^2(\Omega), \dot{W}_\lambda^{1,2}(\Omega))$ -asymptotically compact.*

Proof. By Lemma 3.6, we know the semigroup $\{S(t)\}_{t \geq 0}$ has a $(L^2(\Omega), \dot{W}_\lambda^{1,2}(\Omega))$ -bounded absorbing set B_1 . Let u_1, u_2 be the solution of equation (1.1) with initial data $u_{01}, u_{02} \in B_1$. From Theorem 4.3, we know the solutions u_1, u_2 can be decomposed as

$$u_i = S(t)u_0 = w_i + v, \quad i = 1, 2,$$

where $w_i \in B_\delta$, $v(x)$ is a fixed solution of (4.1).

Set $\omega = u_1 - u_2$, then ω satisfies the following equation in distribution sense:

$$\begin{cases} \partial_t \omega(x, t) = \Delta_\lambda \omega(x, t) + f(u_1(x, t)) - f(u_2(x, t)) & \text{in } \Omega \times (0, \infty), \\ \omega(x, t) = 0 & \text{on } \partial\Omega \times [0, \infty), \\ \omega(x, 0) = u_{01}(x) - u_{02}(x), & x \in \Omega. \end{cases} \quad (5.6)$$

Taking the L^2 -inner product between (5.6) and ω , we have

$$\|\nabla_\lambda \omega\|_2^2 = -(\partial_t \omega, \omega) + (f(u_1) - f(u_2), \omega).$$

Thanks to (1.2) and (5.1) in Lemma 5.1, we can deduce that

$$\|\nabla_\lambda \omega\|_2^2 \leq \|\partial_t \omega\|_2 \|\omega\|_2 + I \|\omega\|_2^2 \leq (C_{\rho_2} + I) \|\omega\|_2, \quad (5.7)$$

where the constant C_{ρ_2} is related to ρ_2 , ρ_2 comes from (5.1).

By Lemma 3.5, we know the semigroup $\{S(t)\}_{t \geq 0}$ is $(L^2(\Omega), L^2(\Omega))$ -asymptotically compact. Then by virtue of (5.7), we know the semigroup $\{S(t)\}_{t \geq 0}$ is $(L^2(\Omega), \dot{W}_\lambda^{1,2}(\Omega))$ -asymptotically compact. \square

Combining with Lemma 3.6 and Theorem 5.2, by the theory of dynamical system (e.g., see [1–3, 20, 21]), we immediately obtain the following conclusion.

Theorem 5.3. *Under the assumptions of Lemma 3.4, the semigroup $\{S(t)\}_{t \geq 0}$ defined by (3.1) has a $(L^2(\Omega), \dot{W}_\lambda^{1,2}(\Omega))$ -global attractor, which is nonempty, compact, invariant in $\dot{W}_\lambda^{1,2}(\Omega)$ and attracts every bounded subset of $\dot{W}_\lambda^{1,2}(\Omega)$ with respect to the $\dot{W}_\lambda^{1,2}$ -norm.*

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