DE GRUYTER Open Mathematics 2021; 19: 87–100

Research Article

Emanuel Guariglia*

Fractional calculus, zeta functions and Shannon entropy

https://doi.org/10.1515/math-2021-0010
received August 11, 2020; accepted January 11, 2021

Abstract: This paper deals with the fractional calculus of zeta functions. In particular, the study is focused on the Hurwitz ζ function. All the results are based on the complex generalization of the Grünwald-Letnikov fractional derivative. We state and prove the functional equation together with an integral representation by Bernoulli numbers. Moreover, we treat an application in terms of Shannon entropy.

Keywords: Hurwitz ζ function, fractional derivative, functional equation, Bernoulli numbers, Shannon entropy

MSC 2020: 11M35, 26A33, 11B68, 34K37, 49K99

1 Introduction

In recent years, there has been a relevant interest in fractional calculus of complex variables (see, e.g., [1,2]). Nevertheless, there are only a few papers concerning the fractional calculus of zeta functions [3–5]. This depends on the different unsolved problems in the theory of zeta functions. Fractional calculus of complex functions entails several problems [6], thus it has not grown as fast as real fractional calculus. Nevertheless, the fractional derivative of zeta functions is fairly easy to compute. In addition to this, fractional calculus of zeta functions has unveiled new results and applications both in dynamical system theory and signal processing [7].

The fractional derivative of the Riemann ζ function allows several generalizations. These include, *inter alia*, the fractional derivative of the Hurwitz ζ function and that of a Dirichlet series [8]. Following the approach of Apostol [9], this paper follows up on fractional analysis of zeta functions. Further studies on the link between fractional calculus of Riemann zeta function and the distribution of prime numbers can be found in [5,7].

In what follows, we deal with the fractional derivative of the Hurwitz ζ function computing the functional equation. For the fractional derivative of the Riemann ζ function and the relative functional equation, we refer the reader to [4,5]. More precisely, the generalized Leibniz rule enables the computation of the functional equation sought. Thus, the fractional derivative of zeta functions seems to play the same relevant role in fractional calculus that the class of zeta functions plays in pure and applied mathematics. In addition to this, zeta functions make the definition of a probability distribution on $\mathbb{N}$ allowable. Moreover, the nontrivial zeros of the Riemann zeta function are nowadays one of the most relevant unsolved problems in mathematics. The distribution of these zeros seems to obey to no mathematical law. Several attempts to prove the Riemann hypothesis have failed in recent years. Quite recently, some analytic results shed new light on the Riemann hypothesis [10]. On the other hand, random matrix theory is a fundamental tool in modern number theory and, more precisely, in the class of zeta functions. For more details we refer the reader

* **Corresponding author: Emanuel Guariglia,** Institute of Biosciences, Letters and Exact Sciences, São Paulo State University (UNESP), Rua Cristóvão Colombo 2265, 15054-000 São José do Rio Preto, SP, Brazil, e-mail: emanuel.guariglia@gmail.com, emanuel.guariglia@unesp.br

to [11,12]. This fact suggests to apply the theory of zeta functions to statistical problems (e.g., Shannon entropy [13]).

In this paper, we show that fractional calculus of zeta functions can provide applications in different mathematical fields, according to recent results [7,8]. In particular, we derive a functional equation for the fractional derivative of the Hurwitz ζ function. Likewise, we give an integral representation of this derivative. Moreover, we prove a connection between fractional calculus of zeta functions and Bernoulli numbers. Finally, we show that our analytic results find application in the concept of Shannon entropy.

The remainder of the paper is organized into three sections. Section 2 presents some preliminaries on fractional calculus of zeta functions and Bernoulli numbers. Section 3 is devoted to analytic results on fractional calculus of zeta functions. Finally, Section 4 concludes the paper with an application in information theory.

2 Notation and background

This section is devoted to introduce some notation and definitions needed throughout the rest of the paper. Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_0$ with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We use the notation $\alpha^{\underline{n}}$ to denote the nth falling factorial of α [4]. Moreover, $\lfloor \alpha \rfloor$ stands for the integer part of a real number α. Here and subsequently, s denotes a complex variable.

The Riemann ζ function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re} s > 1, \tag{2.1}$$

admits several generalizations. In particular, two of the most important ones are, respectively, the Hurwitz ζ function and Dirichlet series defined by

$$\begin{cases} \zeta(s, a) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, & \operatorname{Re} s > 1, \;\; a \in \mathbb{R} : 0 < a \leq 1, \\ F(s) = \displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^s}, & f : \mathbb{N} \to \mathbb{C}. \end{cases} \tag{2.2}$$

Clearly, $\zeta(s, 1) = \zeta(s)$ and $F(s; f(n) = 1) = \zeta(s)$. Fractional calculus of Dirichlet series can be found in [8] and this topic exceeds the scope of this paper. It is worth noting that (2.1) and $(2.2)_1$ exhibit the same analytic behavior. In fact, these zeta functions either converge in the half-plane $\operatorname{Re} s > 1$. Furthermore, (2.1) and $(2.2)_1$ own a unique analytical continuation to the entire complex plane, except a simple pole (with residue 1) in $s = 1$, given [14] by

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s), \tag{2.3}$$

and

$$\zeta\left(s, \frac{p}{q}\right) = 2(2\pi q)^{s-1}\Gamma(1-s)\sum_{m=1}^{q} \sin\left(\frac{\pi s}{2} + \frac{2\pi m p}{q}\right)\zeta\left(1-s, \frac{m}{q}\right). \tag{2.4}$$

Note that (2.3) and (2.4) hold for any $s \in \mathbb{C}$. In the functional equation (2.4), also known as Rademacher's formula [15], p and q are integers such that $1 \leq p \leq q$. Clearly, (2.4) reduces to (2.3) for $p = q = 1$.

Let f be a function analytic inside the region $D \subseteq \mathbb{C}$ and continuous on its contour C_d. The forward Grünwald-Letnikov fractional derivative of f is defined as follows:

$$D_f^{\alpha} f(s) = \lim_{h \to 0^+} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k}(-1)^k f(s-kh)}{h^{\alpha}}, \quad \alpha \in \mathbb{R}. \tag{2.5}$$

Remark 2.1. The choice of the fractional derivative (2.5) depends on a relevant prerequisite. In fact, this derivative satisfies the generalized Leibniz rule:

$$D_f^\alpha(f(s)g(s)) = \sum_{n=0}^{\infty}\binom{\alpha}{n}f^{(n)}(s)g^{(\alpha-n)}(s), \quad \alpha \in \mathbb{R}_{>0}\backslash\mathbb{N}, \tag{2.6}$$

where at least one function between f and g in (2.6) is analytic in the region $D \subseteq \mathbb{C}$. A proof of (2.6) can be found in [4].

The fractional derivative (2.5) covers a fundamental role in fractional calculus of holomorphic functions. In fact, (2.6) implies [4] that

$$\zeta^{(\alpha)}(s) = 2(2\pi)^{s-1}e^{i\pi\alpha}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}A_{n,j,k}^{\alpha}\zeta^{(n)}(1-s)\left(-\frac{\pi}{2}\right)^{j}\sin\left(\frac{\pi}{2}(s+j)\right)\frac{\Gamma^{(k)}(1-s)}{\log^{n+j+k-\alpha}(2\pi)}, \tag{2.7}$$

where $A_{n,j,k}^{\alpha} := \frac{\alpha^{n+j+k}}{n!\,j!\,k!}$. Note that the functional equation (2.7) can easily be written in terms of sines and cosines and entails high computational cost. The author dealt with these problems in [5]. In particular, the method proposed in [5] reduces the computational cost of (2.7) to only one infinite series. This technique involves the auxiliary function ψ defined as follows:

$$\psi(s,w,z) = \Gamma(s)\zeta(s)e^{sw+z}, \quad z \in \mathbb{C}, \ w \in \mathbb{C} : \operatorname{Re} w < 0. \tag{2.8}$$

Accordingly,

$$D_f^\alpha\psi(1-s,w,z) = \sum_{h=0}^{\infty}\binom{\alpha}{h}e^{i\pi(\alpha-h)}w^{\alpha-h}e^{(1-s)w+z}(\Gamma(1-s)\zeta(1-s))^{(h)}. \tag{2.9}$$

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a function such that the fractional incremental ratio in (2.5) is uniformly convergent in D. Then, for any $\alpha \in \mathbb{R}\backslash\mathbb{Z} : \lfloor\alpha\rfloor = m-1$, we have

$$\begin{cases} D_f^\alpha f(s) \xrightarrow{\alpha\to m^-} f^{(m)}(s), \\ D_f^\alpha f(s) \xrightarrow{\alpha\to(m-1)^+} f^{(m-1)}(s). \end{cases} \tag{2.10}$$

whose proof can be found in [4]. Note that (2.1) and (2.2) fulfill all hypotheses of (2.10).

Let us now recall the main properties of Bernoulli numbers with respect to zeta functions. Bernoulli polynomials $B_n(s)$ of the complex variable s are defined by the following equation:

$$\frac{ze^{sz}}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n(s)}{n!}z^n, \quad z \in \mathbb{C} : |z| < 2\pi.$$

All the numbers $B_n(0)$ are called Bernoulli numbers and simply denoted by B_n. Therefore,

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n, \quad z \in \mathbb{C} : |z| < 2\pi. \tag{2.11}$$

It is worth noting that the Riemann ζ function is closely linked to Bernoulli numbers, as next result points out.

Proposition 2.2. [14,16] *Let $n \in \mathbb{N}_0$. The values of the Riemann ζ function for non-positive integers and positive even numbers are given, respectively, by*

$$\zeta(-n) = -\frac{B_{n+1}(1)}{n+1} = \begin{cases} -\frac{1}{2}, & n = 0, \\ -\frac{B_{n+1}}{n+1}, & n \geq 1, \end{cases}$$

and

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!}.$$

Furthermore, we get

$$\zeta'(-2n) = (-1)^{n}\frac{(2n)!}{2(2\pi)^{2n}}\zeta(2n+1), \quad n \geq 1. \tag{2.12}$$

Remark 2.3. From (2.11), we see at once that $B_0 = 1$ and $B_1 = -1/2$. The other Bernoulli numbers are given by the following expansion:

$$B_n = \sum_{k=0}^{n}\binom{n}{k}B_k, \quad n \geq 2. \tag{2.13}$$

Indeed, the recursive implementation of (2.13) gives $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, etc. Moreover, the functional equation (2.3) gives no information on $\zeta(2n+1)$ (both members vanish). In the current literature, no simple formula for positive odd values of ζ is known (see, e.g., [14,17]).

Finally, we note that (2.12) does not hold for $n = 0$. Thus, we recall that

$$\zeta'(0) = -\frac{\log(2\pi)}{2}. \tag{2.14}$$

whose proof can be found in Appendix. Equality (2.14) is often used to show the coherence of new results with the theory of zeta functions.

3 Analytic results

In this section, we focus on analytic properties of the derivative $(3.1)_2$. In particular, we state and prove the functional equation. Moreover, we give an integral representation of $(3.1)_1$ in terms of Bernoulli numbers.

It is worth noting that fractional derivatives of (2.1) and (2.2) have already been computed in [3,8] using a different definition of fractional derivative [6]. Thus, we begin by proving that the definition of fractional derivative in (2.5) gives the same results.

Theorem 3.1. *Let* $\alpha \in \mathbb{R}\setminus\mathbb{Z}$ *and* $0 < a \leq 1$. *Then*

$$\begin{cases} \zeta^{(\alpha)}(s) = \mathrm{e}^{i\pi\alpha}\displaystyle\sum_{n=2}^{\infty}\frac{\log^{\alpha} n}{n^s}, \\ \zeta^{(\alpha)}(s,a) = \mathrm{e}^{i\pi\alpha}\displaystyle\sum_{n=0}^{\infty}\frac{\log^{\alpha}(n+a)}{(n+a)^s}, \\ F^{(\alpha)}(s) = \mathrm{e}^{i\pi\alpha}\displaystyle\sum_{n=2}^{\infty}f(n)\frac{\log^{\alpha} n}{n^s}. \end{cases} \tag{3.1}$$

Proof. We proved $(3.1)_1$ in [4]. The rest of the proof can be handled in much the same way. Indeed, let us prove $(3.1)_3$. From (2.5), we get

$$D_f^{\alpha}F(s) = \lim_{h\to 0^+}\frac{\sum_{k=0}^{\infty}\binom{\alpha}{k}(-1)^k F(s-kh)}{h^{\alpha}} = \sum_{n=1}^{\infty}\frac{f(n)}{n^s}\lim_{h\to 0^+}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}\binom{\alpha}{k}(-1)^k n^{kh}.$$

Binomial series expansion gives

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}(-1)^k n^{kh} = (1-n^h)^\alpha,$$

and so

$$D_f^\alpha F(s) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s}\lim_{h\to 0^+}\frac{(1-n^h)^\alpha}{h^\alpha}. \tag{3.2}$$

We observe that the limit in (3.2) gives us the indeterminate form $\frac{0}{0}$. L'Hôpital's rule implies that

$$\lim_{h\to 0^+}\left(\frac{1-n^h}{h}\right)^\alpha = \left(\lim_{h\to 0^+}\frac{1-n^h}{h}\right)^\alpha = \mathrm{e}^{i\pi\alpha}\log^\alpha n. \tag{3.3}$$

Substituting (3.3) into (3.2) we get $(3.1)_3$. The proof of $(3.1)_2$ is similar and we thus leave it to the reader. □

Remark 3.2. Convergence of both $(3.1)_1$ and $(3.1)_2$ depends on α (see [3,8]). Accordingly, Theorem 3.1 implies that convergence of $(3.1)_3$ depends on both α and f.

3.1 Functional equation

Theorem 3.3. *Let $\alpha \in \mathbb{R}_{>0}\backslash\mathbb{N}$ and $p, q \in \mathbb{Z} : 1 \le p \le q$. Then, for any $s \in \mathbb{C}$,*

$$\begin{aligned}\zeta^{(\alpha)}\left(s,\frac{p}{q}\right) &= 2(2\pi q)^{s-1}\mathrm{e}^{i\pi\alpha}\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}A_{h,j,n}^{\alpha}\frac{\Gamma^{(h)}(1-s)}{\log^{h+j+n-\alpha}(2\pi q)}\\ &\quad\cdot\left(-\frac{\pi}{2}\right)^j\sum_{m=1}^{q}\sin\left(\frac{\pi}{2}(s+j)+\frac{2\pi mp}{q}\right)\zeta^{(n)}\left(1-s,\frac{m}{q}\right).\end{aligned} \tag{3.4}$$

Proof. First, we note that this result can be proved in much the same way as [4, Theorem 9]. In fact, from (2.4), we have

$$\begin{aligned}\zeta^{(\alpha)}\left(s,\frac{p}{q}\right) &= 2\sum_{h=0}^{\infty}\binom{\alpha}{h}\frac{\mathrm{d}^h}{\mathrm{d}s^h}(\Gamma(1-s))\sum_{j=0}^{\infty}\sum_{m=1}^{q}\binom{\alpha-h}{j}\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\sin\left(\frac{\pi s}{2}+\frac{2\pi mp}{q}\right)\right)\\ &\quad\cdot\sum_{n=0}^{\infty}\binom{\alpha-h-j}{n}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\zeta\left(1-s,\frac{m}{q}\right)\right)D_f^{\alpha-h-j-n}((2\pi q)^{s-1}).\end{aligned} \tag{3.5}$$

We see at once that

$$\begin{cases}\dfrac{\mathrm{d}^h}{\mathrm{d}s^h}(\Gamma(1-s)) = \mathrm{e}^{i\pi h}\Gamma^{(h)}(1-s),\\[2mm] \dfrac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\sin\left(\dfrac{\pi s}{2}+\dfrac{2\pi mp}{q}\right)\right) = \left(\dfrac{\pi}{2}\right)^j\sin\left(\dfrac{\pi}{2}(s+j)+\dfrac{2\pi mp}{q}\right),\\[2mm] \dfrac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\zeta\left(1-s,\dfrac{m}{q}\right)\right) = \mathrm{e}^{i\pi n}\zeta\left(1-s,\dfrac{m}{q}\right).\end{cases}$$

Moreover, in view of Corollary 4 in [4], we have

$$D_f^\alpha(2\pi)^s = (2\pi)^s\mathrm{e}^{i\pi\alpha}\log^\alpha(2\pi),$$

and so

$$\begin{cases} D_f^{\alpha-h-j-n}(2\pi q)^{s-1} = (2\pi q)^{s-1} e^{i\pi(\alpha-h-j-n)} \log^{\alpha-h-j-n}(2\pi q), \\ \binom{\alpha}{h}\binom{\alpha-h}{j}\binom{\alpha-h-j}{n} = \dfrac{\alpha^{\underline{h+j+n}}}{h!\, j!\, n!} = A^{\alpha}_{h,j,n}. \end{cases}$$

Therefore,

$$\zeta^{(\alpha)}\left(s, \frac{p}{q}\right) = 2 \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} A^{\alpha}_{h,j,n} \frac{\Gamma^{(h)}(1-s)}{\log^{h+j+n-\alpha}(2\pi q)} (2\pi q)^{s-1} e^{i\pi(\alpha-j)} \left(\frac{\pi}{2}\right)^j \cdot \sum_{m=1}^{q} \sin\left(\frac{\pi}{2}(s+j) + \frac{2\pi m p}{q}\right) \zeta\left(1-s, \frac{m}{q}\right),$$

as desired. □

Remark 3.4. Let us now restrict our attention to the consistency of (3.4). Note that (2.10) yields

$$f^{(\alpha)}(s) \xrightarrow{\alpha \to 0^+} f^{(0)}(s) = f(s).$$

In particular, the Hurwitz ζ function fulfills all hypotheses of (2.10), and thus

$$\zeta^{(\alpha)}\left(s, \frac{p}{q}\right) \xrightarrow{\alpha \to 0^+} \zeta\left(s, \frac{p}{q}\right).$$

Of course, the proof of Theorem 3.3 can be read backwards until (3.5). Therefore, the right-hand side of (3.5) converges to that of (2.4) as $\alpha \to 0^+$. As a consequence, (3.4) is consistent with the theory of zeta functions.

Now, we are in position to characterize the functional equation (3.4). More precisely, Corollary 3.5 allows us to express (3.4) in terms of sines and cosines. Furthermore, Theorem 3.6 reduces its computational cost.

Corollary 3.5. *Let $\alpha \in \mathbb{R}_{>0} \backslash \mathbb{N}$ and $p, q \in \mathbb{Z} : 1 \le p \le q$. Then, for any $s \in \mathbb{C}$,*

$$\zeta^{(\alpha)}\left(s, \frac{p}{q}\right) = 2(2\pi q)^{s-1} e^{i\pi\alpha} \sum_{h=0}^{\infty} \sum_{n=0}^{\infty} \Gamma^{(h)}(1-s) \sum_{m=1}^{q} \left(a^{p,q}_{hamn} \sin\frac{\pi s}{2} + b^{p,q}_{hamn} \cos\frac{\pi s}{2}\right) \zeta^{(n)}\left(s, \frac{m}{q}\right),$$

where the coefficients $a^{p,q}_{hamn}$ and $b^{p,q}_{hamn}$ are given by

$$\begin{cases} a^{p,q}_{hamn} = \displaystyle\sum_{j=0}^{\infty} \frac{A^{\alpha}_{h,j,n}}{\log^{h+j+n-\alpha}(2\pi q)} \left(-\frac{\pi}{2}\right)^j \cos\left(\frac{\pi j}{2} + \frac{2\pi m p}{q}\right), \\ b^{p,q}_{hamn} = \displaystyle\sum_{j=0}^{\infty} \frac{A^{\alpha}_{h,j,n}}{\log^{h+j+n-\alpha}(2\pi q)} \left(-\frac{\pi}{2}\right)^j \sin\left(\frac{\pi j}{2} + \frac{2\pi m p}{q}\right). \end{cases} \tag{3.6}$$

Proof. Substituting the trigonometric identity

$$\sin\left(\frac{\pi}{2}(s+j) + \frac{2\pi m p}{q}\right) = \sin\frac{\pi s}{2} \cos\left(\frac{\pi j}{2} + \frac{2\pi m p}{q}\right) + \cos\frac{\pi s}{2} \sin\left(\frac{\pi j}{2} + \frac{2\pi m p}{q}\right),$$

into (3.4) the proof is straightforward. □

Note that the coefficients $a^{p,q}_{hamn}$ and $b^{p,q}_{hamn}$ in (3.6) are independet of s, as in the case of integer order [9].

Theorem 3.6. *Let $\alpha \in \mathbb{R}_{>0} \backslash \mathbb{N}$, $w_q = -\log(2\pi q) - i\pi/2$ and $p, q \in \mathbb{Z} : 1 \le p \le q$. Then, for any $s \in \mathbb{C}$,*

$$\zeta^{(\alpha)}\left(s, \frac{p}{q}\right) = \sum_{m=1}^{q} \sum_{h=0}^{\infty} \binom{\alpha}{h} e^{i\pi(\alpha-h)} \left(e^{(1-s)w_q + i\frac{2\pi m p}{q}} w_q^{\alpha-h} - e^{(1-s)\overline{w_q} + i\left(\pi - \frac{2\pi m p}{q}\right)} (\overline{w_q})^{\alpha-h}\right) \left(\Gamma(1-s)\zeta\left(1-s, \frac{m}{q}\right)\right)^{(h)}.$$

Proof. Let us begin by writing (2.4) in a different form. We observe that

$$2(2\pi q)^{s-1}\sin\left(\frac{\pi s}{2}+\frac{2\pi mp}{q}\right)=\mathrm{e}^{(1-s)\left(-\log(2\pi q)-i\frac{\pi}{2}\right)+i\frac{2\pi mp}{q}}-\mathrm{e}^{(1-s)\left(-\log(2\pi q)+i\frac{\pi}{2}\right)-i\left(\pi+\frac{2\pi mp}{q}\right)},$$

thus the $2\pi i$-periodicity of the complex exponential implies

$$\zeta\left(s,\frac{p}{q}\right)=\sum_{m=1}^{q}\Gamma(1-s)\zeta\left(1-s,\frac{m}{q}\right)\left(\mathrm{e}^{(1-s)\left[-\log(2\pi q)-i\frac{\pi}{2}\right]+i\frac{2\pi mp}{q}}-\mathrm{e}^{(1-s)\left[-\log(2\pi q)+i\frac{\pi}{2}\right]+i\left(\pi-\frac{2\pi mp}{q}\right)}\right).$$

The auxiliary function in (2.8) allows us to reduce the computational cost of (2.7) (see [5]). In the same spirit, the function ψ_q defined by

$$\psi_q\left(s,\frac{m}{q},w_q,z\right)=\Gamma(s)\zeta\left(s,\frac{m}{q}\right)\mathrm{e}^{sw_q+z},\quad z\in\mathbb{C},\ w_q\in\mathbb{C}:\operatorname{Re}w_q<0,\tag{3.7}$$

implies that (2.4) can be written as follows:

$$\zeta\left(s,\frac{p}{q}\right)=\sum_{m=1}^{q}\left(\psi_q\left(1-s,\frac{m}{q},w_q,i\frac{2\pi mp}{q}\right)-\psi_q\left(1-s,\frac{m}{q},\overline{w_q},i\left(\pi-\frac{2\pi mp}{q}\right)\right)\right).$$

On the other hand,

$$\zeta^{(\alpha)}\left(s,\frac{p}{q}\right)=\sum_{m=1}^{q}\left(D_f^{\alpha}\psi_q\left(1-s,\frac{m}{q},w_q,i\frac{2\pi mp}{q}\right)-D_f^{\alpha}\psi_q\left(1-s,\frac{m}{q},\overline{w_q},i\left(\pi-\frac{2\pi mp}{q}\right)\right)\right).\tag{3.8}$$

From (2.6) we get

$$D_f^{\alpha}\psi_q\left(1-s,\frac{m}{q},w_q,z\right)=\sum_{h=0}^{\infty}\binom{\alpha}{h}\mathrm{e}^{i\pi(\alpha-h)}\,w_q^{\alpha-h}\mathrm{e}^{(1-s)w_q+z}\left(\Gamma(1-s)\zeta\left(1-s,\frac{m}{q}\right)\right)^{(h)}.\tag{3.9}$$

Therefore, the desired result plainly follows from (3.8) and (3.9). □

Remark 3.7. We note that Theorem 3.6 for $p=q=1$ gives

$$\zeta^{(\alpha)}(s)=\sum_{h=0}^{\infty}\binom{\alpha}{h}\mathrm{e}^{i\pi(\alpha-h)}\left(\mathrm{e}^{(1-s)w}w^{\alpha-h}-\mathrm{e}^{(1-s)\overline{w}-i\pi}(\overline{w})^{\alpha-h}\right)\left(\Gamma(1-s)\zeta(1-s)\right)^{(h)},\quad w=-\log(2\pi)-i\pi/2.\tag{3.10}$$

The author proved that (3.10) is a simplified form of (2.7). For more details we refer the reader to [5]. Hence, Theorem 3.6 coherently generalizes recent results in fractional calculus of zeta functions. Moreover, we observe that the introduction of ψ_q relies on the fact that

$$D_f^{\alpha}\mathrm{e}^{(1-s)w}=\mathrm{e}^{i\pi\alpha}\mathrm{e}^{(1-s)w}\,w^{\alpha},\quad \alpha\in\mathbb{R},\ w\in\mathbb{C}:\operatorname{Re}w<0.$$

As in (3.7), the restriction $\operatorname{Re}w<0$ in (2.8) implies convergence of $D_f^{\alpha}\mathrm{e}^{(1-s)w}$. Finally, we see that replacing $\psi(1-s,w,z)$ by $\psi_q(1-s,m/q,w_q,z)$, w by w_q and $\zeta(1-s)$ by $\zeta(1-s,m/q)$ in (2.9), we get the expansion in (3.9).

3.2 Integral representation

The approach proposed here is based [9] on the following representation:

$$\zeta(s,a)=a^{-s}\left(\frac{1}{2}+\frac{a}{s-1}\right)-s(s+1)\int_0^{\infty}\frac{\varphi_2(x)}{(x+a)^{s+2}}\mathrm{d}x,\quad \operatorname{Re}s>-1,\tag{3.11}$$

where the function φ_2 defined by

$$\varphi_2(x) = \int_0^x (t - \lfloor t \rfloor - 1)\,\mathrm{d}t, \tag{3.12}$$

is 1-periodic satisfying the condition

$$\varphi_2(x) = \frac{1}{2}x(x-1), \quad 0 \le x \le 1.$$

Note that (3.11) is a direct consequence of Euler's summation formula. Now, we are ready to give an integral representation of $\zeta^{(\alpha)}(s,a)$, as next result points out.

Theorem 3.8. *Let $\alpha \in \mathbb{R}_{>0} \backslash \mathbb{N}$ and $a \in \mathbb{Z} : 0 < a \le 1$. In the half-plane* $\operatorname{Re} s > -1$, *we have*

$$\begin{aligned}\zeta^{(\alpha)}(s,a) = \mathrm{e}^{i\pi\alpha}\Bigg(&\frac{\log^\alpha a}{2a^s} + a^{1-s}\sum_{j=0}^{\infty}\alpha^{\underline{j}}\frac{\log^{\alpha-j}a}{(s-1)^{j+1}} - s(s+1)\int_0^\infty \frac{\varphi_2(x)\log^\alpha(x+a)}{(x+a)^{s+2}}\mathrm{d}x\\ &+ \alpha(2s+1)\int_0^\infty \frac{\varphi_2(x)\log^{\alpha-1}(x+a)}{(x+a)^{s+2}}\mathrm{d}x - \alpha(\alpha-1)\int_0^\infty \frac{\varphi_2(x)\log^{\alpha-2}(x+a)}{(x+a)^{s+2}}\mathrm{d}x\Bigg),\end{aligned} \tag{3.13}$$

where φ_2 is defined by (3.12).

Proof. From (3.11), we get

$$\zeta^{(\alpha)}(s,a) = D_f^\alpha\left(\frac{1}{2a^s}\right) + D_f^\alpha\left(\frac{a^{1-s}}{s-1}\right) - D_f^\alpha\left(s(s+1)\int_0^\infty \frac{\varphi_2(x)}{(x+a)^{s+2}}\mathrm{d}x\right). \tag{3.14}$$

The proof consists in computing the three fractional derivatives in the right-hand side of (3.14). First, replacing n with a in $(3.1)_1$ gives

$$D_f^\alpha\left(\frac{1}{2a^s}\right) = \mathrm{e}^{i\pi\alpha}\frac{\log^\alpha a}{2a^s}.$$

Moreover, (2.6) implies

$$D_f^\alpha\left(\frac{a^{1-s}}{s-1}\right) = \sum_{j=0}^{\infty}\binom{\alpha}{j}\left(\frac{1}{s-1}\right)^{(j)}\left(\frac{1}{a^{s-1}}\right)^{(\alpha-j)}.$$

On the other hand,

$$\left(\frac{1}{s-1}\right)^{(j)} = \mathrm{e}^{i\pi j}\frac{j!}{(s-1)^{j+1}},$$

and

$$\left(\frac{1}{a^{s-1}}\right)^{(\alpha-j)} = \mathrm{e}^{i\pi(\alpha-j)}\frac{\log^{\alpha-j}a}{a^{s-1}}.$$

Thus,

$$D_f^\alpha\left(\frac{a^{1-s}}{s-1}\right) = \mathrm{e}^{i\pi\alpha}a^{1-s}\sum_{j=0}^{\infty}\alpha^{\underline{j}}\frac{\log^{\alpha-j}a}{(s-1)^{j+1}}.$$

Analogously, we have

$$D_f^\alpha\left(s(s+1)\int_0^\infty \frac{\varphi_2(x)}{(x+a)^{s+2}}\mathrm{d}x\right) = \sum_{m=0}^{\infty}\binom{\alpha}{m}(s(s+1))^{(m)}\int_0^\infty \varphi_2(x)\left(\frac{1}{(x+a)^{s+2}}\right)^{(\alpha-m)}\mathrm{d}x.$$

The series above reduces to only three terms since $(s(s+1))^{(m)} = 0$ for any $m > 2$. As a consequence,

$$D_f^\alpha\left(s(s+1)\int_0^\infty \frac{\varphi_2(x)}{(x+a)^{s+2}}\mathrm{d}x\right) = s(s+1)\mathrm{e}^{i\pi\alpha}\int_0^\infty \frac{\varphi_2(x)\log^\alpha(x+a)}{(x+a)^{s+2}}\mathrm{d}x - \alpha(2s+1)\mathrm{e}^{i\pi\alpha}\int_0^\infty \frac{\varphi_2(x)\log^{\alpha-1}(x+a)}{(x+a)^{s+2}}\mathrm{d}x$$
$$+ \alpha(\alpha-1)\mathrm{e}^{i\pi\alpha}\int_0^\infty \frac{\varphi_2(x)\log^{\alpha-2}(x+a)}{(x+a)^{s+2}}\mathrm{d}x.$$

This concludes the proof. □

It is worth noting that Apostol proved an integral representation similar to (3.13) for the integer derivative of the Hurwitz ζ function [9]. Thus, Theorem 3.8 gives a fractional generalization of this result. Now, we are in position to deal with the consistency of (3.13). To check this, we observe that

$$\zeta^{(\alpha)}(0,a) = \mathrm{e}^{i\pi\alpha}\left(\frac{\log^\alpha a}{2} + a\sum_{j=0}^\infty \alpha^{\underline{j}}\frac{\log^{\alpha-j} a}{(-1)^{j+1}} + \alpha\int_0^\infty \frac{\varphi_2(x)\log^{\alpha-1}(x+a)}{(x+a)^2}\mathrm{d}x - \alpha(\alpha-1)\int_0^\infty \frac{\varphi_2(x)\log^{\alpha-2}(x+a)}{(x+a)^2}\mathrm{d}x\right).$$

Since $\zeta^{(\alpha)}(s,1) = \zeta^{(\alpha)}(s)$, we have

$$\zeta^{(\alpha)}(0) = \alpha\mathrm{e}^{i\pi\alpha}\int_0^\infty \frac{\varphi_2(x)\log^{\alpha-1}(x+1)}{(x+1)^2}\Big(1-(\alpha-1)\log^{-1}(x+1)\Big)\mathrm{d}x,$$

which leads to

$$\zeta^{(\alpha)}(0) \xrightarrow{\alpha\to 1^+} \zeta'(0) = -1 - \int_0^\infty \frac{\varphi_2(x)}{(x+1)^2}\mathrm{d}x = -\frac{\log(2\pi)}{2}, \tag{3.15}$$

where the last equality follows Euler's summation formula and Stirling's formula (see Appendix). We conclude that (3.13) reduces to (2.14) when $s = 0$, $a = 1$ and $\alpha \to 1^+$. Accordingly, Theorem 3.8 is consistent with the theory of zeta functions.

3.3 Link between $\zeta^{(\alpha)}$ and Bernoulli numbers

In the half-plane $\operatorname{Re} s > -2n$ with $n \in \mathbb{N}$, the Riemann ζ function can be expressed as [9] follows:

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{r=1}^n \frac{B_{2r}}{2r}\binom{s+2r-2}{2r-1} - \binom{s+2n}{2n+1}\int_1^\infty \frac{P_{2n+1}(x)}{x^{s+2n+1}}\mathrm{d}x. \tag{3.16}$$

Of course, B_{2r} are Bernoulli numbers and

$$P_{2n+1}(x) = (-1)^{n+1}\frac{2(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^\infty \frac{\sin 2k\pi x}{k^{2n+1}},$$

is the periodic Bernoulli function [14]. Note that (3.16) is a consequence of Euler's summation formula. For the sake of simplicity, we set

$$\begin{cases} Q_m(s) := \dbinom{s+m-1}{m}, \\ I_m(s) := \displaystyle\int_1^\infty \frac{P_m(x)}{x^{s+m}}\mathrm{d}x, \end{cases} \tag{3.17}$$

therefore, (3.16) can be rewritten as

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{r=1}^{n} \frac{B_{2r}}{2r} Q_{2r-1}(s) - Q_{2n+1}(s)\, I_{2n+1}(s). \tag{3.18}$$

Obviously, $Q_m(s)$ is a polynomial in s of degree m, thus $m \in \mathbb{N}_0$ in (3.17). In 1985, Apostol showed [9] the following link between the integer derivative $\zeta^{(k)}$ and Bernoulli numbers:

$$\zeta^{(k)}(s) = \frac{(-1)^k k!}{(s-1)^{k+1}} + \sum_{r=1}^{n} \frac{B_{2r}}{2r} Q_{2r-1}^{(k)}(s) - \sum_{j=0}^{k} \binom{k}{j} Q_{2n+1}^{(j)}(s)\, I_{2n+1}^{(k-j)}(s), \tag{3.19}$$

which holds in the half-plane $\operatorname{Re} s > -2n$ with $n \in \mathbb{N}$. In the special case $n = 1$ we get

$$\zeta'(0) = -1 + \frac{1}{12} - \frac{1}{3} I_3(0), \tag{3.20}$$

and similarly for any $k \geq 2$ it follows

$$\zeta^{(k)}(0) = -k! - \frac{k}{3} I_3^{(k-1)}(0) - \frac{k(k-1)}{2} I_3^{(k-2)}(0) - k(k-1)(k-2) I_3^{(k-3)}(0). \tag{3.21}$$

Thus, (3.20) and (3.21) imply successive closed form evaluations for the family of integrals $\left(I_3^{(m)}(0)\right)_{m \in \mathbb{N}_0}$. For more details, we refer the reader to [9]. Now, we are able to generalize (3.19) for a fractional order of differentiation.

Theorem 3.9. *Let $\alpha \in \mathbb{R}_{>0} \backslash \mathbb{N}$. In the half-plane $\operatorname{Re} s > -2n$ with $n \in \mathbb{N}$, we have*

$$\zeta^{(\alpha)}(s) = e^{i\pi\alpha} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{r=1}^{n} \frac{B_{2r}}{2r} Q_{2r-1}^{(\alpha)}(s) - \sum_{j=0}^{\infty} \binom{\alpha}{j} Q_{2n+1}^{(j)}(s)\, I_{2n+1}^{(\alpha-j)}(s), \tag{3.22}$$

with Q_m and I_m as in (3.17).

Proof. From (3.18), the linearity of D_f^α implies that

$$\zeta^{(\alpha)}(s) = D_f^\alpha\left(\frac{1}{s-1}\right) + D_f^\alpha\left(\frac{1}{2}\right) + \sum_{r=1}^{n} \frac{B_{2r}}{2r} Q_{2r-1}^{(\alpha)}(s) - \sum_{j=0}^{\infty} \binom{\alpha}{j} Q_{2n+1}^{(j)} I_{2n+1}^{(\alpha-j)}. \tag{3.23}$$

We claim that

$$D_f^\alpha\left(\frac{1}{2}\right) = 0, \quad \alpha > 0. \tag{3.24}$$

In fact, we have

$$\sum_{n=0}^{k} (-1)^n \binom{\alpha}{n} = (-1)^k \binom{\alpha-1}{k} = \frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(-\alpha+k+1)}{\Gamma(k+1)} \sim \frac{1}{\Gamma(1-\alpha)} \frac{1}{k^\alpha}, \quad k \to \infty,$$

which leads us to (3.24). It remains to compute the term $D_f^\alpha\left(\frac{1}{s-1}\right)$ in (3.23). We recall that the binomial series expansion implies that $D_f^\alpha(e^{-st}) = e^{i\pi\alpha} e^{-st}$ for $\alpha \in \mathbb{R}_{>0} \backslash \mathbb{N}$. Thus,

$$D_f^\alpha\left(\frac{1}{s-1}\right) = \int_0^\infty e^t D_f^\alpha\,[e^{-st}]\,dt = e^{i\pi\alpha} \int_0^\infty e^t t^\alpha e^{-st} dt = e^{i\pi\alpha} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}}, \quad \alpha > -1. \tag{3.25}$$

Note that (3.25) holds for $\alpha > -1$. This condition agrees with the hypothesis of positivity on α. Finally, substituting (3.25) and (3.24) into (3.23), we get (3.22). This completes the proof. □

Remark 3.10. We note that (3.19) differs from (3.22) in only two respects: the order of differentiation and the upper limit of second summation, where k is replaced by ∞, respectively.

The importance of Theorem 3.9 lies in the link between $\zeta^{(\alpha)}$ and Bernoulli numbers. Similar to Theorem 3.8, the consistency of (3.22) plainly follows from (3.19).

4 An application in information theory

We conclude this paper with an application in terms of Shannon entropy. It is worth pointing out that the results of this section are based on results due to Guiasu [13].

Let $\Pi = \{\pi(n) : n \in \mathbb{N}\}$ be a probability distribution on $\mathbb{N}$, i.e. $\pi(n) \geq 0$ for any $n \in \mathbb{N}$ and

$$\sum_{n\in\mathbb{N}} \pi(n) = 1. \tag{4.1}$$

A measure of the global amount of uncertainty related to the probability distribution Π is given by Shannon entropy H defined as follows:

$$H = -\sum_{n\in\mathbb{N}} \pi(n) \log \pi(n).$$

Clearly $H = H(\Pi)$. Many modern techniques in number theory are based on statistics. In particular, non-trivial zeros of the Riemann ζ function can be viewed as a statistical distribution (see, e.g., [18,19]). The Riemann ζ function can be used to maximize H subject to constraints (4.1) and

$$\sum_{n\in\mathbb{N}} \pi(n) \log n = \chi, \quad \chi > 0. \tag{4.2}$$

More precisely, Guiasu proved that the unique solution of the previous problem is given by

$$\pi(n) = \frac{n^{-x}}{\zeta(x)}, \quad n \in \mathbb{N}, \tag{4.3}$$

where x is the unique real number such that $x > 1$ and

$$\chi = \sum_{p\in\mathbb{P}} \frac{\log p}{p^x - 1}. \tag{4.4}$$

The proof is based on the properties of Shannon entropy [13]. Analogously, fractional calculus of zeta functions can also be used to maximize H, as stated below.

Theorem 4.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Z}$. The maximization of Shannon entropy H subject to the constraints in* (4.1) *and*

$$\sum_{n\in\mathbb{N}} \pi(n) \log D_f^\alpha n^{-x} = \chi_\alpha, \quad x > 1 + \alpha, \tag{4.5}$$

has a solution given by

$$\pi(n) = \frac{D_f^\alpha n^{-x}}{\zeta^{(\alpha)}(x)}, \quad n \in \mathbb{N}. \tag{4.6}$$

The proof of Theorem 4.1 is similar in spirit to [13], that is (4.3). Furthermore, $(3.1)_1$ implies that

$$D_f^\alpha n^{-x} = e^{i\pi\alpha} \frac{\log^\alpha n}{n^x}, \quad \alpha \in \mathbb{R}\backslash\mathbb{Z}, \; n \in \mathbb{N}. \tag{4.7}$$

As a consequence, the right-hand side of (4.7) is complex. Put differently, (4.7) generalizes the integer case. In fact, let $m \in \mathbb{N}$. Replacing α by m, the right-hand side of (4.7) gives the derivative $(n^{-x})^{(m)}$. Being $e^{i\pi m} = \pm 1$, the integer derivative of n^{-x} is real. Conversely, the fractional derivative of real-valued functions can also be complex.

Remark 4.2. We note that $x > 1$ implies that $\chi > 0$ in (4.4). Moreover,

$$\pi(n) = \frac{\frac{\log^\alpha n}{n^x}}{\sum_{k\in\mathbb{N}} \frac{\log^\alpha k}{k^x}}, \quad n \in \mathbb{N}.$$

Accordingly, $\pi(n) \geq 0$ for any $n \in \mathbb{N}$ and (4.1) holds. Moreover, $x > 1 + \alpha$ implies convergence of (4.6). Obviously, (4.5) shows that $\chi_\alpha \in \mathbb{C}$, thus making impossible to replace the positivity condition in (4.2).

Remark 4.3. Let $\mathbb{E}$ be the operator of statistical mean with respect to Π. It is worth noting that $\mathbb{E}(\log n) = \chi$. Likewise, $\mathbb{E}(\log D_f^\alpha n^{-x}) = \chi_\alpha$.

Clearly, the solution in (4.6) holds for any fixed $x \in \mathbb{R} : x > 1 + \alpha$. Note that Theorem 4.1 does not prove uniqueness of the solution (4.6). This depends on the lack of a relation equivalent to (4.4) for χ_α. We recall that uniqueness of (4.3) follows [13] from

$$\sum_{p\in\mathbb{P}} \frac{\log p}{p^x - 1} = -\frac{\zeta'(x)}{\zeta(x)}, \quad x > 1.$$

Let us compute the values of H when π is given by (4.3) and (4.6). For the sake of simplicity, we denote the solution in (4.3) and (4.6) by π_ζ and π_α, respectively. Of course, we have

$$H(\pi_\zeta) = \log\zeta(x) - x\frac{\zeta'(x)}{\zeta(x)} = \log\zeta(x) + x\chi. \tag{4.8}$$

and, on the other hand,

$$H(\pi_\alpha) = -\frac{1}{\zeta^{(\alpha)}(x)} \sum_{n\in\mathbb{N}} D_f^\alpha n^{-x}\left(\log D_f^\alpha\, n^{-x} - \log\zeta^{(\alpha)}\right) = -\frac{1}{\zeta^{(\alpha)}(x)} \sum_{n\in\mathbb{N}} D_f^\alpha\, n^{-x} \log D_f^\alpha\, n^{-x} + \log\zeta^{(\alpha)},$$

that is,

$$H(\pi_\alpha) = \log\zeta^{(\alpha)} - \chi_\alpha. \tag{4.9}$$

Moreover,

$$\chi_\alpha \xrightarrow{\alpha\to 0^+} x\frac{\zeta'(x)}{\zeta(x)} = -x\chi,$$

therefore, (4.8) and (4.9) imply that

$$H(\pi_\alpha) \xrightarrow{\alpha\to 0^+} H(\pi_\zeta).$$

We finally note that there is a lack of study on the relation between the distribution of prime numbers and $\zeta^{(\alpha)}$. The author proved [5] that

$$\zeta^{(\alpha)}(s) \sim \sum_{p\in\mathbb{P}} \sum_{t=0}^{\infty} \frac{\log^\alpha p^t}{p^{-st}}, \quad 1 + \alpha < \operatorname{Re} s < 0, \ \alpha < -1,$$

where the symbol $\sim$ means that both sides above converge or diverge together. Thus, fractional calculus of zeta functions and classical theory of zeta functions seem to have similar behaviors with respect to the prime distribution.

Conflict of interest: The author declares no conflict of interest.

References

[1] H. M. Srivastava, *Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis*, Iran. J. Sci. Technol. Trans. A Sci. **44** (2020), no. 1, 327–344, DOI: https://doi.org/10.1007/s40995-019-00815-0.

[2] P. Závada, *Operator of fractional derivative in the complex plane*, Comm. Math. Phys. **192** (1998), no. 2, 261–285, DOI: https://doi.org/10.1007/s002200050299.

[3] E. Guariglia, *Fractional derivative of the Riemann zeta function*, in: C. Cattani, H. M. Srivastava, and X.-J. Yang (eds), Fractional Dynamics, De Gruyter, Warsaw/Berlin, 2015, pp. 357–368, DOI: https://doi.org/10.1515/9783110472097-022.

[4] E. Guariglia and S. Silvestrov, *A functional equation for the Riemann zeta fractional derivative*, AIP Conference Proceedings **1798** (2017), no. 1, 020063, DOI: https://doi.org/10.1063/1.4972738.

[5] E. Guariglia, *Riemann zeta fractional derivative – functional equation and link with primes*, Adv. Differ. Equ. **2019** (2019), no. 1, 261, DOI: https://doi.org/10.1186/s13662-019-2202-5.

[6] C. Li, X. Dao, and P. Guo, *Fractional derivatives in complex planes*, Nonlinear Anal. **71** (2009), no. 5–6, 1857–1869, DOI: https://doi.org/10.1016/j.na.2009.01.021.

[7] E. Guariglia, *Harmonic symmetry of the Riemann zeta fractional derivative*, AIP Conference Proceedings **2046** (2018), no. 1, 020035, DOI: https://doi.org/10.1063/1.5081555.

[8] C. Cattani and E. Guariglia, *Fractional derivative of the Hurwitz ζ-function and chaotic decay to zero*, J. King Saud Univ. Sci. **28** (2016), no. 1, 75–81, DOI: https://doi.org/10.1016/j.jksus.2015.04.003.

[9] T. M. Apostol, *Formulas for higher derivatives of the Riemann zeta function*, Math. Comp. **44** (1985), no. 169, 223–232, DOI: https://doi.org/10.1090/S0025-5718-1985-0771044-5.

[10] A. Torres-Hernandez and F. Brambila-Paz, *An approximation to zeros of the Riemann zeta function using fractional calculus*, arXiv:2006.14963 (2020).

[11] J. P. Keating and N. C. Snaith, *Random matrix theory and* $\zeta(1/2. it)$, Commun. Math. Phys. **214** (2000), no. 1, 57–89, DOI: https://doi.org/10.1007/s002200000261.

[12] J. P. Keating and N. C. Snaith, *Random matrix theory and L-functions at* $s = 1/2$, Commun. Math. Phys. **214** (2000), no. 1, 91–100, DOI: https://doi.org/10.1007/s002200000262.

[13] S. Guiasu, *An optimization problem related to the zeta function*, Canad. Math. Bull. **29** (1986), no. 1, 70–73, DOI: https://doi.org/10.4153/CMB-1986-013-7.

[14] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York/Heidelberg, 2010, pp. 249–267.

[15] J. Choi and H. M. Srivastava, *The multiple Hurwitz zeta function and the multiple Hurwitz-Euler eta function*, Taiwanese J. Math. **15** (2011), no. 2, 501–522, DOI: https://doi.org/10.11650/twjm/1500406218.

[16] B. K. Choudhury, *The Riemann zeta-function and its derivatives*, Proc. R. Soc. Lond. A **450** (1995), no. 1940, 477–499, DOI: https://doi.org/10.1098/rspa.1995.0096.

[17] J. M. Borwein, D. M. Bradley, and R. E. Crandall, *Computational strategies for the Riemann zeta function*, J. Comput. Appl. Math. **121** (2000), no. 1–2, 247–296, DOI: https://doi.org/10.1016/S0377-0427(00)00336-8.

[18] M. M. Rao, *Harmonic and probabilistic approaches to zeros of Riemann's zeta function*, Stoch. Anal. Appl. **30** (2012), no. 5, 906–915, DOI: https://doi.org/10.1080/07362994.2012.704856.

[19] V. Kargin, *Statistical properties of zeta functions' zeros*, Probab. Surv. **11** (2014), 121–160, DOI: https://doi.org/10.1214/13-ps214.

Appendix

Proof of (3.15)

Proof. Note that (3.15) immediately follows from

$$1 + \int_1^\infty \frac{\varphi_2(x)}{x^2} \mathrm{d}x = \frac{\log(2\pi)}{2}.$$

Euler's summation formula [9] implies

$$\sum_{k=1}^n \log k = \int_1^n \log x \mathrm{d}x + \frac{\log n}{2} + \int_1^n \frac{\varphi_2(x)}{x^2} \mathrm{d}x,$$

and so

$$\log n! = \left(n + \frac{1}{2}\right) \log n - n + 1 + \int_1^n \frac{\varphi_2(x)}{x^2} \mathrm{d}x.$$

Some algebraic manipulations give

$$1 + \int_1^n \frac{\varphi_2(x)}{x^2} \mathrm{d}x = \log \frac{n! \mathrm{e}^n}{n^{n+1/2}}.$$

Getting the limit in the last equality as $n \to \infty$, we have

$$1 + \int_1^\infty \frac{\varphi_2(x)}{x^2} \mathrm{d}x = \log \lim_{n \to \infty} \frac{n! \mathrm{e}^n}{n^{n+1/2}}.$$

Definition (3.12) implies that $|\varphi_2(x)| \leq \frac{1}{8}$, thus the improper integral above converges absolutely. Moreover, Stirling's formula entails that

$$\lim_{n \to \infty} \frac{n! \mathrm{e}^n}{n^{n+1/2}} = \sqrt{2\pi}.$$

The proof is complete. □