



## Research Article

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# Resolving resolution dimensions in triangulated categories

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**Abstract:** Let  $\mathcal{T}$  be a triangulated category with a proper class  $\xi$  of triangles and  $\mathcal{X}$  be a subcategory of  $\mathcal{T}$ . We first introduce the notion of  $\mathcal{X}$ -resolution dimensions for a resolving subcategory of  $\mathcal{T}$  and then give some descriptions of objects having finite  $\mathcal{X}$ -resolution dimensions. In particular, we obtain Auslander-Buchweitz approximations for these objects. As applications, we construct adjoint pairs for two kinds of inclusion functors and characterize objects having finite  $\mathcal{X}$ -resolution dimensions in terms of a notion of  $\xi$ -cellular towers. We also construct a new resolving subcategory from a given resolving subcategory and reformulate some known results.

**Keywords:** triangulated categories, a proper class of triangles, resolving resolution dimensions, resolving subcategories, Auslander-Buchweitz approximations

**MSC 2020:** 18G20, 18G25, 18G10

## 1 Introduction

Approximation theory is the main part of relative homological algebra and representation theory of algebras, and its starting point is to approximate arbitrary objects by a class of suitable subcategories. In particular, resolving subcategories play important roles in approximation theory (e.g., [1–3]). As an important example of resolving subcategories, Auslander and Buchweitz [4] studied the approximation theory of the subcategory consisting of maximal Cohen-Macaulay modules over an artin algebra, and Hernández et al. [5] developed an analogous theory for triangulated categories. Using the approximation triangles established by Hernández et al. [5, Theorem 5.4], Di and Wang [6] constructed additive functors (adjoint pairs) between additive quotient categories. On the other hand, Zhu [7] studied the resolution dimension with respect to a resolving subcategory in an abelian category, and Huang [8] introduced relative preresolving subcategories in an abelian category and defined homological dimensions relative to these subcategories, which generalized many known results (see [4,9,10]).

In analogy to relative homological algebra in abelian categories, Beligiannis [11] developed a relative version of homological algebra in a triangulated category  $\mathcal{T}$ , that is, a pair  $(\mathcal{T}, \xi)$ , in which  $\xi$  is a proper class of triangles (see Definition 2.4). Under this notion, a triangulated category is just equipped with a proper class consisting of all triangles. However, there are lots of non-trivial cases, for example, let  $\mathcal{T}$  be a compactly generated triangulated category, then the class  $\xi$  consisting of pure triangles is a proper class ([12]), and the pair  $(\mathcal{T}, \xi)$  is no longer triangulated in general. Later on, this theory has been paid more attentions and developed (e.g., [13–17]). It is natural to ask how the approximation theory acts on this relative setting of triangulated categories. In [18], Ma et al., introduced the notions of (pre)resolving

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subcategories and homological dimensions relative to these subcategories in this relative setting, which gives a parallel theory analogy to that of abelian categories [8]. In this paper, we devote to further studying relative homological dimensions in triangulated categories with respect to a resolving subcategory. The paper is organized as follows:

In Section 2, we give some terminology and some preliminary results.

In Section 3, some homological properties of resolving subcategories are obtained. In particular, we obtain Auslander-Buchweitz approximation triangles (see Proposition 3.10) for objects having finite resolving resolution dimensions. Our main result is the following:

**Theorem.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  and  $\mathcal{H}$ , a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ . Assume that  $\mathcal{H}$  is closed under hokernels of  $\xi$ -proper epimorphisms or closed under direct summands. For any  $M \in \mathcal{T}$ , if  $M \in \widehat{\mathcal{X}}$ , then the following statements are equivalent:*

- (1)  $\mathcal{X}$ -res.dim  $M \leq m$ .
- (2)  $\Omega^n(M) \in \mathcal{X}$  for all  $n \geq m$ .
- (3)  $\Omega_{\mathcal{X}}^n(M) \in \mathcal{X}$  for all  $n \geq m$ .
- (4)  $\xi\text{xt}_{\xi}^n(M, H) = 0$  for all  $n > m$  and all  $H \in \mathcal{H}$ .
- (5)  $\xi\text{xt}_{\xi}^n(M, L) = 0$  for all  $n > m$  and all  $L \in \widehat{\mathcal{H}}$ .
- (6)  $M$  admits a right  $\mathcal{X}$ -approximation  $\varphi : X \rightarrow M$ , where  $\varphi$  is  $\xi$ -proper epic, such that  $K = \text{Hoker } \varphi$  satisfying  $\mathcal{H}$ -res.dim  $K \leq m - 1$ .
- (7) There are two triangles

$$W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M$$

and

$$M \longrightarrow W^M \longrightarrow X^M \longrightarrow \Sigma M$$

in  $\xi$  such that  $X_M$  and  $X^M$  are in  $\mathcal{X}$  and  $\mathcal{H}$ -res.dim  $W_M \leq m - 1$ ,  $\mathcal{H}$ -res.dim  $W^M = \mathcal{X}$ -res.dim  $W^M \leq m$ .

In Section 4, we will further study objects having finite resolution dimensions with respect to a resolving subcategory  $\mathcal{X}$ . We first construct adjoint pairs for two kinds of inclusion functors. Then we characterize objects having finite resolution dimensions in terms of a notion of  $\xi$ -cellular towers.

As an application, in Section 5, given a resolving subcategory  $\mathcal{X}$  of  $\mathcal{T}$ , we construct a new resolving subcategory  $\mathcal{GP}_{\mathcal{X}}(\xi)$  with a  $\xi$ xt-injective  $\xi$ -cogenerator  $\mathcal{X} \cap {}^{\perp}\mathcal{X}$ , which generalizes the Gorenstein projective subcategory  $\mathcal{GP}(\xi)$  given by Asadollahi and Salarian [13]. Applying the obtained results to  $\mathcal{GP}_{\mathcal{X}}(\xi)$ , we generalize some known results in [13–15].

Throughout this paper, all subcategories are full, additive, and closed under isomorphisms.

## 2 Preliminaries

Let  $\mathcal{T}$  be an additive category and  $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$  an additive functor. One defines the category  $\text{Diag}(\mathcal{T}, \Sigma)$  as follows:

- An object of  $\text{Diag}(\mathcal{T}, \Sigma)$  is a diagram in  $\mathcal{T}$  of the form  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ .
- A morphism in  $\text{Diag}(\mathcal{T}, \Sigma)$  between  $X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Z_i \xrightarrow{w_i} \Sigma X_i$ ,  $i = 1, 2$ , is a triple  $(\alpha, \beta, \gamma)$  of morphisms in  $\mathcal{T}$  such that the following diagram:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_1} & Y_1 & \xrightarrow{v_1} & Z_1 & \xrightarrow{w_1} & \Sigma X_1 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \Sigma \alpha \\ X_2 & \xrightarrow{u_2} & Y_2 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & \Sigma X_2 \end{array}$$

commutes.

A triangulated category is a triple  $(\mathcal{T}, \Sigma, \Delta)$ , where  $\mathcal{T}$  is an additive category and  $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$  is an auto-equivalence of  $\mathcal{T}$  (called suspension functor), and  $\Delta$  is a full subcategory of  $\text{Diag}(\mathcal{T}, \Sigma)$  which is closed under isomorphisms and satisfies the axioms  $(T_1)$ – $(T_4)$  in [11, Section 2.1] (also see [19]), where the axiom  $(T_4)$  is called the octahedral axiom. The elements in  $\Delta$  are called *triangles*.

The following result is well known, which is an efficient tool in studying triangulated categories.

**Remark 2.1.** [11, Proposition 2.1] Let  $\mathcal{T}$  be an additive category and  $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$  an autoequivalence of  $\mathcal{T}$ , and  $\Delta$  a full subcategory of  $\text{Diag}(\mathcal{T}, \Sigma)$  which is closed under isomorphisms. Suppose that the triple  $(\mathcal{T}, \Sigma, \Delta)$  satisfies all the axioms of a triangulated category except possibly of the octahedral axiom. Then, the following statements are equivalent:

- (1) **Octahedral axiom.** For any two morphisms  $u : X \rightarrow Y$  and  $v : Y \rightarrow Z$ , there exists a commutative diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{u} & Y & \xrightarrow{u'} & Z' & \xrightarrow{u''} & \Sigma X \\
 \parallel \downarrow & & \downarrow v & & \downarrow \alpha & & \downarrow \parallel \\
 X & \xrightarrow{vu} & Z & \xrightarrow{w} & Y' & \xrightarrow{w'} & \Sigma X \\
 \downarrow u & & \parallel \downarrow & & \downarrow \beta & & \downarrow \Sigma u \\
 Y & \xrightarrow{v} & Z & \xrightarrow{v'} & X' & \xrightarrow{v''} & \Sigma Y \\
 \downarrow & & \downarrow 0 & & \downarrow (\Sigma u')v'' & & \downarrow \\
 0 & \longrightarrow & \Sigma Z' & \xrightarrow{=} & \Sigma Z' & \longrightarrow & 0,
 \end{array}$$

in which all rows and the third column are triangles in  $\Delta$ .

- (2) **Base change.** For any triangle  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$  in  $\Delta$  and any morphism  $\alpha : Z' \rightarrow Z$ , there exists the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X' & \xrightarrow{=} & X' & \longrightarrow & 0 \\
 \downarrow & & \downarrow \beta' & & \downarrow \beta & & \downarrow \\
 X & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X \\
 \parallel \downarrow & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \parallel \\
 X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
 \downarrow & & \downarrow \gamma' & & \downarrow \gamma & & \downarrow \\
 0 & \longrightarrow & \Sigma X' & \xrightarrow{=} & \Sigma X' & \longrightarrow & 0,
 \end{array}$$

in which all rows and columns are triangles in  $\Delta$ .

- (3) **Cobase change.** For any triangle  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$  in  $\Delta$  and any morphism  $\beta : X \rightarrow X'$ , there exists the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Sigma^{-1} Z' & \xrightarrow{=} & \Sigma^{-1} Z' & \longrightarrow & 0 \\
 \downarrow & & \downarrow -\Sigma^{-1} \gamma & & \downarrow -\Sigma^{-1} \gamma' & & \downarrow \\
 \Sigma^{-1} Z & \xrightarrow{-\Sigma^{-1} w} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\
 \parallel \downarrow & & \downarrow \beta & & \downarrow \beta' & & \downarrow \parallel \\
 \Sigma^{-1} Z & \xrightarrow{-\Sigma^{-1} w'} & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z \\
 \downarrow & & \downarrow \alpha & & \downarrow \alpha' & & \downarrow \\
 0 & \longrightarrow & Z' & \xrightarrow{=} & Z' & \longrightarrow & 0,
 \end{array}$$

in which all rows and columns are triangles in  $\Delta$ .

Throughout this paper,  $\mathcal{T} = (\mathcal{T}, \Sigma, \Delta)$  is a triangulated category.

**Definition 2.2.** [11] A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is called *split* if it is isomorphic to the triangle

$$X \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} X \oplus Z \xrightarrow{(0,1)} Z \xrightarrow{0} \Sigma X.$$

We use  $\Delta_0$  to denote the full subcategory of  $\Delta$  consisting of all split triangles.

**Definition 2.3.** [11] Let  $\xi$  be a class of triangles in  $\mathcal{T}$ .

(1)  $\xi$  is said to be *closed under base change* (resp. *cobase change*) if for any triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in  $\xi$  and any morphism  $\alpha : Z' \rightarrow Z$  (resp.  $\beta : X \rightarrow X'$ ) as in Remark 2.1(2) (resp. Remark 2.1(3)), the triangle

$$X \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X \quad (\text{resp. } X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} \Sigma X')$$

is in  $\xi$ .

(2)  $\xi$  is said to be *closed under suspension* if for any triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in  $\xi$  and any  $i \in \mathbb{Z}$  (the set of all integers), the triangle

$$\Sigma^i X \xrightarrow{(-1)^i \Sigma^i u} \Sigma^i Y \xrightarrow{(-1)^i \Sigma^i v} \Sigma^i Z \xrightarrow{(-1)^i \Sigma^i w} \Sigma^{i+1} X$$

is in  $\xi$ .

(3)  $\xi$  is called *saturated* if in the situation of base change as in Remark 2.1(2), whenever the third vertical and the second horizontal triangles are in  $\xi$ , then the triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is in  $\xi$ .

**Definition 2.4.** [11] A class  $\xi$  of triangles in  $\mathcal{T}$  is called *proper* if the following conditions are satisfied.

- (1)  $\xi$  is closed under isomorphisms, finite coproducts and  $\Delta_0 \subseteq \xi$ .
- (2)  $\xi$  is closed under suspensions and is saturated.
- (3)  $\xi$  is closed under base and cobase change.

Throughout this paper, we always assume that  $\xi$  is a proper class of triangles in  $\mathcal{T}$ .

**Definition 2.5.** [11] Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

be a triangle in  $\xi$ . Then, the morphism  $u$  (resp.  $v$ ) is called  $\xi$ -*proper monic* (resp.  $\xi$ -*proper epic*), and  $u$  (resp.  $v$ ) is called the *hokernel* of  $v$  (resp. the *hocokernel* of  $u$ ).

We use  $\text{Hoker } v$  to denote the hokernel of  $v : Y \rightarrow Z$ . Dually, we use  $\text{Hocok } u$  to denote the hocokernel of  $u : X \rightarrow Y$ . For any triangle,

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

in  $\xi$ . We say that  $\mathcal{X}$  is *closed under  $\xi$ -extensions* if  $X, Z \in \mathcal{X}$ , it holds that  $Y \in \mathcal{X}$ . We say that  $\mathcal{X}$  is *closed under hokernels of  $\xi$ -proper epimorphisms* (resp. *hocokernels of  $\xi$ -proper monomorphisms*) if  $Y, Z \in \mathcal{X}$  (resp.  $X, Y \in \mathcal{X}$ ), it holds that  $X \in \mathcal{X}$  (resp.  $Z \in \mathcal{X}$ ).

**Definition 2.6.** (see [11, 4.1]) An object  $P$  (resp.  $I$ ) in  $\mathcal{T}$  is called  $\xi$ -projective (resp.  $\xi$ -injective) if for any triangle  $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$  in  $\xi$ , the induced complex

$$0 \rightarrow \text{Hom}_{\mathcal{T}}(P, X) \rightarrow \text{Hom}_{\mathcal{T}}(P, Y) \rightarrow \text{Hom}_{\mathcal{T}}(P, Z) \rightarrow 0$$

$$(\text{resp. } 0 \rightarrow \text{Hom}_{\mathcal{T}}(Z, I) \rightarrow \text{Hom}_{\mathcal{T}}(Y, I) \rightarrow \text{Hom}_{\mathcal{T}}(X, I) \rightarrow 0)$$

is exact. We use  $\mathcal{P}(\xi)$  (resp.  $\mathcal{I}(\xi)$ ) to denote the full subcategory of  $\mathcal{T}$  consisting of  $\xi$ -projective (resp.  $\xi$ -injective) objects.

We say that  $\mathcal{T}$  has enough  $\xi$ -projective objects if for any object  $M \in \mathcal{T}$ , there exists a triangle  $K \rightarrow P \rightarrow M \rightarrow \Sigma K$  in  $\xi$  with  $P \in \mathcal{P}(\xi)$ . Dually, we say that  $\mathcal{T}$  has enough  $\xi$ -injective objects if for any object  $M \in \mathcal{T}$ , there exists a triangle  $M \rightarrow I \rightarrow K \rightarrow \Sigma M$  in  $\xi$  with  $I \in \mathcal{I}(\xi)$ .

**Remark 2.7.**  $\mathcal{P}(\xi)$  is closed under direct summands, hokernels of  $\xi$ -proper epimorphisms, and  $\xi$ -extensions. Dually,  $\mathcal{I}(\xi)$  is closed under direct summands, hocokernels of  $\xi$ -proper monomorphisms, and  $\xi$ -extensions.

**Definition 2.8.** Let  $\mathcal{E}$  be a subcategory of  $\mathcal{T}$ .

(1) A triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

in  $\xi$  is called  $\text{Hom}_{\mathcal{T}}(\mathcal{E}, -)$ -exact (resp.  $\text{Hom}_{\mathcal{T}}(-, \mathcal{E})$ -exact) if for any object  $E$  in  $\mathcal{E}$ , the induced complex

$$0 \rightarrow \text{Hom}_{\mathcal{T}}(E, X) \rightarrow \text{Hom}_{\mathcal{T}}(E, Y) \rightarrow \text{Hom}_{\mathcal{T}}(E, Z) \rightarrow 0$$

$$(\text{resp. } 0 \rightarrow \text{Hom}_{\mathcal{T}}(Z, E) \rightarrow \text{Hom}_{\mathcal{T}}(Y, E) \rightarrow \text{Hom}_{\mathcal{T}}(X, E) \rightarrow 0)$$

is exact.

(2) [13] A  $\xi$ -exact complex is a complex

$$\dots \rightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \rightarrow \dots \tag{2.1}$$

in  $\mathcal{T}$  such that for any  $n \in \mathbb{Z}$ , there exists a triangle

$$K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1} \tag{2.2}$$

in  $\xi$  and the differential  $d_n$  is defined as  $d_n = g_{n-1}f_n$ . A  $\xi$ -exact complex as (2.1) is called  $\text{Hom}_{\mathcal{T}}(\mathcal{E}, -)$ -exact (resp.  $\text{Hom}_{\mathcal{T}}(-, \mathcal{E})$ -exact) if the triangle (2.2) is  $\text{Hom}_{\mathcal{T}}(\mathcal{E}, -)$ -exact (resp.  $\text{Hom}_{\mathcal{T}}(-, \mathcal{E})$ -exact) for any  $n \in \mathbb{Z}$ .

Asadollahi and Salarian [13] introduced the notion of  $\xi$ -Gorenstein projective objects.

**Definition 2.9.** [13, Definition 3.6] Let  $\mathcal{T}$  be a triangulated category with enough  $\xi$ -projective objects and  $X$  an object in  $\mathcal{T}$ . A complete  $\xi$ -projective resolution is a  $\text{Hom}_{\mathcal{T}}(-, \mathcal{P}(\xi))$ -exact  $\xi$ -exact complex

$$\dots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow \dots$$

in  $\mathcal{T}$  with all  $P_i \xi$ -projective objects. The objects  $K_n$  as in (2.2) are called  $\xi$ -Gorenstein projective objects. We use  $\mathcal{GP}(\xi)$  to denote the full subcategory of  $\mathcal{T}$  consisting of all  $\xi$ -Gorenstein projective objects.

Throughout this paper, we always assume that  $\mathcal{T}$  is a triangulated category with enough  $\xi$ -projective objects and  $\xi$ -injective objects.

Let  $M$  be an object in  $\mathcal{T}$ . Beligiannis [11] defined the  $\xi$ -extension groups  $\xi \text{xt}_{\xi}^n(-, M)$  to be the  $n$ th right  $\xi$ -derived functor of the functor  $\text{Hom}_{\mathcal{T}}(-, M)$ , that is,

$$\xi \text{xt}_{\xi}^n(-, M) := \mathcal{R}_{\xi}^n \text{Hom}_{\mathcal{T}}(-, M).$$

**Remark 2.10.** Let

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

be a triangle in  $\xi$ . By [11, Corollary 4.12], there exists a long exact sequence

$$\begin{aligned} 0 \longrightarrow \xi\text{xt}_\xi^0(Z, M) \longrightarrow \xi\text{xt}_\xi^0(Y, M) \longrightarrow \xi\text{xt}_\xi^0(X, M) \longrightarrow \\ \xi\text{xt}_\xi^1(Z, M) \longrightarrow \xi\text{xt}_\xi^1(Y, M) \longrightarrow \xi\text{xt}_\xi^1(X, M) \longrightarrow \dots \end{aligned}$$

of “ $\xi\text{xt}$ ” functor. If  $\mathcal{T}$  has enough  $\xi$ -injective objects and  $N$  is an object in  $\mathcal{T}$ , then there exists a long exact sequence

$$\begin{aligned} 0 \longrightarrow \xi\text{xt}_\xi^0(N, X) \longrightarrow \xi\text{xt}_\xi^0(N, Y) \longrightarrow \xi\text{xt}_\xi^0(N, Z) \longrightarrow \\ \xi\text{xt}_\xi^1(N, X) \longrightarrow \xi\text{xt}_\xi^1(N, Y) \longrightarrow \xi\text{xt}_\xi^1(N, Z) \longrightarrow \dots \end{aligned}$$

of “ $\xi\text{xt}$ ” functor.

Following Remark 2.10, we usually use the strategy of “dimension shifting,” which is an important tool in relative homological theory of triangulated categories.

Now, we set

$$\begin{aligned} \mathcal{X}^\perp &= \{M \in \mathcal{T} \mid \xi\text{xt}_\xi^{n \geq 1}(X, M) = 0 \text{ for all } X \in \mathcal{X}\}, \\ {}^\perp \mathcal{X} &= \{M \in \mathcal{T} \mid \xi\text{xt}_\xi^{n \geq 1}(M, X) = 0 \text{ for all } X \in \mathcal{X}\}. \end{aligned}$$

For two subcategories  $\mathcal{H}$  and  $\mathcal{X}$  of  $\mathcal{T}$ , we say  $\mathcal{H} \perp \mathcal{X}$  if  $\mathcal{H} \subseteq {}^\perp \mathcal{X}$  (equivalently,  $\mathcal{X} \subseteq \mathcal{H}^\perp$ ).

Taking  $\mathcal{C} = \mathcal{E} = \mathcal{P}(\xi)$  in [18, Definitions 3.1 and 3.2], we have the following definitions.

**Definition 2.11.** (cf. [18, Definition 3.1]) Let  $\mathcal{H}$  and  $\mathcal{X}$  be two subcategories of  $\mathcal{T}$  with  $\mathcal{H} \subseteq \mathcal{X}$ . Then,  $\mathcal{H}$  is called a  $\xi$ -cogenerator of  $\mathcal{X}$  if for any object  $X$  in  $\mathcal{X}$ , there exists a triangle

$$X \longrightarrow H \longrightarrow Z \longrightarrow \Sigma X$$

in  $\xi$  with  $H$  an object in  $\mathcal{H}$  and  $Z$  an object in  $\mathcal{X}$ . In particular, a  $\xi$ -cogenerator  $\mathcal{H}$  is called  $\xi\text{xt}$ -injective if  $\mathcal{X} \perp \mathcal{H}$ .

**Definition 2.12.** (cf. [18, Definition 3.2]) Let  $\mathcal{T}$  be a triangulated category with enough  $\xi$ -projective objects and  $\mathcal{X}$  a subcategory of  $\mathcal{T}$ . Then,  $\mathcal{X}$  is called a *resolving* subcategory of  $\mathcal{T}$  if the following conditions are satisfied.

- (1)  $\mathcal{P}(\xi) \subseteq \mathcal{X}$ .
- (2)  $\mathcal{X}$  is closed under  $\xi$ -extensions.
- (3)  $\mathcal{X}$  is closed under hokernels of  $\xi$ -proper epimorphisms.

### 3 Resolution dimensions with respect to a resolving subcategory

Taking  $\mathcal{E} = \mathcal{P}(\xi)$  in [18, Definition 3.5], we first have the following definition.

**Definition 3.1.** Let  $\mathcal{X}$  be a subcategory of  $\mathcal{T}$  and  $M$  an object in  $\mathcal{T}$ . The  $\mathcal{X}$ -resolution dimension of  $M$ , written  $\mathcal{X}\text{-res.dim } M$ , is defined by

$$\begin{aligned} \mathcal{X}\text{-res.dim } M &= \inf\{n \geq 0 \mid \text{there exists a } \xi\text{-exact complex} \\ &0 \longrightarrow X_n \longrightarrow \dots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0 \text{ in } \mathcal{T} \text{ with all } X_i \text{ objects in } \mathcal{X}\}. \end{aligned}$$

For a  $\xi$ -exact complex

$$\dots \xrightarrow{f_{n+1}} X_n \longrightarrow \dots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \longrightarrow 0$$

with all  $X_i \in \mathcal{X}$ . The Hoker  $f_{n-1}$  is called an  $n$ th  $\xi$ - $\mathcal{X}$ -syzygy of  $M$ , denoted by  $\Omega_{\mathcal{X}}^n(M)$ . In case for  $\mathcal{X} = \mathcal{P}(\xi)$ , we write  $\xi$ -pd  $M := \mathcal{X}$ -res.dim  $M$  and  $\Omega^n(M) := \Omega_{\mathcal{P}(\xi)}^n(M)$ . In case for  $\mathcal{X} = \mathcal{GP}(\xi)$ ,  $\mathcal{X}$ -res.dim  $M$  coincides with  $\xi$ - $\mathcal{GP}$ d  $M$  defined in [13] as  $\xi$ -Gorenstein projective dimension. We use  $\widehat{\mathcal{X}}$  to denote the full subcategory of  $\mathcal{T}$  whose objects have finite  $\mathcal{X}$ -resolution dimension.

**Lemma 3.2.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{X}$  a resolving subcategory of  $\mathcal{T}$ . For any object  $M \in \mathcal{T}$ , if*

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow Y_n \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow M \longrightarrow 0$$

are  $\xi$ -exact complexes with all  $X_i$  and  $Y_i$  in  $\mathcal{X}$  for  $0 \leq i \leq n-1$ , then  $X_n \in \mathcal{X}$  if and only if  $Y_n \in \mathcal{X}$ .

**Proof.** For  $M \in \mathcal{T}$ , there exists a  $\xi$ -exact complex

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with  $P_i \in \mathcal{P}(\xi)$  for  $0 \leq i \leq n-1$ .

Consider the following triangle:

$$K_1^M \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K_1^M$$

in  $\xi$ . As a similar argument to that of [11, Proposition 4.11], we get the following  $\xi$ -exact complex

$$0 \longrightarrow K_n \longrightarrow X_n \oplus P_{n-1} \longrightarrow X_{n-1} \oplus P_{n-2} \longrightarrow \cdots \longrightarrow X_2 \oplus P_1 \longrightarrow X_1 \oplus P_0 \longrightarrow X_0 \longrightarrow 0.$$

Similarly, we have the following  $\xi$ -exact complex

$$0 \longrightarrow K_n \longrightarrow Y_n \oplus P_{n-1} \longrightarrow Y_{n-1} \oplus P_{n-2} \longrightarrow \cdots \longrightarrow Y_2 \oplus P_1 \longrightarrow Y_1 \oplus P_0 \longrightarrow Y_0 \longrightarrow 0.$$

Set

$$X := \text{Hoker}(X_{n-1} \oplus P_{n-2} \longrightarrow X_{n-2} \oplus P_{n-3})$$

and

$$Y := \text{Hoker}(Y_{n-1} \oplus P_{n-2} \longrightarrow Y_{n-2} \oplus P_{n-3}).$$

Since  $\mathcal{X}$  is resolving, we have that  $X$  and  $Y$  are objects in  $\mathcal{X}$ . Consider the following triangles:

$$K_n \longrightarrow X_n \oplus P_{n-1} \longrightarrow X \longrightarrow \Sigma K_n$$

and

$$K_n \longrightarrow Y_n \oplus P_{n-1} \longrightarrow Y \longrightarrow \Sigma K_n$$

in  $\xi$ , we have that  $X_n \oplus P_{n-1} \in \mathcal{X}$  if and only if  $K_n \in \mathcal{X}$  if and only if  $Y_n \oplus P_{n-1} \in \mathcal{X}$ .

But from the following triangles in  $\xi$

$$X_n \longrightarrow X_n \oplus P_{n-1} \longrightarrow P_{n-1} \xrightarrow{0} \Sigma X_n \quad \text{and} \quad Y_n \longrightarrow Y_n \oplus P_{n-1} \longrightarrow P_{n-1} \xrightarrow{0} \Sigma Y_n,$$

we have that  $X_n \in \mathcal{X}$  if and only if  $X_n \oplus P_{n-1} \in \mathcal{X}$ , and  $Y_n \in \mathcal{X}$  if and only if  $Y_n \oplus P_{n-1} \in \mathcal{X}$ . Thus,  $X_n \in \mathcal{X}$  if and only if  $Y_n \in \mathcal{X}$ .  $\square$

Using the above, we can get:

**Proposition 3.3.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  and  $M \in \mathcal{T}$ . Then, the following statements are equivalent:*

- (1)  $\mathcal{X}$ -res.dim  $M \leq m$ .
- (2)  $\Omega^n(M) \in \mathcal{X}$  for  $n \geq m$ .
- (3)  $\Omega_{\mathcal{X}}^n(M) \in \mathcal{X}$  for  $n \geq m$ .

**Proof.** Apply Lemma 3.2.  $\square$

Now we can compare resolution dimensions in a given triangle in  $\xi$  as follows.

**Proposition 3.4.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$ , and let*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

*be a triangle in  $\xi$ . Then, we have the following statements:*

- (1)  $\mathcal{X}\text{-res.dim } B \leq \max\{\mathcal{X}\text{-res.dim } A, \mathcal{X}\text{-res.dim } C\}$ .
- (2)  $\mathcal{X}\text{-res.dim } A \leq \max\{\mathcal{X}\text{-res.dim } B, \mathcal{X}\text{-res.dim } C - 1\}$ .
- (3)  $\mathcal{X}\text{-res.dim } C \leq \max\{\mathcal{X}\text{-res.dim } A + 1, \mathcal{X}\text{-res.dim } B\}$ .

**Proof.** For any  $A \in \mathcal{T}$ , if  $\mathcal{X}\text{-res.dim } A = m$ , by Proposition 3.3, we have the following  $\xi$ -exact complex

$$0 \longrightarrow P_m^A \longrightarrow P_{m-1}^A \longrightarrow \cdots \longrightarrow P_1^A \longrightarrow P_0^A \longrightarrow A \longrightarrow 0$$

in  $\mathcal{T}$  with  $P_i^A \in \mathcal{P}(\xi)$  for  $0 \leq i \leq m - 1$  and  $P_m^A \in \mathcal{X}$ .

(1) Assume  $\mathcal{X}\text{-res.dim } A = m$  and  $\mathcal{X}\text{-res.dim } C = n$ . We proceed it by induction on  $m$  and  $n$ . The case  $m = n = 0$  is trivial. Without loss of generality, we assume  $m \leq n$ , then we can let  $P_i^A = 0$  for  $i > m$ . As a similar argument to that of [11, Proposition 4.11], we get the following  $\xi$ -exact complex:

$$0 \longrightarrow P_n^A \oplus P_n^C \longrightarrow P_{n-1}^A \oplus P_{n-1}^C \longrightarrow \cdots \longrightarrow P_0^A \oplus P_0^C \longrightarrow B \longrightarrow 0$$

in  $\mathcal{T}$ . Thus,  $\mathcal{X}\text{-res.dim } B \leq n$  and the desired assertion are obtained.

(2) Assume  $\mathcal{X}\text{-res.dim } B = m$  and  $\mathcal{X}\text{-res.dim } C = n$ . We proceed it by induction on  $m$  and  $n$ . The case  $m = n = 0$  is trivial. Without loss of generality, we assume  $m \leq n - 1$ , then we can let  $P_i^B = 0$  for  $i > m$ . By [18, Theorem 3.7], there exist a  $\xi$ -exact complex

$$0 \longrightarrow P_n^C \oplus P_{n-1}^B \longrightarrow P_{n-1}^C \oplus P_{n-2}^B \longrightarrow \cdots \longrightarrow P_2^C \oplus P_1^B \longrightarrow K \longrightarrow A \longrightarrow 0$$

and a triangle

$$K \longrightarrow P_1^C \oplus P_0^B \longrightarrow P_0^C \longrightarrow K[1]$$

in  $\xi$ , it follows that  $K \in \mathcal{P}(\xi)$  by Remark 2.7. Thus,  $\mathcal{X}\text{-res.dim } A \leq n - 1$  and the desired assertion is obtained.

(3) Assume  $\mathcal{X}\text{-res.dim } A = m$  and  $\mathcal{X}\text{-res.dim } B = n$ . We proceed it by induction on  $m$  and  $n$ . The case  $m = n = 0$  is trivial. Without loss of generality, we assume  $m + 1 \leq n$ , then we can let  $P_i^A = 0$  for  $i > m$ . By [18, Theorem 3.8], we have the following  $\xi$ -exact complex

$$0 \longrightarrow P_n^B \oplus P_{n-1}^A \longrightarrow \cdots \longrightarrow P_2^B \oplus P_1^A \longrightarrow P_1^B \oplus P_0^A \longrightarrow P_0^B \longrightarrow C \longrightarrow 0$$

in  $\mathcal{T}$ , thus  $\mathcal{X}\text{-res.dim } A \leq n$  and the desired assertion is obtained.  $\square$

As direct results, we have the following closure properties for the subcategory  $\widehat{\mathcal{X}}$ .

**Remark 3.5.** If  $\mathcal{X}$  is a resolving subcategory of  $\mathcal{T}$ , then  $\widehat{\mathcal{X}}$  is closed under hokernels of  $\xi$ -proper epimorphisms, hokernels of  $\xi$ -proper monomorphisms, and  $\xi$ -extensions.

**Corollary 3.6.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$ , and let*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

*be a triangle in  $\xi$ . Then, we have the following statements:*

- (1) (cf. [18, Proposition 3.11]) *Assume that  $C$  is an object in  $\mathcal{X}$ . Then,  $\mathcal{X}\text{-res.dim } A = \mathcal{X}\text{-res.dim } B$ .*
- (2) *Assume that  $B$  is an object in  $\mathcal{X}$ . Then, either  $A \in \mathcal{X}$  or else  $\mathcal{X}\text{-res.dim } A = \mathcal{X}\text{-res.dim } C - 1$ .*
- (3) (cf. [18, Proposition 3.13]) *Assume that  $A$  is an object in  $\mathcal{X}$  and neither  $B$  nor  $C$  in  $\mathcal{X}$ . Then,  $\mathcal{X}\text{-res.dim } B = \mathcal{X}\text{-res.dim } C$ .*

**Proposition 3.7.** *Let  $\mathcal{H}$  and  $\mathcal{X}$  be two subcategories of  $\mathcal{T}$  with  $\mathcal{H} \subseteq \mathcal{X}$ .*

- (1)  $\widehat{\mathcal{H}} \subseteq \widehat{\mathcal{X}}$ .
- (2) *If  $\mathcal{X}$  is resolving, then for any  $M \in \widehat{\mathcal{H}}$ ,  $\mathcal{H}$ -res.dim  $M = \mathcal{X}$ -res.dim  $M$  if and only if  $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$ .  
In particular, if  $\mathcal{X} \perp \mathcal{H}$ , and  $\mathcal{H}$  is closed under hokernels of  $\xi$ -proper epimorphisms or closed under direct summands, then  $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$ .*

**Proof.**

- (1) It is clear.
- (2) ( $\Rightarrow$ ) Clearly,  $\mathcal{H} \subseteq \widehat{\mathcal{H}} \cap \mathcal{X}$ . Let  $M \in \widehat{\mathcal{H}} \cap \mathcal{X}$ . By the assumption, we have  $\mathcal{H}$ -res.dim  $M = \mathcal{X}$ -res.dim  $M = 0$ , then  $M \in \mathcal{H}$ , so  $\widehat{\mathcal{H}} \cap \mathcal{X} \subseteq \mathcal{H}$ . Thus,  $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$ .

( $\Leftarrow$ ) Let  $M \in \widehat{\mathcal{H}}$ . Suppose  $\mathcal{H}$ -res.dim  $M = n$  and  $\mathcal{X}$ -res.dim  $M = m$ . Clearly,  $m \leq n$ . Consider the following  $\xi$ -exact complexes:

$$0 \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow X_m \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with  $H_i \in \mathcal{H}$  and  $X_j \in \mathcal{X}$  for all  $0 \leq i \leq n$  and  $0 \leq j \leq m$ . Since  $\mathcal{H} \subseteq \mathcal{X}$ , we have  $\Omega_{\mathcal{H}}^m(M) \in \mathcal{X}$  by Lemma 3.2. Then,  $\Omega_{\mathcal{H}}^m(M) \in \widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$ , and thus,  $\mathcal{H}$ -res.dim  $M \leq m$  and the desired equality is obtained.

Now, we assume that  $\mathcal{X} \perp \mathcal{H}$  and  $\widehat{\mathcal{H}}$  is closed under hokernels of  $\xi$ -proper epimorphisms or closed under direct summands. Clearly,  $\mathcal{H} \subseteq \widehat{\mathcal{H}} \cap \mathcal{X}$ . Conversely, let  $M \in \widehat{\mathcal{H}} \cap \mathcal{X}$ . There exists a  $\xi$ -exact complex

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \cdots \longrightarrow H_0 \longrightarrow M \longrightarrow 0.$$

Set  $K_i = \text{Hoker}(H_i \rightarrow H_{i-1})$  for  $0 \leq i \leq n-2$ , where  $H_{-1} = M$ . Since  $\mathcal{X}$  is resolving, we have  $K_i \in \mathcal{X}$ , and hence,  $K_i \in \widehat{\mathcal{H}} \cap \mathcal{X}$ . Consider the following triangle:

$$H_n \longrightarrow H_{n-1} \longrightarrow K_{n-2} \longrightarrow \Sigma H_n \quad (1)$$

in  $\xi$ . Since  $\xi \text{xt}_{\xi}^1(K_{n-2}, H_n) = 0$  by the assumption that  $\mathcal{X} \perp \mathcal{H}$ , we have that the triangle (1) is split. It follows that  $H_{n-1} \cong H_n \oplus K_{n-2}$  and there exists a triangle

$$K_{n-2} \longrightarrow H_{n-1} \longrightarrow H_n \xrightarrow{0} \Sigma K_{n-2}$$

in  $\xi$ . Since  $\mathcal{H}$  is closed under hokernels of  $\xi$ -proper epimorphisms or closed under direct summands by assumption, we have  $K_{n-2} \in \mathcal{H}$ . Repeating this process, we can obtain each  $K_i \in \mathcal{H}$ , hence,  $M \in \mathcal{H}$  and  $\widehat{\mathcal{H}} \cap \mathcal{X} \subseteq \mathcal{H}$ . Thus,  $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$ .  $\square$

Now we give the following definition.

**Definition 3.8.** Let  $\mathcal{X}$  be a subcategory of  $\mathcal{T}$  and  $M$  an object in  $\mathcal{T}$ . A  $\xi$ -proper epimorphism  $X \rightarrow M$  is said to be a right  $\mathcal{X}$ -approximation of  $M$  if  $\text{Hom}_{\mathcal{T}}(\widetilde{X}, X) \rightarrow \text{Hom}_{\mathcal{T}}(\widetilde{X}, M) \rightarrow 0$  is exact for any  $\widetilde{X} \in \mathcal{X}$ . In this case, there is a triangle  $K \rightarrow X \rightarrow M \rightarrow \Sigma K$  in  $\xi$ .

We need the following easy and useful observation.

**Lemma 3.9.** *Let  $\mathcal{H}$  and  $\mathcal{X}$  be two subcategories of  $\mathcal{T}$ .*

- (1) *If  $\mathcal{X} \perp \mathcal{H}$ , then  $\mathcal{X} \perp \widehat{\mathcal{H}}$ . In particular, if  $\mathcal{H} \perp \mathcal{H}$ , then  $\mathcal{H} \perp \widehat{\mathcal{H}}$ .*
- (2) *If  $M \in {}^{\perp}\mathcal{H}$ , then  $M \in {}^{\perp}\widehat{\mathcal{H}}$ .*

**Proof.** Apply Remark 2.10.  $\square$

The following is an analogous theory of Auslander-Buchweitz approximations (see [4,5]).

**Proposition 3.10.** *Let  $\mathcal{X}$  be a subcategory of  $\mathcal{T}$  closed under  $\xi$ -extensions, and let  $\mathcal{H}$  be a subcategory of  $\mathcal{T}$  such that  $\mathcal{H}$  is a  $\xi$ -cogenerator of  $\mathcal{X}$ . Then, for each  $M \in \mathcal{T}$  with  $\mathcal{X}$ -res.dim  $M = n < \infty$ , there exist two triangles*

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K \tag{2}$$

and

$$M \longrightarrow W \longrightarrow X' \longrightarrow \Sigma M \tag{3}$$

in  $\xi$ , where  $X, X' \in \mathcal{X}$ ,  $\mathcal{H}$ -res.dim  $K \leq n - 1$  and  $\mathcal{H}$ -res.dim  $W \leq n$  (if  $n = 0$ , this should be interpreted as  $K = 0$ ).

In particular, if  $\mathcal{X} \perp \mathcal{H}$ , then the  $\xi$ -proper epimorphism  $X \longrightarrow M$  is a right  $\mathcal{X}$ -approximation of  $M$ .

**Proof.** We proceed by induction on  $n$ . The case for  $n = 0$  is trivial. If  $n = 1$ , there exists a triangle

$$X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma X_1 \tag{4}$$

in  $\xi$  with  $X_0, X_1 \in \mathcal{X}$ . Since  $\mathcal{H}$  is a  $\xi$ -cogenerator of  $\mathcal{X}$ , there is a triangle

$$X_1 \longrightarrow H \longrightarrow X'_1 \longrightarrow \Sigma X_1$$

in  $\xi$  with  $H \in \mathcal{H}$  and  $X'_1 \in \mathcal{X}$ . Applying cobase change for the triangle (4) along the morphism  $X_1 \longrightarrow H$ , we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^{-1}X'_1 & \xrightarrow{=} & \Sigma^{-1}X'_1 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1}M & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M \\ \parallel \downarrow & & \downarrow & & \downarrow & & \downarrow \parallel \\ \Sigma^{-1}M & \longrightarrow & H & \xrightarrow{u} & X'_0 & \longrightarrow & M \\ \downarrow & & \downarrow \alpha & & \downarrow \alpha' & & \downarrow \\ 0 & \longrightarrow & X'_1 & \xrightarrow{=} & X'_1 & \longrightarrow & 0. \end{array}$$

Since  $\xi$  is closed under cobase changes, we obtain that the triangle

$$H \longrightarrow X'_0 \longrightarrow M \longrightarrow \Sigma H \tag{5}$$

is in  $\xi$  with  $\mathcal{H}$ -res.dim  $H = 0$ . Note that  $\alpha'u = \alpha$  is  $\xi$ -proper epic, so we have that  $\alpha'$  is  $\xi$ -proper epic by [16, Proposition 2.7]; hence, the triangle

$$X_0 \longrightarrow X'_0 \longrightarrow X'_1 \longrightarrow \Sigma X_0$$

is in  $\xi$ . Since  $\mathcal{X}$  is closed under  $\xi$ -extensions by assumption, we have  $X'_0 \in \mathcal{X}$ . So, (5) is the first desired triangle.

For  $X'_0$ , there is a triangle

$$X'_0 \longrightarrow H_0 \longrightarrow X''_0 \longrightarrow \Sigma X'_0$$

in  $\xi$  with  $H_0 \in \mathcal{H}$  and  $X''_0 \in \mathcal{X}$ . Applying cobase change for the triangle (5) along the morphism  $X'_0 \longrightarrow H_0$ , we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^{-1}X''_0 & \xrightarrow{=} & \Sigma^{-1}X''_0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H & \xrightarrow{u} & X'_0 & \longrightarrow & M & \longrightarrow & \Sigma H \\ \parallel \downarrow & & \downarrow \beta & & \downarrow & & \downarrow \parallel \\ H & \xrightarrow{u'} & H_0 & \xrightarrow{v'} & U & \longrightarrow & \Sigma H \\ \downarrow & & \downarrow \gamma & & \downarrow \gamma' & & \downarrow \\ 0 & \longrightarrow & X''_0 & \xrightarrow{=} & X''_0 & \longrightarrow & 0. \end{array} \tag{6}$$

Note that  $u' = \beta u$  is  $\xi$ -proper monic by [16, Proposition 2.6], so the third horizontal triangle is in  $\xi$ . Since  $\gamma'v' = \gamma$  is  $\xi$ -proper epic,  $\gamma'$  is  $\xi$ -proper epic by [16, Proposition 2.7]. So the triangle

$$M \longrightarrow U \longrightarrow X_0'' \longrightarrow \Sigma M$$

is in  $\xi$  with  $\mathcal{H}\text{-res.dim } U \leq 1$  and  $X_0'' \in \mathcal{X}$ , which is the second desired triangle.

Now suppose  $n \geq 2$ . Then, there is a triangle

$$K' \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K' \quad (7)$$

in  $\xi$  with  $\mathcal{X}\text{-res.dim } K' \leq n - 1$  and  $X_0 \in \mathcal{X}$ . For  $K'$ , by the induction hypothesis, we get a triangle

$$K' \longrightarrow K \longrightarrow X_2 \longrightarrow \Sigma K'$$

in  $\xi$  with  $\mathcal{H}\text{-res.dim } K \leq n - 1$  and  $X_2 \in \mathcal{X}$ . Applying cobase change for the triangle (7) along the morphism  $K' \longrightarrow K$ , we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^{-1}X_2 & \xrightarrow{=} & \Sigma^{-1}X_2 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1}M & \longrightarrow & K' & \longrightarrow & X_0 & \longrightarrow & M \\ \parallel \downarrow & & \downarrow & & \downarrow & & \parallel \downarrow \\ \Sigma^{-1}M & \longrightarrow & K & \xrightarrow{\kappa} & X & \longrightarrow & M \\ \downarrow & & \downarrow \lambda & & \downarrow \lambda' & & \downarrow \\ 0 & \longrightarrow & X_2 & \xrightarrow{=} & X_2 & \longrightarrow & 0. \end{array}$$

Note that  $\lambda'\kappa = \lambda$  is  $\xi$ -proper epic, then  $\lambda'$  is  $\xi$ -proper epic by [16, Proposition 2.7], so the triangle

$$X_0 \longrightarrow X \longrightarrow X_2 \longrightarrow \Sigma X_0$$

is in  $\xi$ . It follows that  $X \in \mathcal{X}$  from the assumption that  $\mathcal{X}$  is closed under  $\xi$ -extensions. Since  $\xi$  is closed under cobase changes, we obtain the first desired triangle

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K \quad (8)$$

in  $\xi$  with  $\mathcal{H}\text{-res.dim } K \leq n - 1$  and  $X \in \mathcal{X}$ .

For  $X$ , since  $\mathcal{H}$  is a  $\xi$ -cogenerator of  $\mathcal{X}$ , we get the following triangle

$$X \longrightarrow H_1 \longrightarrow X' \longrightarrow \Sigma X$$

in  $\xi$  with  $H_1 \in \mathcal{H}$  and  $X' \in \mathcal{X}$ .

Applying cobase change for the triangle (8) along the morphism  $X \longrightarrow H_1$ , we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^{-1}X' & \xrightarrow{=} & \Sigma^{-1}X' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & X & \longrightarrow & M & \longrightarrow & \Sigma K \\ \parallel \downarrow & & \downarrow & & \downarrow & & \parallel \downarrow \\ K & \longrightarrow & H_1 & \longrightarrow & W & \longrightarrow & \Sigma K \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X' & \xrightarrow{=} & X' & \longrightarrow & 0. \end{array}$$

As a similar argument to that of the diagram (6), we obtain that the triangles

$$K \longrightarrow H_1 \longrightarrow W \longrightarrow \Sigma K$$

and

$$M \longrightarrow W \longrightarrow X' \longrightarrow \Sigma M \quad (9)$$

are in  $\xi$ . Thus, (9) is the second desired triangle in  $\xi$  with  $\mathcal{H}\text{-res.dim } W \leq n$  and  $X' \in \mathcal{X}$ .

In particular, suppose  $\mathcal{X} \perp \mathcal{H}$ , by Lemma 3.9, we have  $\mathcal{X} \perp \widehat{\mathcal{H}}$ . Then,  $\xi \text{xt}_{\xi}^1(\overline{X}, K) = 0$  for any  $\overline{X} \in \mathcal{X}$ , it follows that  $\text{Hom}_{\mathcal{T}}(\overline{X}, X) \longrightarrow \text{Hom}_{\mathcal{T}}(\overline{X}, M) \longrightarrow 0$  is exact. Thus, the  $\xi$ -proper epimorphism  $X \longrightarrow M$  is a right  $\mathcal{X}$ -approximation of  $M$ .  $\square$

**Proposition 3.11.** *Keep the notion as Proposition 3.10. Assume  $M \in \widehat{\mathcal{X}}$  with  $\mathcal{X}$ -res.dim  $M = n < \infty$ .*

(1) *If  $\mathcal{X}$  is resolving, then in the triangles (2) and (3), we have  $\mathcal{H}$ -res.dim  $K = n - 1$  and  $\mathcal{H}$ -res.dim  $W = \mathcal{X}$ -res.dim  $W = n$ .*

*In particular, if  $\mathcal{X} \perp \mathcal{H}$ , then the  $\xi$ -proper epimorphism  $X \rightarrow M$  in the triangle (2) is a right  $\mathcal{X}$ -approximation of  $M$ , such that  $\mathcal{H}$ -res.dim  $K = n - 1$  (if  $n = 0$ , it should be interpreted  $K = 0$ ).*

(2) *If  $\mathcal{X} \perp \mathcal{H}$  and  $\mathcal{X}$  is resolving, then there is a triangle*

$$M \longrightarrow M' \longrightarrow X \longrightarrow \Sigma M$$

*in  $\xi$  with  $M' \in \mathcal{X}^{\perp}$ ,  $X \in \mathcal{X}$  and  $\mathcal{X}$ -res.dim  $M = \mathcal{X}$ -res.dim  $M'$ .*

(3) (a) *Let  $\omega_{\mathcal{H}} = \mathcal{H}^{\perp} \cap \mathcal{H}$ . If  $\omega_{\mathcal{H}}$  is a  $\xi$ -cogenerator of  $\mathcal{H}$  and  $\mathcal{H}$  is closed under  $\xi$ -extensions, then  $\mathcal{X} \perp \omega_{\mathcal{H}}$  if and only if  $\mathcal{X} \perp (\mathcal{H}^{\perp} \cap \widehat{\mathcal{H}})$ .*

(b) *If  $\mathcal{X}$  is a resolving and  $\omega_{\mathcal{X}} = \mathcal{X} \cap \mathcal{X}^{\perp}$  is a  $\xi$ -cogenerator of  $\mathcal{X}$  and  $M \in \mathcal{X}^{\perp}$ , then  $\mathcal{X}$ -res.dim  $M = \omega_{\mathcal{X}}$ -res.dim  $M$ .*

(4) *Suppose that  $\mathcal{H}$  and  $\mathcal{X}$  are resolving. If  $\omega_{\mathcal{H}} = \mathcal{H} \cap \mathcal{H}^{\perp}$  is a  $\xi$ -cogenerator of  $\mathcal{H}$  and  $\mathcal{X} \perp \omega_{\mathcal{H}}$ , then  $M$  admits a right  $\mathcal{X}$ -approximation  $X' \longrightarrow M$  such that  $K'' \longrightarrow X' \longrightarrow M \longrightarrow \Sigma K''$  is a triangle in  $\xi$ , where  $\mathcal{H}$ -res.dim  $K'' = n - 1$ . In fact, we have  $\omega_{\mathcal{H}}$ -res.dim  $K'' = n - 1$ .*

**Proof.**

(1) Suppose  $\mathcal{X}$  is resolving. Applying Corollary 3.6(2) to the triangle (2) yields that  $\mathcal{X}$ -res.dim  $K = n - 1$ . On the other hand, since  $\mathcal{H} \subseteq \mathcal{X}$ , we have  $n - 1 = \mathcal{X}$ -res.dim  $K \leq \mathcal{H}$ -res.dim  $K \leq n - 1$ . Thus,  $\mathcal{H}$ -res.dim  $K = n - 1$ .

Moreover, applying Corollary 3.6(1) to the triangle (3) implies  $\mathcal{X}$ -res.dim  $W = \mathcal{X}$ -res.dim  $M = n$ . So,  $n = \mathcal{X}$ -res.dim  $W \leq \mathcal{H}$ -res.dim  $W \leq n$ . Hence,  $\mathcal{H}$ -res.dim  $W = \mathcal{X}$ -res.dim  $W = n$ .

The last assertion follows from the above argument and Proposition 3.10.

(2) Since  $\mathcal{X} \perp \mathcal{H}$ , we have  $\mathcal{X} \perp \widehat{\mathcal{H}}$  by Lemma 3.9, and so the result immediately follows from (1).

(3) (a)  $(\Leftarrow)$  Suppose  $\mathcal{X} \perp (\mathcal{H}^{\perp} \cap \widehat{\mathcal{H}})$ . Clearly,  $\omega_{\mathcal{H}} = \mathcal{H}^{\perp} \cap \mathcal{H} \subseteq \mathcal{H}^{\perp} \cap \widehat{\mathcal{H}} \subseteq \mathcal{X}^{\perp}$ , that is,  $\mathcal{X} \perp \omega_{\mathcal{H}}$ .

$(\Rightarrow)$  Suppose  $\mathcal{X} \perp \omega_{\mathcal{H}}$ . Let  $L \in \mathcal{H}^{\perp} \cap \widehat{\mathcal{H}}$ . By Proposition 3.10, there exists a triangle

$$K' \longrightarrow H_0 \longrightarrow L \longrightarrow \Sigma K'$$

in  $\xi$  with  $H_0 \in \mathcal{H}$  and  $\omega_{\mathcal{H}}$ -res.dim  $K' \leq \mathcal{H}$ -res.dim  $L - 1 < \infty$ . Note that  $K' \in \mathcal{H}^{\perp}$  by Lemma 3.9, so  $L \in \mathcal{H}^{\perp}$  implies  $H_0 \in \mathcal{H}^{\perp}$ . Then,  $H_0 \in \omega_{\mathcal{H}}$ , and so,  $L \in \widehat{\omega_{\mathcal{H}}}$ . Since  $\mathcal{X} \perp \omega_{\mathcal{H}}$ , we have  $L \in \mathcal{X}^{\perp}$  by Lemma 3.9. Thus,  $\mathcal{X} \perp (\mathcal{H}^{\perp} \cap \widehat{\mathcal{H}})$ .

(b) Suppose  $\mathcal{X}$ -res.dim  $M = n$ , by (1), there exists a triangle

$$K \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K$$

in  $\xi$  with  $X_0 \in \mathcal{X}$  and  $\omega_{\mathcal{X}}$ -res.dim  $K = n - 1$ . Note that  $M \in \mathcal{X}^{\perp}$  and  $K \in \mathcal{X}^{\perp}$ , so  $X_0 \in \mathcal{X}^{\perp}$ , and hence,  $X_0 \in \omega_{\mathcal{X}}$ . It follows that  $\omega_{\mathcal{X}}$ -res.dim  $M \leq n$ . But  $n = \mathcal{X}$ -res.dim  $M \leq \omega_{\mathcal{X}}$ -res.dim  $M \leq n$ , thus  $\mathcal{X}$ -res.dim  $M = \omega_{\mathcal{X}}$ -res.dim  $M$ .

(4) Suppose  $\mathcal{X}$ -res.dim  $M = n$ , by (1), there exists a triangle

$$K \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K \tag{10}$$

in  $\xi$  with  $X_0 \in \mathcal{X}$  and  $\mathcal{H}$ -res.dim  $K = n - 1$ . By (2), there is a triangle

$$K \longrightarrow K'' \longrightarrow H \longrightarrow \Sigma K$$

in  $\xi$  with  $H \in \mathcal{H}$ ,  $K'' \in \mathcal{H}^\perp$  and  $\mathcal{H}\text{-res.dim } K'' = \mathcal{H}\text{-res.dim } K$ . Then,  $K'' \in \mathcal{H}^\perp \cap \widehat{\mathcal{H}}$ . Applying cobase change for the triangle (10) along the morphism  $K \rightarrow K''$ , we get the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Sigma^{-1}H & \xrightarrow{=} & \Sigma^{-1}H & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Sigma^{-1}M & \longrightarrow & K & \longrightarrow & X_0 & \longrightarrow & M \\
 \parallel \downarrow & & \downarrow & & \downarrow & & \parallel \downarrow \\
 \Sigma^{-1}M & \longrightarrow & K'' & \longrightarrow & X' & \longrightarrow & M \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H & \xrightarrow{=} & H & \longrightarrow & 0.
 \end{array}$$

One can see that the triangle

$$K'' \longrightarrow X' \longrightarrow M \longrightarrow \Sigma K'' \tag{11}$$

is in  $\xi$  and  $X' \in \mathcal{X}$ . Note that  $\mathcal{X} \perp \omega_{\mathcal{H}}$ , so  $\mathcal{X} \perp \mathcal{H}^\perp \cap \widehat{\mathcal{H}}$  by (3)(a). Then,  $\xi \text{xt}_\xi^1(\widehat{\mathcal{X}}, K'') = 0$  for any  $\widehat{\mathcal{X}} \in \mathcal{X}$ , and so,  $\text{Hom}_{\mathcal{T}}(\widehat{\mathcal{X}}, X') \rightarrow \text{Hom}_{\mathcal{T}}(\widehat{\mathcal{X}}, M) \rightarrow 0$  is exact. Thus, the  $\xi$ -proper epimorphism  $X' \rightarrow M$  is a right  $\mathcal{X}$ -approximation of  $M$  and  $\mathcal{H}\text{-res.dim } K'' = n - 1$  in the triangle (11). Note that  $K'' \in \mathcal{H}^\perp$ , so we have  $\omega_{\mathcal{H}}\text{-res.dim } K'' = \mathcal{H}\text{-res.dim } K'' = n - 1$  by (3)(b).  $\square$

**Lemma 3.12.** *Let  $\mathcal{H}$  be a subcategory of  $\mathcal{T}$  with  $\mathcal{H} \perp \mathcal{H}$ . Assume that  $\mathcal{H}$  is closed under hokernels of  $\xi$ -proper epimorphisms or closed under direct summands. Then,  $\mathcal{H} = \widehat{\mathcal{H}} \cap {}^\perp \mathcal{H}$ .*

**Proof.** Clearly,  $\mathcal{H} \subseteq \widehat{\mathcal{H}} \cap {}^\perp \mathcal{H}$ .

Conversely, let  $M \in \widehat{\mathcal{H}} \cap {}^\perp \mathcal{H}$ . Consider the following  $\xi$ -exact complex:

$$0 \longrightarrow H_n \longrightarrow H_{n-1} \longrightarrow \dots \longrightarrow H_0 \longrightarrow M \longrightarrow 0.$$

Set  $K_i = \text{Hoker}(H_i \rightarrow H_{i-1})$  for  $0 \leq i \leq n - 2$ , where  $H_{-1} = M$ . Then,  $M \in {}^\perp \mathcal{H}$  yields  $K_i \in {}^\perp \mathcal{H}$ , and so the triangle

$$H_n \longrightarrow H_{n-1} \longrightarrow K_{n-2} \longrightarrow \Sigma H_n$$

is split. It follows that  $H_{n-1} \cong H_n \oplus K_{n-2}$  and there exists a triangle

$$K_{n-2} \longrightarrow H_{n-1} \longrightarrow H_n \xrightarrow{0} \Sigma K_{n-2}$$

in  $\xi$ . Since  $\mathcal{H}$  is closed under hokernels of  $\xi$ -proper epimorphisms or closed under direct summands by assumption, we have  $K_{n-2} \in \mathcal{H}$ . Repeating this process, we can obtain  $K_i \in \mathcal{H}$ , hence  $M \in \mathcal{H}$  and  $\widehat{\mathcal{H}} \cap {}^\perp \mathcal{H} \subseteq \mathcal{H}$ . Thus,  $\widehat{\mathcal{H}} \cap {}^\perp \mathcal{H} = \mathcal{H}$ .  $\square$

**Proposition 3.13.** *Let  $\mathcal{X}$  be a resolving subcategory and  $\mathcal{H}$  a  $\xi \text{xt}$ -injective  $\xi$ -cogenerator of  $\mathcal{X}$ . Assume that  $\mathcal{H}$  is closed under hokernels of  $\xi$ -proper epimorphisms or closed under direct summands. Then,  $\mathcal{X} = \widehat{\mathcal{X}} \cap {}^\perp \widehat{\mathcal{H}} = \widehat{\mathcal{X}} \cap {}^\perp \mathcal{H}$ .*

**Proof.** Clearly,  $\mathcal{X} \subseteq \widehat{\mathcal{X}} \cap {}^\perp \mathcal{H}$  and  $\widehat{\mathcal{X}} \cap {}^\perp \widehat{\mathcal{H}} \subseteq \widehat{\mathcal{X}} \cap {}^\perp \mathcal{H}$ .

Now, let  $M \in \widehat{\mathcal{X}} \cap {}^\perp \mathcal{H}$ . Then, by Lemma 3.9, we have  $M \in \widehat{\mathcal{X}} \cap {}^\perp \widehat{\mathcal{H}}$ , and hence,  $\widehat{\mathcal{X}} \cap {}^\perp \mathcal{H} \subseteq \widehat{\mathcal{X}} \cap {}^\perp \widehat{\mathcal{H}}$ .

On the other hand, by Proposition 3.10, there is a triangle

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K \tag{12}$$

in  $\xi$  with  $X \in \mathcal{X}$  and  $\mathcal{H}\text{-res.dim } K < \infty$ . Note that  $M \in {}^\perp \mathcal{H}$  implies  $K \in {}^\perp \mathcal{H}$ , and hence,  $K \in \widehat{\mathcal{H}} \cap {}^\perp \mathcal{H} = \mathcal{H}$  by Lemma 3.12. Note that  $\xi \text{xt}_\xi^1(M, K) = 0$ , so the triangle (12) is split; hence,  $X \cong K \oplus M$ . Consider the following triangle

$$M \longrightarrow X \longrightarrow K \xrightarrow{0} \Sigma M$$

in  $\xi$ . It follows that  $M \in \mathcal{X}$  from the assumption that  $\mathcal{X}$  is resolving. Thus,  $\widehat{\mathcal{X}} \cap {}^\perp \mathcal{H} \subseteq \mathcal{X}$ .  $\square$

Our main result is the following.

**Theorem 3.14.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  and  $\mathcal{H}$  a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ . Assume that  $\mathcal{H}$  is closed under hokernels of  $\xi$ -proper epimorphisms or closed under direct summands. For any  $M \in \mathcal{T}$ , if  $M \in \widehat{\mathcal{X}}$ , then the following statements are equivalent:*

- (1)  $\mathcal{X}$ -res.dim  $M \leq m$ .
- (2)  $\Omega^n(M) \in \mathcal{X}$  for all  $n \geq m$ .
- (3)  $\Omega_{\mathcal{X}}^n(M) \in \mathcal{X}$  for all  $n \geq m$ .
- (4)  $\xi \text{xt}_{\xi}^n(M, H) = 0$  for all  $n > m$  and all  $H \in \mathcal{H}$ .
- (5)  $\xi \text{xt}_{\xi}^n(M, L) = 0$  for all  $n > m$  and all  $L \in \widehat{\mathcal{H}}$ .
- (6)  $M$  admits a right  $\mathcal{X}$ -approximation  $\varphi : X \rightarrow M$ , where  $\varphi$  is  $\xi$ -proper epic, such that  $K = \text{Hoker } \varphi$  satisfying  $\mathcal{H}$ -res.dim  $K \leq m - 1$ .
- (7) There are two triangles

$$W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M$$

and

$$M \longrightarrow W^M \longrightarrow X^M \longrightarrow \Sigma M$$

in  $\xi$  such that  $X_M, X^M \in \mathcal{X}$  and  $\mathcal{H}$ -res.dim  $W_M \leq m - 1$ ,  $\mathcal{H}$ -res.dim  $W^M = \mathcal{X}$ -res.dim  $W^M \leq m$ .

**Proof.** (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3) It follows from Proposition 3.3.

(1)  $\Leftrightarrow$  (6) It follows from Proposition 3.11(1).

(1)  $\Leftrightarrow$  (7) It follows from Proposition 3.11(1).

(1)  $\Rightarrow$  (4) Suppose  $\mathcal{X}$ -res.dim  $M \leq m$ . There is a  $\xi$ -exact complex

$$0 \longrightarrow X_m \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with all  $X_i$  in  $\mathcal{X}$ . Since  $\mathcal{H}$  is a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ , we have  $\xi \text{xt}_{\xi}^{k \geq 1}(X_i, H) = 0$  for all  $H \in \mathcal{H}$ . So,  $\xi \text{xt}_{\xi}^n(M, H) \cong \xi \text{xt}_{\xi}^{n-m}(X_m, H) = 0$  for  $n > m$ .

(4)  $\Rightarrow$  (5) It follows from Lemma 3.9.

(5)  $\Rightarrow$  (4) It is clear.

(4)  $\Rightarrow$  (1) Since  $M \in \widehat{\mathcal{X}}$ , by Proposition 3.11(1), there is a triangle  $K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K$  in  $\xi$  with  $\mathcal{H}$ -res.dim  $K < \infty$  and  $X \in \mathcal{X}$ . Then,  $\xi \text{xt}_{\xi}^i(K, H) \cong \xi \text{xt}_{\xi}^{i+1}(M, H)$  for  $H \in \mathcal{H}$  and  $i \geq 1$  since  $\xi \text{xt}_{\xi}^{i \geq 1}(X, H) = 0$ . So,  $\xi \text{xt}_{\xi}^{i \geq m}(K, H) = 0$ . Note that  $\mathcal{H}$ -res.dim  $K < \infty$ , so we have the following  $\xi$ -exact complex

$$0 \longrightarrow H_n \longrightarrow \cdots \longrightarrow H_0 \longrightarrow K \longrightarrow 0$$

with all  $H_i \in \mathcal{H}$ . Then,

$$\xi \text{xt}_{\xi}^i(\Omega_{\mathcal{H}}^{m-1}(K), H) \cong \xi \text{xt}_{\xi}^{i+m-1}(K, H) = 0$$

for  $i \geq 1$  and all  $H \in \mathcal{H}$ , which means  $\Omega_{\mathcal{H}}^{m-1}(K) \in {}^\perp \mathcal{H}$ . Note that  $\mathcal{H}$ -res.dim  $\Omega_{\mathcal{H}}^{m-1}(K) < \infty$ , hence,  $\Omega_{\mathcal{H}}^{m-1}(K) \in \widehat{\mathcal{H}} \cap {}^\perp \mathcal{H}$ . It follows that  $\Omega_{\mathcal{H}}^{m-1}(K) \in \mathcal{H}$  from Lemma 3.12, so  $\mathcal{H}$ -res.dim  $K \leq m - 1$ . Thus,  $\mathcal{X}$ -res.dim  $M \leq m$ .  $\square$

## 4 Additive quotient categories and $\xi$ -cellular towers with respect to a resolving subcategory

In this section, we will further study objects having finite resolution dimension with respect to a resolving subcategory  $\mathcal{X}$ . We first construct adjoint pairs for two kinds of inclusion functors. Then, we characterize objects having finite resolution dimension in terms of a notion of  $\xi$ -cellular towers.

### 4.1 Adjoint pairs

Suppose that  $\mathcal{D}$  and  $\mathcal{X}$  are two subcategories of  $\mathcal{T}$ . Denote by  $[\mathcal{D}]$  the ideal of  $\mathcal{X}$  consisting of morphisms factoring through some object in  $\mathcal{D}$ . Thus, we have a quotient category  $\mathcal{X}/[\mathcal{D}]$ , which is also an additive category.

**Lemma 4.1.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  and  $\mathcal{H}$  a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ . Assume that  $f : X \rightarrow M$  is a morphism in  $\mathcal{T}$  with  $X \in \mathcal{X}$  and  $M \in \widehat{\mathcal{X}}$ , then the following statements are equivalent:*

- (1)  $f$  factors through an object in  $\mathcal{H}$ .
- (2)  $f$  factors through an object in  $\widehat{\mathcal{H}}$ .

**Proof.** It suffices to show that (2)  $\Rightarrow$  (1). Suppose that  $f$  factors through an object  $L \in \widehat{\mathcal{H}}$ . Then,  $f = gh$ , where  $h : X \rightarrow L$  and  $g : L \rightarrow M$ . Consider the following triangle

$$L' \longrightarrow H \longrightarrow L \longrightarrow \Sigma L'$$

in  $\xi$  with  $H \in \mathcal{H}$  and  $L' \in \widehat{\mathcal{H}}$ . Note that  $\mathcal{H}$  is a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ , by Lemma 3.9, we have  $\xi \text{xt}_{\xi}^1(X, L') = 0$ . So,  $h$  factors through  $H$ , it follows that  $f$  factors through  $H$ .  $\square$

**Lemma 4.2.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  and  $\mathcal{H}$  a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ , and let  $M, N \in \widehat{\mathcal{X}}$ . Assume that  $f : M \rightarrow N$  is a morphism in  $\mathcal{T}$ , consider two triangles*

$$W_M \xrightarrow{\alpha} X_M \xrightarrow{p} M \longrightarrow \Sigma W_M \quad \text{and} \quad W_N \xrightarrow{\beta} X_N \xrightarrow{q} N \longrightarrow \Sigma W_N$$

in  $\xi$  with  $X_M, X_N \in \mathcal{X}$  and  $W_M, W_N \in \widehat{\mathcal{H}}$  (see Proposition 3.10), then we have the following statements:

- (1) There exists a morphism  $g : X_M \rightarrow X_N$  such that  $qg = fp$ .
- (2) If  $g, g' : X_M \rightarrow X_N$  are two morphisms such that  $qg = fp$  and  $qg' = fp$ , then  $[g] = [g']$  in  $\text{Hom}_{\mathcal{X}/[\mathcal{H}]}(X_M, X_N)$ .

**Proof.**

(1) Apply Proposition 3.10.

(2) Suppose  $g, g' : X_M \rightarrow X_N$  are two morphisms such that  $qg = fp$  and  $qg' = fp$ , then  $q(g' - g) = qg' - qg = 0$ , and so there exists a morphism  $h : X_M \rightarrow W_N$  such that  $g' - g = \beta h$ , that is, there is a commutative diagram as follows:

$$\begin{array}{ccccc} & & X_M & & \\ & \swarrow h & \downarrow g' - g & & \\ W_N & \xrightarrow{\beta} & X_N & \xrightarrow{q} & N \longrightarrow \Sigma W_N \end{array}$$

Note that  $W_N \in \widehat{\mathcal{H}}$ , so  $g' - g : X_M \rightarrow X_N$  factors through an object in  $\mathcal{H}$  by Lemma 4.1. Thus,  $[g] = [g']$  in  $\text{Hom}_{\mathcal{X}/[\mathcal{H}]}(X_M, X_N)$ .  $\square$

By Lemma 4.2, there exists a well-defined additive functor

$$F : \widehat{\mathcal{X}} \rightarrow \mathcal{X}/[\mathcal{H}],$$

which maps an object  $M \in \widehat{\mathcal{X}}$  to  $X_M$  and a morphism  $f : M \rightarrow N \in \text{Hom}_{\widehat{\mathcal{X}}}(M, N)$  to  $[g] \in \text{Hom}_{\mathcal{X}/[\mathcal{H}]}(X_M, X_N)$  as described in Lemma 4.2.

Clearly, we have  $F(H) = 0$  for any object  $H \in \mathcal{H}$ . Hence,  $F$  factors through  $\widehat{\mathcal{X}}/[\mathcal{H}]$ . That is, there exists an additive functor  $\mu : \widehat{\mathcal{X}}/[\mathcal{H}] \rightarrow \mathcal{X}/[\mathcal{H}]$  making the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathcal{X}} & \xrightarrow{F} & \mathcal{X}/[\mathcal{H}] \\ & \searrow \pi & \nearrow \mu \\ & \widehat{\mathcal{X}}/[\mathcal{H}] & \end{array}$$

where  $\pi$  is the canonical quotient functor.

Now we show that the additive functor  $\mu$  defined above and the inclusion functor between additive quotients  $\mathcal{X}/[\mathcal{H}]$  and  $\widehat{\mathcal{X}}/[\mathcal{H}]$  are adjoint.

**Theorem 4.3.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  and  $\mathcal{H}$  a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ . Then, the additive functor  $\mu : \widehat{\mathcal{X}}/[\mathcal{H}] \rightarrow \mathcal{X}/[\mathcal{H}]$  defined above is right adjoint to the inclusion functor  $\mathcal{X}/[\mathcal{H}] \rightarrow \widehat{\mathcal{X}}/[\mathcal{H}]$ .*

**Proof.** Let  $X \in \mathcal{X}$  and  $N \in \widehat{\mathcal{X}}$ . By Proposition 3.10, there is a triangle

$$W_N \xrightarrow{\beta} X_N \xrightarrow{q} N \rightarrow \Sigma W_N$$

in  $\xi$  with  $W_N \in \widehat{\mathcal{H}}$  and  $X_N \in \mathcal{X}$ . Note that the additive map

$$[q]_* : \text{Hom}_{\mathcal{X}/[\mathcal{H}]}(X, \mu(N)) \rightarrow \text{Hom}_{\widehat{\mathcal{X}}/[\mathcal{H}]}(X, N)$$

is natural in both  $X$  and  $N$  by Lemma 4.2. We claim that  $[q]_*$  is an isomorphism.

Indeed, since  $\mathcal{H}$  is a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ , by Lemma 3.9, we have  $\xi \text{xt}_{\xi}^1(X, W_N) = 0$ , and hence,  $\text{Hom}_{\mathcal{T}}(X, X_N) \rightarrow \text{Hom}_{\mathcal{T}}(X, N)$  is an epimorphism, so  $[q]_*$  is still an epimorphism.

Now, assume that  $g : X \rightarrow X_N$  is a morphism such that  $[qg] = [q][g] = [q]_*[g] = [0] \in \text{Hom}_{\mathcal{X}/[\mathcal{H}]}(X, N)$ . Then, there exists an object  $H \in \mathcal{H}$  such that  $qg = ts$  as the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{s} & H & & \\ \downarrow g & \nearrow \theta & \downarrow t & & \\ W_N & \xrightarrow{\beta} & X_N & \xrightarrow{q} & N \rightarrow \Sigma W_N \end{array}$$

Note that  $\xi \text{xt}_{\xi}^1(H, W_N) = 0$  by assumption, so there exists a morphism  $\theta : H \rightarrow X_N$  such that  $t = q\theta$ . Since  $q(g - \theta s) = qg - q\theta s = ts - ts = 0$ , so  $g - \theta s$  factors through  $W_N$ . By Lemma 4.1,  $g - \theta s$  factors through an object in  $\mathcal{H}$ . It follows that  $[g - \theta s] = 0 \in \text{Hom}_{\mathcal{X}/[\mathcal{H}]}(X, N)$ . Since  $\theta s = 0 \in \text{Hom}_{\mathcal{X}/[\mathcal{H}]}(X, N)$ , we have  $0 = [g] \in \text{Hom}_{\mathcal{X}/[\mathcal{H}]}(X, N)$ . So  $[q]_*$  is a monomorphism, and thus,  $[q]_*$  is an isomorphism.  $\square$

**Corollary 4.4.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  and  $\mathcal{H}$  a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ . Assume that  $\mathcal{H}$  is closed under direct summands. For any  $N \in \widehat{\mathcal{X}}$ , the following statements are equivalent:*

- (1)  $N \in \widehat{\mathcal{H}}$ .
- (2) There is a triangle

$$W_N \rightarrow X_N \xrightarrow{q} N \rightarrow \Sigma W_N$$

in  $\xi$  with  $W_N \in \widehat{\mathcal{H}}$  and  $X_N \in \mathcal{X}$  such that  $[q] = [0] \in \text{Hom}_{\widehat{\mathcal{X}}/[\mathcal{H}]}(X, N)$ .

**Proof.** The assertion (1)  $\Rightarrow$  (2) follows from Lemma 4.1. It suffices to show (2)  $\Rightarrow$  (1). Note that the adjunction isomorphism established in Theorem 4.3 implies that the additive map

$$[q]_* : \text{Hom}_{\mathcal{X}/[\mathcal{H}]}(X_N, X_N) \longrightarrow \text{Hom}_{\widehat{\mathcal{X}}/[\mathcal{H}]}(X_N, N)$$

is isomorphic. Since  $[q]_* [\text{id}_{X_N}] = [q \text{id}_{X_N}] = [q] = [0] \in \text{Hom}_{\widehat{\mathcal{X}}/[\mathcal{H}]}(X_N, N) = 0$ , so  $[\text{id}_{X_N}] = [0] \in \text{Hom}_{\widehat{\mathcal{X}}/[\mathcal{H}]}(X_N, X_N)$ , and thus,  $\text{id}_{X_N}$  factors through an object  $H \in \mathcal{H}$ . It follows that  $X_N$  is a direct summand of  $W_N$ . Since  $\mathcal{H}$  is closed under direct summands, we have  $X_N \in \mathcal{H}$ . Thus,  $N \in \widehat{\mathcal{H}}$ .  $\square$

Next, we compare additive quotients  $\widehat{\mathcal{H}}/[\mathcal{X}]$  and  $\widehat{\mathcal{X}}/[\mathcal{X}]$ .

**Lemma 4.5.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  and  $\mathcal{H}$  a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ , and let  $M, N \in \widehat{\mathcal{X}}$ . Assume that  $f : M \rightarrow N$  is a morphism in  $\mathcal{T}$ , consider two triangles*

$$M \xrightarrow{s} W^M \xrightarrow{l} X^M \longrightarrow \Sigma M \quad \text{and} \quad N \xrightarrow{t} W^N \xrightarrow{r} X^N \longrightarrow \Sigma N$$

in  $\xi$  with  $X^M, X^N \in \mathcal{X}$  and  $W^M, W^N \in \widehat{\mathcal{H}}$  (see Proposition 3.10), then, we have the following statements:

- (1) *There exists a morphism  $g : W^M \rightarrow W^N$  such that  $gs = tf$ .*
- (2) *If  $g, g' : W^M \rightarrow W^N$  are two morphisms such that  $gs = tf$  and  $g's = tf$ , then  $[g] = [g']$  in  $\text{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(X_M, X_N)$ .*

**Proof.**

(1) Since  $\mathcal{X} \perp \mathcal{H}$  by assumption, we have  $\xi \text{xt}_\xi^1(X^M, W^N) = 0$  by Lemma 3.9. So, there exists a morphism  $g : W^M \rightarrow W^N$  such that  $gs = tf$ .

(2) Suppose  $g, g' : W^M \rightarrow W^N$  are two morphisms such that  $gs = tf$  and  $g's = tf$ , then  $(g' - g)s = g's - gs = 0$ , and so there exists a morphism  $h' : X^M \rightarrow W^N$  such that  $g' - g = h'l$ , that is, there is a commutative diagram as follows:

$$\begin{array}{ccccc} M & \xrightarrow{s} & W^M & \xrightarrow{l} & X^M & \longrightarrow & \Sigma M \\ & & \downarrow g'-g & \searrow h & & & \\ & & W^N & & & & \end{array}$$

Note that  $X^M \in \mathcal{X}$ , so  $g' - g : W^M \rightarrow W^N$  factors through an object in  $\mathcal{X}$ . Thus,  $[g] = [g']$  in  $\text{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^M, W^N)$ .  $\square$

By Lemma 4.5, there exists a well-defined additive functor

$$G : \widehat{\mathcal{X}} \rightarrow \widehat{\mathcal{H}}/[\mathcal{X}],$$

which maps an object  $M \in \widehat{\mathcal{X}}$  to  $W^M$  and a morphism  $f : M \rightarrow N \in \text{Hom}_{\widehat{\mathcal{X}}}(M, N)$  to  $[g] \in \text{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^M, W^N)$  as described in Lemma 4.5.

Clearly, we have  $G(X) = 0$  for any object  $X \in \mathcal{X}$ . Hence,  $G$  factors through  $\widehat{\mathcal{X}}/[\mathcal{X}]$ . That is, there exists an additive functor  $\eta : \widehat{\mathcal{X}}/[\mathcal{X}] \rightarrow \widehat{\mathcal{H}}/[\mathcal{X}]$  making the following diagram commutes

$$\begin{array}{ccc} \widehat{\mathcal{X}} & \xrightarrow{G} & \widehat{\mathcal{H}}/[\mathcal{X}] \\ \pi \searrow & & \nearrow \eta \\ & \widehat{\mathcal{X}}/[\mathcal{X}] & \end{array}$$

where  $\eta$  is the canonical quotient functor.

Now we show that the additive functor  $\eta$  defined above and the inclusion functor between additive quotients  $\widehat{\mathcal{H}}/[\mathcal{X}]$  and  $\widehat{\mathcal{X}}/[\mathcal{X}]$  are adjoint.

**Theorem 4.6.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  and  $\mathcal{H}$  a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ . Then, the additive functor  $\eta : \widehat{\mathcal{X}}/[\mathcal{X}] \rightarrow \widehat{\mathcal{H}}/[\mathcal{X}]$  defined above is left adjoint to the inclusion functor  $\widehat{\mathcal{H}}/[\mathcal{X}] \rightarrow \widehat{\mathcal{X}}/[\mathcal{X}]$ .*

**Proof.** Let  $K$  be an object in  $\widehat{\mathcal{H}}$  and  $M$  an object in  $\widehat{\mathcal{X}}$ . By Proposition 3.10, there is a triangle

$$M \xrightarrow{s} W^M \xrightarrow{l} X^M \longrightarrow \Sigma M$$

in  $\xi$  with  $W^M \in \widehat{\mathcal{H}}$  and  $X^M \in \mathcal{X}$ . Note that the additive map

$$[s]^* : \text{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(\eta(M), K) \longrightarrow \text{Hom}_{\widehat{\mathcal{X}}/[\mathcal{X}]}(M, K)$$

is natural in both  $M$  and  $K$  by Lemma 4.5. We claim that  $[s]^*$  is an isomorphism.

Indeed, since  $\mathcal{H}$  is a  $\xi$ xt-injective cogenerator of  $\mathcal{X}$ , by Lemma 3.9, we have  $\xi \text{xt}_{\xi}^1(X^M, K) = 0$ , and hence,  $\text{Hom}_{\mathcal{T}}(W^M, K) \rightarrow \text{Hom}_{\mathcal{T}}(M, K)$  is an epimorphism, so  $[s]^*$  is still an epimorphism.

Now, assume that  $g : W^M \rightarrow K$  is a morphism such that  $[gs] = [g][s] = [s]^*[g] = [0] \in \text{Hom}_{\widehat{\mathcal{X}}/[\mathcal{X}]}(M, K)$ . Then, there exists an object  $X \in \mathcal{X}$  such that  $gs = kv$ . Since  $\mathcal{H}$  is a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ , there exists a triangle

$$X \longrightarrow H \longrightarrow X' \longrightarrow \Sigma X$$

in  $\xi$  with  $H \in \mathcal{H}$  and  $X' \in \mathcal{X}$ . Note that  $\xi \text{xt}_{\xi}^1(X^M, H) = 0$  and  $\xi \text{xt}_{\xi}^1(X', K) = 0$ , so we get the following commutative diagram:

$$\begin{array}{ccccccc} M & \xrightarrow{s} & W^M & \xrightarrow{l} & X^M & \longrightarrow & \Sigma M \\ \downarrow v & & \downarrow v' & & & & \\ X & \longrightarrow & H & \longrightarrow & X' & \longrightarrow & \Sigma X \\ \downarrow k & \swarrow v'' & & & & & \\ & & K & & & & \end{array}$$

It follows that  $[v''v'] = [0] \in \text{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^M, K)$  as  $H \in \mathcal{X}$ . Since  $v''v's = kv = gs \in \text{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(M, K)$ , by Lemma 4.5(2), we have  $[g] = [v''v'] \in \text{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^M, K)$ , and hence,  $[g] = 0$ . So  $[s]^*$  is a monomorphism, and thus,  $[s]^*$  is an isomorphism.  $\square$

**Corollary 4.7.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  and  $\mathcal{H}$  a  $\xi$ xt-injective  $\xi$ -cogenerator of  $\mathcal{X}$ . Assume that  $\mathcal{X}$  is closed under direct summands. For any  $N \in \widehat{\mathcal{X}}$ , the following statements are equivalent:*

- (1)  $N \in \mathcal{X}$ .
- (2) *There is a triangle*

$$N \xrightarrow{s} W^N \longrightarrow X^N \longrightarrow \Sigma N$$

*in  $\xi$  with  $W^N \in \widehat{\mathcal{H}}$  and  $X^N \in \mathcal{X}$  such that  $[s] = [0] \in \text{Hom}_{\widehat{\mathcal{X}}/[\mathcal{X}]}(N, W^N)$ .*

**Proof.** The assertion (1)  $\Rightarrow$  (2) is obvious. It suffices to show (2)  $\Rightarrow$  (1). Note that the adjunction isomorphism established in Theorem 4.6 implies that the additive map

$$[s]^* : \text{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^N, W^N) \longrightarrow \text{Hom}_{\widehat{\mathcal{X}}/[\mathcal{X}]}(N, W^N)$$

is isomorphic. Since  $[s]^*[\text{id}_{W^N}] = [\text{id}_{W^N}s] = [s] = [0] \in \text{Hom}_{\widehat{\mathcal{X}}/[\mathcal{X}]}(N, W^N) = 0$ , so  $[\text{id}_{W^N}] = [0] \in \text{Hom}_{\widehat{\mathcal{H}}/[\mathcal{X}]}(W^N, W^N)$ , and thus,  $\text{id}_{W^N}$  factors through an object  $X' \in \mathcal{X}$ . It follows that  $W^N$  is a direct summand of  $X'$ . Since  $\mathcal{X}$  is closed under direct summands, we have  $W^N \in \mathcal{X}$ . Thus,  $N \in \mathcal{X}$ .  $\square$

## 4.2 A characterization of finite resolution dimension via $\xi$ -cellular towers

For  $M \in \widehat{\mathcal{X}}$ , there exists a triangle

$$K_1 \xrightarrow{f_0} X_0 \xrightarrow{g_0} M \xrightarrow{h_0} \Sigma K_1 \quad (13)$$

in  $\xi$  with  $X_0 \in \mathcal{X}$  and  $K_1 \in \widehat{\mathcal{X}}$ . Similarly, there exists a triangle

$$K_2 \xrightarrow{f_1} X_1 \xrightarrow{g_1} K_1 \xrightarrow{h_1} \Sigma K_2$$

in  $\xi$  with  $X_1 \in \mathcal{X}$  and  $K_2 \in \widehat{\mathcal{X}}$ . Continuing the above procedure for  $K_n$ , there exists a triangle

$$K_{n+1} \xrightarrow{f_n} X_n \xrightarrow{g_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$$

in  $\xi$  with  $X_n \in \mathcal{X}$  and  $K_{n+1} \in \widehat{\mathcal{X}}$ .

Applying cobase change for the triangle (13) along the morphism  $h_1 : K_1 \rightarrow \Sigma K_2$ , we get the following commutative diagram:

$$\begin{array}{ccccccc} \Sigma^{-1}M & \longrightarrow & K_1 & \xrightarrow{f_0} & X_0 & \xrightarrow{g_0} & M \\ \parallel & & \downarrow h_1 & & \downarrow \gamma_1 & & \parallel \\ \Sigma^{-1}M & \longrightarrow & \Sigma K_2 & \xrightarrow{u_2} & C_2 & \xrightarrow{v_2} & M, \end{array}$$

where the triangle

$$\Sigma K_2 \xrightarrow{u_2} C_2 \xrightarrow{v_2} M \longrightarrow \Sigma^2 K_2 \quad (14)$$

is in  $\xi$ . Next consider the triangle (14) along the morphism  $-\Sigma h_2 : \Sigma K_2 \rightarrow \Sigma^2 K_3$ , we get the following commutative diagram:

$$\begin{array}{ccccccc} \Sigma^{-1}M & \longrightarrow & \Sigma K_2 & \xrightarrow{u_2} & C_2 & \xrightarrow{v_2} & M \\ \parallel & & \downarrow -\Sigma h_2 & & \downarrow \gamma_2 & & \parallel \\ \Sigma^{-1}M & \longrightarrow & \Sigma^2 K_3 & \xrightarrow{u_3} & C_3 & \xrightarrow{v_3} & M, \end{array}$$

where the triangle  $\Sigma^2 K_3 \xrightarrow{u_3} C_3 \xrightarrow{v_3} M \rightarrow \Sigma^3 K_3$  is in  $\xi$ .

Continuing in this manner, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} K_1 & \xrightarrow{f_0} & X_0 & \xrightarrow{g_0} & M & \xrightarrow{h_0} & \Sigma K_1 \\ \downarrow & & \downarrow \gamma_1 & & \parallel & & \downarrow \\ \Sigma K_2 & \xrightarrow{u_2} & C_2 & \xrightarrow{v_2} & M & \longrightarrow & \Sigma^2 K_2 \\ \downarrow & & \downarrow \gamma_2 & & \parallel & & \downarrow \\ \Sigma^2 K_3 & \xrightarrow{u_3} & C_3 & \xrightarrow{v_3} & M & \longrightarrow & \Sigma^3 K_3 \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ \vdots & & \vdots & & \parallel & & \vdots \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ \Sigma^{n-1} K_n & \xrightarrow{u_n} & C_n & \xrightarrow{v_n} & M & \longrightarrow & \Sigma^n K_n \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ \vdots & & \vdots & & \parallel & & \vdots \end{array}$$

where all the horizontal triangles are in  $\xi$ .

Set  $C_0 = 0$  and  $C_1 = X_0$ . The above construction produces a tower

$$0 \longrightarrow C_1 \xrightarrow{Y_1} C_2 \xrightarrow{Y_2} \cdots \longrightarrow C_{n-1} \xrightarrow{Y_{n-1}} C_n \cdots,$$

which we call the  $\xi$ -cellular tower of  $M$  with respect to  $\mathcal{X}$ .

According to the above construction, one can obtain the following result by Proposition 3.3.

**Theorem 4.8.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$ . For any  $M \in \mathcal{T}$ , if  $M \in \widehat{\mathcal{X}}$ , then the following statements are equivalent:*

- (1)  $\mathcal{X}$ -res.dim  $M \leq n$ .
- (2) For each  $i > 0$ , the morphisms  $v_{n+i} : C_{n+i} \rightarrow M$  of the  $\xi$ -cellular tower of  $M$  with respect to  $\mathcal{X}$  constructed above are isomorphisms.

## 5 Applications

In this section, we will construct a new resolving subcategory from a given resolving subcategory, which generalizes the notion of  $\xi$ -Gorenstein projective objects given by Asadollahi and Salarian [13]. By applying the previous results to this subcategory, we obtain some known results in [13–15].

**Definition 5.1.** Let  $\mathcal{X}$  be a subcategory of  $\mathcal{T}$  and  $M$  an object in  $\mathcal{T}$ . A complete  $\mathcal{P}(\xi)\mathcal{X}$ -resolution of  $M$  is a  $\text{Hom}_{\mathcal{T}}(-, \mathcal{X})$ -exact  $\xi$ -exact complex

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

in  $\mathcal{T}$  with all  $P_i \in \mathcal{P}(\xi)$ ,  $X^i \in \mathcal{X} \cap {}^\perp \mathcal{X}$  such that both

$$K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow \Sigma K_1 \quad \text{and} \quad M \longrightarrow X^0 \longrightarrow K^1 \longrightarrow \Sigma M$$

are corresponding triangles in  $\xi$ . The  $\mathcal{GP}_{\mathcal{X}}(\xi)$ -Gorenstein category is defined as

$$\mathcal{GP}_{\mathcal{X}}(\xi) = \{M \in \mathcal{T} \mid M \text{ admits a complete } \mathcal{P}(\xi)\mathcal{X}\text{-resolution}\}.$$

**Remark 5.2.**

- (1) Since  $\mathcal{X}$  is a resolving subcategory of  $\mathcal{T}$ , we have  $\mathcal{P}(\xi) \subseteq \mathcal{X}$ , so  $\mathcal{P}(\xi) \subseteq \mathcal{X} \cap {}^\perp \mathcal{X}$ . Then, we have  $K_1 \in \mathcal{GP}_{\mathcal{X}}(\xi)$ .
- (2) If  $M \in \mathcal{GP}_{\mathcal{X}}(\xi)$ , then  $\xi \text{xt}_{\xi}^0(M, X) \cong \text{Hom}_{\mathcal{T}}(M, X)$  and  $\xi \text{xt}_{\xi}^1(M, X) = 0$  for any  $X \in \mathcal{X}$ . In fact, the following  $\xi$ -exact complex:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is a  $\xi$ -projective resolution of  $M$  (see [11]), which is  $\text{Hom}_{\mathcal{T}}(-, \mathcal{X})$ -exact.

Evidently,  $M \in \mathcal{GP}_{\mathcal{X}}(\xi)$  if and only if  $\xi \text{xt}_{\xi}^0(M, X) \cong \text{Hom}_{\mathcal{T}}(M, X)$  and  $\xi \text{xt}_{\xi}^1(M, X) = 0$  for any  $X \in \mathcal{X}$ , and  $M$  admits a  $\text{Hom}_{\mathcal{T}}(-, \mathcal{X})$ -exact  $\xi$ -exact complex

$$0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$

with  $X^i \in \mathcal{X} \cap {}^\perp \mathcal{X}$ .

- (3) If  $\mathcal{X} = \mathcal{P}(\xi)$ , then we have  $\mathcal{X} \cap {}^\perp \mathcal{X} = \mathcal{P}(\xi)$  by Lemma 3.12, and thus,  $\mathcal{GP}_{\mathcal{X}}(\xi)$  coincides with  $\mathcal{GP}(\xi)$  defined in [13].

We have the following result.

**Theorem 5.3.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$ . Then,  $\mathcal{GP}_{\mathcal{X}}(\xi)$  is a resolving subcategory of  $\mathcal{T}$ .*

**Proof.** Let  $P$  be a  $\xi$ -projective object. Consider the following  $\xi$ -exact complex:

$$\dots \longrightarrow 0 \xrightarrow{0} P \xrightarrow{\text{id}_P} P \xrightarrow{0} 0 \longrightarrow \dots$$

in  $\mathcal{T}$ . Clearly, it is  $\text{Hom}_{\mathcal{T}}(-, \mathcal{X})$ -exact. In particular,

$$0 \xrightarrow{0} P \xrightarrow{\text{id}_P} P \xrightarrow{0} 0 \quad \text{and} \quad P \xrightarrow{\text{id}_P} P \xrightarrow{0} 0 \xrightarrow{0} \Sigma P$$

are corresponding triangles in  $\xi$ . Since  $P \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$  by Remark 5.2(1), we have  $\mathcal{P}(\xi) \subseteq \mathcal{GP}_{\mathcal{X}}(\xi)$ .

As a similar argument to the proof of [18, Theorem 4.3(1)], we obtain that  $\mathcal{GP}_{\mathcal{X}}(\xi)$  is closed under  $\xi$ -extensions and hokernels of  $\xi$ -proper epimorphisms. Thus,  $\mathcal{GP}_{\mathcal{X}}(\xi)$  is a resolving subcategory of  $\mathcal{T}$ .  $\square$

**Lemma 5.4.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  satisfying  $\mathcal{X} \cap {}^{\perp}\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{X}}(\xi)$ . Then,  $\mathcal{X} \cap {}^{\perp}\mathcal{X}$  is a  $\xi\text{xt}$ -injective  $\xi$ -cogenerator of  $\mathcal{GP}_{\mathcal{X}}(\xi)$  and is closed under hokernels of  $\xi$ -proper epimorphisms.*

**Proof.** Let  $M \in \mathcal{GP}_{\mathcal{X}}(\xi)$ . There is a  $\text{Hom}_{\mathcal{T}}(-, \mathcal{X})$ -exact triangle

$$M \longrightarrow X^0 \longrightarrow K^1 \longrightarrow \Sigma M \tag{15}$$

in  $\xi$  with  $X^0 \in \mathcal{X} \cap {}^{\perp}\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{X}}(\xi)$ . For any  $\tilde{X} \in \mathcal{X}$ , applying the functor  $\text{Hom}_{\mathcal{T}}(-, \tilde{X})$  to the triangle (15) yields the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{\mathcal{T}}(K^1, \tilde{X}) & \longrightarrow & \text{Hom}_{\mathcal{T}}(X^0, \tilde{X}) & \longrightarrow & \text{Hom}_{\mathcal{T}}(M, \tilde{X}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \xi\text{xt}_{\xi}^0(K^1, \tilde{X}) & \longrightarrow & \xi\text{xt}_{\xi}^0(X^0, \tilde{X}) & \longrightarrow & \xi\text{xt}_{\xi}^0(M, \tilde{X}) \longrightarrow \xi\text{xt}_{\xi}^1(K^1, \tilde{X}) \longrightarrow \xi\text{xt}_{\xi}^1(X^0, \tilde{X}) (= 0), \end{array}$$

where the two isomorphisms follow from the assumption that  $X^0, M \in \mathcal{GP}_{\mathcal{X}}(\xi)$  and Remark 5.2(2). It follows that  $\xi\text{xt}_{\xi}^1(K^1, \tilde{X}) = 0$  and  $\xi\text{xt}_{\xi}^0(K^1, \tilde{X}) \cong \text{Hom}_{\mathcal{T}}(K^1, \tilde{X})$ , so  $K^1 \in \mathcal{GP}_{\mathcal{X}}(\xi)$  by Remark 5.2(2), then  $\mathcal{X} \cap {}^{\perp}\mathcal{X}$  is a  $\xi$ -cogenerator of  $\mathcal{GP}_{\mathcal{X}}(\xi)$ . Obviously,  $\mathcal{X} \cap {}^{\perp}\mathcal{X}$  is a  $\xi\text{xt}$ -injective  $\xi$ -cogenerator of  $\mathcal{GP}_{\mathcal{X}}(\xi)$ .

It is obvious that  $\mathcal{X} \cap {}^{\perp}\mathcal{X}$  is closed under hokernels of  $\xi$ -proper epimorphisms.  $\square$

As an application of Theorem 3.14, we have:

**Proposition 5.5.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$  satisfying  $\mathcal{X} \cap {}^{\perp}\mathcal{X} \subseteq \mathcal{GP}_{\mathcal{X}}(\xi)$  and  $M \in \mathcal{T}$ . If  $M \in \widehat{\mathcal{GP}_{\mathcal{X}}(\xi)}$ , then the following statements are equivalent:*

- (1)  $\mathcal{GP}_{\mathcal{X}}(\xi)\text{-res.dim } M \leq m$ .
- (2)  $\Omega^n(M) \in \mathcal{GP}_{\mathcal{X}}(\xi)$  for all  $n \geq m$ .
- (3)  $\Omega_{\mathcal{GP}_{\mathcal{X}}(\xi)}^n(M) \in \mathcal{GP}_{\mathcal{X}}(\xi)$  for all  $n \geq m$ .
- (4)  $\xi\text{xt}_{\xi}^n(M, H) = 0$  for all  $n > m$  and all  $H \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$ .
- (5)  $\xi\text{xt}_{\xi}^n(M, L) = 0$  for all  $n > m$  and all  $L \in \widehat{\mathcal{X} \cap {}^{\perp}\mathcal{X}}$ .
- (6)  $M$  admits a right  $\mathcal{GP}_{\mathcal{X}}(\xi)$ -approximation  $\varphi : X \rightarrow M$ , where  $\varphi$  is  $\xi$ -proper epic, such that  $K = \text{Hoker } \varphi$  satisfying  $\mathcal{H}\text{-res.dim } K \leq m - 1$ .
- (7) There are two triangles

$$W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M$$

and

$$M \longrightarrow W^M \longrightarrow X^M \longrightarrow \Sigma M$$

in  $\xi$  such that  $X_M, X^M \in \mathcal{GP}_{\mathcal{X}}(\xi)$  and  $\mathcal{X} \cap {}^{\perp}\mathcal{X}\text{-res.dim } W_M \leq m - 1$ ,  $\mathcal{X} \cap {}^{\perp}\mathcal{X}\text{-res.dim } W^M = \mathcal{GP}_{\mathcal{X}}(\xi)\text{-res.dim } W^M \leq m$ .

Immediately, we have:

**Corollary 5.6.** *Let  $\mathcal{T}$  be a triangulated category and  $M \in \mathcal{T}$ . If  $M \in \widehat{\mathcal{GP}(\xi)}$ , then the following statements are equivalent:*

- (1)  $\mathcal{GP}(\xi)$ -res.dim  $M \leq m$ .
- (2)  $\Omega^n(M) \in \mathcal{GP}(\xi)$  for all  $n \geq m$ .
- (3)  $\Omega^n_{\mathcal{GP}(\xi)}(M) \in \mathcal{GP}(\xi)$  for all  $n \geq m$ .
- (4)  $\xi \text{xt}_\xi^n(M, H) = 0$  for all  $n > m$  and all  $H \in \mathcal{P}(\xi)$ .
- (5)  $\xi \text{xt}_\xi^n(M, L) = 0$  for all  $n > m$  and all  $L \in \widehat{\mathcal{P}(\xi)}$ .
- (6)  $M$  admits a  $\mathcal{GP}(\xi)$ -approximation  $\varphi : X \rightarrow M$ , where  $\varphi$  is  $\xi$ -proper epic, such that  $K = \text{Hoker } \varphi$  satisfying  $\xi$ -pd  $K \leq m - 1$ .
- (7) There are two triangles

$$W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M$$

and

$$M \longrightarrow W^M \longrightarrow X^M \longrightarrow \Sigma M$$

in  $\xi$  such that  $X_M$  and  $X^M$  are in  $\mathcal{X}$  and  $\xi$ -pd  $W_M \leq m - 1$ ,  $\xi$ -pd  $W^M = \mathcal{GP}(\xi)$ -res.dim  $W^M \leq m$ .

**Remark 5.7.** As in Corollary 5.6, (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (6) is [13, Theorem 4.6 (ii)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv)], (1)  $\Leftrightarrow$  (5) is [13, Proposition 3.19]. (1)  $\Leftrightarrow$  (4) is [14, Remark 2.14].

Following Theorems 4.8 and 5.3, we have the following result, which is a generalization of [15, Proposition 5.1].

**Proposition 5.8.** *Let  $\mathcal{X}$  be a resolving subcategory of  $\mathcal{T}$ . For any  $M \in \mathcal{T}$ , if  $M \in \widehat{\mathcal{GP}_{\mathcal{X}}(\xi)}$ , then the following statements are equivalent:*

- (1)  $\mathcal{GP}_{\mathcal{X}}(\xi)$ -res.dim  $M \leq n$ .
- (2) For each  $i > 0$ , the morphisms  $v_{n+i} : C_{n+i} \rightarrow M$  of the  $\xi$ -cellular tower of  $M$  with respect to  $\mathcal{GP}_{\mathcal{X}}(\xi)$  constructed above are isomorphisms.

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