Research Article

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Multiple solutions and ground state solutions for a class of generalized Kadomtsev-Petviashvili equation

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Abstract: In this paper, we study the following generalized Kadomtsev-Petviashvili equation

$$u_t + u_{xxx} + (h(u))_x = D_x^{-1}\Delta_y u,$$

where $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{N-1}$, $N \geq 2$, $D_x^{-1}f(x, y) = \int_{-\infty}^{x} f(s, y)\mathrm{d}s$, $f_t = \frac{\partial f}{\partial t}$, $f_x = \frac{\partial f}{\partial x}$ and $\Delta_y = \sum_{i=1}^{N-1} \frac{\partial^2}{\partial^2_{y_i}}$. We get the existence of infinitely many nontrivial solutions under certain assumptions in bounded domain without Ambrosetti-Rabinowitz condition. Moreover, by using the method developed by Jeanjean [13], we establish the existence of ground state solutions in $\mathbb{R}^N$.

Keywords: generalized Kadomtsev-Petviashvili equation, ground state solutions, multiplicity of solutions

MSC 2020: 35J60, 35J20

1 Introduction

This article is concerned with the following generalized Kadomtsev-Petviashvili equation:

$$u_t + u_{xxx} + (h(u))_x = D_x^{-1}\Delta_y u, \tag{1.1}$$

where $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{N-1}$, $N \geq 2$, $D_x^{-1}f(x, y) = \int_{-\infty}^{x} f(s, y)\mathrm{d}s$, $f_t = \frac{\partial f}{\partial t}$, $f_x = \frac{\partial f}{\partial x}$ and $\Delta_y = \sum_{i=1}^{N-1} \frac{\partial^2}{\partial^2_{y_i}}$.

To find a solitary wave for (1.1), it needs us to get a solution u of the form $u(t, x, y) = u(x - \tau t, y)$, with $\tau \geq 0$. Hence, equation (1.1) can be rewritten as:

$$-\tau u_x + u_{xxx} + (h(u))_x = D_x^{-1}\Delta_y u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

If we choose $h(s) = s^2$ in (1.1), then equation (1.1) is a two-dimensional generalization of the Korteweg-de Vries equation, which describes long dispersive waves in mathematical models, see [1]. When $h(s) = |s|^p s$ with $p = \frac{m}{n}$, where m and n are relative prime numbers, and n is odd, Bouard and Saut [2,3] proved that there is a solitary wave for (1.1) with $1 \leq p < 4$, if $N = 2$, or $1 \leq p < \frac{4}{3}$, if $N = 3$, via the concentration com-

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pactness principle from [4,5]. In [6], Willem proved the existence of solitary waves of (1.1) as $N = 2$ and $h \in C^1(\mathbb{R}, \mathbb{R})$. In [7], Xuan extended the results obtained by [6] to higher dimension. In [8], $h(u)$ was replaced by $Q(x, y)|u|^{p-2}u$, and Liang and Su had obtained nontrivial solutions of (1.1). In [9], Xu and Wei studied infinitely many solutions for $u_{xxx} + (h(u))_x = D_x^{-1}\Delta_y u$ with the Ambrosetti-Rabinowitz condition in bounded domain. For related contributions to study of solitary waves of the generalized Kadomtsev-Petviashvili equations, we refer to previous studies [10,11].

The aim of this paper is to prove the existence of multiple solutions of (1.3) in bounded domain without condition (AR), which is to ensure the boundedness of the (PS) sequences of the corresponding functional, and obtain the ground state solutions of (1.2) in $\mathbb{R}^N$. In what follows, we assume that the function $h : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(h1) $h \in C(\mathbb{R})$, $h(0) = 0$;

(h2) for some $p \in (1, \bar{N} - 1)$, where $\bar{N} = \frac{4N-2}{2N-3}$, $\lim_{|t|\to+\infty}\frac{h(t)}{|t|^p} = \lim_{t\to 0}\frac{h(t)}{|t|} = 0$;

(h3) $h(t) = -h(-t)$, $\lim_{|t|\to+\infty}\frac{H(t)}{|t|^2} = +\infty$, where $H(t) = \int_0^t h(r)\mathrm{d}r$;

(h4) there exist $\mu > 2$, $\kappa > 0$ such that $\mu H(t) \le th(t) + \kappa t^2$;

(h5) there exists $\mu > 2$ such that $0 \le \mu H(t) \le h(t)t$.

Consider the following system,

$$\begin{cases} -\tau u_x + u_{xxx} + (h(u))_x = D_x^{-1}\Delta_y u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain.

Now, we can state our first result.

Theorem 1.1. *Assume that* (h_1)–(h_4) *are satisfied, then equation* (1.3) *possesses infinitely many nontrivial solutions in* Ω, *where* $\Omega \subset \mathbb{R}^N$ *is a bounded domain.*

Our second result is as follows.

Theorem 1.2. *Assume that* (h_1)–(h_2) *and* (h_5) *are satisfied, then equation* (1.2) *has a ground state solution.*

Notations. Throughout the paper, we denote by $\|\cdot\|_p$ the usual norm of Lebesgue space $L^p(\mathbb{R}^N)$. X^* is the dual space of X. The symbol C denotes a positive constant and may vary from line to line.

2 Preliminary

In this section, we want to introduce the functional setting and some main results. At first, we present the functional setting (see [7,11]).

Definition 2.1. [7] On $Y = \{g_x : g \in C_0^\infty(\mathbb{R}^N)\}$, define the inner product

$$(u, v) = \int_{\mathbb{R}^N} (u_x v_x + D_x^{-1}\nabla_y u D_x^{-1}\nabla_y v + \tau uv)\mathrm{d}V, \quad \tau > 0,$$

and the norm is

$$\|u\| = \left(\int_{\mathbb{R}^N} (|u_x|^2 + |D_x^{-1}\nabla_y u|^2 + \tau|u|^2)\mathrm{d}V\right)^{\frac{1}{2}}, \quad \tau > 0$$

where $\nabla_y = \left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{N-1}}\right)$ and $dV = dxdy$.

If there exists a sequence $\{u_n\} \subset Y$ such that $u_n \to u$ a.e. on $\mathbb{R}^N$, and $\|u_j - u_k\| \to 0$ as $j, k \to \infty$, then we say that $u : \mathbb{R}^N \to \mathbb{R}$ belongs to X.

Definition 2.2. [7] On $Y = \{g_x : g \in C_0^\infty(\mathbb{R}^N)\}$, define the inner product

$$(u, v)_0 = \int_{\mathbb{R}^N} (u_x v_x + D_x^{-1}\nabla_y u D_x^{-1}\nabla_y v)\, dV,$$

and the norm is

$$\|u\|_0 = \left(\int_{\mathbb{R}^N} (|u_x|^2 + |D_x^{-1}\nabla_y u|^2)\, dV\right)^{\frac{1}{2}},$$

where $\nabla_y = \left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{N-1}}\right)$ and $dV = dxdy$.

If there exists a sequence $\{u_n\} \subset Y$ such that $u_n \to u$ a.e. on $\mathbb{R}^N$, and $\|u_j - u_k\|_0 \to 0$ as $j, k \to \infty$, then we say that $u : \mathbb{R}^N \to \mathbb{R}$ belongs to X_0.

Lemma 2.1. [7,11,12] *The following continuous embeddings hold.*
- (*i*) *the embeddings* $X \hookrightarrow X_0$ *are continuous*;
- (*ii*) *the embeddings* $X \hookrightarrow L^q(\mathbb{R}^N)$, *for* $1 \le q \le \bar{N}$ *are continuous*;
- (*iii*) *the embeddings* $X \hookrightarrow L^q_{\mathrm{loc}}(\mathbb{R}^N)$, *for* $1 \le q < \bar{N}$ *are compact*;
- (*iv*) *the embeddings* $X_0 \hookrightarrow L^{\bar{N}}(\mathbb{R}^N)$ *are continuous.*

Lemma 2.2. [14] *Let X be an infinite dimensional Banach space, and there exists a finite dimensional space W such that* $X = W \oplus V$. $I \in C^1(\mathbb{R})$ *satisfies the (PS) condition, and*
- (*i*) $I(u) = I(-u)$ *for all* $u \in X$, $I(0) = 0$;
- (*ii*) *there exist* $\rho > 0$, $\alpha > 0$ *such that* $I|_{\partial B_\rho} \cap V \ge \alpha$;
- (*iii*) *for any finite dimensional subspace* $Y \subset X$, *there is* $R = R(Y) > 0$ *such that* $I(u) \le 0$ *on* $Y \backslash B_R$.

Then I possesses an unbounded sequence of critical values.

Lemma 2.3. [7] *Assume that* $\{u_n\}$ *is a bounded sequence in X. If*

$$\lim_{n\to+\infty} \sup_{(x,y)\in\mathbb{R}^N} \int_{B_r((x,y))} |u_n|^2 dV = 0,$$

then $u_n \to 0$ *in* $L^q(\mathbb{R}^N)$ *for all* $q \in (2, \bar{N})$.

Lemma 2.4. [13] *Let* $(X, \|\cdot\|)$ *be a Banach space and* $T \subset \mathbb{R}^+$ *be an interval. Consider a family of* C^1 *functionals on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u) \quad \forall \lambda \in T,$$

with $B(u) \ge 0$ *and either* $A(u) \to +\infty$ *or* $B(u) \to +\infty$ *as* $\|u\| \to +\infty$. *If there are two points* $v_1, v_2 \in X$ *such that*

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\} \quad \forall \lambda \in T,$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in T$, *there exists a bounded* $(PS)_{c_\lambda}$ *sequence in X, and the mapping* $\lambda \to c_\lambda$ *is non-increasing and left continuous.*

3 Proof of Theorem 1.1

In this section, we consider the boundary value problem (1.3). The energy functional $I : X \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega H(u)\mathrm{d}V$$

and

$$I'(u)[v] = \int_\Omega (u_x v_x + D_x^{-1}\nabla_y u D_x^{-1}\nabla_y v + \tau uv)\mathrm{d}V - \int_\Omega h(u)v\mathrm{d}V.$$

Lemma 3.1. *Suppose h satisfies (h_1)–(h_4). If $\{u_n\} \subset X$ satisfies*
(i) $\{I(u_n)\}$ is bounded;
(ii) $\langle I'(u_n), u_n\rangle \to 0$,
then $\{u_n\}$ is bounded in X.

Proof. If $\{u_n\}$ is unbounded in X, we can find a subsequence still denoted by $\{u_n\}$ such that $\{u_n\} \to +\infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, we have $\|v_n\| = 1$. Thus, we may assume that $v_n \rightharpoonup v$ in X. As the embedding $X \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^N)$ is compact, we have $v_n \to v$ in $L^2(\Omega)$. By (h_4) and (i), there exists $c > 0$ such that

$$c + 1 \geq I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \geq \frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{\kappa}{\mu}\|u_n\|_2^2 = \frac{\mu-2}{2\mu}\|u_n\|^2\|v_n\|^2 - \frac{\kappa}{\mu}\|u_n\|^2\|v_n\|_2^2,$$

as $n \to +\infty$, which implies $1 \leq \frac{2\kappa}{\mu-2}\lim_{n\to+\infty}\sup\|v_n\|_2^2$. Therefore, $v \not\equiv 0$. By (h_3) and Fatou's Lemma, one has

$$0 = \lim_{n\to+\infty}\frac{c}{\|u_n\|^2} = \lim_{n\to+\infty}\frac{I(u_n)}{\|u_n\|^2} = \lim_{n\to+\infty}\left(\frac{1}{2} - \int_\Omega \frac{H(u)}{u_n^2}v_n^2\right) = -\infty,$$

which is a contradiction. Hence, $\{u_n\}$ is bounded in X. □

Lemma 3.2. *Suppose h satisfies (h_1)–(h_4). Then the functional I satisfies the (PS) condition.*

Proof. To prove that I satisfies the (PS) condition, we only need to prove $\{u_n\} \subset X$ has a convergent subsequence, where $\{u_n\}$ obtained by Lemma 3.1. As $\{u_n\}$ is bounded in X, there exists a subsequence still denoted by $\{u_n\}$ and $u_0 \in X$ such that $u_n \rightharpoonup u_0$ in X and $u_n \to u_0$ in $L^q(\Omega)$ for $1 \leq q < \bar{N}$. From (h_2), we have

$$|h(u_n)| \leq \varepsilon|u_n| + C_\varepsilon|u_n|^p, \quad \forall\varepsilon > 0.$$

then

$$\left(\int_\Omega |h(u_n)|^{\frac{p+1}{p}}\mathrm{d}V\right)^{\frac{p}{p+1}} \leq \left(\int_\Omega (\varepsilon|u_n| + C_\varepsilon|u_n|^p)^{\frac{p+1}{p}}\mathrm{d}V\right)^{\frac{p}{p+1}} \leq C(\|u_n\| + \|u_n\|^p) < +\infty.$$

Applying the Hölder inequality, for $1 < p < \bar{N} - 1$, one has

$$\begin{aligned}\int_\Omega ((h(u_n) - h(u_0))(u_n - u_0))\mathrm{d}V &\leq \left(\int_\Omega |h(u_n) - h(u_0)|^{\frac{p+1}{p}}\mathrm{d}V\right)^{\frac{p}{p+1}}\left(\int_\Omega |u_n - u_0|^{p+1}\mathrm{d}V\right)^{\frac{1}{p+1}} \\ &\leq C\left(\int_\Omega\left(|h(u_n)|^{\frac{p+1}{p}} + |h(u_0)|^{\frac{p+1}{p}}\right)\mathrm{d}V\right)^{\frac{p}{p+1}}\left(\int_\Omega |u_n - u_0|^{p+1}\mathrm{d}V\right)^{\frac{1}{p+1}} \to 0.\end{aligned}$$

It follows from $u_n \rightharpoonup u_0$ in X and $I'(u_0) \in X^*$ that $\langle I'(u_0), u_n - u_0\rangle \to 0$. And as $I'(u_n) \to 0$ in X^*, it is easy to obtain

$$\langle I'(u_n), u_n - u_0\rangle \le \|I'(u_n)\|_{X^*}\|u_n - u_0\|_{X(\Omega)} \to 0.$$

Therefore,

$$\langle I'(u_n) - I'(u_0), u_n - u_0\rangle = \langle I'(u_0), u_n - u_0\rangle - \langle I'(u_n), u_n - u_0\rangle \to 0,$$

as $n \to +\infty$.

Thus, we have

$$\|u_n - u_0\|^2 = \langle I'(u_n) - I'(u_0), u_n - u_0\rangle + \int_\Omega ((h(u_n) - h(u_0))(u_n - u_0))\mathrm{d}V \to 0,$$

as $n \to +\infty$. □

Proof of Theorem 1.1. We have verified that I satisfies the (PS) condition. It follows from (h_5) that I is an even function. As X is a separable space, X has orthonormal basis $\{e_i\}$. Define $X_j := \mathbb{R}e_j$, $W_k := \oplus_{j=1}^k X_j$, $V_k := \overline{\oplus_{j=k+1}^\infty X_j}$. Let $W = W_k$, $V = V_k$, clearly $X = W \oplus V$ and $\dim W < \infty$.

Next, we verify that I satisfies (ii) in Lemmas 2.2. By Lemma 2.1, for all $u \in V$, we have

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega H(u)\mathrm{d}V \ge \frac{1}{2}\|u\|^2 - \left(\frac{\varepsilon}{2}\|u\|_2^2 + \frac{C_\varepsilon}{p}\|u\|_{p+1}^{p+1}\right) \ge \frac{1}{2}\|u\|^2 - C(\varepsilon\|u\|^2 + C_\varepsilon\|u\|^{p+1}).$$

Then, there exists $\rho > 0$ small enough, $\alpha > 0$ such that $I(u) \ge \alpha > 0$ as $\|u\| = \rho$.

Now, we verify that I satisfies (iii) in Lemma 2.2. For any finite dimensional subspace $Y \subset X$, since $\lim_{|t|\to+\infty}\frac{H(t)}{|t|^2} = +\infty$, for $u \ne 0$,

$$I(ru) = \frac{r^2}{2}\|u\|^2 - \int_\Omega H(ru)\mathrm{d}V = \frac{r^2}{2}\left(\|u\|^2 - 2\int_\Omega \frac{H(ru)}{(ru)^2}u^2\mathrm{d}V\right) \to -\infty,$$

as $r \to +\infty$. Thus, there exists $r_0 > 0$ such that $I(ru) < 0$ for all $r \ge r_0 > 0$. So, we can conclude that there exists a $R(Y) > 0$ such that $I(u) \le 0$ on $Y\backslash B_{R(Y)}$.

Hence, according to Lemma 2.2, equation (1.3) possesses infinitely many nontrivial solutions. □

4 Proof of Theorem 1.2

In this section, the weak solutions of (1.2) are the critical points of the energy functional I, where $I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} H(u)\mathrm{d}V$. As h satisfies (h_1)–(h_2) and (h_5), it is clear that I is of class $C^1(X, \mathbb{R})$. To apply Jeanjean's trick [13], we give a family of energy functions

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda\int_{\mathbb{R}^N} H(u)\mathrm{d}V, \quad \forall\lambda \in \left[\frac{1}{2}, 1\right].$$

Lemma 4.1. *Suppose that h satisfies (h_1)–(h_2) and (h_5). Then*

(i) there exists $v \in X\backslash\{0\}$ such that $I_\lambda(v) < 0$ for all $\lambda \in \left[\frac{1}{2}, 1\right]$;

(ii) $c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(v)\}$ for all $\lambda \in \left[\frac{1}{2}, 1\right]$, where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0,\ \gamma(1) = v\}.$$

Proof. (i) By (h_5), we have $\lim_{s\to+\infty}\frac{H(s)}{s^2} = +\infty$. Furthermore, for some $u \in X$

$$I_\lambda(tu) = \frac{t^2}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} H(tu)\mathrm{d}V \le \frac{t^2}{2}\left(\|u\|^2 - \int_{\mathbb{R}^N} \frac{H(tu)}{(tu)^2}u^2\mathrm{d}V\right) \to -\infty \quad \text{as } t \to +\infty.$$

Thus, there exists $t_0 > 0$ such that $I_\lambda(t_0u) < 0$. By taking $v = t_0u$, we have $I_\lambda(v) < 0$.

(ii) By virtue of (h_2), for any $\varepsilon > 0$ and some $p \in (1, \bar{N} - 1)$, there exists $C_\varepsilon > 0$ such that

$$|H(t)| \le \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{p}|t|^{p+1} \quad \forall t \in \mathbb{R}.$$

By Lemma 2.3, we have

$$I_\lambda(u) \ = \ \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} H(u)\mathrm{d}V \ge \ \frac{1}{2}\|u\|^2 - \left(\frac{\varepsilon}{2}\|u\|_2^2 + \frac{C_\varepsilon}{p}\|u\|_{p+1}^{p+1}\right) \ge \ \frac{1}{2}\|u\|^2 - C\left(\varepsilon\|u\|^2 + C_\varepsilon\|u\|^{p+1}\right).$$

Then, there exists $\rho > 0$ small enough such that

$$b := \inf_{\|u\|=\rho} I_\lambda(u) > 0 = I_\lambda(0) > I_\lambda(v).$$

Therefore, $c_\lambda > \max\{I_\lambda(0), I_\lambda(v)\}$. □

Combining Lemma 4.1 with Theorem 2.6, we have the following conclusion.

Lemma 4.2. *Suppose h satisfies (h_1)–(h_2) and (h_5). For almost every $\lambda \in \left[\frac{1}{2}, 1\right]$, there is a bounded sequence $\{v_m\}$, such that $I_\lambda(v_m) \to c_\lambda$ in X and $I'_\lambda(v_m) \to 0$ in the dual X^* of X.*

Lemma 4.3. *If $\{v_m\}$ is a bounded sequence in X and $\lim_{m\to+\infty}\sup_{(x,y)\in\mathbb{R}^N}\int_{B_1((x,y))}|v_m|^2\mathrm{d}V = 0$, then $\lim_{m\to+\infty}\int_{\mathbb{R}^N} G(v_m) = 0$, where $G(v_m) = \frac{1}{2}h(v_m)v_m - H(v_m)$.*

Proof. On one hand, by simple calculations, we derive

$$\int_{\mathbb{R}^N} H(v_m)\mathrm{d}V \le \frac{\varepsilon}{2}\|v_m\|_2^2 + \frac{C_\varepsilon}{p+1}\|v_m\|_{p+1}^{p+1},$$

$$\int_{\mathbb{R}^N} h(v_m)v_m\mathrm{d}V \le \varepsilon\|v_m\|_2^2 + C_\varepsilon\|v_m\|_{p+1}^{p+1}.$$

On the other hand, by Lemma 2.3, we have $v_m \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (2, \bar{N})$. Hence, we can conclude that

$$\lim_{m\to+\infty}\int_{\mathbb{R}^N} H(v_m)\mathrm{d}V = 0,$$

$$\lim_{m\to+\infty}\int_{\mathbb{R}^N} h(v_m)v_m\mathrm{d}V = 0.$$

Thus, $\lim_{m\to+\infty}\int_{\mathbb{R}^N} G(v_m)\mathrm{d}V = 0$. □

Lemma 4.4. *If $\{v_m\} \subset X$ is the sequence obtained by* Lemma 4.2, *then for a.e. $\lambda \in \left[\frac{1}{2}, 1\right]$, there exists a sequence of points $\{(x_m, y_m)\} \subset \mathbb{R} \times \mathbb{R}^{N-1}$, $u_m(x, y) := v_m(x - x_m, y - y_m)$, such that*
(i) $u_m \rightharpoonup u_\lambda \neq 0$ *in* X;
(ii) $I'_\lambda(u_\lambda) = 0$ *in* X^*;
(iii) $I_\lambda(u_\lambda) \le c_\lambda$ *in* X; and
(iv) *there exists* $M > 0$ *such that* $I_\lambda(u_\lambda) \ge M$.

Proof. By Lemma 4.2, we know that for almost every $\lambda \in \left[\frac{1}{2}, 1\right]$, there exists a bounded sequence $\{v_m\}$ that satisfy $I_\lambda(v_m) \to c_\lambda$ in X and $I'_\lambda(v_m) \to 0$ in X^* as $m \to +\infty$. Furthermore,

$$\int_{\mathbb{R}^N} G(v_m) = I_\lambda(v_m) - \frac{1}{2}\langle I'_\lambda(v_m), v_m\rangle \to c_\lambda > 0 \quad \text{as } m \to +\infty.$$

By Lemma 4.3, there exist a sequence of points $\{(x_m, y_m)\} \subset \mathbb{R} \times \mathbb{R}^{N-1}$ and $\alpha > 0$, such that

$$\int_{B_1(x_m, y_m)} v_m^2 \mathrm{d}V \geq \alpha > 0.$$

Let $u_m(x, y) := v_m(x - x_m, y - y_m)$. By the invariance translations of I_λ, as $m \to +\infty$, we have that $I_\lambda(u_m) \to c_\lambda$ in X and $I'_\lambda(u_m) \to 0$ in X^*. Since $\{u_m\}$ is bounded, there exists $u_\lambda \in X$ such that $u_m \rightharpoonup u_\lambda$ in X.

In the following, we complete the proof of this lemma.

(i) It follows from Lemma 2.1 that

$$C\|u_\lambda\|^2 \geq \|u_\lambda\|_2^2 \geq \int_{B_1(0)} u_\lambda^2 \mathrm{d}V = \lim_{m\to+\infty} \int_{B_1(0)} u_m^2 \mathrm{d}V \geq \alpha > 0,$$

and thus obtain $u_\lambda \not\equiv 0$ in X.

(ii) As $C_0^\infty(\mathbb{R}^N)$ is dense in X, we only need to check that $\langle I'_\lambda(u_\lambda), \varphi\rangle = 0$ for any $\varphi \in X$. We have

$$\langle I'_\lambda(u_m), \varphi\rangle - \langle I'_\lambda(u_\lambda), \varphi\rangle = (u_m - u_\lambda, \varphi) - \lambda \int_{\mathbb{R}^N} [h(u_m) - h(u_\lambda)]\varphi \mathrm{d}V \to 0,$$

since $u_m \rightharpoonup u_\lambda$ in X, $u_m \to u_\lambda$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for $1 \leq p \leq \bar{N}$. It follows from $I'_\lambda(u_m) \to 0$ that $I'_\lambda(u_\lambda) = 0$.

(iii) By (h_5) and Fatou's Lemma, we get

$$\begin{aligned} c_\lambda &= \lim_{m\to+\infty}\left[I_\lambda(u_m) - \frac{1}{2}\langle I'_\lambda(u_m), u_m\rangle\right] = \lambda \lim_{m\to+\infty} \int_{\mathbb{R}^N} G(u_m)\mathrm{d}V \\ &\geq \lambda \int_{\mathbb{R}^N} G(u_\lambda)\mathrm{d}V = I_\lambda(u_\lambda) - \frac{1}{2}\langle I'_\lambda(u_\lambda), u_\lambda\rangle = I_\lambda(u_\lambda). \end{aligned}$$

(iv) Combining (ii) with (h_2) and Lemma 2.1, we obtain that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|u_\lambda\|^2 = \lambda \int_{\mathbb{R}^N} h(u_\lambda)u_\lambda \mathrm{d}V \leq \int_{\mathbb{R}^N} h(u_\lambda)u_\lambda \mathrm{d}V \leq C(\varepsilon\|u_\lambda\|^2 + C_\varepsilon\|u_\lambda\|^{p+1}).$$

Then, there exists $\beta > 0$ such that $\|u_\lambda\| \geq \beta > 0$. Therefore,

$$\begin{aligned} I_\lambda(u_\lambda) &= I_\lambda(u_\lambda) - \frac{1}{\mu}\langle I'_\lambda(u_\lambda), u_\lambda\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_\lambda\|^2 + \int_{\mathbb{R}^N} \frac{1}{\mu}h(u_\lambda)u_\lambda - H(u_\lambda)\mathrm{d}V \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_\lambda\|^2 \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\beta^2 := M > 0. \end{aligned}$$

This completes the proof. □

Now, according to Lemmas 4.2 and 4.4, there exists a sequence $\{(\lambda_n, u_{\lambda_n})\} \subset \left[\frac{1}{2}, 1\right] \times X$, such that

(i) $\lambda_n \to 1$ as $n \to +\infty$; (ii) $u_{\lambda_n} \not\equiv 0$, $M \leq I_{\lambda_n}(u_{\lambda_n}) \leq c_{\lambda_n}$ and $I'_{\lambda_n}(u_{\lambda_n}) = 0$.

Lemma 4.5. (Pohozaev identity, [7]) *Suppose h satisfies* (h_1)–(h_2). *If* $u \in X$ *is a weak solution of the equation*:

$$-\tau u_x + u_{xxx} + \lambda(h(u))_x = D_x^{-1}\Delta_y u \quad \text{in } \mathbb{R}^N,$$

then we have the following Pohozaev identity:

$$\mathcal{P}_\lambda(u) := \frac{2N-3}{2}\|u\|_0^2 + (2N-1)\int\limits_{\mathbb{R}^N}\left(\frac{\tau}{2}u^2 - \lambda H(u)\right)\mathrm{d}V = 0.$$

Proof of Theorem 1.2. By Lemma 4.5, if $\{u_{\lambda_n}\}$ is nontrivial solution of equation

$$-\tau u_x + u_{xxx} + \lambda_n(h(u))_x = D_x^{-1}\Delta_y u \quad \text{in } \mathbb{R}^N,$$

then $\{u_{\lambda_n}\}$ satisfies the following equation:

$$\mathcal{P}_{\lambda_n}(u_{\lambda_n}) = \frac{2N-3}{2}\|u_{\lambda_n}\|_0^2 + \frac{(2N-1)\tau}{2}\int\limits_{\mathbb{R}^N} u_{\lambda_n}^2\mathrm{d}V - (2N-1)\lambda_n\int\limits_{\mathbb{R}^N} H(u_{\lambda_n})\mathrm{d}V = 0.$$

Remember that

$$c_{\lambda_n} \ge I_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2N-1}\mathcal{P}_{\lambda_n}(u_{\lambda_n}) = \frac{1}{2N-1}\|u_{\lambda n}\|_0^2.$$

So,

$$\|u_{\lambda_n}\|_0^2 \le (2N-1)c_{\lambda_n} \le (2N-1)c_{\frac{1}{2}},$$

it follows from Lemma 2.1 that $\{u_{\lambda_n}\}$ is bounded in X_0 and also in $L^{\bar{N}}$.

Since $I'_{\lambda_n}(u_{\lambda_n}) = 0$, we have

$$\langle I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n}\rangle = \|u_{\lambda_n}\|^2 - \lambda_n\int\limits_{\mathbb{R}^N} h(u_{\lambda_n})u_{\lambda_n}\mathrm{d}V = 0.$$

Moreover, by Lemma 2.1, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|u_{\lambda_n}\|^2 = \lambda_n\int\limits_{\mathbb{R}^N} h(u_{\lambda_n})u_{\lambda_n}\mathrm{d}V \le \varepsilon C\|u_{\lambda_n}\|^2 + C_\varepsilon\|u_{\lambda_n}\|_{\bar{N}}^{\bar{N}}.$$

Then, for ε small enough, there exists a constant $C > 0$ such that $\|u_{\lambda_n}\|^2 \le C$, since $\{u_{\lambda_n}\}$ is bounded in $L^{\bar{N}}$. Thus, $\{u_{\lambda_n}\}$ is bounded in X. By the facts that for any $\varphi \in X$,

$$\langle I'(u_{\lambda_n}), \varphi\rangle = \langle I'_{\lambda_n}(u_{\lambda_n}), \varphi\rangle + (\lambda_n - 1)\int\limits_{\mathbb{R}^N} h(u_{\lambda_n})\varphi\mathrm{d}V,$$

$$I(u_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1)\int\limits_{\mathbb{R}^N} H(u_{\lambda_n})\mathrm{d}V,$$

and $\{u_{\lambda_n}\}$ is bounded in X, it follows that $M \le \lim_{n\to+\infty} I(u_{\lambda_n}) \le c_1$ and $\lim_{n\to+\infty} I'(u_{\lambda_n}) = 0$. Up to a subsequence, there exists a subsequence still denoted by $\{u_{\lambda_n}\}$ and $u_0 \in X$ such that $u_{\lambda_n} \rightharpoonup u_0$ in X. By using the method in Lemma 4.4, we can obtain the existence of a nontrivial solution u_0 for I such that $I'(u_0) = 0$ and $I(u_0) \le c_1$. Thus, u_0 is a nontrivial solution of (1.2). Define $m := \inf\{I(u) : u \ne 0, I'(u) = 0\}$. Let $\{u_n\}$ be a sequence such that $I'(u_n) = 0$ and $I(u_n) \to m$. Similar to arguments in Lemma 4.4, we can prove that there exists $\bar{u} \in X$ such that $I'(\bar{u}) = 0$ and $I(\bar{u}) \le m$. By the definition of m, we have $m \le I(\bar{u})$. Hence, $\bar{I}(\bar{u}) = m$, which shows that $\bar{u}$ is a ground state solution of (1.2). □

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